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# An example of non-uniqueness for Radon transforms with continuous positive rotation invariant weights

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## Abstract

We consider weighted Radon transforms  $R_W$  along hyperplanes in  $\mathbb{R}^3$  with strictly positive weights  $W$ . We construct an example of such a transform with non-trivial kernel  $\text{Ker}R_W$  in the space of infinitely smooth compactly supported functions and with continuous weight. Moreover, in this example the weight  $W$  is rotation invariant. In particular, by this result we continue studies of Quinto (1983), Markoe, Quinto (1985), Boman (1993) and Goncharov, Novikov (2017). We also extend our example to the case of weighted Radon transforms along two-dimensional planes in  $\mathbb{R}^d$ ,  $d \geq 3$ .

**Keywords:** Radon transforms, integral geometry, injectivity, non-injectivity

**AMS Mathematics Subject Classification:** 44A12, 53C65, 65R32

## 1 Introduction

We consider weighted Radon transforms  $R_W$  in  $\mathbb{R}^d$  defined by

$$R_W f(s, \theta) = \int_{x\theta=s} W(x, \theta) f(x) dx, \quad (1.1)$$
$$s \in \mathbb{R}, \theta \in \mathbb{S}^{d-1}, x \in \mathbb{R}^d, d \geq 2,$$

where  $W = W(x, \theta)$  is the weight,  $f = f(x)$  is a test function on  $\mathbb{R}^d$ .

We assume that

$$W = \overline{W} \geq c > 0, W \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1}), \quad (1.2)$$

where  $\overline{W}$  denotes the complex conjugate of  $W$ ,  $c$  is a constant.

The aforementioned transforms  $R_W$  arise in various domains of pure and applied mathematics; see, e.g., [Bey84], [Bey85], [BQ87], [Bom93], [Fi86], [LB73], [GN16], [Gon17], [GN17], [Kun92], [Natt01], [Nov14], [Qui83], [Qui83Err] and references therein.

In particular, studies on the transforms  $R_W$  under assumptions (1.2) were recently continued in [GN16], [Gon17], [GN17] for  $d \geq 3$ .

Note that the works [GN16], [Gon17] extend to the case of  $R_W$ ,  $d \geq 3$ , the two-dimensional injectivity and reconstruction results of [Kun92], [Nov11], [Nov14], [GuiNov14].

On the other hand, under assumptions (1.2), the work [GN17] gives an example of  $R_W$ ,  $d \geq 3$ , with non-trivial kernel in  $C_0^\infty(\mathbb{R}^d)$  (infinitely smooth functions with compact support). This example was constructed in [GN17] proceeding from the example of non-uniqueness of [Bom93] for  $R_W$  in  $\mathbb{R}^2$  and a recent result of [GN16].

In the two-dimensional example of non-uniqueness of [Bom93] the weight  $W$  satisfies (1.2), for  $d = 2$ , and is infinitely smooth everywhere. In the multidimensional example of non-uniqueness of [GN17] the weight satisfies (1.2), for  $d \geq 3$ , is infinitely smooth almost everywhere but is not yet continuous at some points.

In the present work we construct an example of  $R_W$ , for  $d = 3$ , with non-trivial kernel in  $C_0^\infty(\mathbb{R}^3)$ , where  $W$  satisfies (1.2) and is continuous everywhere. Moreover, in this example  $W$  is rotation invariant and  $R_W f \equiv 0$  for some non-zero spherically symmetric  $f \in C_0^\infty(\mathbb{R}^3)$ .

The rotation invariance of the latter example is its principal advantage in comparison with the aforementioned examples of [Bom93] and [GN17].

By our rotation invariant example of non-uniqueness we also continue studies of [MQ85], where an example of non-uniqueness for  $R_W$  was constructed for  $d = 2$ . In the example of [MQ85] the weight  $W$  is

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bounded and positive but is not yet continuous and strictly positive. The continuity and strict positivity of  $W$  is the principal advantage of the example of the present work in comparison with the example of [MQ85].

Following [Qui83] we say that  $W$  is rotation invariant if and only if

$$W(x, \theta) = U(|x - (x\theta)\theta|, x\theta), \quad x \in \mathbb{R}^d, \theta \in \mathbb{S}^{d-1}, \quad (1.3)$$

for some positive and continuous  $U$  such that

$$U(r, s) = U(-r, s) = U(r, -s), \quad r \in \mathbb{R}, s \in \mathbb{R}. \quad (1.4)$$

On the other hand, we recall that weighted Radon transforms  $R_W$  in  $\mathbb{R}^d$  with smooth weights  $W$  satisfying properties (1.2), (1.3), (1.4) are injective for  $f \in L_0^2(\mathbb{R}^d)$  (square integrable functions on  $\mathbb{R}^d$  with compact support); see [Qui83]. Here the smoothness of  $W$  can be specified, at least, as  $C^1$  for  $d = 2$  and  $d = 3$ . In view of the aforementioned counterexamples of [MQ85] and of the present work, some smoothness of  $W$  is crucial for these injectivity results.

In the present work we also extend our rotation invariant example of non-uniqueness for  $R_W$  in  $\mathbb{R}^3$  to the case of weighted Radon transforms  $R_W^{d,2}$  along two-dimensional planes in  $\mathbb{R}^d$ ,  $d > 3$ ; see Section 4. In this case  $R_W^{d,2}f$  is defined on  $\mathcal{P}^{d,2}$  (manifold of all oriented two-dimensional planes in  $\mathbb{R}^d$ ) and is overdetermined already. That is

$$\dim \mathcal{P}^{d,2} = 3d - 6 > \dim \mathbb{R}^d = d \text{ for } d > 3. \quad (1.5)$$

Nevertheless,  $R_W^{d,2}f \equiv 0$  on  $\mathcal{P}^{d,2}$  in our result.

We expect that the results of the present work admit generalizations to the weighted Radon transforms  $R_W^{d,n}$  along  $n$ -dimensional planes in  $\mathbb{R}^d$  for arbitrary  $d$  and  $n$  such that  $1 \leq n < d$ ,  $d \geq 2$ . For  $n = 1$  such results are already obtained in [GonNov17].

Note also that the construction of the present work was developed in a large extent in the process of adopting the Boman's construction of the aforementioned work [Bom93].

In Section 2 we give some preliminaries.

Our main results are formulated in detail in Sections 3, 4.

Proofs are given in Sections 5-8.

## 2 Some preliminaries

**Notations for  $d = 3$ .** Let

$$B = \{x \in \mathbb{R}^3 : |x| < 1\}, \bar{B} = \{x \in \mathbb{R}^3 : |x| \leq 1\}, \quad (2.1)$$

$$\mathcal{P} = \mathbb{R} \times \mathbb{S}^2, \quad (2.2)$$

$$\mathcal{P}_0(\delta) = \{(s, \theta) \in \mathcal{P} : |s| > \delta\}, \quad (2.3)$$

$$\mathcal{P}_1(\delta) = \mathcal{P} \setminus \mathcal{P}_0(\delta) = \{(s, \theta) \in \mathcal{P} : |s| \leq \delta\}, \delta > 0, \quad (2.4)$$

$$\mathcal{P}(\Lambda) = \{(s, \theta) \in \mathcal{P} : s \in \Lambda\}, \Lambda \subset \mathbb{R}, \quad (2.5)$$

$$\Omega(\Lambda) = \{(x, \theta) \in \mathbb{R}^3 \times \mathbb{S}^2 : x\theta \in \Lambda\}, \Lambda \subset \mathbb{R}, \quad (2.6)$$

$$\mathcal{J}_{s,\varepsilon} = \mathcal{J}_{|s|,\varepsilon} = (-|s| - \varepsilon, -|s| + \varepsilon) \cup (|s| - \varepsilon, |s| + \varepsilon) \subset \mathbb{R}, s \in \mathbb{R}, \varepsilon > 0. \quad (2.7)$$

Note that  $\mathcal{P}_0(\delta), \mathcal{P}_1(\delta)$  of (2.3), (2.4) are particular cases of  $\mathcal{P}(\Lambda)$  of formula (2.5).

In addition, we interpret  $\mathcal{P}$  as the set of all oriented planes in  $\mathbb{R}^3$ . If  $P = (s, \theta) \in \mathcal{P}$ , then

$$P = P_{(s,\theta)} = \{x \in \mathbb{R}^3 : x\theta = s\} \text{ (modulo orientation)} \quad (2.8)$$

and  $\theta$  gives the orientation of  $P$  (in the sense that ordered tuple  $(e_1, e_2, \theta)$  is positively oriented in  $\mathbb{R}^3$  with any orthonormal positively oriented basis  $e_1, e_2$  on  $P$ ).

The set  $\mathcal{P}_0(\delta)$  in (2.3) is considered as the set of all oriented planes in  $\mathbb{R}^3$  which are positioned at distance greater than  $\delta$  from the origin.

The set  $\mathcal{P}_1(\delta)$  in (2.4) is considered as the set of all oriented planes in  $\mathbb{R}^3$  which are located at distance less or equal than  $\delta$ .

**Rotation invariancy for  $d = 3$ .** Symmetries (1.3), (1.4) of  $W$  can be also written as

$$W(x, \theta) = \tilde{U}(|x|, x\theta), \quad x \in \mathbb{R}^3, \theta \in \mathbb{S}^2, \quad (2.9)$$

$$\tilde{U}(r, s) = \tilde{U}(r, -s), \quad \tilde{U}(r, s) = \tilde{U}(-r, s), \quad r \in \mathbb{R}, s \in \mathbb{R}, \quad (2.10)$$

where  $\tilde{U}$  is positive and continuous on  $\mathbb{R} \times \mathbb{R}$ . Using the formula  $|x|^2 = |x\theta|^2 + |x - (x\theta)\theta|^2$ ,  $\theta \in \mathbb{S}^2$ , one can see that symmetries (1.3), (1.4) and symmetries (2.9), (2.10) of  $W$  are equivalent.

**Additional notations.** For a function  $f$  on  $\mathbb{R}^d$  we denote its restriction to a subset  $\Sigma$  by  $f|_\Sigma$ .

By  $C_0, C_0^\infty$  we denote continuous compactly supported and infinitely smooth compactly supported functions, respectively.

**Partition of unity.** We recall the following classical result (see Theorem 5.6 in [MCar92]):  
Let  $\mathcal{M}$  be a  $C^\infty$ -manifold, which is Hausdorff and satisfies second countability axiom (i.e. has countable base). Let also  $\{U_i\}_{i=1}^\infty$  be the open locally-finite cover of  $\mathcal{M}$ .

Then there exists a  $C^\infty$ -smooth locally-finite partition of unity  $\{\psi_i\}_{i=1}^\infty$  on  $\mathcal{M}$ , such that

$$\text{supp } \psi_i \subset U_i. \quad (2.11)$$

In particular, any open interval  $(a, b) \subset \mathbb{R}$  and  $\mathcal{P} \simeq \mathbb{R} \times \mathbb{S}^2$  satisfy conditions of the aforementioned statement. It will be used in Subsection 3.4.

### 3 Main results for $d = 3$

**Theorem 1.** *There exist a non-zero spherically symmetric function  $f \in C_0^\infty(\mathbb{R}^3)$  with support in  $\overline{B}$ , and  $W$  satisfying (1.2)-(1.4) such that*

$$R_W f \equiv 0, \quad (3.1)$$

where  $R_W$  is defined in (1.1).

The construction of  $f$  and  $W$  proving Theorem 1 is presented below in this section. This construction adopts considerations of [Bom93]. In particular, we construct  $f$ , first, and then  $W$ .

#### 3.1 Construction of $f$

The function  $f$  is constructed as follows:

$$f = \sum_{k=1}^{\infty} \frac{f_k}{k!}, \quad (3.2)$$

$$f_k(x) = f_k(|x|) = \Phi(2^k(1 - |x|)) \cos(8^k |x|^2), x \in \mathbb{R}^3, k = 1, 2, \dots, \quad (3.3)$$

for arbitrary  $\Phi \in C^\infty(\mathbb{R})$  such that

$$\text{supp } \Phi = [4/5, 6/5], \quad (3.4)$$

$$0 < \Phi(t) \leq 1 \text{ for } t \in (4/5, 6/5), \quad (3.5)$$

$$\Phi(t) = 1 \text{ for } t \in [9/10, 11/10]. \quad (3.6)$$

Properties (3.4), (3.5) imply that functions  $f_k$  in (3.3) have disjoint supports and series (3.2) converges for every fixed  $x \in \mathbb{R}^3$ .

**Lemma 1.** *Let  $f$  be defined by (3.2)-(3.6). Then  $f$  is spherically symmetric,  $f \in C_0^\infty(\mathbb{R}^3)$  and  $\text{supp } f \subseteq \overline{B}$ . In addition, if  $P \in \mathcal{P}$ ,  $P \cap B \neq \emptyset$ , then  $f|_P \not\equiv 0$  and  $f|_P$  has non-constant sign.*

Lemma 1 is proved in Section 5.

#### 3.2 Construction of $W$

In our example  $W$  is of the following form:

$$\begin{aligned} W(x, \theta) &= \sum_{i=0}^N \xi_i(|x\theta|) W_i(x, \theta) \\ &= \xi_0(|x\theta|) W_0(x, \theta) + \sum_{i=1}^N \xi_i(|x\theta|) W_i(x, \theta), x \in \mathbb{R}^3, \theta \in \mathbb{S}^2, \end{aligned} \quad (3.7)$$

where

$$\{\xi_i(s), s \in \mathbb{R}\}_{i=0}^N \text{ is a } C^\infty\text{-smooth partition of unity on } \mathbb{R}, \quad (3.8)$$

$$\xi_i(s) = \xi_i(-s), s \in \mathbb{R}, i = \overline{0, N}, \quad (3.9)$$

$$\begin{aligned} W_i(x, \theta) \text{ are bounded continuous strictly positive and} \\ \text{rotation invariant (according to (1.3), (1.4)) on } \text{supp } \xi_i(|x\theta|), i = \overline{0, N}, \text{ respectively.} \end{aligned} \quad (3.10)$$

From (3.7)-(3.10) it follows that  $W$  of (3.7) satisfies the conditions (1.2)-(1.4).

The weight  $W_0$  is constructed in Subsection 3.3 and has the following properties:

$$W_0 \text{ is bounded, continuous and rotation invariant on } \{(x, \theta) : |x\theta| > 1/2\}, \quad (3.11)$$

there exists  $\delta_0 \in (1/2, 1)$  such that:

$$W_0(x, \theta) \geq 1/2 \text{ if } |x\theta| > \delta_0, \quad (3.12)$$

$$W_0(x, \theta) = 1 \text{ if } |x\theta| \geq 1,$$

$$R_{W_0} f(s, \theta) = 0 \text{ for } |s| > 1/2, \theta \in \mathbb{S}^2, \quad (3.13)$$

where  $R_{W_0}$  is defined according to (1.1) for  $W = W_0$ ,  $f$  is given by (3.2), (3.3).

In addition,

$$\text{supp } \xi_0 \subset (-\infty, -\delta_0) \cup (\delta_0, +\infty), \quad (3.14)$$

$$\xi_0(s) = 1 \text{ for } |s| \geq 1, \quad (3.15)$$

where  $\delta_0$  is the number of (3.12).

In particular, from (3.8), (3.12), (3.14) it follows that

$$W_0(x, \theta)\xi_0(|x\theta|) > 0 \text{ if } \xi_0(|x\theta|) > 0. \quad (3.16)$$

**Remark 1.** The result of (3.11)-(3.13) can be considered as a counterexample to the Cormack-Helgason support theorem (see Theorem 3.1 in [Natt01]) in the framework of the theory of weighted Radon transforms under assumptions (1.2) and even under assumptions (1.2)-(1.4).

In addition,

$$\xi_i(|x\theta|)W_i(x, \theta) \text{ are bounded, continuous and rotation invariant on } \mathbb{R}^3 \times \mathbb{S}^2, \quad (3.17)$$

$$W_i(x, \theta) \geq 1/2 \text{ if } \xi_i(|x\theta|) \neq 0, \quad (3.18)$$

$$R_{W_i}f(s, \theta) = 0 \text{ if } \xi_i(|s|) \neq 0, \quad (3.19)$$

$$i = \overline{1, N}, x \in \mathbb{R}^3, \theta \in \mathbb{S}^2, s \in \mathbb{R}. \quad (3.20)$$

Weights  $W_1, \dots, W_N$  of (3.7) and  $\{\xi_i\}_{i=0}^N$  are constructed in Subsection 3.4.

Result of Theorem 1 follows from Lemma 1 and formulas (3.7)-(3.9), (3.11)-(3.13), (3.16)-(3.20).

We point out that the construction of  $W_0$  of (3.7) is substantially different from the construction of  $W_1, \dots, W_N$ . In particular, the weight  $W_0$  is defined on the planes  $P \in \mathcal{P}$  which can be close to the boundary  $\partial B$  of  $B$  which results in restrictions on the smoothness of  $W_0$ .

### 3.3 Construction of $W_0$

Let

$$\{\psi_k\}_{k=1}^\infty \text{ be a } C^\infty \text{ partition of unity on } (1/2, 1), \text{ such that } \text{supp } \psi_k \subset (1 - 2^{-k+1}, 1 - 2^{-k-1}), k \in \mathbb{N}. \quad (3.21)$$

Note that

$$1 - 2^{-(k-2)-1} < 1 - 2^{-k}(6/5), k \geq 3. \quad (3.22)$$

Therefore,

$$\forall s_0, t_0 \in \mathbb{R} : s_0 \in \text{supp } \psi_{k-2}, t_0 \in \text{supp } \Phi(2^k(1-t)) \Rightarrow s_0 < t_0, k \geq 3. \quad (3.23)$$

The weight  $W_0$  is defined by the following formulas

$$W_0(x, \theta) = \begin{cases} 1 - G(x, \theta) \sum_{k=3}^\infty k! f_k(x) \frac{\psi_{k-2}(|x\theta|)}{H_k(x, \theta)}, & 1/2 < |x\theta| < 1, \\ 1, & |x\theta| \geq 1 \end{cases}, \quad (3.24)$$

$$G(x, \theta) = \int_{y\theta=x\theta} f(y) dy, H_k(x, \theta) = \int_{y\theta=x\theta} f_k^2(y) dy, \quad (3.25)$$

$$x \in \mathbb{R}^3, \theta \in \mathbb{S}^2,$$

where  $f_k$  are defined in (3.3).

Formula (3.24) implies that  $W_0$  is defined on  $\mathcal{P}_0(1/2) \subset \mathcal{P}$ . Due to (3.3) and (3.23), in (3.25) we have that  $H_k(x, \theta) \neq 0$  if  $\psi_{k-2}(|x\theta|) \neq 0$ .

Also, for any fixed  $(x, \theta) \in \mathbb{R}^3 \times \mathbb{S}^2$ ,  $1/2 < |x\theta| < 1$ , the series in the right hand-side of (3.24) has only a finite number of non-zero terms (in fact, no more than two) and, hence,  $W_0$  is well-defined.

By the spherical symmetry of  $f$ , functions  $G, H_k$  in (3.24) are of the type (2.9), (2.10). Therefore,  $W_0$  is rotation invariant (in the sense (2.9), (2.10)).

Actually, formula (3.13) follows from (3.2)-(3.4), (3.24), (3.25) (see Subsection 5.2 for details).

Using the construction of  $W_0$  and the assumption that  $|x\theta| > 1/2$  (implying that  $\text{sign}(x\theta)$  is locally constant) one can see that  $W_0$  is  $C^\infty$  on its domain of definition, possibly, except points with  $|x\theta| = 1$ .

**Lemma 2.** Let  $W_0$  be defined by (3.24), (3.25). Then the following estimate holds:

$$|1 - W_0(x, \theta)| \leq C_0 \rho(|x\theta|) \left( \log_2 \frac{1}{\rho(|x\theta|)} \right)^4, \quad (3.26)$$

$$W_0(x, \theta) \rightarrow 1 \text{ as } |x\theta| \rightarrow 1, \quad (3.27)$$

$$x \in \mathbb{R}^3, \theta \in \mathbb{S}^2, 1/2 < |x\theta| < 1.$$

where  $\rho = \rho(s) = 1 - s$ ,  $s \in (1/2, 1)$ ,  $C_0$  is a positive constant depending on  $\Phi$ .

Lemma 2 is proved in Section 6.

The result of Lemma 2 completes the proof of (3.12).

This completes the description of  $W_0$  and  $\delta_0$ .

### 3.4 Construction of $W_1, \dots, W_N$ and $\xi_0, \dots, \xi_N$

**Lemma 3.** Let  $f \in C_0(\mathbb{R}^3)$  be spherically symmetric,  $P_{(s_0, \theta_0)} \in \mathcal{P}$ ,  $f|_{P_{(s_0, \theta_0)}} \neq 0$  and  $f|_{P_{(s_0, \theta_0)}}$  changes the sign. Then:

(i) there exist  $\varepsilon > 0$  and weight  $W_{f, s_0, \varepsilon}$  such that

$$RW_{f, s_0, \varepsilon} f(s, \theta) = 0 \text{ for } s \in \mathcal{J}_{s_0, \varepsilon}, \theta \in \mathbb{S}^2, \quad (3.28)$$

where  $\mathcal{J}_{s, \varepsilon}$  is defined in (2.7),  $W_{f, s_0, \varepsilon}$  is defined on the open set  $\Omega(\mathcal{J}_{s_0, \varepsilon})$ , defined by (2.6);

(ii) weight  $W_{f, s_0, \varepsilon}$  is bounded, continuous, strictly positive and rotation invariant on  $\Omega(\mathcal{J}_{s_0, \varepsilon})$ .

Lemma 3 is proved in Section 7.

Let  $f$  be the function of (3.2), (3.3). Then, using Lemmas 1, 3 one can see that

$$\begin{aligned} \forall \delta \in (0, 1) \text{ there exist } \{J_i = \mathcal{J}_{s_i, \varepsilon_i}, W_i = W_{f, s_i, \varepsilon_i}\}_{i=1}^N \\ \text{such that } J_i, i = \overline{1, N}, \text{ is an open cover of } [-\delta, \delta] \\ \text{and } W_i \text{ satisfy (i) and (ii) (of Lemma 3) on } \Omega(J_i). \end{aligned} \quad (3.29)$$

Actually, we consider (3.29) for the case of  $\delta = \delta_0$  of (3.12).

Note that in this case  $\{\mathcal{P}(J_i)\}_{i=1}^N$  for  $J_i$  of (3.29) is an open cover of  $\mathcal{P}_1(\delta_0)$ .

To the set  $\mathcal{P}_0(\delta_0)$  we associate the open set

$$J_0 = (-\infty, \delta_0) \cup (\delta_0, +\infty) \subset \mathbb{R}. \quad (3.30)$$

Therefore, the collection of intervals  $\{J_i, i = \overline{0, N}\}$  is an open cover of  $\mathbb{R}$ .

We construct the partition of unity  $\{\xi_i\}_{i=0}^N$  on  $\mathbb{R}$  as follows:

$$\xi_i(s) = \xi_i(|s|) = \frac{1}{2}(\tilde{\xi}_i(s) + \tilde{\xi}_i(-s)), \quad s \in \mathbb{R}, \quad (3.31)$$

$$\text{supp } \xi_i \subset J_i, \quad i = \overline{0, N}, \quad (3.32)$$

where  $\{\tilde{\xi}_i\}_{i=0}^N$  is a partition of unity for the open cover  $\{J_i\}_{i=0}^N$  (see Section 2, Partition of unity, for  $U_i = J_i$ ).

Properties (3.14), (3.32) follow from (2.11) for  $\{\tilde{\xi}_i\}_{i=0}^N$  (with  $U_i = J_i$ ), the symmetry of  $J_i = \mathcal{J}_{s_i, \varepsilon_i}$ ,  $i = \overline{1, N}$ , choice of  $J_0$  in (3.30) and from (3.31).

In addition, (3.15) follows from (3.30) and the construction of  $J_i, i = \overline{1, N}$ , from (3.29) (see the proof of Lemma 3 and properties (3.29) in Section 7 for details).

Properties (3.17)-(3.20) follow from (3.29) for  $\delta = \delta_0$  and from (3.30)-(3.32).

## 4 Extension to the case of $R_W^{d,2}$

We consider the weighted Radon transforms  $R_W^{d,2}$  along two-dimensional planes in  $\mathbb{R}^d$ , defined by

$$R_W^{d,2} f(P) = \int_P W(x, P) f(x) dx, \quad P \in \mathcal{P}^{d,2}, \quad x \in P, \quad d \geq 3, \quad (4.1)$$

where  $W = W(x, P)$  is the weight,  $f = f(x)$  is a test function on  $\mathbb{R}^d$ ,

$$\mathcal{P}^{d,2} \text{ is the manifold of all oriented two-dimensional planes } P \text{ in } \mathbb{R}^d. \quad (4.2)$$

Note that the transform  $R_W^{d,2}$  is reduced to  $R_W$  of (1.1) for  $d = 3$ .

We say that  $W$  in (4.1) is rotation invariant if and only if

$$W(x, P) = \tilde{U}(|x|, \text{dist}(P, \{0\})), \quad (4.3)$$

$$\tilde{U}(r, s) = \tilde{U}(r, -s), \quad \tilde{U}(r, s) = \tilde{U}(-r, s), \quad r \in \mathbb{R}, \quad s \in \mathbb{R}, \quad (4.4)$$

where  $\tilde{U}$  is some positive and continuous function on  $\mathbb{R} \times \mathbb{R}$ ,  $\text{dist}(P, \{0\})$  denotes the distance from the origin  $\{0\} \in \mathbb{R}^d$  to the plane  $P$ . Note that  $W(x, P)$  is independent of the orientation of  $P$  in this case.

Consider  $\tilde{U}$  and  $\tilde{f}$  such that

$$W(x, \theta) = \tilde{U}(|x|, |x\theta|), \quad f(x) = \tilde{f}(|x|), \quad x \in \mathbb{R}^3, \quad \theta \in \mathbb{S}^2, \quad (4.5)$$

for  $W$  and  $f$  of Theorem 1 of Section 3.

Theorem 1 implies the following corollary:

**Corollary 1.** Let  $W$  and  $f$  be defined as

$$W(x, P) = \tilde{U}(|x|, \text{dist}(P, \{0\})), P \in \mathcal{P}^{d,2}, x \in P, \quad (4.6)$$

$$f(x) = \tilde{f}(|x|), x \in \mathbb{R}^d, \quad (4.7)$$

where  $\tilde{U}, \tilde{f}$  are the functions of (4.5),  $d > 3$ . Then

$$R_W^{d,2} f \equiv 0 \text{ on } \mathcal{P}^{d,2}. \quad (4.8)$$

In addition, the weight  $W$  is continuous strictly positive and rotation invariant,  $f$  is infinitely smooth compactly supported on  $\mathbb{R}^d$  and  $f \not\equiv 0$ .

Formula (4.8) is proved as follows:

$$R_W^{d,2}(P) = \int_P \tilde{U}(|x|, \text{dist}(P, \{0\})) \tilde{f}(|x|) dx = I \stackrel{\text{def}}{=} \int_{P'} \tilde{U}(|x|, s) \tilde{f}(|x|) dx, \quad (4.9)$$

$$P' = \{se_3 + x_1e_1 + x_2e_2 : x = (x_1, x_2) \in \mathbb{R}^2\}, s = \text{dist}(P, \{0\}),$$

where  $(e_1, \dots, e_d)$  is the standard basis in  $\mathbb{R}^d$ . In addition,  $I = 0$  by Theorem 1.

Properties of  $W$  and  $f$  mentioned in Corollary 1 follow from definitions (4.6), (4.7) and properties of  $\tilde{U}$  and  $\tilde{f}$  (arising in Theorem 1).

## 5 Proofs of Lemma 1 and formula (3.13)

### 5.1 Proof of Lemma 1

The spherical symmetry of  $f$  follows from (3.2), (3.3).

The series in (3.2) converges uniformly with all derivatives of  $f_k$ . Therefore,  $f \in C^\infty(\mathbb{R}^3)$ . Due to (3.2), (3.3), (3.4), (3.5) we have that  $\text{supp } f_k \subset B$ ,  $k \geq 1$ . Therefore,  $\text{supp } f \subseteq \overline{B}$ .

It remains to show that  $f$  restricted to any straight line  $l$  in  $\mathbb{R}^3$  intersecting  $B$  changes the sign. This implies change of the sign for  $f|_P$  for any plane  $P$  such that  $P \cap B \neq \emptyset$ .

We consider

$$D_k = \{x \in \mathbb{R}^3 : |x| \in (1 - 2^{-k}(6/5), 1 - 2^{-k}(4/5))\}, k \geq 1, \quad (5.1)$$

$$l(x_0, \omega) = \{x \in \mathbb{R}^3 : x = x(t) = x_0 + \omega t, t \in \mathbb{R}\}, \omega \in \mathbb{S}^2, x_0 \in \mathbb{R}^3, x_0 \omega = 0. \quad (5.2)$$

Note that  $\text{supp } f_k = \overline{D}_k \subset B$ . Note also that the line  $l(x_0, \omega)$  intersects  $B$  if and only if  $|x_0| < 1$ .

Assuming that

$$|x_0| < 1 - 2^{-k}(6/5), \quad (5.3)$$

we consider  $D_k \cap l(x_0, \omega) = I_k^- \sqcup I_k^+$  (see Figure 1):

$$I_k^- = \{x(t) : t \in (-t_1, -t_0)\}, \quad (5.4)$$

$$I_k^+ = \{x(t) : t \in (t_0, t_1)\}, \quad (5.5)$$

$$t_0 := t_0(k), t_1 := t_1(k).$$

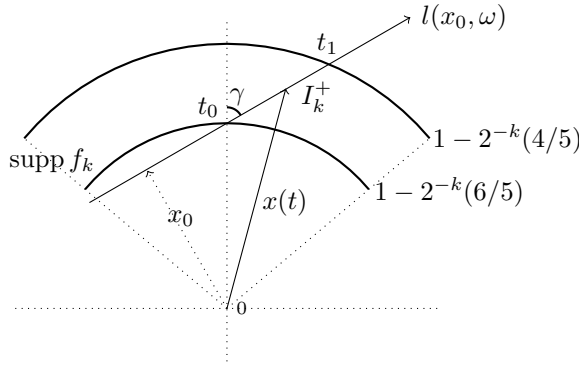


Figure 1.

One can see that assumption (5.3) holds for all  $k \geq k_0(|x_0|) = -\ln(\frac{5}{6}(1 - |x_0|))$ .

By the Cosine theorem we have (see Figure 1):

$$\varphi(t) := |x(t)|^2 = (t - t_0)^2 + |x(t_0)|^2 - 2|x(t_0)|(t - t_0) \cos(\pi - \gamma) \text{ for } t \in [t_0, t_1]. \quad (5.6)$$

One can see also that

$$\gamma \in [0, \pi/2], \cos(\pi - \gamma) \leq 0. \quad (5.7)$$

Let

$$g_k(t) := \cos(8^k \varphi(t)), t \in [t_1, t_2], \quad (5.8)$$

where  $\varphi(t)$  is defined in (5.6).

It is sufficient to show that  $g_k$  changes the sign on  $(t_0, t_1)$  for sufficiently large  $k$ .

Due to (3.2)-(3.5) this implies that  $f$  changes the sign on  $I_k^+$ .

From (5.6), (5.7) we obtain the inequality

$$\varphi'(t) = 2(t - t_0) - 2|x(t_0)| \cos(\pi - \gamma) \geq 0 \text{ for } t \in (t_0, t_1), \gamma \in [0, \pi/2], \quad (5.9)$$

which implies the phase in (5.8) is monotonously increasing on  $t \in (t_0, t_1)$ .

The full variation  $V_{(t_0, t_1)}(\varphi)$  of the monotonous phase  $\varphi(t)$  on  $(t_0, t_1)$  is given by the formula

$$V_{(t_0, t_1)}(\varphi) = (t_1 - t_0)^2 - 2|x(t_0)|(t_1 - t_0) \cos(\pi - \gamma). \quad (5.10)$$

From (5.7), (5.10) we obtain the following inequality

$$V_{(t_0, t_1)}(\varphi) \geq (t_1 - t_0)^2. \quad (5.11)$$

From (5.8), (5.11) it follows that  $g_k$  changes the sign on  $t \in (t_0, t_1)$ , for example, if

$$8^k V_{(t_0, t_1)}(\varphi) \geq 2\pi \text{ or } (t_1 - t_0) \geq \sqrt{2\pi} 4^{-k}. \quad (5.12)$$

On the other hand,  $(t_1 - t_0)$  is exactly the length of the segment  $I_k^+$  (see Figure 1). Therefore,

$$(t_1 - t_0) \geq (2/5)2^{-k}. \quad (5.13)$$

Inequality (5.13) implies that (5.12) holds for  $k \geq 3$ . Therefore,  $g_k$  of (5.8) changes the sign on  $(t_0, t_1)$  starting from  $k \geq \max(3, k_0(|x_0|))$ .

Lemma 1 is proved.

## 5.2 Proof of formula (3.13)

From (1.1), (3.2)-(3.5), (3.21), (3.24), (3.25) it follows that:

$$R_{W_0} f(s, \theta) = \int_{x\theta=s} f(x) dx - G(s\theta, \theta) \sum_{k=3}^{\infty} k! \psi_{k-2}(|s|) \frac{\int f(x) f_k(x) dx}{H_k(s\theta, \theta)} \quad (5.14)$$

$$= \int_{x\theta=s} f(x) dx - \int_{x\theta=s} f(x) dx \sum_{k=3}^{\infty} \psi_{k-2}(|s|) \frac{\int f_k^2(x) dx}{\int_{x\theta=s} f_k^2(x) dx} \quad (5.15)$$

$$= \int_{x\theta=s} f(x) dx - \int_{x\theta=s} f(x) dx \sum_{k=3}^{\infty} \psi_{k-2}(|s|) = 0, |s| > 1/2, \theta \in \mathbb{S}^2. \quad (5.16)$$

Formula (3.13) is proved.

## 6 Proof of Lemma 2

Let

$$\Lambda_k := \{(x, \theta) \in \mathbb{R}^3 \times \mathbb{S}^2 : |x\theta| \in (1 - 2^{-k+3}, 1 - 2^{-k+1})\}, k \in \mathbb{N}, k \geq 4. \quad (6.1)$$

From (3.21) it follows that, for  $k \geq 4$ :

$$\text{supp } \psi_{k-1} \subset (1 - 2^{-k+2}, 1 - 2^{-k}), \quad (6.2)$$

$$\text{supp } \psi_{k-2} \subset (1 - 2^{-k+3}, 1 - 2^{-k+1}), \quad (6.3)$$

$$\text{supp } \psi_{k-3} \subset (1 - 2^{-k+4}, 1 - 2^{-k+2}). \quad (6.4)$$

Formulas (3.24), (3.25), (6.2)-(6.4) imply the following expression for  $W_0(x, \theta)$ :

$$W_0(x, \theta) = 1 - G(x, \theta) \left( (k-1)! f_{k-1}(x) \frac{\psi_{k-3}(|x\theta|)}{H_{k-1}(x, \theta)} + k! f_k(x) \frac{\psi_{k-2}(|x\theta|)}{H_k(x, \theta)} + (k+1)! f_{k+1}(x) \frac{\psi_{k-1}(|x\theta|)}{H_{k+1}(x, \theta)} \right), (x, \theta) \in \Lambda_k, k \geq 4. \quad (6.5)$$



**Lemma 4.** *There are positive constants  $c_1, c_2, k_1$  depending on  $\Phi$ , such that*

$$|f_k(x)| \leq c_1, \text{ for } k \in \mathbb{N}, \quad (6.6)$$

$$\left| \frac{\psi_{k-2}(|x\theta|)}{H_k(x, \theta)} \right| \leq c_2 2^k \quad \text{for } k \geq k_1 \text{ and } |x\theta| \leq 1 - 2^{-k+1}, \quad (6.7)$$

$$\begin{aligned} |G(x, \theta)| &\leq c_1 \frac{4^{-k}}{k!} \quad \text{for } k \geq 3 \text{ and } |x\theta| \geq 1 - 2^{-k}, \\ x \in \mathbb{R}^3, \theta \in \mathbb{S}^2, 1/2 < |x\theta| < 1, \end{aligned} \quad (6.8)$$

where  $f_k, G, H_k$  are defined in (3.3), (3.25).

Lemma 4 is proved in Section 8.

From definition (6.1) and estimates (6.7), (6.8) it follows that

$$|G(x, \theta)| \leq c_1 4^{-k+3}/(k-3)!, \quad (6.9)$$

$$\left| \frac{\psi_{k-2}(|x\theta|)}{H_k(x, \theta)} \right| \leq c_2 2^k, \quad (6.10)$$

for  $(x, \theta) \in \Lambda_k, k \geq \max(4, k_1)$ .

In addition, properties (6.2)-(6.4) and estimate (6.7) imply that:

$$\begin{cases} \psi_{k-1}(|x\theta|) = 0, \\ \left| \frac{\psi_{k-3}(|x\theta|)}{H_{k-1}(x, \theta)} \right| \leq c_2 2^{k-1} \end{cases} \quad \text{if } |x\theta| \in (1 - 2^{-k+3}, 1 - 2^{-k+2}), \quad (6.11)$$

$$\begin{cases} \psi_{k-2}(|x\theta|) = 0, \\ \left| \frac{\psi_{k-1}(|x\theta|)}{H_{k+1}(x, \theta)} \right| \leq c_2 2^{k+1} \end{cases} \quad \text{if } |x\theta| \in (1 - 2^{-k+2}, 1 - 2^{-k+1}), \quad (6.12)$$

$$\begin{cases} \psi_{k-1}(|x\theta|) = 0, \\ \psi_{k-3}(|x\theta|) = 0 \end{cases} \quad \text{if } |x\theta| = 1 - 2^{-k+2}, \quad (6.13)$$

for  $(x, \theta) \in \Lambda_k, k \geq \max(4, k_1)$ .

Note that the condition  $(x, \theta) \in \Lambda_k$  is splitted into the assumptions of (6.11), (6.12), (6.13).

Due to formulas (6.5), (6.9)-(6.13), we obtain the following estimates:

$$\begin{aligned} |1 - W_0(x, \theta)| &= |G(x, \theta)| \left| (k-1)! f_{k-1}(x) \frac{\psi_{k-3}(|x\theta|)}{H_{k-1}(x, \theta)} + k! f_{k-2}(x) \frac{\psi_{k-2}(|x\theta|)}{H_k(x, \theta)} \right| \\ &\leq c_1 4^{-k+3} (c_1 c_2 (k-2)(k-1) 2^{k-1} + c_1 c_2 (k-2)(k-1) k 2^k) \\ &\leq 2^6 c_1^2 c_2 2^{-k} k^3 \quad \text{if } |x\theta| \in (1 - 2^{-k+3}, 1 - 2^{-k+2}), \end{aligned} \quad (6.14)$$

$$\begin{aligned} |1 - W_0(x, \theta)| &= |G(x, \theta)| \left| k! f_k(x) \frac{\psi_{k-2}(|x\theta|)}{H_k(x, \theta)} + (k+1)! f_{k+1}(x) \frac{\psi_{k-1}(|x\theta|)}{H_{k+1}(x, \theta)} \right| \\ &\leq c_1 4^{-k+3} (c_1 c_2 2^k (k-1)(k-2) + c_1 c_2 2^{k+1} (k-2)(k-1) k (k+1)) \\ &\leq 2^{12} c_1^2 c_2 2^{-k} k^4 \quad \text{if } |x\theta| \in (1 - 2^{-k+2}, 1 - 2^{-k+1}), \end{aligned} \quad (6.15)$$

$$\begin{aligned} |1 - W_0(x, \theta)| &= |G(x, \theta)| \left| k! f_k(x) \frac{\psi_{k-2}(|x\theta|)}{H_k(x, \theta)} \right| \\ &\leq 2^6 c_1^2 c_2 2^{-k} k^3 \quad \text{if } |x\theta| = 1 - 2^{-k+2}. \end{aligned} \quad (6.16)$$

Estimates (6.14)-(6.16) imply that

$$|1 - W_0(x, \theta)| \leq C \cdot 2^{-k} k^4, \quad (x, \theta) \in \Lambda_k, k \geq \max(4, k_1). \quad (6.17)$$

where  $C$  is a positive constant depending on  $c_1, c_2$  of Lemma 4.

In addition, for  $(x, \theta) \in \Lambda_k$  we have that  $2^{-k+1} < \rho(|x\theta|) < 2^{-k+3}$ , which together with (6.17) imply (3.26).

Lemma 2 is proved.

## 7 Proof of Lemma 3

Let  $(e_1, e_2)$  be an orthonormal basis on  $P_{(s, \theta)} \in \mathcal{P}$  and the origin of the coordinate system on  $P_{(s, \theta)}$  is located at  $s\theta \in P_{(s, \theta)}$ .

By  $u = (u_1, u_2), u \in \mathbb{R}^2$ , we denote the coordinates on  $P_{(s, \theta)}$  with respect to  $(e_1, e_2)$ .

Using Lemma 1 one can see that

$$f|_{P(s,\theta)} \in C_0^\infty(\mathbb{R}^2), f|_{P(s,\theta)}(u) = f|_{P(s,\theta)}(|u|), u \in \mathbb{R}^2. \quad (7.1)$$

By our assumptions  $f|_{P(s_0,\theta_0)}(u)$  changes the sign.

Using this assumption and (7.1) one can see that there exist  $\psi_{1,s_0}, \psi_{2,s_0}$ , such that:

$$\psi_{1,s_0} \in C([0, +\infty)), \psi_{1,s_0} \geq 0, \psi_{2,s_0}(u) := \psi_{1,s_0}(|u|), u \in \mathbb{R}^2, \quad (7.2)$$

$$\int_{P(s_0,\theta_0)} f \psi_{2,s_0} d\sigma \neq 0. \quad (7.3)$$

and if

$$\int_{P(s_0,\theta_0)} f d\sigma \neq 0 \quad (7.4)$$

then also

$$\operatorname{sgn}\left(\int_{P(s_0,\theta_0)} f d\sigma\right) \operatorname{sgn}\left(\int_{P(s_0,\theta_0)} f \psi_{2,s_0} d\sigma\right) = -1, \quad (7.5)$$

where  $d\sigma = du_1 du_2$  (i.e.,  $\sigma$  is the standard Euclidean measure on  $P(s,\theta)$ ).

Let

$$W_{f,s_0}(x, \theta) = 1 - \psi_{1,s_0}(|x - (x\theta)\theta|) \frac{\int_{P(x\theta,\theta)} f d\sigma}{\int_{P(x\theta,\theta)} f \psi_{2,s_0} d\sigma}, x \in \mathbb{R}^3, \theta \in \mathbb{S}^2, \quad (7.6)$$

where  $d\sigma = du_1 du_2$  and  $(u_1, u_2)$  are the coordinates on  $P(s,\theta)$ ,  $s = x\theta$ , defined at the beginning of this proof.

Results of Lemma 1 and property (7.2) imply that

$$\int_{P(x\theta,\theta)} f d\sigma \text{ and } \int_{P(x\theta,\theta)} f \psi_{2,s_0} d\sigma \text{ depend only on } |x\theta|, \text{ where } x \in \mathbb{R}^3, \theta \in \mathbb{S}^2. \quad (7.7)$$

From (7.6), (7.7) it follows that  $W_{f,s_0}$  is rotation-invariant in the sense (1.3), (1.4).

Formulas (7.3), (7.6), (7.7) and properties of  $f$  and  $\psi_{2,s_0}$  of Lemma 1 and (7.2) imply that

$$\exists \varepsilon_1 > 0 : \int_{P(x\theta,\theta)} f \psi_{2,s_0} d\sigma \neq 0, \text{ for } (x, \theta) \in \Omega(\mathcal{J}_{s_0,\varepsilon_1}), \quad (7.8)$$

where the sets  $\mathcal{J}_{s,\varepsilon}, \Omega(\mathcal{J})$  are defined in (2.6), (2.7), respectively.

In addition, using (7.6), (7.8), one can see that

$$W_{f,s_0} \text{ is continuous on } (x, \theta) \in \Omega(\mathcal{J}_{s_0,\varepsilon_1}). \quad (7.9)$$

In addition, from (7.1)-(7.7) it follows that

$$\begin{aligned} \text{if } |x\theta| = |s_0| \text{ then } W_{f,s_0}(x, \theta) &= 1 - \psi_{1,s_0}(|x - (x\theta)\theta|) \frac{\int_{P(s_0,\theta)} f d\sigma}{\int_{P(s_0,\theta)} f \psi_{2,s_0} d\sigma} \\ &= 1 - \psi_{1,s_0}(|x - (x\theta)\theta|) \frac{\int_{P(s_0,\theta_0)} f d\sigma}{\int_{P(s_0,\theta_0)} f \psi_{2,s_0} d\sigma} \geq 1. \end{aligned} \quad (7.10)$$

From properties of  $f, \psi_{1,s_0}, \psi_{2,s_0}$  of Lemma 1 and (7.2) and from formulas (7.6), (7.7), (7.9), (7.10) it follows that

$$\exists \varepsilon_0 > 0 (\varepsilon_0 < \varepsilon_1) : W_{f,s_0}(x, \theta) \geq 1/2, \text{ for } (x, \theta) \in \Omega(\mathcal{J}_{s_0,\varepsilon_0}), \quad (7.11)$$

which implies strict positiveness for  $W_{f,s_0}$  on  $\Omega(\mathcal{J}_{s_0,\varepsilon})$ .

Properties (7.7), (7.9), (7.11) imply item (ii) of Lemma 3 for  $W_{f,s_0,\varepsilon} := W_{f,s_0}$ , defined on  $\Omega(\mathcal{J}_{s_0,\varepsilon_0})$ .

From (1.1), (7.6), (7.8) it follows that

$$\begin{aligned} R_{W_{f,s_0}} f(s, \theta) &= \int_{P(s,\theta)} W_{f,s_0}(\cdot, \theta) f d\sigma \\ &= \int_{P(s,\theta)} f d\sigma - \frac{\int_{P(s,\theta)} f d\sigma}{\int_{P(s,\theta)} f \psi_{2,s_0} d\sigma} \int_{P(s,\theta)} f \psi_{2,s_0} d\sigma = 0 \text{ for } s \in \mathcal{J}_{s_0,\varepsilon_0}, \theta \in \mathbb{S}^2. \end{aligned} \quad (7.12)$$

Item (i) of Lemma 3 follows from (7.12).

Lemma 3 is proved.

## 8 Proof of Lemma 4

### 8.1 Proof of estimate (6.6)

Estimate (6.6) follows from (3.3) and properties (3.4)-(3.6).

### 8.2 Proof of estimate (6.8)

From definitions (3.2), (3.25) we have that

$$G = \sum_{k=1}^{\infty} \frac{G_k}{k!}, \quad (8.1)$$

$$G_k(x, \theta) = \int_{y\theta=x\theta} f_k(y) dy, \quad x \in \mathbb{R}^3, \theta \in \mathbb{S}^2. \quad (8.2)$$

Parametrization of the points  $y(r, \phi)$  on  $P_{(s, \theta)} \in \mathcal{P}$ ,  $s \in \mathbb{R}$ ,  $\theta \in \mathbb{S}^2$ , is given by the formula

$$y(r, \phi) = s\theta + r(e_1 \cos \phi + e_2 \sin \phi), \quad r \in [0, +\infty), \phi \in [0, 2\pi], \quad (8.3)$$

where  $(e_1, e_2)$  is some fixed orthonormal basis on  $P_{(s, \theta)}$ .

On the other hand,

$$r = r(\gamma) = |s| \tan(\gamma), \quad \gamma \in [0, \pi/2), \quad (8.4)$$

where  $\gamma$  is the angle between  $s\theta$  and the radius-vector  $y(r, \phi)$  of (8.3).

It is convenient to rewrite  $y(r, \phi)$  of (8.3) as  $y = y(r(\gamma), \phi) \stackrel{def}{=} y(\gamma, \phi)$ ,  $\gamma \in [0, \pi/2)$ ,  $\phi \in [0, 2\pi]$ .

The standard Lebesgue measure  $\sigma$  on  $P_{(s, \theta)}$  is given by the following formula:

$$\begin{aligned} d\sigma(\gamma, \phi) &= r(\gamma, \phi) d\phi dr(\gamma) = |s| \tan \gamma d\phi dr(\gamma) \\ &= |s|^2 \frac{\sin \gamma}{\cos^3 \gamma} d\phi d\gamma. \end{aligned} \quad (8.5)$$

From (3.3), (8.2)-(8.5) we obtain

$$\begin{aligned} G_k(x, \theta) &= s^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \Phi \left( 2^k \left( 1 - \frac{|s|}{\cos \gamma} \right) \right) \cos \left( 8^k \frac{|s|^2}{\cos^2 \gamma} \right) \frac{\sin \gamma}{\cos^3 \gamma} d\gamma \\ &= -2\pi |s|^2 \int_0^{\pi/2} \Phi \left( 2^k \left( 1 - \frac{|s|}{\cos \gamma} \right) \right) \cos \left( 8^k \frac{|s|^2}{\cos^2 \gamma} \right) \frac{d(\cos \gamma)}{\cos^3 \gamma} \\ &= \{t = \cos \gamma\} = -2\pi |s|^2 \int_1^0 \Phi \left( 2^k \left( 1 - \frac{|s|}{t} \right) \right) \cos \left( 8^k \frac{|s|^2}{t^2} \right) \frac{dt}{t^3} \\ &= \{u = \frac{1}{t^2}\} = \pi |s|^2 \int_1^{+\infty} \Phi(2^k(1 - |s|\sqrt{u})) \cos(8^k |s|^2 u) du, \quad s = x\theta. \end{aligned} \quad (8.6)$$

From (3.4)-(3.6), (8.6) it follows that

$$\begin{aligned} G_k(x, \theta) &= 8^{-k} \pi \int_1^{+\infty} \Phi(2^k(1 - |s|\sqrt{u})) d(\sin(8^k |s|^2 u)) \\ &= 8^{-k} \pi \left( -\Phi(2^k(1 - |s|)) \sin(8^k |s|^2) - \int_1^{+\infty} \left( \frac{d}{du} \Phi(2^k(1 - |s|\sqrt{u})) \right) \sin(8^k |s|^2 u) du \right), \end{aligned} \quad (8.7)$$

$$|\Phi(2^k(1 - |s|)) \sin(8^k |s|^2)| \leq 1, \quad (8.8)$$

$$\begin{aligned} \left| \int_1^{+\infty} \left( \frac{d}{du} \Phi(2^k(1 - |s|\sqrt{u})) \right) \sin(8^k |s|^2 u) du \right| &\leq 2^k \max_{t \in \mathbb{R}} |\Phi'(t)| \int_{\Lambda_{k, |s|}} du \\ &\leq 2^k \max_{t \in \mathbb{R}} |\Phi'(t)|, \end{aligned} \quad (8.9)$$

$$\Lambda_{k, |s|} = \{u \geq 1 : 2^k(1 - |s|\sqrt{u}) \in [4/5, 6/5]\}, \quad (8.10)$$

where  $1/2 < |s| < 1$ ,  $s = x\theta$ ,  $k \in \mathbb{N}$ .

Note that

$$|\Lambda_{k, |s|}| \leq 1 \text{ for } 1/2 < |s| < 1, \quad (8.11)$$

where  $|\Lambda|$  denotes the length of  $\Lambda$ .

Formulas (8.7)-(8.11) imply that

$$|G_k(x, \theta)| \leq 4^{-k} \pi \max_{t \in \mathbb{R}} |\Phi'(t)| \text{ for } 1/2 < |s| < 1, s = x\theta, k \in \mathbb{N}. \quad (8.12)$$

Note that for  $y \in P_{(s, \theta)}$ , the following inequality holds:

$$2^k(1 - |y|) \leq 2^k(1 - |s|) \leq 2^{k-m} \leq 4/5 \text{ for } 1 - 2^{-m} \leq |s| < 1, k < m, m \geq 3. \quad (8.13)$$

Formulas (3.3), (3.4), (8.13) imply that

$$P_{(s, \theta)} \cap \text{supp } f_k = \emptyset \text{ if } |s| \geq 1 - 2^{-m}, k < m. \quad (8.14)$$

In turn, (8.2), (8.14) imply that

$$G_k(x, \theta) = 0 \text{ for } k < m, |x\theta| \geq 1 - 2^{-m}. \quad (8.15)$$

Due to (8.1), (8.12), (8.15) we have that:

$$\begin{aligned} |G(x, \theta)| &\leq \sum_{k=1}^{\infty} |G_k(x, \theta)|/k! = \sum_{k=m}^{\infty} |G_k(x, \theta)|/k! \\ &\leq \max_{t \in \mathbb{R}} |\Phi'(t)| \pi 4^{-m}/m! \sum_{k=0}^{\infty} 4^{-k} = c_1 \frac{4^{-m}}{m!}, c_1 = \frac{4\pi}{3} \max_{t \in \mathbb{R}} |\Phi'(t)| \\ &\text{for } |x\theta| \geq 1 - 2^{-m}, m \geq 3. \end{aligned} \quad (8.16)$$

Estimate (6.8) follows from (8.16).

### 8.3 Proof of estimate (6.7)

For each  $\psi_k$  from (3.21) we have that:

$$|\psi_k| \leq 1. \quad (8.17)$$

Therefore, it is sufficient to show that

$$H_k \geq C_2 2^{-k} \text{ for } k \geq k_1, C_2 = c_2^{-1}. \quad (8.18)$$

Due to formula (3.25) and in a similar way with (8.6) we obtain

$$H_k(x, \theta) = |s|^2 \pi \int_1^{\infty} \Phi^2(2^k(1 - |s|\sqrt{u})) \cos^2(8^k |s|^2 u) du = H_{k,1}(x, \theta) + H_{k,2}(x, \theta), s = x\theta, \quad (8.19)$$

$$H_{k,1}(x, \theta) = \frac{\pi |s|^2}{2} \int_1^{+\infty} \Phi^2(2^k(1 - |s|\sqrt{u})) du, \quad (8.20)$$

$$H_{k,2}(x, \theta) = \frac{\pi |s|^2}{2} \int_1^{+\infty} \Phi^2(2^k(1 - |s|\sqrt{u})) \cos(2 \cdot 8^k |s|^2 u) du. \quad (8.21)$$

Note that

$$2^k(1 - |s|) \geq 2^k \cdot 2^{-k+1} \geq 2 > 6/5 \text{ for } |s| \leq 1 - 2^{-k+1}, k \geq 3. \quad (8.22)$$

In turn, (3.4), (8.22) imply that

$$\Phi(2^k(1 - |s|\sqrt{u})) = 0 \text{ for } u \leq 1, |s| \leq 1 - 2^{-k+1}, k \geq 3. \quad (8.23)$$

Using (8.22) one can see that

$$\exists u_1 \geq 1, u_2 \geq 1, u_2 > u_1 \text{ such that } \begin{cases} 2^k(1 - |s|\sqrt{u_1}) = 11/10, \\ 2^k(1 - |s|\sqrt{u_2}) = 9/10, \end{cases} \quad (8.24)$$

$$|u_2 - u_1| \geq (\sqrt{u_2} - \sqrt{u_1}) = \frac{2^{-k}}{5} |s|^{-1} \geq \frac{2^{-k}}{5}, \quad (8.25)$$

$$\text{for } 1/2 < |s| \leq 1 - 2^{-k+1}, k \geq 3.$$

Using (3.4), (3.6), (8.20), (8.23), (8.25) we obtain

$$H_{k,1}(x, \theta) \geq \frac{\pi}{8} \int_{u_1}^{u_2} du \geq 2^{-k} \frac{\pi}{40}, \text{ for } 1/2 < |x\theta| < 1 - 2^{-k+1}, k \geq 3. \quad (8.26)$$

On the other hand, using (3.4), (8.21), (8.23), in a similar way with (8.7)-(8.11), we obtain

$$\begin{aligned}
|H_{k,2}(x, \theta)| &= \frac{\pi |s|^2}{2} \left| \int_1^{+\infty} \Phi^2(2^k(1 - |s|\sqrt{u})) \cos(2 \cdot 8^k |s|^2 u) du \right| \\
&= \frac{\pi}{4} 8^{-k} |s|^{-2} \left| \int_1^{+\infty} \sin(2 \cdot 8^k |s|^2 u) \left( \frac{d}{du} \Phi^2(2^k(1 - |s|\sqrt{u})) \right) du \right| \\
&\leq \frac{\pi}{4} 8^{-k} |s|^{-1} \max_{t \in \mathbb{R}} |\Phi(t)| \cdot \max_{t \in \mathbb{R}} |\Phi'(t)| \cdot 2^k \int_{\Lambda_{k,|s|}} du \\
&\leq \frac{\pi}{2} 4^{-k} \max_{t \in \mathbb{R}} |\Phi(t)| \cdot \max_{t \in \mathbb{R}} |\Phi'(t)|, \quad s = x\theta,
\end{aligned} \tag{8.27}$$

for  $1/2 < |x\theta| < 1 - 2^{-k+1}$ ,  $k \geq 3$ .

From (8.19)-(8.21), (8.26), (8.27) it follows that

$$\begin{aligned}
|H_k(x, \theta)| &\geq |H_{k,1}(x, \theta)| - |H_{k,2}(x, \theta)| \\
&\geq \frac{\pi}{40} 2^{-k} - \frac{\pi}{2} 4^{-k} \max_{t \in \mathbb{R}} |\Phi(t)| \cdot \max_{t \in \mathbb{R}} |\Phi'(t)| \\
&\geq C_2 2^{-k} \quad \text{for } 1/2 < |x\theta| < 1 - 2^{-k+1}, \quad k \geq k_1, \\
C_2 &= \frac{\pi}{40} - 2^{-k_1} \frac{\pi}{2} \max_{t \in \mathbb{R}} |\Phi(t)| \max_{t \in \mathbb{R}} |\Phi'(t)|,
\end{aligned} \tag{8.28}$$

where  $k_1$  is arbitrary constant such that  $k_1 \geq 3$  and  $C_2$  is positive.

Estimate (6.7) follows from (8.28).

Lemma 4 is proved.

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