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Exponential growth of torsion in abelian coverings

JEAN RAIMBAULT

We show exponential growth of torsion numbers for links whose first nonzero Alexander polynomial has positive logarithmic Mahler measure. This extends a theorem of Silver and Williams to the case of a null first Alexander polynomial and provides a partial solution for a conjecture of theirs.

57M10; 57M25, 57Q10

Introduction

Let M be a compact three–manifold; the homology groups $H_i(M)$ can be written as the direct sums $H_i(M)_{tors} \oplus H_i(M)_{free}$ of a finite abelian group with a finite-rank free abelian group. The torsion summand is nontrivial only for $i = 1$: H_0 and H_3 are $\mathbb{Z}$ or 0, and the universal coefficients theorem for cohomology implies that $H^1(M, \partial M)$ is free, and by Poincaré duality it follows that $H_2(M)$ is also torsion-free. On the other hand, the torsion in H_1 can be arbitrarily large (e.g., for lens spaces; see below for hyperbolic examples) and it is believed that "most" 3-manifolds should have a rather large torsion. For example E. Kowalski shows in [Kow08, Proposition 7.19] that the first homology group of a "Dunfield-Thurston random 3-manifold" typically has a large torsion subgroup. This paper is concerned with the growth rate of the order of $H_1(M_N)_{tors}$ in a sequence of finite coverings M_N of a manifold M. The least precise question that can be asked is whether it is exponential in the degree or not, i.e. whether the sequence $\frac{\log |H_1(M_N)_{tors}|}{[\pi_1(M):\pi_1(M_N)]}$ has a positive limit (or limit superior). This shall be partially answered here in the case where the M_N are abelian coverings converging to a free abelian covering of M. The main motivation to study this question was to provide a partial result towards a conjecture of Silver and Williams on the growth rate of torsion numbers of abelian coverings of complements of links (see Conjecture 6.1 in [SW02a] or (0.1) below).

Historically, the first context where the growth of torsion in the homology of coverings has been studied is that of cyclic coverings of a knot complement. Let K be an open knotted solid torus in the three-sphere and $M = \mathbb{S}^3 - K$; then M is a compact three-manifold with $H_1(M) = \mathbb{Z}$. Thus we can consider the infinite cyclic covering $\widehat{M}$

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of M and its finite quotients M_N, which are the finite coverings of M corresponding to the maps $\pi_1(M) \to \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$. If Δ is the Alexander polynomial of K (the first Alexander polynomial of the $\mathbb{Z}[\mathbb{Z}]$-module $H_1(\widehat{M})$-see 1.2) then the so-called Fox formula says that for all N such that $H_1(M_N)$ has rank one we have:

$$|H_1(M_N)_{tors}| = \prod_{\zeta^N=1} |\Delta(\zeta)|.$$

It is known that Δ is always nonzero. A diophantine inequality due to Gelfond then allows to show that when N tends to infinity $\frac{1}{N}\sum_{\zeta^N=1} \log|\Delta(\zeta)|$ converges to the logarithmic Mahler measure of Δ, that is:

$$m(\Delta) = \int_{\mathbb{T}^1} \log|\Delta(z)|dz$$

(here $\mathbb{T}^1$ is the unit circle in $\mathbb{C}$ endowed with the Lebesgue probability measure); see Lemma 2.6 below or Proposition 2.8 of [SW02b]. Let d be the product of all integers l such that the lth cyclotomic polynomial divides Δ: it is known ([SW02b, Proposition 2.2] or [Gor72, Theorem 4.1(ii)]) that $H_1(M_N, \mathbb{Q}) \cong H_1(M_{N+d}, \mathbb{Q})$. We get that

$$\lim_{\substack{N\to\infty \\ d|N-1}} (\frac{1}{N}\log|H_1(M_N)_{tors}|) = m(\Delta);$$

in particular, if $m(\Delta) > 0$ an infinite subsequence of the torsion numbers tends to infinity with exponential growth. This result answered a question of Gordon (who proved in [Gor72] that when $m(\Delta) = 0$ the torsion numbers are periodic in N) and has first been proved by González-Acuña and Short in [GAS91] and independently by Robert Riley in [Ril90]. Since there exists hyperbolic knots whose Alexander polynomial has positive Mahler measure (for example the figure-eight) this proves in particular that there exists (noncompact with finite volume) hyperbolic manifolds with arbitrarily large torsion in their H_1.

The similar setting for a collection of $m > 1$ linked solid tori L in the three-sphere is to consider the coverings M_H of $M = \mathbb{S}^3 - L$ given by the maps $\pi_1(M) \to H_1(M) \cong \mathbb{Z}^m \to \mathbb{Z}^m/H$ where H is a finite-index subgroup. We want to study the torsion numbers as the quantity:

$$\alpha(H) = \min\{\max_i |v_i|;\ 0 \neq v = (v_1, \dots, v_m) \in H\}$$

tends to infinity. Fox's formula admits a generalization due to Mayberry and Murasugi ([MM82]; see also [Por04] for a proof using Reidemeister torsion). However, because the Betti numbers may very well tend to infinity as $\alpha(H)$ does, it may not be applicable to an infinite sequence of coverings. This forbids any naïve application of the preceding

scheme of proof to this case. Moreover, two other obstructions arise when considering links: the Alexander polynomial may be zero, and the generalisation of Gelfond's estimate needed to prove the convergence of Riemann sums to the Mahler measure is not known to hold for polynomials in several variables.

By using methods from algebraic dynamical systems Daniel Silver and Susan Williams, in [SW02a], were able to show that if the first Alexander polynomial $\Delta(L)$ of L is nonzero then:

$$\limsup_{\alpha(H)\to\infty} \frac{\log |H_1(M_H)_{tors}|}{[G:H]} = m(\Delta(L)) := \int_{\mathbb{T}^m} \log |\Delta(L)|$$

and that this holds with a limit in the case of a knot (this is also proved in [SW02b]). They also conjecture that, in the general case, the following limit should hold where Δ_i is the first nonzero polynomial in the sequence of Alexander polynomials of L:

$$\lim_{\alpha(H)\to\infty} \frac{\log |H_1(M_H)_{tors}|}{[G:H]} = m(\Delta_i(L)). \tag{0.1}$$

The Mahler measure of the Alexander polynomial of a knot can be interpreted as the ℓ^2-torsion of the infinite cyclic covering $\widehat{M}$ of its exterior M (see for example (1.5) below). In [BV] Nicolas Bergeron and Akshay Venkatesh used this to give a new proof of the theorem of Silver and Williams in the case of a knot: the main point is that the nonnullity of the Alexander polynomial guarantees that $\widehat{M}$ is ℓ^2-acyclic, and in this case the approximation of ℓ^2-torsion by Reidemeister torsions is known. They then show that the growth of torsion numbers is the same as that of Reidemeister torsion and obtain a generalization of the theorem for knots (see [BV, Theorem 7.3]). For links with nonzero first Alexander polynomial their proof can be adapted; however the result obtained is slightly weaker than Silver and Williams'.

The aim of this work is to generalise their results to the case of a null first Alexander polynomial, i.e. that of a non-ℓ^2-acyclic infinite covering. This yields the following result, which follows from the more general Theorem 3.1.

Theorem 0.1 *With notation as in* (0.1) *we have:*

$$\limsup_{\alpha(H)\to\infty} \frac{\log |H_1(M_H)_{tors}|}{[G:H]} \geq m(\Delta_i(L)).$$

The same scheme of proof can be applied to homology with coefficients in a local system. Let V be a free $\mathbb{Z}$-module of finite rank and χ a representation of $\pi_1(M)$ to $GL(V)$. Then we can define homology twisted by χ for all coverings of M, also called

homology with coefficients in the local system defined by χ. See [Hat02, Chapter 3.H] for definitions. Let $\Delta_i(L,\chi)$ be the first nonzero Alexander polynomial of $H_1(\widehat{M};V)$ (the Alexander polynomials with nontrivial coefficients are usually called twisted Alexander polynomials). We get (see also [SW09, Theorem 3.10]):

$$\limsup_{\alpha(H)\to\infty} \frac{\log|H_1(M_H;V)_{tors}|}{[G:H]} \geq m(\Delta_i(L,\chi)).$$

In the case of a sequence of cyclic coverings converging to a possibly non-ℓ^2-acyclic infinite cyclic covering the proofs yield a more precise result, generalising Theorem 2.10 of [SW02b].

Theorem 0.2 *If M is a compact n-manifold (with boundary) with a surjection $\pi_1(M)\to\mathbb{Z}$ let M_N be the cyclic covering of M associated to $\pi_1(M)\to\mathbb{Z}/N\mathbb{Z}$ and $\widehat{M}$ the infinite cyclic covering. Then we have for all $i=1,\dots,n-1$:*

$$\lim_{N\to\infty} \frac{\log|H_i(M_N)_{tors}|}{N} = m(\Delta_i)$$

where Δ_i is the first nonzero Alexander polynomial of $H_i(\widehat{M})$.

Thang Le has recently proved the equality lacking in Theorem 0.1 in the preprint [Le]. His proof uses methods slightly different from ours, namely a Bourbaki theorem on "pseudo-zero modules" and the dynamical version of approximation (due to Lind; see [Sch95, Theorem 21.1]). He also gives a different construction of the approximating sequences contructed in Proposition 2.7 below.

Finally, note that throughout this paper we work with three-manifolds with boundary, that is, we consider regular coverings of $\mathbb{S}^3-L$ instead of the correspondings branched (over core circles of L) coverings of $\mathbb{S}^3$, which are closed three-manifolds. However, the growth of torsion homology in abelian coverings is the same for the two sequences of manifolds, as shown in [Le, 4.4] (for knots the two homologies differ only by a infinite cyclic direct factor).

Warnings and outline

Most of the techniques and results used here are fairly elementary with one notable exception: I have found more practical and efficient to use the Fuglede-Kadison determinant as defined in the book [Lüc02] of Wolfgang Lück, and all its properties proved therein instead of trying to do everything solely in terms of Mahler measures (in

any case, it is the right tool to use to define ℓ^2-torsion). However, the reader need not be familiar with the Fuglede-Kadison determinant since all those of its properties that we use are recalled. One warning about notations: I have chosen to denote by $\mathcal{M}$ the multiplicative Mahler measure given by $\exp(m)$ since I already use the casual M to denote modules and manifolds.

The paper is organised as follows: the first section reviews various classical and ℓ^2-invariants for $\mathbb{Z}[G]$-modules. In particular, I give there the computation of the ℓ^2-torsion in terms of Mahler measures of Alexander polynomials. In the second section I give a proof of the approximation of the Fuglede-Kadison determinant in the free Abelian case since there was (to the best of my knowledge) no complete proof of this result available in the literature. In the third section we prove Theorems 0.1 and 0.2. The fourth section contains some further remarks about three-manifolds. The appendices contain proofs that I judged too cumbersome for a first reading; I hope that their being relegated there will make the main thrust of the paper more apparent and intelligible.

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1 Review of classical and ℓ^2-invariants

In this section G shall denote a free Abelian group of rank m; we identify the group ring $\mathbb{Z}[G]$ with the Laurent polynomials $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ through the choice of a free generating set $t_1, \dots, t_m$ for G. In this section, we review various invariants for G-modules and G-complexes.

1.1 Rank, von Neumann dimension and ℓ^2-Betti numbers

Recall that a finitely generated Hilbert $\mathcal{N}(G)$-module is a Hilbert space with a unitary action of G, which admits a G-equivariant surjection of $\ell^2(G)^n$ for some integer n (we shall not use explicitly the von Neumann algebra $\mathcal{N}(G)$ in the sequel, so we shall simply call these finitely generated $\mathcal{N}(G)$-modules). Given a submodule $M \subset \mathbb{Z}[G]^n$, the closure of $M \otimes \mathbb{C}$ is a $\mathcal{N}(G)$-submodule of $\ell^2(G)^n$, in particular it is a finitely generated $\mathcal{N}(G)$-module. The von Neumann dimension of a $\mathcal{N}(G)$-module is defined for any discrete group ([Lüc02, Definition 1.10]): it assigns to a nonzero finitely generated $\mathcal{N}(G)$-module a positive real number and satisfies the same basic properties that the dimension of a vector space does (see Theorem 1.12 in [Lüc02]). In general it is not easy to compute. However, in our case the group ring $\mathbb{Z}[G]$ has a fraction field $\mathbb{Q}(G)$ identified with the field of rational functions in m variables $\mathbb{Q}(t_1, \dots, t_m)$. We can then define the rank of M as the dimension of the $\mathbb{Q}(G)$-vector space $M \otimes \mathbb{Q}(G)$.

Lemma 1.1 *Let M be a submodule of $\mathbb{C}[G]^n$; then the von Neumann dimension of its closure in $\ell^2(G)^n$ is equal to its rank.*

Proof This is easy to deduce from the two lemmas below: according to Lemma 1.2 we can pick a free module $L \subset M$ such that M/L is torsion; then the rank of M is equal to that of L, and by Lemma 1.3 the completion of L is equal to that of M. Thus it suffices to show the lemma for a free submodule L. Let r be the rank of L; then L is the image of an injective morphism from $\mathbb{C}[G]^r$ to $\mathbb{C}[G]^n$, which extends to an isomorphism of $\mathcal{N}(G)$-modules from $\ell^2(G)^r$ to the closure of L, so that its von Neumann dimension is r. □

Lemma 1.2 *Let M be a finitely generated $\mathbb{Z}[G]$-module. If $L \subset M$ is a maximal rank free submodule (which always exists) then the quotient M/L is torsion.*

Proof If $a_1, \dots, a_r$ is a basis for the finite-dimensional vector space $M \otimes \mathbb{C}(G)$ such that $a_i \in M$ for all i, then the module $L = \sum_i \mathbb{Z}[G]a_i$ is a maximal rank free submodule. If f is the lowest common multiple of the denominators of the coordinates of a generating family for M in the basis $(a_1, \dots, a_r)$ we have $M \subset fL$; thus f annihilates M/L. □

Lemma 1.3 *The quotient of a $\mathbb{C}[G]$-module M by a submodule M' is torsion if and only M' is dense in M.*

Proof If the quotient is torsion then for any $x \in M$ the intersection of $\mathbb{C}[G]x$ with M' is equal to Ix for some nonzero ideal I; thus it suffices to prove that an ideal is dense in $\mathbb{C}[G]$. This is true: any ideal I contains a nonzero principal ideal (f) (since we are in a Noetherian UFD) and the operator of $\ell^2(G)$ induced by multiplication by f is injective, so it has dense image by [Lüc02, Lemma 1.14].

If M' is dense then its ℓ^2 completion equals that of M and so its rank, which equals its Von Neumann dimension, is the same as that of M. Now it is clear that the quotient of a module by a submodule of maximal rank is torsion (see the proof of the above lemma). □

In the same way, given a complex C_*, d_* of free finitely generated $\mathbb{Z}[G]$-modules we can define its reduced ℓ^2-homology: the $C_i \otimes \mathbb{C}$ can be endowed with a G-invariant prehilbertian inner product (where we pick a $\mathbb{Z}[G]$-basis and its images under G to be an orthonormal basis) and the diferentials d_i extend to bounded operators $d_i^{(2)}$ from $C_i^{(2)}$ to $C_{i-1}^{(2)}$. The reduced ith ℓ^2-homology module $H_i^{(2)}(C)$ of C_* is then defined to be the $\mathcal{N}(G)$-module $\ker(d_i^{(2)})/\overline{\mathrm{Im}(d_{i+1}^{(2)})}$ and the ith ℓ^2-Betti number $b_i^{(2)}(C)$ to be its dimension. It follows from the Lemma above that we have $b_i^{(2)}(C) = \mathrm{rk}(H_i(C))$ (see also [Lüc02, Lemma 1.34]). The complex C_* is thus ℓ^2-acyclic if and only all $H_i(C)$ are $\mathbb{Z}[G]$-torsion modules.

1.2 Alexander polynomials and Fuglede-Kadison determinant

We now present the analogue for $\mathbb{Z}[G]$-modules of the order of the torsion subgroup for $\mathbb{Z}$-modules, and the Fuglede-Kadison determinant which is linked to the Mahler measure.

Alexander polynomials of $\mathbb{Z}[G]$-modules

Let M be a finitely generated $\mathbb{Z}[G]$-module and $A \in M_{m,n}(\mathbb{Z}[G])$ a presentation matrix for M, i.e. $M = \mathbb{Z}[G]^m/A\mathbb{Z}[G]^n$. We define the l-th elementary ideal of M as the ideal of $\mathbb{Z}[G]$ generated by the $(m-l)$-minors of A. This definition makes sense since a given presentation matrix for M can be made into any other one by a sequence of elementary transformations that do not affect the elementary ideals (cf. [Tur01, Lemma 4.4]). Since $\mathbb{Z}[G]$ is a Noetherian UFD, we can take the greatest common divisor of an ideal; we thus put $\Delta_l(M) = gcd(I_l)$, and call it the l-th Alexander polynomial of the module M. The Laurent polynomial $\Delta_l(M)$ is of course defined only up to

multiplication by a unit of $\mathbb{Z}[G]$ and it is zero for all $l \leq \mathrm{rk}(M)$. It depends only on the $\mathbb{Z}[G]$-torsion submodule of M; in fact we have ([Tur01, Chapter 4]):

$$\Delta_{\mathrm{rk}(M)+l}(M) = \Delta_l(M_{tors}).$$

In particular, it follows from this equality that $\Delta_r(M) = 1$ for a torsion-free module M of rank r. We remark that Δ_0 is a multiplicative invariant, i.e. if we have a short exact sequence $0 \to M' \to M \to M'' \to 0$ of modules, then $\Delta_0(M) = \Delta_0(M')\Delta_0(M'')$. To prove this we can suppose the modules are torsion (if not both sides of the equality are zero). Thus M', M'' have presentation matrices A', A'' which are of maximal ranks m', m''; then M has a presentation matrix of the form:

$$A = \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix},$$

whose nonzero $(m' + m'')$-minors are products of m'-minors of A' with m''-minors of A''.

Note that the Mahler measure of the Alexander polynomial $\Delta_l(M)$ is a well-defined positive real number since the units of $\mathbb{Z}[G]$ are the monomials $\pm t^v$ for $v \in \mathbb{Z}^m$, whose Mahler measure is equal to one.

Fuglede-Kadison determinant

The Fuglede-Kadison determinant is defined for any $\mathcal{N}(G)$-module morphism, cf. [Lüc02, Definition 3.11]; it is an infinite-dimensional analogue of the $\det'$ introduced in the next subsection. Here we only need it for those morphisms that are induced from morphisms between free $\mathbb{Z}[G]$-modules. It is always nonzero when the modules are finitely generated. If A is a $(n \times n')$-matrix with coefficients in $\mathbb{Z}[G]$ we abusively denote by A the continuous operator from $\ell^2(G)^{n'}$ to $\ell^2(G)^n$ that it induces, and by $\det_{\mathcal{N}(G)}(A)$ its Fuglede-Kadison determinant. We list some of its properties below for the reader's convenience:

(i) If A is square with nonzero determinant, then we have:

$$\det_{\mathcal{N}(G)}(A) = \mathcal{M}(\det A). \tag{1.1}$$

(ii) ([Lüc02, Theorem 3.14(1)]) If $U \xrightarrow{f} V \xrightarrow{g} W$, f has dense image and g is injective then

$$\det_{\mathcal{N}(G)}(g \circ f) = \det_{\mathcal{N}(G)}(f) \det_{\mathcal{N}(G)}(g).$$

(iii) ([Lüc02, Theorem 3.14(2)]) If f_1 has dense image and f_2 is injective then

$$\det_{\mathcal{N}(G)} \begin{pmatrix} f_1 & f_3 \\ 0 & f_2 \end{pmatrix} = \det_{\mathcal{N}(G)}(f_1) \det_{\mathcal{N}(G)}(f_2).$$

(iv) ([Lüc02, Lemma 3.15(3)]) We have:

$$\det_{\mathcal{N}(G)}(f|_{\ker(f)^\perp}^{\overline{\mathrm{Im}(f)}}) = \det_{\mathcal{N}(G)}(f)$$

(v) ([Lüc02, Lemma 3.15(4)]) Let f^* be the adjoint of f, then:

$$\det_{\mathcal{N}(G)}(f^*) = \det_{\mathcal{N}(G)}(f) = \sqrt{\det_{\mathcal{N}(G)}(ff^*)}.$$

We prove (i), which is left as an exercise in [Lüc02]. Let $A = LPU$ be a Bruhat decomposition of A over the field $\mathbb{C}(G)$, i.e. P is a permutation matrix, L is a lower triangular matrix with diagonal coefficients all equal to 1 and U is an upper triangular matrix with diagonal coefficients $u_1, \dots, u_n \in \mathbb{C}(G)$. Let q be the lcm of the denominators of the coefficients of L and U. We get $\det_{\mathcal{N}(G)}(q^2 A) = \det_{\mathcal{N}(G)}(qLPqU)$; The permutation matrix P is diagonalisable over $\mathbb{C}$ with modulus one eigenvalues, and thus $\det_{\mathcal{N}(G)}(P) = 1$. From properties (ii) and (iii) above follow first:

$$\det_{\mathcal{N}(G)}(q^2 Id) \det_{\mathcal{N}(G)}(A) = \det_{\mathcal{N}(G)}(qL) \det_{\mathcal{N}(G)}(P) \det_{\mathcal{N}(G)}(qU),$$

and then:

$$\begin{aligned} \det_{\mathcal{N}(G)}(A) &= \mathcal{M}(q)^{-2n} \mathcal{M}(q^n) \mathcal{M}(q^n u_1 \dots u_n) \\ &= \mathcal{M}(u_1 \dots u_n) = \mathcal{M}(\det(A)). \end{aligned}$$

1.3 Reidemeister torsion and ℓ^2-torsion

Reidemeister torsion

Let us first recall some elementary definitions: if f is a map between two finite-dimensional hermitian spaces V, W we define $\det'(f) = det(f^* f|_{\ker(f)^\perp})$ (the product of all nonzero eigenvalues of the self-adjoint map $f^* f$). A lattice of V is defined to be a $\mathbb{Z}$-submodule L of rank $\dim(V)$ spanning V over $\mathbb{C}$; its volume is defined to be the absolute value of the determinant of any map sending an orthonormal basis of V to a $\mathbb{Z}$-basis of L. If V' is a subspace of V such that $V' \cap L$ spans V', then this is also true for its orthogonal complement V'' and we have

$$\mathrm{vol}(L) = \mathrm{vol}(L \cap V') \,\mathrm{vol}(L'') = \mathrm{vol}(L') \,\mathrm{vol}(L \cap V'') \tag{1.2}$$

where L', L'' are the orthogonal projections of L on V', V''. From (1.2) it is easy to deduce the "metric rank formula"

$$\mathrm{vol}(\mathrm{Im}(f)) = \frac{\det{}'(f)\,\mathrm{vol}(L)}{\mathrm{vol}(\ker f)}. \tag{1.3}$$

Now let C_*, d_* be a finite complex of free $\mathbb{Z}[G]$-modules; we pick for each i a $\mathbb{Z}$-basis of C_i and endow $C_* \otimes \mathbb{C}$ with the inner product where all these bases are orthonormal. The free part $H_i(C)_{free}$ of the homology is a lattice in $H_i(C \otimes \mathbb{C})$, which itself can be seen as the orthogonal of $\mathrm{Im}(d_{i+1})$ in $\ker(d_i) \otimes \mathbb{C}$. Then the Reidemeister torsion defined by:

$$\tau(C_*) = \prod_i \det{}'(d_i)^{(-1)^i}$$

does not depend on the choice of the bases and is in fact given by the following equality:

$$\tau(C_*) = \prod_i \left(\frac{|H_i(C)_{tors}|}{\mathrm{vol}(H_i(C)_{free})}^{(-1)^{i+1}} \right). \tag{1.4}$$

The proof of this equality is a simple exercise using (1.3).

ℓ^2-torsion

Let C_*, d_* be a finite complex of free $\mathbb{Z}[G]$-modules. By analogy with the finite-dimensional case we put:

$$\tau^{(2)}(C) = \prod_i \det{}_{\mathcal{N}(G)}(d_i)^{(-1)^i}.$$

In our particular case this can be computed:

Proposition 1.4 *In the situation above, suppose that C_* is ℓ^2-acyclic. Then we have:*

$$\tau^{(2)}(C) = \prod_i \mathcal{M}\big(\Delta_0(H_i(C))\big) \tag{1.5}$$

Proof This can be deduced without too much effort from the equality [Tur01, Theorem 4.7]. We do not do so here, and instead prove a result which is also valid for non ℓ^2-acyclic complexes (Lemma B.3), following the lines of the proof of (1.4). The equality (1.5) follows immediately from this result. □

Example: link complements

Let X be a CW-structure on the complement of a regular neighbourhood L of a link in the three-sphere, and put $G = H_1(X)$. Let $\widehat{X}$ be the maximal abelian cover of X; then $\widehat{X}$ is a G-CW-complex. The only $H_i(\widehat{X})$ that can contain $\mathbb{Z}[G]$-torsion is H_1, and it is known that in the case of a knot K, H_1 is in fact a torsion module. In this case we get from Proposition 1.4 that

$$\tau^{(2)}(\widehat{X}) = \mathcal{M}(\Delta(K))$$

where $\Delta(K)$ is the Alexander polynomial of K. For a link with nonzero first Alexander polynomial $\Delta(L)$ we get in the same way that $\tau^{(2)}(\widehat{X}) = \mathcal{M}(\Delta(K))$; if $\Delta(L) = 0$ then the ℓ^2-torsion depends on the CW-structure; the precise dependance is computed in Lemma B.3.

2 Approximation for the Fuglede-Kadison determinant

Let $H \subset G$ be a subgroup and M a $\mathbb{Z}[G]$-module. We denote by M_H the $\mathbb{Z}[G/H]$-module obtained by restricting from G to G/H:

$$M_H = \mathbb{Z}[G/H] \otimes_{\mathbb{Z}[G]} M.$$

There is a G/H-equivariant isomorphism from M_H to the quotient module $M/(H-1)M$. If $f : \mathbb{Z}[G]^n \to \mathbb{Z}[G]^{n'}$ is a linear map of free modules, f_H is the map from $\mathbb{Z}[G/H]^n \to \mathbb{Z}[G/H]^{n'}$ that it induces. The restricted determinant $\det'(f_H)$ is always taken with respect to the canonical metric on $\mathbb{C}[G/H]$, for which the elements of G/H are an orthonormal basis. Let C_*, d_* be a complex of $\mathbb{Z}[G]$-modules; then C_H is the $\mathbb{Z}[G/H]$-complex with $C_{*,H} = (C_*)_H$ and $d_{*,H} = (d_*)_H$.

It is a well-known result of Lück (cf. [Lüc94] or chapter 13 of [Lüc02]) that for any complex C_* of free $\mathbb{Z}[G]$-modules the normalised Betti numbers converge to the ℓ^2 Betti number, i.e.

$$\lim_{\alpha(H)\to\infty} \frac{\dim(H_i(C_H) \otimes \mathbb{C})}{[G:H]} = b_i^{(2)}(C_*). \tag{2.1}$$

This result is true for any residually finite group and any exhaustive sequence of finite index normal subgroups. It is a natural question to ask whether a similar result holds for torsions, i.e. does the sequence $\det'(f_H)^{\frac{1}{[G:H]}}$ converge to $\det_{\mathcal{N}(G)}(f)$. We shall prove the following result.

Proposition 2.1 *Let $A \in M_{n,n'}(\mathbb{Z}[G])$ be any matrix. Then:*

$$\limsup_{\alpha(H)\to\infty} \frac{\log(\det(A_H))}{[G:H]} = \log(\det{}_{\mathcal{N}(G)}(A)).$$

More precisely, the result we prove and use in the next section is:

Proposition 2.2 *Let $A_1, \dots, A_n$ be a finite collection of matrices with coefficients in $\mathbb{Z}[G]$; there exists a sequence G_N of subgroups of finite index in G such that $\alpha(G_N) \underset{N\to\infty}{\longrightarrow} \infty$ and for all i we have:*

(i)

$$\lim_{N\to\infty} \frac{\log\det{}'(A_{i,G_N})}{[G:G_N]} = \log\det{}_{\mathcal{N}(G)}(A_i)$$

(ii)

$$|\dim(\ker(A_{i,G_N})\otimes\mathbb{C}) - [G:G_N]\operatorname{rk}(\ker(A_i))| = O(\log[G:G_N]).$$

Proposition 2.1 follows from this since arguing as in the proof of Lemma 2.6 below it is easy to see that:

$$\limsup_{\alpha(H)\to\infty} (\det{}'(A_H)^{\frac{1}{[G:H]}}) \leq \det{}_{\mathcal{N}(G)}(A)$$

(this is a general fact, see Theorem 3.4(2) in [Lüc94]).

2.1 Growth of Betti numbers

We shall need precise estimates on the speed of convergence in (2.1). For a subgroup $H \subset \mathbb{Z}^m$ we set:

$$H^\perp = \{(\zeta_1, \dots, \zeta_m) \in \mathbb{T}^m,\ \forall (v_1, \dots, v_m) \in H,\ \zeta_1^{v_1} \dots \zeta_m^{v_m} = 1\},$$

The result we want is the following (we count the multiplicities when computing the number of zeroes of some polynomial):

Proposition 2.3 *We remind the reader that $G = \mathbb{Z}^m$ and that we identify the group ring $\mathbb{Z}[G]$ with the ring of Laurent polynomials $\mathbb{Z}[t_1^{\pm 1}, \dots; t_m^{\pm 1}]$. Let M be a finitely generated $\mathbb{Z}[G]$-module (resp. C_* a finite complex of finitely generated free $\mathbb{Z}[G]$-modules). Then there exists a polynomial $P \in \mathbb{Z}[t_1, \dots, t_m]$ such that for all subgroups $H \subset G$ we have:*

$$|\dim(M_H \otimes \mathbb{C}) - [G:H]\operatorname{rk}(M)| \leq |\{\zeta \in H^\perp,\ P(\zeta) = 0\}|$$

(resp.

$$|b_i(C_H) - [G:H]b_i^{(2)}(C_*)| \leq |\{\zeta \in H^\perp,\ P(\zeta) = 0\}|).$$

Proof Let M be a finitely generated $\mathbb{Z}[G]$-module; let $i : L \hookrightarrow M$ be an embedding of a free module of maximal rank in M and T the torsion module M/L. We see that:

$$|\dim(M_H \otimes \mathbb{C}) - [G:H]\,\mathrm{rk}(M)| \leq \dim(T_H \otimes \mathbb{C}) + \dim(\ker(i_H) \otimes \mathbb{C}).$$

We need to bound the right-hand side; this is done in the two next lemmas.

Lemma 2.4 *If T is a finitely generated torsion $\mathbb{Z}[G]$-module there is a polynomial P such that we have:*

$$\dim(T_H \otimes \mathbb{C}) \leq |\{\zeta \in H^\perp,\ P(\zeta) = 0\}|.$$

Proof Let $a_1, \dots, a_s$ be generators for T, since they are torsion elements there exists a nonzero $P \in \mathbb{Z}[t_1, \dots, t_m]$ such that $Pa_i = 0$ for all i. Then there is a surjection from $(\mathbb{C}[G]/(P))^k$ to T, and we have:

$$\dim(T_H \otimes \mathbb{C}) \leq k \dim(\mathbb{C}[G]/(P) \otimes \mathbb{C}[G/H]).$$

The dimension on the right is equal to $|H^\perp \cap \{P = 0\}|$: we have a surjection $\mathbb{C}[G/H] \to (\mathbb{C}[G]/(P))_H$, and in the decomposition $\mathbb{C}[G/H] = \sum_{\zeta \in H^\perp} \mathbb{C}_\zeta$ (where $\mathbb{C}_\zeta$ is a complex line on which G/H acts by the character associated with ζ) the kernel of this surjection is $\bigoplus_{\zeta, P(\zeta) \neq 0} \mathbb{C}_\zeta$. Thus $\dim(T_H)$ is bounded by the number of zeroes of P^k (recall that we count multiplicities). □

Lemma 2.5 *Let $i : M_1 \to M_2$ be an injective map of $\mathbb{Z}[G]$-modules. Then there is a polynomial Q such that for all H we have:*

$$\dim(\ker(i_H) \otimes \mathbb{C}) \leq |\{\zeta \in H^\perp,\ Q(\zeta) = 0\}|.$$

Proof We first consider the case where $M \subset \mathbb{Z}[G]^n$ is a submodule and i its embedding. We begin by proving there is a Q_1 such that:

$$\dim(M_H \otimes \mathbb{C}) \leq [G:H]\,\mathrm{rk}(M) + |\{\zeta \in H^\perp,\ Q_1(\zeta) = 0\}|. \tag{2.2}$$

To prove this take a torsion module $T = M/L$ where $L \subset M$ is a free maximal rank submodule: the image of $L_H \otimes \mathbb{C}$ in $M_H \otimes \mathbb{C}$ has dimension less than $\dim(L_H \otimes \mathbb{C}) =$

$\mathrm{rk}(M)[G:H]$ and codimension equal to $\dim(T_H \otimes \mathbb{C})$ so that we can take for Q_1 the polynomial associated to T by Lemma 2.4.

Now let $M' \subset \mathbb{Z}[G]^n$ be a submodule such that $M \cap M' = 0$ and $\mathrm{rk}(M) + \mathrm{rk}(M') = n$ and i' its embedding. The quotient $T' := \mathbb{Z}[G]^n / M \oplus M'$ is a torsion module and so from Lemma 2.4 we get a polynomial Q_2 such that

$$\dim(T'_H \otimes \mathbb{C}) \leq |\{\zeta \in H^\perp,\ Q_2(\zeta) = 0\}|. \tag{2.3}$$

On the other hand we have an exact sequence:

$$0 \to M \oplus M' \to \mathbb{Z}[G]^n \to T' \to 0$$

which becomes after tensorization with $\mathbb{Z}[G/H]$:

$$M_H \oplus M'_H \to \mathbb{Z}[G/H]^n \to T'_H \to 0.$$

We get:

$$\dim(\ker(i_H) \otimes \mathbb{C}) + \dim(\ker(i'_H) \otimes \mathbb{C}) = \dim(T'_H \otimes \mathbb{C}) + \dim(M_H \oplus M'_H \otimes \mathbb{C}) - n[G:H].$$

We can apply the reasoning leading to (2.2) to M' and this yields a polynomial Q_3 such that $\dim(M'_H \otimes \mathbb{C}) - [G:H]\,\mathrm{rk}(M')$ is less than the number of zeroes of Q_3 on $H^\perp$; letting $Q = Q_1 Q_2 Q_3$, (2.2) and (2.3) above and this imply that the right hand side is bounded above by the number of zeroes of Q in $H^\perp$.

In the general case we have an embedding of $M_1/(M_1)_{tors}$ in a free module $\mathbb{Z}[G]^n$: let i' be the embedding of $M_2/(M_2)_{tors}$ in $\mathbb{Z}[G]^n$ this yields. We get Lemma 2.5 by applying the preceding discussion to i' and Lemma 2.4 to $(M_2)_{tors}$. □

Let C_* be a finite complex of free $\mathbb{Z}[G]$-modules; we use standard arguments to deduce the second point of the proposition from the first one. Let $n_i = \mathrm{rk}_{\mathbb{Z}[G]}(C_i)$; then we have $\mathrm{rk}_{\mathbb{Z}}(C_{i,H}) = [G:H]n_i$, and it follows that:

$$\begin{aligned} b_i(C_H) &= \mathrm{rk}_{\mathbb{Z}}(\ker(d_{i,H})) - \mathrm{rk}_{\mathbb{Z}}(\mathrm{Im}(d_{i+1,H})) \\ &= \mathrm{rk}_{\mathbb{Z}}(\ker(d_{i,H})) + n_{i+1}[G:H] - \mathrm{rk}_{\mathbb{Z}}(\ker(d_{i+1,H})). \end{aligned}$$

On the other hand,

$$b_i^{(2)}(C) = \mathrm{rk}_{\mathbb{Z}[G]}(\ker(d_i)) + n_{i+1} - \mathrm{rk}_{\mathbb{Z}[G]}(\ker(d_{i+1}))$$

so that we need only prove that for a matrix $A \in M_{n,n'}(\mathbb{Z}[G])$ the difference

$$\mathrm{rk}_{\mathbb{Z}}(\ker(A_H)) - [G:H]\,\mathrm{rk}_{\mathbb{Z}[G]}(\ker(A))$$

is bounded by the number of zeroes in $H^\perp$ of some polynomial. We know from the case of a module and Lemma 2.5 that

$$\mathrm{rk}_{\mathbb{Z}}\,\mathrm{Im}(A_H) = [G:H]\,\mathrm{rk}_{\mathbb{Z}[G]}(\mathrm{Im}\,A) + e$$

where e is the number of zeroes in $H^\perp$ of some polynomial, and since

$$\mathrm{rk}_{\mathbb{Z}}\,\mathrm{Im}(A_H) + \mathrm{rk}_{\mathbb{Z}}\,\ker(A_H) = n'[G:H] = (\mathrm{rk}_{\mathbb{Z}[G]}\,\mathrm{Im}(A) + \mathrm{rk}_{\mathbb{Z}[G]}\,\ker(A))[G:H]$$

we get the result we want. □

There is a majoration of the number of zeroes of a polynomial on the m-torus: given an affine subvariety $X \subset \mathbb{C}^m$ we denote by $X[H]$ the finite set $H^\perp \cap X$; then there exists a constant C depending only on X such that:

$$|X[H]| \leq C\frac{[G:H]}{\alpha(H)}. \tag{2.4}$$

We can retrieve from Proposition 2.3 and this inequality a generalisation to the non ℓ^2-acyclic case of Theorem 2.1 in [CW03] (where a proof of the above bound can be found).

Corollary *Let M be a finitely generated $\mathbb{Z}[G]$-module (resp. C_* a finite complex of finitely generated free $\mathbb{Z}[G]$-modules). Then there exists a constant $C > 0$ such that:*

$$|\dim(M_H \otimes \mathbb{Q}) - [G:H]\,\mathrm{rk}(M)| \leq C\frac{[G:H]}{\alpha(H)}$$

(resp.

$$|b_i(C_H) - [G:H]b_i^{(2)}(C_*)| \leq C\frac{[G:H]}{\alpha(H)}).$$

2.2 Convergence of Riemann sums

We shall soon give a method to construct from a finite number of polynomials in m variables a sequence of subgroups G_N of G for which the Riemann sums over the subsets $G_N^\perp$ converge to the Mahler measure, and the polynomials have "few" zeroes in $G_N^\perp$. We will also need a lower bound on the nonzero values of the polynomials at points in $G_N^\perp$. In the case where G is cyclic these are two well-known facts which we recall below.

Lemma 2.6 *Let $P \in \mathbb{Z}[t]$ be a nonzero polynomial;*

(i) $\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{\zeta^N=1 \\ P(\zeta)\neq 0}} \log|P(\zeta)| = m(P)$.

(ii) *There is a $C_P > 0$ such that for all Nth roots of unity ζ such that $P(\zeta) \neq 0$ we have:* $\log|P(\zeta)| \geq -C_P \log N$.

Proof From the additivity of the logarithmic Mahler measure and the decomposition of P into irreducible factors over $\mathbb{C}$ we see that the only case we need consider is that of a degree one polynomial of the form $t-a$ for some algebraic number a; if the absolute value of a is not 1 then the result is trivial. If $|a|=1$ then it is easy to see that the superior limit of the Riemann sums is less than 0 (just cut off $\log|z-a|$ near a to replace it with a continuous function with arbitrarily small mean value over $\mathbb{T}^1$). Jensen's formula implies that the Mahler mesure of $z-a$ is 0, so we have to prove that:

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{\substack{\zeta^N=1\\ \zeta\neq a}}\log|\zeta-a|\geq 0.$$

From a diophantine estimate due to Baker it is deduced in Lemma 1.11 of [EW99] that there exists a constant $C_a>0$ such that:

$$\prod_{\substack{\zeta^N=1\\ \zeta\neq a}}|\zeta-a|\geq\frac{1}{N^{C_a}}. \tag{2.5}$$

we get for all N

$$\frac{1}{N}\sum_{\substack{\zeta^N=1\\ \zeta\neq a}}\log|\zeta-a|\geq\frac{-C_a\log N}{N}$$

and since the right-hand side converges to zero we get the desired result.

Statement (ii) is a "local" version of (2.5) and is easily deduced from it. □

If $m>1$ the same argument as above shows that:

$$\limsup_{\alpha(H)\to\infty}\sum_{\substack{\zeta\in H^\perp\\ P(\zeta)\neq 0}}\log|P(\zeta)|\leq m(P).$$

However, we cannot give good lower bounds for the values of P at roots of unity and thus cannot prove the convergence. As a palliative, we use in the next subsection an approximation result due to Boyd and Lawton to be able to apply the above results to multivariate polynomials.

2.3 Construction of the sequence G_N

Proposition 2.7 *Let $\mathcal{P}$ be a finite collection of polynomials in $\mathbb{Z}[t_1,\dots,t_m]$. There exists a sequence of finite index subgroups $G_N\subset G=\mathbb{Z}^m$ such that $\alpha(G_N)\underset{N\to\infty}{\longrightarrow}\infty$*

and for any polynomial $P \in \mathcal{P}$ we have:

(i) $$\frac{1}{[G:G_N]} \sum_{\substack{\zeta \in G_N^\perp \\ P(\zeta)\neq 0}} \log|P(\zeta)| \underset{N\to\infty}{\longrightarrow} m(P);$$

(ii) $$|\{\zeta \in G_N^\perp,\ P(\zeta)=0\}| \leq \log[G:G_N];$$

(iii) $$\forall \zeta \in G_N^\perp,\ \log|P(\zeta)| \geq -(\log[G:G_N])^2.$$

Proof It is clear from the nature of the arguments below that we need only consider the case of a single polynomial $P \in \mathbb{Z}[t_1,\dots,t_m]$.

For any vector $v=(v_1,\dots,v_m) \in \mathbb{Z}^m$ we denote by v^* the subgroup given by $\{(u_1,\dots,u_m) \in \mathbb{Z}^m,\ \sum_i u_i v_i = 0\}$. We put $q(v)=\alpha(v^*)$ and for a polynomial $P \in \mathbb{Z}[t_1,\dots,t_m]$, $P_v(X)=P(X^{v_1},\dots,X^{v_m})$. The main result of [Law83] is that:

$$m(P_v) \underset{q(v)\to\infty}{\longrightarrow} m(P).$$

For a prime p we denote by $p=p_1<p_2<\dots<p_m$ the m primes following p; we put $r_i=\prod_{j\neq i} p_j$ for $i=1,\dots,m$ and $r=(r_1;\dots,r_m)$. It is easy to see that $q(r)\geq p$: if $\sum_i s_i r_i = 0$ for some nonzero $s \in \mathbb{Z}^m$ and if $s_j \neq 0$ we get $-ts_jr_j = p_j \sum_{i\neq j} \frac{r_i}{p_j} s_j$. Since the prime p_j does not divide r_j we must have $p_j | s_j$ and thus $q(r) \geq |s_j| \geq p_j \geq p$. Thus $m(P_r) \underset{p\to\infty}{\longrightarrow} m(P)$. For an integer $M \geq 0$ we define a subgroup $G_{p,M}$ of G by:

$$G_{p,M} = \{v \in G, \sum_i v_i r_i \equiv 0 \pmod{M}\};$$

this is equal to $r^* \oplus \mathbb{Z}Mv$ for any v generating a supplement to r^*, and has index M in G. It is not hard to see that for $M > mp_1\dots p_m$ we have $\alpha(G_{p,m}) \geq p$: let $v \in G_{p,M}$, $v \notin r^*$. Then $|\sum_i r_i v_i| \geq M$ and so we must have $|v_j| > M/(m\prod_{i\neq j} p_i)$ for some j. This is bigger than p for M as above and the claim follows. Now given an $\varepsilon > 0$, choose p big enough so that $|m(P_r)-m(P)|<\varepsilon$; by Lemma 2.6 there exists an integer M_0 such that for all $M \geq M_0$:

$$\left| m(P_r) - \frac{1}{M} \sum_{\zeta^M=1,\, P_r(\zeta)\neq 0} \log|P_r(\zeta)| \right| < \varepsilon.$$

We see that $G_{p,M}^\perp = \{(\zeta^{r_1},\dots,\zeta^{r_m}),\ \zeta^M=1\}$ and thus we get:

$$\left| m(P) - \frac{1}{[G:G_{p,M}]} \sum_{\zeta \in G_{p,M}^\perp,\, P(\zeta)\neq 0} \log|P(\zeta)| \right| < 2\varepsilon$$

for any $M \geq M_0$.

Now the number of zeroes of P in $G_{p,M}^{\perp}$ is less than the degree of P_r, which is itself less than:

$$\max_i |r_i| \times \deg(P) \leq \deg(P) p_1 \dots p_m \leq \deg(P) 2^{\frac{m(m+1)}{2}} p^m$$

since we have $p_i \leq 2p_{i-1}$ by a well-known result of Chebyshev. Thus there exists an integer M_1 such that the number of zeroes of P in $G_{p,M}$ is less than $\log(M)$ for all $M \geq M_1$.

From Lemma 2.6 we get that for all p there is a constant C_p such that the nonzero values of P at the points $\zeta \in G_{p,M}$ satisfy:

$$\log |P(\zeta)| \geq -C_p \log(M).$$

It follows that for $M \geq M_2 = \exp(C_p)$ we have $\log |P(\zeta)| \geq -\log(M)^2$.

We can construct a sequence $G_N = G_{p_N, M_N}$ where p_N is the Nth prime number and M_N a well-chosen positive integer: we take $M_N > m p_N \dots p_{N+m-1}$ so that $\alpha(G_{p_N,M_N}) \geq p_N$, $M_N \geq M_0, M_1, M_2$ (M_0 as above for, say, $\varepsilon = 1/p$) so that (i), (ii) and (iii) hold. □

2.4 Proof of Proposition 2.2

We shall need the following elementary lemma:

Lemma 2.8 *Let f be a linear map between Hermitian spaces V, W and V' a subspace in V with codimension d; suppose that any singular value λ of f satisfies $c \leq \lambda \leq C$. Putting $f' = f_{|V'}$, we have:*

$$d \log(c) \leq \log(\det{}'(f)) - \log(\det{}'(f')) \leq d \log(C).$$

Proof If $\lambda_1 \geq \dots \geq \lambda_{\dim V}$ are the eigenvalues of an invertible positive self-adjoint endomorphism g of V, we have that:

$$\lambda_i = \max_{F \leq V, \dim(F)=i} \min_{x \in F} \frac{||gx||}{||x||};$$

so if $g = f^* f|_{\ker(f)^\perp}$ and $\lambda'_1, \dots, \lambda'_{\dim V - d}$ are its eigenvalues in restriction to V' (the singular values of f'), we have $\lambda_i \geq \lambda'_i \geq \lambda_{i+d}$ for all $i \geq 0$ and it follows that:

$$\log(\det{}'(f)) - \log(\det{}'(f')) \leq \sum_{i=1}^{d} \log(\lambda_i) \leq d \log(C)$$

and that

$$\log(\det{}'(f)) - \log(\det{}'(f')) \geq \sum_{i=\dim V-d}^{\dim V} \log(\lambda_i) \geq d\log(c).$$

□

Proof of Proposition 2.2 As in the proof of proposition 2.7 it is clear that we can restrict ourselves to the case of a single matrix. The second thing to be acknowledged is that it suffices to prove the result for self-adjoint matrices since for any matrix $A \in M_{n,m}(\mathbb{C}[G])$ we have $\det_{\mathcal{N}(G)}(A) = \sqrt{\det_{\mathcal{N}(G)}(A^*A)}$. Thus we suppose that $A \in M_n(\mathbb{Z}[G])$ is a self-adjoint matrix.

We want to show that:

$$\lim_{N\to\infty} \frac{\log\det{}'(A_{G_N})}{[G:G_N]} = \det{}_{\mathcal{N}(G)}(A)$$

for a sequence G_N obtained from Proposition 2.7. The strategy is to use Lemma 2.8 above to reduce to the case when A has a nonzero determinant. The proposition is then a straightforward consequence of (i) of Proposition 2.7 applied to $\det(A)$. Let L be a free maximal rank $\mathbb{Z}[G]$-submodule in $\operatorname{Im}(A)$, L' its image by A and h, g be isomorphisms of $\mathbb{Z}[G]^r$ ($r = \operatorname{rk}(A)$) with L, L' so that the following diagram commute:

$$\begin{array}{ccc} \mathbb{Z}[G]^r & \xrightarrow{A'} & \mathbb{Z}[G]^r \\ {\scriptstyle h}\downarrow & & {\scriptstyle g}\downarrow \\ \mathbb{Z}[G]^n & \xrightarrow{A} & \mathbb{Z}[G]^n \end{array}$$

with A' a square matrix with nonzero determinant. Since $L \otimes \mathbb{C}$ is dense in $\ker(A)^\perp$ Lemma 3.15(3) in [Lüc02] yields the equality:

$$\det{}_{\mathcal{N}(G)}(A) = \det{}_{\mathcal{N}(G)}(g)\det{}_{\mathcal{N}(G)}(A')\det{}_{\mathcal{N}(G)}(h^{-1}).$$

We write the characteristic polynomial $\det(t-A)$ as $\sum_{i=i_0}^n P_i(t_1,\dots,t_m)t^i$ where i_0 is the rank of $\ker(A)$ and thus all P_i are nonzero. Let $\mathcal{P}$ be the collection of all P_i together with the nonzero coefficients of the characteristic polynomials of A', hh^* and gg^* and all the polynomials appearing in Lemma 2.5 for h and g and the polynomials from Lemma 2.4 for $\operatorname{Im}(A)/\operatorname{Im}(h)$ and $\operatorname{Im}(A)/\operatorname{Im}(g)$. Let G_N be the sequence associated to $\mathcal{P}$ by Proposition 2.7. Since $\det{}' A(\zeta) = P_i(\zeta)$ for $i = \dim\ker(A(\zeta)) \geq i_0$ (and similarly for the other matrices) we get by (iii) of Proposition 2.7 that for any singular value λ of $A_{G_N}, A'_{G_N}, h_{G_N}$ or g_{G_N} we have:

$$\log\lambda \geq -\log[G:G_N]^2. \tag{2.6}$$

From Lemma 2.5 and (ii) of Proposition 2.7 we see that the subspace $\ker(h_{G_N})+\ker(g_{G_N})$ of $\mathbb{C}[G/G_N]^r$ has dimension $\leq \log[G:G_N]$. In the same way Lemma 2.4 yields that $\mathrm{Im}(h_{G_N})$ and $\mathrm{Im}(g_{G_N}) = A_H \operatorname{Im} h_{G_N}$ have codimension $\leq \log[G:G_N]$ in $\ker(A_H)^\perp$. We define subspaces $V_1 \subset \mathbb{C}[G/G_N]^r$ and $V_2 \subset \mathbb{C}[G/G_N]^n$ by:

$$V_1 = \ker(h_{G_N})^\perp \cap \ker(A'_{G_N}) \cap A'^{-1}_{G_N}(\ker(g_{G_N})^\perp)$$
$$V_2 = h_{G_N}(V_1)$$

so that we have a commutative diagram:

$$\begin{array}{ccc} V_1 & \xrightarrow{A'_{G_N}} & A'_{G_N}(V_1) \\ {\scriptstyle h_{G_N}}\downarrow & & {\scriptstyle g_{G_N}}\downarrow \\ V_2 & \xrightarrow{A_{G_N}} & A_{G_N}(V_2) \end{array}$$

such that all arrows are isomorphisms. We also get for $i = 1,2$:

$$\dim V_i - r[G:G_N] = O(\log[G:G_N]).$$

In the sequel we use the abbreviation $u_H \sim v_H$ to mean that $\log(u_H) = \log(v_H) + o([G:H])$. From Lemma 2.8 and (2.6) above we see that for any linear map $f : \mathbb{Z}[G]^n \to \mathbb{Z}[G]^{n'}$ and any subspaces $F_N \subset \mathbb{C}[G/G_N]^n$ such that $\dim(F_N) = O([G:G_N]^b)$ for some $b < 1$ we have $\det'(f_{G_N}) \sim \det'(f_{G_N}|_{F_N})$. Applying this to $f = A'$, with Lemma 2.8 applied to $V = W = \mathbb{C}[G/G_N]^r$ and $V' = V_1$ we get:

$$\det{}'(A'_{G_N}) \sim \det{}'(A'_{G_N}|_{V_1}).$$

In the same way we obtain:

$$\begin{aligned} \det{}'(A'_{G_N}|_{V_1}) &= \det{}'((g_{G_N}|^{A_{G_N}V_2}_{A'_{G_N}V_1})^{-1} A_{G_N} h_{G_N}|_{V_1}) \\ &= \det(g_{G_N}|^{A_{G_N}V_2}_{A'_{G_N}V_1})^{-1} \det(A_{G_N}|_{V_2}) \det(h_{G_N}|_{V_1}) \\ &\sim \det{}'(g_{G_N})^{-1} \det{}'(A_{G_N}) \det{}'(h_{G_N}), \end{aligned}$$

where the last line line follows from Lemma 2.8 applied to $f = g_{G_N}$, $V = \mathbb{C}[G/G_N]^r$ and $W = \mathrm{Im}(A)_{G_N}$, and $V' = A'_{G_N}V_1$; then to $f = A_{G_N}$, $V = W = \mathrm{Im}(A)_{G_N}$, $V' = V_2$ and finally to $f = h_{G_N}$, $V = \mathbb{C}[G/G_N]^r$, $W = \mathrm{Im}(A)_{G_N}$ and $V' = V_1$. We conclude that:

$$\frac{\log(\det{}'(A_{G_N}))}{[G:G_N]} = \frac{\log(\det{}'(h_{G_N})^{-1} \det{}'(A'_{G_N}) \det{}'(g_{G_N}))}{[G:G_N]} + o(1)$$

and we see that to prove the proposition it suffices to consider the case with nonzero determinant, since we can then apply it to the matrices A', hh^* and gg^*.

We suppose now that $P = \det(A) \neq 0$; this case is proved as in [Lüc02, Lemma 13.53]: we decompose $\mathbb{C}[G/G_N]^n$ as the sum $\bigoplus_{\zeta \in G_N^\perp} V_\zeta^n$ where G/G_N acts on the line V_ζ through the character induced by ζ so that A_{G_N} is represented on V_ζ^n by the matrix $A(\zeta)$. The number of $\zeta \in G_N^\perp$ with $\det(A)(\zeta) = 0$ is $\leq \log[G : G_N]$ and since the singular values of the A_{G_N} are bounded above by a constant an below by (2.6) we get:

$$\begin{aligned} \frac{1}{[G : G_N]} \log \det{}'(A_{G_N}) &= \frac{1}{[G : G_N]} \sum_{\zeta \in G_N^\perp} \log \det{}' A(\zeta) \\ &\sim \frac{1}{[G : G_N]} \sum_{\zeta \in G_N^\perp,\, \det(A)(\zeta) \neq 0} \log \det(A(\zeta)) \end{aligned}$$

Applying (i) of Proposition 2.7 to the right hand side we see that it converges to $\log \det_{\mathcal{N}(G)}(A)$ as $N \to \infty$. □

Remark

There is a general condition under which it is known that the approximation of the Fuglede-Kadison determinant is true (see [Lüc94, Theorem 3.4]). We could have checked this condition directly for the matrices A_{G_N} using (iii) of Proposition 2.7, but we have chosen to give a direct proof instead.

2.5 Growth of volumes for ℓ^2-acyclic complexes

We treat here the growth of the parasite term $\prod_i \mathrm{vol}(H_i)^{(-1)^i}$ in (1.4). The following proposition is an adaptation of the main result of [BV, section 7].

Proposition 2.9 *Let C_* be a finite complex of finitely generated $\mathbb{Z}[G]$-modules. Suppose that for some i we have $H_i^{(2)}(C) = 0$. Then for a sequence G_N obtained from Proposition 2.7 we have:*

$$|\log \mathrm{vol}(H_i(C_{G_N})_{free})| = O(\log[G : G_N]^2).$$

Proof For convenience we pick up the notation in [BV] and denote $R_i(A) = \mathrm{vol}(H_i(A)_{free})$ for a finite complex of free finitely generated $\mathbb{Z}$-modules A_*. The two next lemmas are proved there:

Lemma 2.10 *Let a finite group K act by isometries on a finite complex A_* of free $\mathbb{Z}$-modules endowed with metrics. Let M be bigger than all singular values of the differentials of A_* an suppose that the A_i are generated by vectors with length less than ν. Suppose that all irreducible characters of K appearing in the K-vector space $H_i(A)\otimes\mathbb{C}$ are contained in a subset X and denote by D the maximal dimension of the X-isotypical component of the A_j. Then the following inequality holds:*

$$R_i(A) \geq (M\nu|K|^5)^{-D}.$$

Lemma 2.11 *Let A be as above and B_* be the dual complex $B_{n-j} = \hom(A_j,\mathbb{Z})$ with the dual metric. We have:*

$$R_j(A_*)R_{n-j}(B_*) = 1.$$

Now we know that the differentials of the complexes $(C_*)_H$ have their singular values bounded by a constant M depending only on C_* and that the $(C_j)_H$ are generated by vectors of length one. Let X be the set of irreducible characters of G/H appearing in $H_i(C_H)\otimes\mathbb{C}$ and D the maximal dimension of the X-isotypical component of the $(C_j)_H$. Lemma 2.10 yields:

$$R_i(C_H) \geq (M[G:H]^5)^{-D}.$$

Now Lemma 2.11 implies that:

$$(M[G:H]^5)^D \geq R_i(C_H) \geq (M[G:H]^5)^{-D}; \tag{2.7}$$

Since all C_j are contained in $\mathbb{Z}[G]^n$ for some n we see that for any set of characters Y the dimension of the Y-isotypical component of $(C_j)_H\otimes\mathbb{C}$ is less than $n|Y|$. From Proposition 2.3 and (ii) of Proposition 2.7 we get that D in (2.7) is an $O(\log[G:G_N])$ and thus that:

$$|\log R_i(C_{G_N})| = O(\log[G:G_N]^2).$$

□

Corollary *Let C_* be a finite ℓ^2-acyclic complex of free finitely generated $\mathbb{Z}[G]$-modules. There exists a sequence G_N so that we have:*

$$\lim_{N\to\infty}\left(\prod_i |H_i(C_{G_N})_{\mathbb{Z}-tors}|^{(-1)^i}\right)^{\frac{1}{[G:G_N]}} = \tau^{(2)}(C).$$

Proof From Proposition 2.9 we get that we can choose G_N so that for all i we have

$$\frac{|\log \mathrm{vol}(H_i(C_{G_N})_{free})|}{[G:G_N]}$$

tends to 0 for all i. From Proposition 2.2 applied to the d_i we get that we can choose G_N so that furthermore

$$\frac{\log \tau(C_{G_N})}{[G:G_N]} \xrightarrow[N\to\infty]{} \tau^{(2)}(C).$$

The corollary then follows from (1.4). □

3 Proof of the main theorems

We recall notation from the Introduction: if X is a CW-complex with universal covering $\widetilde{X}$ and a surjection $\phi : \pi_1(M) \to G \cong \mathbb{Z}^m$, for any subgroup $H \subset G$ we denote by X_H the Galois covering of X given by $\phi^{-1}(H) \setminus \widetilde{X}$; its Galois group is G/H. We also denote the free abelian covering of X given by $\ker(\phi) \setminus \widetilde{X}$ by $\widehat{X}$. Theorem 0.1 and Theorem 0.2 are immediate consequences of the following result:

Theorem 3.1 *For all i, let $\Delta_j(H_i(\widehat{X}))$ be the first nonzero Alexander polynomial. There exists a sequence of subgroups $G_N \subset G$ with $\alpha(G_N) \xrightarrow[N\to\infty]{} \infty$ such that:*

$$\lim_{N\to\infty} \frac{\log |H_i(X_{G_N})_{tors}|}{[G:G_N]} = m(\Delta_j(H_i(\widehat{X}))).$$

When $m = 1$ we have:

$$\lim_{N\to\infty} \frac{\log |H_i(X_N)_{tors}|}{N} = m(\Delta_j(H_i(\widehat{X}))).$$

We begin by proving this for $m = 1$, and then use Proposition 2.7 and the Bombieri-Zannier theorem to deduce the result for $m > 1$.

3.1 When G is cyclic

Let M be a finitely generated $\mathbb{Z}[t^{\pm 1}]$-module of rank r; put $M_N = M/(t^N - 1)M$. We use the same notation for a finite complex of free modules C_*, d_*. We want to show that:

$$\lim_{N\to\infty} \frac{\log |H_i(C_N)_{\mathbb{Z}-tors}|}{N} = m\big(\Delta_{b_i^{(2)}(C)}(H_i(C))\big) \tag{3.1}$$

and also the corresponding limit for the module M:

$$\lim_{N\to\infty} \frac{\log|(M_N)_{\mathbb{Z}-tors}|}{N} = m\big(\Delta_r(M)\big) \tag{3.2}$$

We shall deduce (3.1) from (3.2). The latter is proved in three steps: first we consider torsion-free modules, then torsion modules and finally we combine those two to prove the general case.

Torsion-free modules

Suppose that M is torsion-free; we want to show that

$$\frac{1}{N}\log|(M_N)_{tors}| \underset{N\to\infty}{\longrightarrow} 0. \tag{3.3}$$

We will in fact show that the torsion in M_N is bounded independantly of N. We can pick an embedding of M into a free module $L \cong \mathbb{Z}[G]^n$. Put $T = L/M$; then the $\mathbb{Z}$-torsion part of T has a finite exponent: the $\mathbb{Z}$-torsion part of T is isomorphic to M'/M where $M' = L \cap M \otimes \mathbb{Q}$. The submodule M' is finitely generated by $x_1, \dots, x_k \in L$, and for each index i there is an integer n_i so that $n_i x_i \in M$. It follows that the exponent of M'/M,and thus that of T, divides $n_1 \dots n_k$. We can use this to bound the exponent of M_N:

Lemma 3.2 *For all N the exponent of $(M_N)_{tors}$ divides that of the $\mathbb{Z}$-torsion part of $T = L/M$.*

Proof Suppose that $x \in M$ descends to an element of order exactly n in M_N; this means that there exists $y \in M$, $y \notin dM$ for any divisor $d > 1$ of n such that $nx = (t^N - 1)y$. In the free module L we see (by applying the Gauss lemma coordinate by coordinate) that n must divide y. It follows that $n^{-1}y$ has order exactly n in L/M. □

For the torsion-free module M, it follows from the above Lemma that the $\mathbb{Z}$-torsion in M_N is always K-torsion with a K not depending on N. Now (3.3) follows from the following lemma.

Lemma 3.3 *Let M be a finitely generated $\mathbb{Z}[t^{\pm 1}]$-module (resp. C_* a finite $\mathbb{Z}[t^{\pm 1}]$-complex) and K a nonzero integer. Suppose that the K-torsion part of M (resp. $H_i(C_*)$ is finite; then the K-torsion in M_N (resp. $H_i(C_N)$) is bounded independantly of N.*

Proof Let $p_1^{k_1}\dots p_l^{k_l}$ be the prime factorization of K. The K-torsion part of M_N can be written as:

$$(M_N)_{K-tors} \cong \bigoplus_{i=1}^{l}\left(\bigoplus_{j=1}^{d_{i,N}} \mathbb{Z}/p^{\alpha_{i,j}}\mathbb{Z}\right)$$

where $d_{i,N} = \dim(M_N \otimes \mathbb{F}_{p_i}) - \mathrm{rk}_{\mathbb{Z}}(M_N)$ and $\alpha_{i,j} \le k_i$. It follows immediately that

$$|(M_N)_{K-tors}| \le \sum_{i=1}^{l} p_i^{k_i}(\dim(M_N \otimes \mathbb{F}_{p_i}) - \dim(M_N \otimes \mathbb{Q})).$$

Thus, we need only prove that for $p = p_1, \dots, p_l$ the difference:

$$\dim(M_N \otimes \mathbb{F}_p) - \dim(M_N \otimes \mathbb{Q})$$

is bounded. This can be seen by examining the proof of Theorem 0.2(i) in [LLS11] in our very special case; we give a short direct proof here. Note that the proof is very similar to that of Proposition 2.3.

Suppose that M is torsion-free, let $r = \mathrm{rk}_{\mathbb{Z}[t^{\pm 1}]}(M)$, and choose an embedding of M into a free module L of rank r. Since we know by Proposition 2.3 that $\mathrm{rk}_{\mathbb{Z}}(M_N) - rN$ is bounded, we need to show that $\dim(M_N \otimes \mathbb{F}_p) - rN$ is bounded. In the sequel we denote abusively M, L the $\mathbb{F}_p$-modules $M, L \otimes \mathbb{F}_p$. There exists a $f \in \mathbb{F}_p[t]$ such that $fL \subset M$. We have $f_N L_N \subset M_N \subset L_N$; thus we only need to show that $\dim_{\mathbb{F}_p} \ker(f_N)$ is bounded. Suppose that $L = \mathbb{F}_p[t^{\pm 1}]$; then we can identify L_N with the polynomials in $\mathbb{F}_p[t]$ of degree less than N. Let $\ell = \deg(f)$. Then if $h \in \mathbb{F}_p[t]$ is nonzero with $\deg(h) < N - \ell$ the product fh is still a polynomial of degree $< N$ and thus fh is not zero in L_N. It follows that f_N is injective on a subspace of dimension $N - \ell$, so that its kernel has dimension less than ℓ. If $L \cong \mathbb{F}_p[t^{\pm 1}]^n$ we can make the same argument coordinate by coordinate.

In general, according to the hypothesis on the p-torsion in M we have $\mathrm{rk}_{\mathbb{F}_p[t^{\pm 1}]} M \otimes \mathbb{F}_p = r$. Thus, to deduce the general case from the torsion-free case we need only show that for a $\mathbb{F}_p[t^{\pm 1}]$-torsion module T the dimension $\dim(T_N)$ is bounded. This is easily deduced from the result above by choosing a surjection from some $\left(\mathbb{F}_p[t^{\pm 1}]/(f)\right)^k$ to T.

The statement about homology is deduced from the statement for modules in the exact same way as in the proof of Proposition 2.3. □

Torsion modules

Let T be a finitely generated torsion module over $\mathbb{Z}[t^{\pm 1}]$; we want to show that:

$$\lim_{N\to\infty} \frac{\log |(T_N)_{tors}|}{N} = m(\Delta_0(T)). \tag{3.4}$$

We first return to the more general situation where $G \cong \mathbb{Z}^m$ to make a number of considerations.

Lemma 3.4 *Let T be a finitely generated $\mathbb{Z}[G]$-torsion module. Then there exists a finite resolution of T by finitely generated free modules, which we write as:*

$$0 \to F_{m+2} \overset{\phi_{m+2}}{\to} \dots \overset{\phi_2}{\to} F_1 \overset{\phi_1}{\to} T \to 0 \tag{3.5}$$

The complex F_*, ϕ_* is a ℓ^2-acyclic complex and its ℓ^2-torsion equals $\mathcal{M}(\Delta_0(T))^{-1}$. Moreover, for all $i > 1$ there exists finitely generated torsion-free modules M_i such that for any subgroup $H \subset G$ the homology $H_i(F_H)$ embeds into $M_{i,H}$.

Proof The ring $\mathbb{Z}[t_1, \dots, t_m]$ is a regular ring with dimension $m+1$ by [Ser00, IV.D, Proposition 25]; since its localizations are the same as those of $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \cong \mathbb{Z}[G]$ the latter is also regular, with the same dimension. Now this means that all finitely generated modules over $\mathbb{Z}[G]$ have a projective resolution of length $\leq m+2$. Since it is known that all projective $\mathbb{Z}[G]$-modules are free (cf. [Lam06, Corollary V.4.12]) we have in fact that this holds with a free resolution, so that we get (3.5). The complex F_* is given by the sequence

$$0 \to F_{m+2} \to \dots \to F_1 \to 0$$

so that its homology is zero for $i > 1$ and T for $i = 1$. Thus it is a finite ℓ^2-acyclic complex of free finitely generated $\mathbb{Z}[G]$-modules, and Proposition 1.5 allows to compute that its ℓ^2 torsion equals $-m(\Delta_0(T))$.

Since $\mathrm{Im}(\phi_{i,H})$ is equal to the image of $\mathrm{Im}(\phi_i)_H = \ker(\phi_i)_H$, the homology $H_i(F_H)$ is equal to the quotient $\ker(\phi_{i,H})/\ker(\phi_i)_H$. The map ϕ_i embeds $M_i = F_i/\ker(\phi_i)$ into F_{i-1}; $\ker(\phi_{i,H})/\ker(\phi_i)_H$ is isomorphic to the kernel of the induced map $M_{i,H} \to F_{i-1,H}$, which is of course embedded in $M_{i,H}$. □

Now we return to $m = 1$: it follows from the above Lemma and (3.3) that for $i > 1$ we have

$$\frac{1}{N} \log |H_i(C_N)_{tors}| \leq \frac{1}{N} \log |(M_{i,N})_{tors}| \underset{N \to \infty}{\longrightarrow} 0.$$

Thus (3.4) follows from the Corollary to Proposition 2.9 applied to the complex F_*, ϕ_* since $H_1(F_N) \cong T_N$.

Conclusion

Now we can prove (3.2): we have the exact sequence

$$0 \to M_{tors} \xrightarrow{i} M \to M' \to 0$$

where $M' = M/M_{tors}$ is torsion free. Tensoring with $\mathbb{Z}[\mathbb{Z}/N\mathbb{Z}]$ yields

$$(M_{tors})_N \xrightarrow{i_N} M_N \to M'_N \to 0.$$

Lemma 3.5 *The map $i_N : (M_{tors})_N \to M_N$ is injective for all N.*

Proof Suppose that $x \in M_{tors}$ and the image of x in $(M_{tors})_N$ is in the kernel of i_N; this means that there exists a $y \in M$ such that $x = (t^N - 1)y$. But it follows from this last equality that $y \in M_{tors}$, so that x is already zero in $(M_{tors})_N$. □

Thus, for the module M we have the inequality

$$1 \leq \frac{|(M_N)_{\mathbb{Z}-tors}|}{|(M_{tors})_N)_{\mathbb{Z}-tors}|} \leq (M'_N)_{\mathbb{Z}-tors}.$$

Since we have already proved the term on the right is bounded it follows that

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \log |(M_N)_{\mathbb{Z}-tors}| &= \lim_{N\to\infty} \frac{1}{N} \log |\big((M_{tors})_N\big)_{\mathbb{Z}-tors}| \\ &= m(\Delta_0(M_{tors})) \\ &= m(\Delta_r(M))\end{aligned}$$

which finishes the proof of (3.2).

If C_*, d_* is a finite complex of free finitely generated $\mathbb{Z}[t^{\pm 1}]$-modules the map from $\ker(d_i)_N$ to $\ker(d_{i,N})$ induces a map $j_N : H_i(C)_N \to H_i(C_N)$. The same proof as that of Lemma 3.5[1] yields that j_N is injective so that we get:

$$1 \leq \frac{|H_i(C_N)_{tors}|}{|(H_i(C)_N)_{tors}|} \leq |\operatorname{coker}(j_N)_{tors}|.$$

To deal with the right-hand side we use the same trick as to prove (3.4): the cokernel of j_N embeds into $\ker(d_{i,N})/\ker(d_i)_N$, which embeds into M_N where $M = C_i/\ker(d_i)$ is torsion-free, so that $|\operatorname{coker}(j_N)_{tors}|$ is bounded. Summing up, we get that

$$\lim_{N\to\infty} \frac{1}{N} |H_i(C_N)_{tors}| = \lim_{N\to\infty} \frac{1}{N} |(H_i(C)_N)_{tors}| = m(\Delta_{b_i^{(2)}(C)}(H_i(C)))$$

where the last equality follows from (3.2) applied to $H_i(C)$.

[1] Suppose that $x \in \ker(d_i)$ is zero in $H_i(C_N)$, i.e. $x = y + (t^N - 1)z$, $y \in \operatorname{Im}(d_{i+1})$, $z \in (t^N - 1)C_i$. We get that $(t^N - 1)d_i(z) = 0$, so that in fact $z \in \ker(d_i)$ and x is already zero in $H_i(C)_N$.

3.2 Torsion-free modules, $m > 1$

Our aim here is to prove the equivalent of (3.3) for some sequence G_N obtained from Proposition 2.7. Recall from the proof of Proposition 2.7 that we have defined, when $m > 1$:

$$G_N = r_N^* \oplus \mathbb{Z} k_N v_N \tag{3.6}$$

where $r_N, v_N \in G$ so that the scalar product (r_N, v_N) is equal to 1 and $\alpha(r_N^*)$ tends to infinity as N does. By taking k_N large enough we can ensure that this sequence satifies the conclusions (i) and (ii) of Proposition 2.2, for any given finite collection of matrices. We want to show that given a torsion-free $\mathbb{Z}[G]$-module M we can also choose k_N so that:

$$\frac{1}{[G : G_N]} \log |(M_{G_N})_{tors}| \underset{N\to\infty}{\longrightarrow} 0 \tag{3.7}$$

holds. We remark that $M_{r_N^*}$ is a $\mathbb{Z}[t^{\pm v_N}]$ module (for $v \in \mathbb{Z}^m$ we use the notation t^v to denote the variable $t_1^{v_1} \dots t_m^{v_m}$). The strategy is then to apply the results of the preceding section to the $\mathbb{Z}[t^{\pm v_N}]$-modules $M \otimes \mathbb{Z}[G/r_N^*]$ together with the following result, the proof of which is postponed until the Appendix A.

Proposition 3.6 *Let M be a $\mathbb{Q}[G]$-module embedded in a free module L; then for N big enough there exists a product of cyclotomic polynomials Φ_N such that the kernel of $M_{r_N^*} \to L_{r_n^*}$ is annihilated by $\Phi_N(t^{\pm v_N})$.*

Let M be a torsion-free $\mathbb{Z}[G]$-module which we embed into a free module L. It follows from the above Proposition that for N big enough the kernel T of the map $M_{r_N^*} \to L_{r_N^*}$ is annihilated by $K\Phi_N$ where K is the exponent of the $\mathbb{Z}$-torsion in L/M and Φ_N is a product of cyclotomic polynomials in the variable t^{v_N}. The submodule T is the $\mathbb{Z}[t^{\pm v_N}]$-torsion part of $M_{r_N^*}$, so that the maps from $T_N = T/(t^{k_N v_N} - 1)T$ to M_{G_N} is injective according to Lemma 3.5. Let M' be the image of $M_{r_N^*}$ in $L_{r_N^*}$, and put $M'_N = M'/(t^{k_N v_N} - 1)M'$; it follows from the exact sequence:

$$0 \to T_N \to M_{G_N} \to M'_N \to 0$$

that the $\mathbb{Z}$-torsion in M_{G_N} is less than the product $|(T_N)_{tors}| \times |(M'_N)_{tors}|$. The $\mathbb{Z}[t^{\pm v_N}]$-module $T' = KT$ is annihilated by the polynomial Φ_N, so that the order of $(T'_k)_{tors}$ is an $o(k)$ according to (3.4). The torsion in T_N is less than the K-torsion times the torsion in T'_{k_N}, and the former is bounded (in k) according to Lemma 3.3; it follows that we can choose k_N so that $|(T_N)_{tors}| < k_N/N$ (say). Since M'_N is $\mathbb{Z}[t^{\pm v_N}]$-torsion-free (3.3) allows us to choose k_N so that $|(M'_N)_{tors}| < k_N/N$. The conclusion (3.7) follows (recall that $k_N = [G : G_N]$).

3.3 Conclusion

We are now in position to prove the following theorem, from which Theorem 3.1 follows immediately.

Theorem 3.7 *(i) Let M be a finitely generated module of rank r; then:*

$$\lim_{N\to\infty} \frac{\log |(M_{G_N})_{tors}|}{[G:G_N]} = m(\Delta_r(M)). \tag{3.8}$$

(ii) Let C_*, d_* be a complex of finitely generated free modules, $r = \operatorname{rk} H_i(C)$; we have:

$$\lim_{N\to\infty} \frac{\log |H_i(C_{G_N})_{tors}|}{[G:G_N]} = m(\Delta_r(H_i(C))) \tag{3.9}$$

Proof We use the same scheme of proof as in the cyclic case: we begin by proving (3.8) for a torsion module, then deduce the case of a finitely generated module, and finally use it to prove (3.9).

Modules

First of all, the proof of (3.8) does not change from the case $m = 1$: Lemma 3.4, the corollary to Proposition 2.9 and (3.7) together imply that for a torsion module T we can choose k_N in (3.6) so that

$$\frac{1}{[G:G_N]} \log |(T_{G_N})_{tors}| \underset{N\to\infty}{\longrightarrow} m(\Delta_0(T)). \tag{3.10}$$

Now let M be any finitely generated $\mathbb{Z}[G]$-module; we have an exact sequence:

$$0 \to M_{tors} \to M \to M' \to 0$$

where $M' = M/M_{tors}$. According to (3.7) we can choose the k_N so that M'_{G_N} has a negligible torsion, so that to deduce (3.8) from (3.10) we need only show that:

$$\frac{\log |\ker\big((M_{tors})_{G_N} \to M_{G_N}\big)_{\mathbb{Z}-tors}|}{[G:G_N]} \underset{N\to\infty}{\longrightarrow} 0. \tag{3.11}$$

The following result is deduced from Proposition 3.6 in Appendix A.

Proposition 3.8 *For N big enough the kernel of the map $(M_{tors})_{r_N^*} \to M_{r_N^*}$ is annihilated by $K\Phi_N$ where Φ_N is a cyclotomic polynomial and K an integer (not depending on N).*

We use this and an argument similar to that used to prove Lemma 3.5 to deduce (3.11). Suppose that $x \in M_{tors}$ descends to an element in $\ker\big((M_{tors})_{G_N} \to M_{G_N}\big)$; then there exists $y \in M, z \in (r_N^* - 1)M$ such that $x = (t^{k_N v_N} - 1)y + z$. It follows that $y \in M_{tors} + (r_N^* - 1)M$, so that we can write $x = (t^{k_N v_N} - 1)y' + z'$ where $y' \in M_{tors}$ and $z' \in (r_N^* - 1)M$ It follows that x and z' have the same image in $(M_{tors})_{G_N}$, so that the kernel $\ker\big((M_{tors})_{G_N} \to M_{G_N}\big)$ is the image of $T = \ker\big((M_{tors})_{r_N^*} \to M_{r_N^*}\big)$. According to Proposition 3.8, (3.3) and (3.4) we can choose k_N so that $(T_{k_N})_{tors}$ has order less than k_N/N. This implies that the torsion in the kernel of the map $(M_{tors})_{G_N} \to M_{G_N}$ also has order less than k_N/N and (3.11) follows.

Complexes

Let N be fixed. We compare the growth of the $\mathbb{Z}$-torsion of $H_i(C)_{r_N^*}/(t^{kv_N} - 1)H_i(C)_{r_N^*}$ and $H_i(C_{r_N^*}/(t^{kv_N} - 1)C_{r_N^*})$; more precisely we show that:

(3.12)
$$\lim_{k\to\infty} \frac{1}{k}\big(\log|(H_i(C)_{r_N^*}/(t^{kv_N}-1)H_i(C)_{r_N^*})_{\mathbb{Z}-tors}| - \log|(H_i(C_{r_N^*}/(t^{kv_N}-1)C_{r_N^*}))_{\mathbb{Z}-tors}|\big) = 0.$$

Let ϕ denote the map from $(H_i(C)_{r_N^*})_{tors}$ to $H_i(C_{r_N^*})_{tors}$; we prove that there exists a cyclotomic polynomial Φ_N which annihilates $\ker(\phi)$. Let $x \in \ker(d_i)$ map to an element in $\ker(\phi)$; we can write $x = y + z$ where $z \in \ker(d_i) \cap (r_N^* - 1)C_i$. By proposition 3.6, since there is no $\mathbb{Z}$-torsion in $C_i/\ker(d_i)$, we get that there exists Φ_N such that $\Phi_N z \in (r_N^* - 1)\ker(d_i)$, so that $\Phi_N x$ maps to 0 in $(H_i(C)_{r_N^*})_{tors}$. As for $\operatorname{coker}(\phi)$, applying Proposition 3.6 to the embedding of $C_i/\ker(d_i)$ into C_{i-1} we get that there is an integer K independant of N and a cyclotomic polynomial Φ'_N such that $K\Phi'_N \operatorname{coker}(\phi) = 0$. It follows from these remarks and (3.2), (3.1) that:

$$\begin{aligned} 0 \le \lim_{k\to\infty} \frac{1}{k}\big(&\log(H_i(C_{r_N^*}/(t^{kv_N} - 1)C_{r_N^*})) - \log(H_i(C)_{r_N^*}/(t^{kv_N} - 1)H_i(C)_{r_N^*})\big) \\ &\le \lim_{k\to\infty} \frac{1}{k}\log|(H_i(C_{r_N^*}/(t^{kv_N} - 1)C_{r_N^*}))_{K-tors}|. \end{aligned}$$

It follows from Lemma 3.3 that the limit on the right is zero, proving (3.12). By choosing k_N big enough we can thus suppose that

$$\lim_{N\to\infty} \frac{\log|(H_i(C)_{G_N})_{\mathbb{Z}-tors}| - \log|H_i(C_{G_N})_{\mathbb{Z}-tors}|}{[G : G_N]} = 0,$$

so that we deduce (3.9) from (3.8) applied to $H_i(C)$.

□

4 Cyclic coverings of 3–manifolds

In the case $m = 1$, Theorem 3.1 yields that for any finite CW-complex X with an infinite cyclic covering $\widehat{X}$ and $X_N = N\mathbb{Z} \setminus \widehat{X}$ we have:

$$\frac{\log |H_i(X_N)_{tors}|}{N} \underset{N\to\infty}{\longrightarrow} \Delta_{b_i^{(2)}(\widehat{X})}(H_i(\widehat{X})).$$

Letting X be a cell structure on a compact n–manifold we get Theorem 0.2.

4.1 A presentation for the Alexander module

It would be nice to have an explicit formula for the Alexander polynomial $\Delta_i(\widehat{M})$. The simplest case is that of a manifold fibering over the circle, which is ℓ^2-acyclic and thus already treated in [BV, Corollary 7.7]: if φ is the diffeomorphism gluing the fiber S and φ_* its action on $H_1(S)$ we have $\Delta_0(\widehat{M}) = \det(1 - t\varphi_*)$. In the general case, the covering $\widehat{M}$ is obtained from an epimorphism $\pi_1(M) \to \mathbb{Z}$, and all such morphisms are given by the intersection form with an embedded incompressible two-sided nonseparating surface in M: see the proof of Lemma 6.6 in [Hem76]. From here on we suppose we have a closed 3-manifold M containing an embedded incompressible two-sided nonseparating surface S. Let $\widehat{M}$ be the infinite cyclic covering of M associated with S. Let $U \cong S \times (-1,1)$ be a tubular neighbourhood of S in M and let $M' = M - U$. We denote by $S_\pm$ the connected components of $\partial M'$ corresponding to $S \times \{\pm 1\}$ in $\overline{U}$. The fundamental group of M is then given by an HNN-extension: we consider $\pi_1(S_\pm)$ as subgroups of $\pi_1(M')$, and there is an isomorphism $\alpha : \pi_1(S_+) \to \pi_1(S_-)$ such that

$$\pi_1(M) = \langle \pi_1(M'), t \mid \forall g \in \pi_1(S_+),\ tgt^{-1} = \alpha(g) \rangle.$$

Let us denote by α_* the induced map $H_1(S_+) \to H_1(S_-)$. We were not able to get a formula as explicit as that above, but only to obtain a presentation of $H_1(\widehat{M})$.

We can give an explicit construction of $\widehat{M}$ using S: $\widehat{M}$ is diffeomorphic to the manifold obtained from $M' \times \mathbb{Z}$ by identifying $S_+ \times \{n\}$ with $S_- \times \{n+1\}$ using α. Let i be the embedding of S in M' corresponding to S_+. We know that $H_0(\widehat{M}) = 0$ and the map from $H_1(S_+)$ to $H_1(M')$ is injective; thus the homology long exact sequence coming from:

$$S \times \mathbb{Z} \underset{(1-t\alpha)i}{\longrightarrow} M' \times \mathbb{Z} \to \widehat{M}$$

yields the short exact sequence:

$$\mathbb{Z}[t^{\pm 1}] \otimes H_1(S) \underset{(1-t\alpha_*)i_*}{\longrightarrow} \mathbb{Z}[t^{\pm 1}] \otimes H_1(M') \to H_1(\widehat{M}) \to 0. \tag{4.1}$$

which is a presentation of $H_1(\widehat{M})$. However, in all generality it seems rather difficult to compute the minors of the matrix $(1-t\alpha_*)i_*$ due to the unknown nature of i_*.

4.2 Examples

Computation for rational homology cylinders

Suppose that $M', i_\pm$ is a rational homology cylinder, that is, the homology map i_* induces an isomorphism between rational homologies. Then we can compute the Alexander polynomial, generalising a result for knot complements:

Lemma 4.1 *In the situation described above, $\widehat{M}$ is ℓ^2-acyclic and $\Delta_0(H_1(\widehat{M}) = \tau(M', S_+)\det(1-t\alpha_*)$ where $\tau(M', S_+)$ denotes the order of $H_1(M')/i_*H_1(S)$.*

Proof This is an immediate consequence of (4.1) since in this case the map $(1-t\alpha_*)\circ i_*$ is represented by a square matrix with nonzero determinant. □

In the case where M is a knot exterior we retrieve the result of [Lic97, Theorem 6.5]. Note also that by the Dehn-Nielsen theorem α corresponds to an element ϕ in the mapping class group of S and we have $\det(1-t\alpha_*) = \det(1-t\phi_*)$, so that we also retrieve the case where M is fibered.

Given an homology cylinder $M', i_\pm$ (i.e. $i_\pm$ are embeddings of S in $\partial M'$ which induce an isomorphism in homology), [CFK11, Section 4.3] construct by surgery on the interior another homology cylinder. Their construction does not affect the Alexander polynomial of the infinite cyclic covering of the 3-manifold $M = M'/\{\forall x \in S,\ i_+(x) = i_-(x)\}$ dual to S.

Positive ℓ^2-Betti number

There indeed are cases where the infinite cyclic covering is not ℓ^2-acyclic. A somewhat artificial example is obtained as follows: suppose that M_0 is a three-manifold having an infinite cyclic covering $\widehat{M}_0$ and that N is another three-manifold with positive b_1. Then the connected sum $M = M_0\sharp N$ has an infinite cyclic covering $\widehat{M}$ difeomorphic to $\widehat{M}_0\sharp(N\times\mathbb{Z})$ (if M_0' is M_0 cut along a surface dual to the covering, it is obtained by attaching a copy of N to each copy of M_0' in $\widehat{M}_0$). Thus we have

$$H_1(\widehat{M}) \cong H_1(\widehat{M}_0) \oplus (\mathbb{Z}[t^{\pm 1}] \otimes H_1(N))$$

and it follows that $b_1^{(2)}(\widehat{M}) \geq b_1(N)$. Note that we also have $\Delta_{b_1(N)+i}(H_1(\widehat{M})) = \Delta_i(\widehat{M}_0)$.

A more interesting example is given by manifolds with large fundamental group. Recall that a group is said to be large when it has a finite-index subgroup surjecting onto a noncyclic free group. Suppose that $\pi_1(M)$ surjects onto the free product $\mathbb{Z} * \mathbb{Z}$; then we can take the surjection from $\pi_1(M)$ to $\mathbb{Z}$ obtained by projection onto the second free factor. Then the associated infinite cyclic covering $\widehat{M}$ has $b_1^{(2)}(\widehat{M}) \geq 1$. Indeed, the kernel of the map $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ is the subgroup freely generated by $a^N, b, aba^{-1}, \dots, a^{N-1}ba^{1-N}$, so that $\pi_1(M_N)$ surjects onto a free group of rank N and thus we have $b_1(M_N) = b_1(\pi_1(M_N)) \geq N$ (any three-manifold with infinite fundamental group is aspherical) so that $b_1^{(2)}(\widehat{M}) \geq 1$ by Lück's theorem.

Lots of arithmetic three-manifolds are known to have a large fundamental group, and it is expected that in fact all hyperbolic three-manifolds have a large π_1 [2]. See for example [Lac10] for recent progress on this and its links with other conjectures in 3-manifold topology.

A Loose ends

We shall prove here the results used in the proof of Theorem 3.7. Recall that, for a subgroup $H \subset \mathbb{Z}^{m'}$, we denote by H^* the subgroup of vectors orthogonal to H in $\mathbb{Z}^{m'}$; $\alpha(H)$ is the smallest length of a nonzero vector in H. We introduce notation we shall use throughout this Appendix: if A is an $m \times m'$ matrix with coefficients in $\mathbb{Z}$ we denote by H_A the subgroup of $\mathbb{Z}^{m'}$ spanned by its lines. We will continue to denote by A the induced morphism from $\mathbb{Z}[\mathbb{Z}^{m'}]$ to $\mathbb{Z}[\mathbb{Z}^m]$. Recall that we identify the group ring $\mathbb{Z}[\mathbb{Z}^{m'}]$ with the Laurent polynomials $\mathbb{Z}[t_1^{\pm 1}, \dots, t_{m'}^{\pm 1}]$; we denote this identification by $v \mapsto t^v$.

A.1 The Bombieri-Schinzel-Zannier theorem

The following result is an immediate generalisation of a theorem by Bombieri and Zannier (improving on a previous result by Schinzel):

[2] Ian Agol recently announced that he had a proof of this conjecture, following a previous work of Dani Wise on the subject.

Theorem A.1 *Let $f_1,\dots,f_k \in \mathbb{Z}[\mathbb{Z}^{m'}]$ be coprime polynomials. Then there exists a constant $c = c(f_1,\dots,f_m)$ such that if the gcd of $Af_1,\dots,Af_k$ is not equal to a cyclotomic polynomial then $\alpha(H_A^*) \leq c$.*

Proof This follows from an easy induction using the theorem as stated in [Sch00, Theorem 45 and page 517]; see also [Le, Proposition 5.2]. □

In the sequel we shall use the term "cyclotomic polynomial" to design multivariate generalised cyclotomic polynomials , i.e. the evaluations of a cyclotomic polynomial at a monomial, as well as their finite products.

A.2 Proof of Proposition 3.6

In the sequel we denote by A^n the map induced by A from $\mathbb{Z}[\mathbb{Z}^{m'}]^n$, i.e. $A^n(f_1,\dots,f_n) = (Af_1,\dots,Af_N)$.

Proposition A.2 *Let M be a submodule in $\mathbb{Q}[\mathbb{Z}^{m'}]^n$; there exists a constant c_M such that for all A with $\alpha(H_A^*) > c_M$, there exists a cyclotomic polynomial Φ_A so that:*

$$\Phi_A \ker(A^n) \cap M \subset \ker(A)M.$$

Proof In this proof we denote the group ring $\mathbb{Q}[\mathbb{Z}^{m'}]$ by R, and $\mathbb{Q}[A\mathbb{Z}^{m'}]$ by R'. It suffices to prove the result when the image of A is of rank one (i.e. A is a line matrix).

Suppose first that $M = fR$ is a nonzero principal ideal. For $\alpha(H)$ big enough the support of f injects into $\mathbb{Z}^{m'}/H$, in particular for $\alpha(H_A^*)$ big enough $Af \neq 0$. The ideal $\ker(A)$ is prime (because $R/\ker(A) \cong R'$ is a domain), so that if $fg \in \ker(A)$ then $g \in \ker(A)$. This means that $fR \cap \ker(A) = f\ker(A)$.

We next take M to be an ideal in R, and prove the result by induction on the number of generators. First we reduce to the case where M is not contained in any principal ideal. Suppose that f is the gcd of M; then we can write $M = fM'$ where M' is not contained in any principal ideal. We have $\ker(A) \cap fM' = f\ker(A) \cap fM'$ by the principal case, so that if the result is true for the embedding of M' in fR it is also true for M.

Thus, let $f_1,\dots,f_k$ be a minimal generating family for M with no common factor. Suppose that $\alpha(H_A^*) > c = c(f_1,\dots,f_k)$ with c the constant from Theorem A.1; then the gcd of $Af_1,\dots,Af_k$ must be equal to a cyclotomic polynomial $\Phi \in AR$ (we identify

it with its evaluation at the monomial t^{v_A} where v_A generates a supplementary for H_A^*). Let $h = \sum_i h_i f_i \in \ker(A) \cap M$; we get that:

$$-Ah_1 Af_1 = \sum_{i \geq 2} Ah_i Af_i$$

and so the gcd of $Af_2, \dots, Af_k$ divides ΦAh_1. We can write this gcd as $\sum_{i \geq 2} Ag_i Af_i$ for some $g_i \in R$ since R' is a principal ring (because $A\mathbb{Z}^{m'}$ is a cyclic group). Thus we get $\Phi Ah_1 = \sum_{i \geq 2} Ah'_i Af_i$ for some $h'_i \in R$. It follows that we have:

$$\Phi h_1 = \sum_{i \geq 2} h'_i f_i + h'_1$$

for some $h'_1 \in \ker(A)$, and from that:

$$\Phi h = \Phi h'_1 f_1 + \sum_{i \geq 2} (h_i + h'_i f_1) f_i.$$

By the induction hypothesis, if $\alpha(H_A^*) > c_{M'}$ (with $M' = (f_2, \dots, f_k)$) then $\sum_{i \geq 2}(h_i + h'_i f_1) f_i \in \Phi' \ker(A) M'$, so we get the result with $c_M = \max(c, c_{M'})$.

Suppose now that $\mathrm{rk}(M) = 1$ and M is embedded in R^n; up to passing to a larger free module (in $R \otimes \mathbb{Q}(\mathbb{Z}^{m'})^n$) we may suppose that M is contained in a copy of R that is a direct factor of R^n; then we can apply the above arguments to this embedding of M in R and get the result.

Now let us prove the general result by induction on the rank. Let M have rank $k > 1$ and be embedded in R^n; let $l < n$ so that the intersection $M_0 = M \cap (R^l \times 0)$ has rank one. Put $M_1 = M/M_0$; this is torsion-free and we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & M_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^l \times 0 & \longrightarrow & R^n & \longrightarrow & R^{n-l} & \longrightarrow & 0
\end{array}$$

Suppose that $x \in \ker(A^n) \cap M$; its projection x_1 in M_1 lies in $\ker(A^{n-l}) \cap M_1$ and, by the induction hypothesis, there exists a cyclotomic Φ such that $\Phi x_1 \in \ker(A) M_1$. It follows that $\Phi x = y + x_0$ for some $y \in \ker(A) M$ and $x_0 \in \ker(A^n) \cap M_0$. By the induction hypothesis we get that there exists a cyclotomic Φ' such that $\Phi' x_0 \in \ker(A) M_0$. Finally, we have that $\Phi \Phi' x \in \ker(A) M$. □

Proof of Proposition 3.6 The kernel $\ker(A)$ is easily seen to be equal to $(H_A^* - 1)R$; indeed, the groups $A\mathbb{Z}^{m'}$ and $\mathbb{Z}^{m'}/H_A^*$ are isomorphic via A so that the kernels of the maps $\mathbb{Z}[\mathbb{Z}^{m'}] \to \mathbb{Z}[A\mathbb{Z}^{m'}]$ and $\mathbb{Z}[\mathbb{Z}^{m'}] \to \mathbb{Z}[\mathbb{Z}^{m'}/H_A^*]$ must be equal. The first is $\ker(A)$

and the second $(H_A^* - 1)\mathbb{Z}[\mathbb{Z}^{m'}]$. Thus if $M \subset L$ is a submodule of a free module, the above Proposition yields that there exists a cyclotomic Φ_A such that Φ_A annihilates the kernel of the map

$$M_{H_A^*} = M/(H_A^* - 1)M \to L/(H_A^* - 1) = L_{H_A^*}.$$

Proposition 3.6 follows by taking A to be the line matrix r_N since $\alpha(r_N^*)$ tends to infinity. □

A.3 Proof of Proposition 3.8

We will in fact prove Proposition 3.8 for any sequence of subgroups of finite index which all contain a direct factor and whose smallest length tends to infinity. The method is to use induction on the rank. Here is the result that allows this:

Lemma A.3 *Let H be a subgroup of $\mathbb{Z}^{m'}$ which contains a direct factor of $\mathbb{Z}^{m'}$ of rank $m'-1$, and such that $\alpha(G) > c$. Then there exists subgroups $H = H_1 \supset H_2 \supset \ldots \supset H_{m'} = 0$ such that all inclusions are of corank one and all H_i, $i \geq 2$ are direct factors. Moreover, they all satisfy $\alpha(H_i^*) \geq c$.*

Proof Let H_2 be a direct factor contained in H. It has rank $m'-1$, so the intersection $H_2 \cap (0 \times \mathbb{Z}^2)$ is not empty. Let $v = (0, a, b)$ be in this intersection; we may suppose that $gcd(a,b) = 1$ since H_2 is a direct factor, and then we have $\min(|a|, |b|) \geq \alpha(H) > c$. Thus we see that $\alpha(v^*) > c$. Now we only have to choose a basis $\{v_1 = v, v_2, \ldots, v_{m'-1}\}$ of H and put $H_i = \sum_{j=1}^{i} \mathbb{Z}v_i$; the H_i all contain v and so $\alpha(H_i^*) \geq \alpha(v^*) > c$, and it is clear that they satisfy the other conclusions. □

In consequence, Proposition 3.8 follows from:

Proposition A.4 *Let M be a finitely generated module over R. There exists a constant c depending on M such that the following holds. If H is a subgroup of $\mathbb{Z}^{m'}$ of corank ≥ 1 having a sequence of subgroups $H_2 \supset \ldots \supset H_{m'}$ such that all inclusions are of corank one, all H_i are direct factors and they all satisfy $\alpha(H_i^*) > c$, then the kernel of the map $(M_{tors})_H \to M_H$ is annihilated by a cyclotomic polynomial.*

Proof If H has rank one, we proved that the kernel is 0 in Lemma 3.5. We need the following result to carry out the induction step:

Lemma A.5 *There is a constant c so that for $\alpha(H_A^*) > c$ there exists a cyclotomic polynomial Φ such that the map $M_{R-tors} \to \Phi_A(AM)_{R'-tors}$ is surjective .*

Proof We put $M' = M/M_{tors}$, embed M' in a free module L and denote by π the map $M \to L$ this yields. Suppose that $x \in M$ maps to a R'-torsion element in AM; this means that there exists a $f \in R$ with $Af \neq 0$ such that $\pi(fx) \in \ker(A)\pi(M)$. Let c be the constant for $M' \subset L$ from Proposition 3.6; we get that for $\alpha(H_A^*) > c$ there exists a cyclotomic Φ such that $\Phi\pi(fx) \in \ker(A)M'$. It follows that $\Phi x = y + x'$ where $y \in M_{R-tors} = \ker(\pi)$ and $x' \in \ker(A)M$, and thus that $A\Phi Ax$ is in the image of M_{R-tors} in AM. □

Write $H = \mathbb{Z}v \oplus H_2$ where H_2 is a direct factor of $\mathbb{Z}^{m'}$. Let $x \in M_{tors}$, suppose that $x \in (H-1)M$. Then we can write $x = (t^v - 1)x' + z$ where $z \in (H_2 - 1)M$. Let A be a matrix with $\ker(A) = H_2$ (this is possible since H_2 is a direct factor); we get that $Ax = (t^{Av} - 1)Ax'$, so that x' is R'-torsion in AM' (we have to suppose that the annihilator of x is not contained in $(H_2 - 1)R$, but this is true for $\alpha(H_2)$ big enough). By the Lemma above we get that $\Phi x' = y + z'$ where $z' \in \ker(A)M = (H-1)M$, $y \in M_{tors}$. It follows that $\Phi x = (t^v - 1)y + z + z'$, and we get that $\Phi'(z + z') \in (H-1)M_{tors}$ for some cyclotomic Φ', by the induction hypothesis. □

B The ℓ^2-volume and ℓ^2-torsion

We introduce here a new ℓ^2-invariant, the aim of which is to be able to mimic the arguments leading to (1.4). In particular, we deduce (1.5) as a particular case.

B.1 Definition

We want to define a ℓ^2-volume for submodules of $\mathbb{Z}[G]^n$ by analogy with the lattices in a finite-dimensional Hermitian space. It is quite natural to do this for free submodules: if $L \subset \mathbb{Z}[G]^n$ is free of rank r, then all morphisms $\mathbb{Z}[G]^r \to \mathbb{Z}[G]^n$ with image L have the same Fuglede-Kadison determinant since if we fix one such morphism f, then any other one can be written $f \circ u$ for a change of basis u in $\mathbb{Z}[G]^r$, and all change of basis morphisms for free $\mathbb{Z}[G]$-modules have a unit for determinant and thus their Fuglede-Kadison determinant is equal to one by equation (1.1). Thus, choosing a matrix $A \in M_{n,r}(\mathbb{Z}[G])$ with image L we can put, as in the finite-dimensional case:

$$\mathrm{vol}^{(2)}(L) = \det{}_{\mathcal{N}(G)}(A) = \mathcal{M}(\det(A^*A)).$$

When M is a submodule of $\mathbb{Z}[G]^n$ with rank r, we can pick a free submodule of maximal rank (lemma 1.2). The quotient M/L is torsion by lemma 1.3, so it has nonzero first

Alexander polynomial. In the finite dimensional case, if we have a finite index sumodule V' of a module V then for any metric on M the equality $\mathrm{vol}(V) = [V : V']\,\mathrm{vol}(V')$ holds. Since we are interested in approximation problems the analogue of the index we shall consider for the maximal rank submodule $L \subset M$ is $\Delta_0(M/L)$ (this is justified by (3.10)). We put:

$$\mathrm{vol}^{(2)}(M) = \frac{\mathrm{vol}^{(2)}(L)}{\mathcal{M}(\Delta_0(M/L))}.$$

First of all, we have of course to check this is well-defined:

Lemma B.1 *The real number* $\frac{\mathrm{vol}^{(2)}(L)}{\mathcal{M}(\Delta_0(M/L))}$ *does not depend on the free submodule of maximal rank* $L \subset M$.

Proof Let L_1, L_2 be two free submodules of maximal rank in M. Then $L_1 \cap L_2$ is a submodule of maximal rank, and thus contains a free submodule of maximal rank. Thus it is enough to prove that the ℓ^2-volume is the same when computed for two free submodules of maximal rank $L \supset L'$. Let Q be the coordinate matrix of some basis of L' in a basis of L. We have:

$$\mathrm{vol}^{(2)}(L') = \det{}_{\mathcal{N}(G)}(Q)\,\mathrm{vol}^{(2)}(L) = \mathcal{M}(\Delta_0(L/L'))\,\mathrm{vol}^{(2)}(L)$$

On the other hand, we have a short exact sequence $0 \to L/L' \to M/L' \to M/L \to 0$ which gives:

$$\mathcal{M}(\Delta_0(M/L')) = \mathcal{M}(\Delta_0(L/L'))\mathcal{M}(\Delta_0(M/L))$$

(by multiplicativity of Δ_0 and the Mahler measure). Thus:

$$\begin{aligned}\frac{\mathrm{vol}^{(2)}(L')}{\mathcal{M}(\Delta_0(M/L'))} &= \frac{\mathrm{vol}^{(2)}(L)}{\mathcal{M}(\Delta_0(M/L'))\mathcal{M}(\Delta_0(L/L'))}\\ &= \frac{\mathrm{vol}^{(2)}(L)}{\mathcal{M}(\Delta_0(M/L))}.\end{aligned}$$

This concludes the proof. □

B.2 Metric rank formula

We can now state the ℓ^2 metric rank formula; the proof is similar to that of the finite-dimensional case.

Lemma B.2 *Let* M, M' *be* $\mathbb{Z}[G]$*-submodules of free modules and* f *a morphism from* M *to* M'*; we have then that:*

$$\mathrm{vol}^{(2)}(f(M)) = \frac{\det{}_{\mathcal{N}(G)}(f)\,\mathrm{vol}^{(2)}(M)}{\mathrm{vol}^{(2)}(\ker(f))}. \tag{B.1}$$

Proof Renormalizing by an homothety we may suppose $\mathrm{vol}^{(2)}(M) = 1$. Let $L' \subset \ker(f)$ and $L'' \subset \ker(f)^{\perp}$ be free submodules of maximal rank; the direct sum $L' \oplus L''$ is then a free submodule of maximal rank in M. Put $T' = \ker(f)/L'$, $T'' = \mathrm{Im}(f)/f(L'') \cong M/\ker(f) \oplus L''$ and $T = M/L' \oplus L''$. We get a short exact sequence $0 \to T' \to T \to T'' \to 0$, for example by applying the Snake Lemma to the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(f) & \longrightarrow & M & \longrightarrow & \mathrm{Im}(f) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle f} & & \\ 0 & \longrightarrow & L' & \longrightarrow & L' \oplus L'' & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

It follows that:

$$\mathcal{M}(\Delta_0(T)) = \mathcal{M}(\Delta_0(T''))\mathcal{M}(\Delta_0(T')).$$

On the other hand, since L' and L'' are orthogonal we have:

$$\mathrm{vol}^{(2)}(L')\,\mathrm{vol}^{(2)}(L'') = \mathrm{vol}^{(2)}(L' + L'');$$

and since M has unit volume this yields:

$$\begin{aligned} \mathrm{vol}^{(2)}(L')\,\mathrm{vol}^{(2)}(L'') &= \mathcal{M}(\Delta_0(T)) \\ &= \mathcal{M}(\Delta_0(T''))\mathcal{M}(\Delta_0(T')). \end{aligned}$$

As $f_{|L''}$ is injective, $f(L'')$ is a free submodule of maximal rank in $\mathrm{Im}(f)$ and thus:

$$\begin{aligned} \mathrm{vol}^{(2)}(\mathrm{Im}(f)) &= \mathrm{vol}^{(2)}(f(L''))/\mathcal{M}(\Delta_0(\mathrm{Im}(f)/f(L''))); \\ &= \mathrm{vol}^{(2)}(f(L''))/\mathcal{M}(\Delta_0(T'')). \end{aligned}$$

Finally we have:

$$\mathrm{vol}^{(2)}(f(L'')) = \det{}_{\mathcal{N}(G)}(f)\,\mathrm{vol}^{(2)}(L'').$$

From the three equalities above it follows that:

$$\begin{aligned} \mathrm{vol}^{(2)}(\mathrm{Im}(f)) &= \frac{\det{}_{\mathcal{N}(G)}(f)\,\mathrm{vol}^{(2)}(L'')}{\mathcal{M}(\Delta_0(T''))} \\ &= \frac{\det{}_{\mathcal{N}(G)}(f)\mathcal{M}(\Delta_0(T'))}{\mathrm{vol}^{(2)}(L')} \\ &= \frac{\det{}_{\mathcal{N}(G)}(f)}{\mathrm{vol}^{(2)}(\ker(f))}, \end{aligned}$$

and this concludes the proof of (B.1). □

B.3 Computation of the ℓ^2-torsion

Let C_*, d_* be a finite complex of free finitely generated $\mathbb{Z}[G]$-modules. Let L be a free maximal rank submodule in $H_i(C)$; then L lifts to a free submodule L' in the orthogonal of $\mathrm{Im}(d_{i+1})$ in $\ker(d_i)$. Let T be the quotient $(H_i(C)/H_i(C)_{tors})/L$; we define:

$$\mathrm{vol}^{(2)}(H_i(C)) = \mathcal{M}(\Delta_0(T))\,\mathrm{vol}^{(2)}(L').$$

In particular, when C_* is ℓ^2-acyclic this is equal to one since the orthogonal of $\mathrm{Im}(d_{i+1})$ in $\ker(d_i)$ is zero.

Lemma B.3 *With notation as above, the following equality holds:*

$$\tau^{(2)}(C) = \prod_i \left(\frac{\mathcal{M}(\Delta_{b_i^{(2)}(C)}(H_i(C))}{\mathrm{vol}^{(2)}(H_i(C))}\right)^{(-1)^i}$$

Proof We denote by M_i the orthogonal of $\mathrm{Im}(d_{i+1})$ in $\ker(d_i)$; this is a submodule of C_i of rank $b_i^{(2)}(C)$. Let L' be a maximal rank free submodule in $\mathrm{Im}(d_{i-1})$ and L'' in M_i; put $L = L' \oplus L''$, which is a free maximal rank submodule in $\ker(d_i)$. Since M_i and $\mathrm{Im}(d_{i-1})$ are orthogonal we get $\mathrm{vol}^{(2)}(L) = \mathrm{vol}^{(2)}(L')\,\mathrm{vol}^{(2)}(L'')$. Putting $T = \ker(d_i)/L$ it follows that:

$$\mathrm{vol}^{(2)}(\ker(d_i)) = \frac{\mathrm{vol}^{(2)}(L)}{\mathcal{M}(\Delta_0(T))} = \frac{\mathrm{vol}^{(2)}(L')\,\mathrm{vol}^{(2)}(L'')}{\mathcal{M}(\Delta_0(T))}.$$

Put $T'' = M_i/L''$ and $T' = \mathrm{Im}(d_{i-1})/L'$; there is a natural injection of $T' \oplus T''$ into T, whose cokernel is naturally isomorphic to $H_i(C)/M_i =: T_0$. Thus we get an exact sequence $0 \to T' \oplus T'' \to T \to T_0 \to 0$, and the equality:

$$\mathcal{M}(\Delta_0(T)) = \mathcal{M}(\Delta_0(T'))\mathcal{M}(\Delta_0(T''))\mathcal{M}(\Delta_0(T_0)).$$

Finally, putting $T_1 = \widehat{H}_i(C)/M_i$ we get the exact sequence $0 \to T_1 \to T_0 \to H_i(C)_{tors} \to 0$ and we can compute:

$$\mathcal{M}(\Delta_0(T_0)) = \mathcal{M}(\Delta_0(T_1))\mathcal{M}(\Delta_0(H_i(C)_{tors})).$$

Putting the three equalities above together gives:

$$\begin{aligned}
\mathrm{vol}^{(2)}(\ker(d_i)) &= \frac{\mathrm{vol}^{(2)}(L')}{\mathcal{M}(\Delta_0(T'))} \times \frac{\mathrm{vol}^{(2)}(L'')}{\mathcal{M}(\Delta_0(T''))} \times \frac{1}{\mathcal{M}(\Delta_0(T_0))} \\
&= \mathrm{vol}^{(2)}(\mathrm{Im}(d_{i-1})) \times \frac{\mathrm{vol}^{(2)}(M_i)}{\mathcal{M}(\Delta_0(T_1))} \times \frac{1}{\mathcal{M}(\Delta_0(H_i(C)_{tors}))} \\
&= \frac{\mathrm{vol}^{(2)}(\mathrm{Im}(d_{i-1}))\,\mathrm{vol}^{(2)}(\widehat{H}_i(C))}{\mathcal{M}(\Delta_0(H_i(C)_{tors}))}.
\end{aligned}$$

From this and (B.1) it follows that:

$$\mathrm{vol}^{(2)}(\ker(d_i)) = \frac{\det_{\mathcal{N}(G)}(d_{i-1})\,\mathrm{vol}^{(2)}(\widehat{H}_i(C))}{\mathcal{M}(\Delta_0(H_i(C)_{tors}))\,\mathrm{vol}^{(2)}(\ker d_{i-1})}.$$

Taking the alternating product over i we obtain:

$$1 = \prod_i \left(\frac{\det_{\mathcal{N}(G)}(d_{i-1})\,\mathrm{vol}^{(2)}(\widehat{H}_i(C))}{\mathcal{M}(\Delta_0(H_i(C)_{tors}))} \right)^{(-1)^i}.$$

□

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