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Polishness of some topologies related to automata (Extended version)

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Abstract

We prove that the Büchi topology, the automatic topology, the alphabetic topology and the strong alphabetic topology are Polish, and provide consequences of this. We also show that this cannot be fully extended to the case of a space of infinite labelled binary trees; in particular the Büchi and the Muller topologies in that case are not Polish.

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1 Introduction

This paper is a contribution to the study of the interactions between descriptive set theory and theoretical computer science. These interactions have already been the subject of many studies, see for instance [21, 40, 24, 37, 32, 30, 35, 8, 34, 9, 13, 11, 7].

In particular, the theory of automata reading infinite words, which is closely related to infinite games, is now a rich theory which is used for the specification and verification of non-terminating systems, see [15, 30]. The space $\Sigma^{\mathbb{N}}$ of infinite words over a finite alphabet Σ being equipped with the usual Cantor topology, a natural way to study the complexity of ω -languages accepted by various kinds of automata is to study their topological complexity, and particularly to locate them with regard to the Borel and the projective hierarchies.

However, as noticed in [31] by Schwarz and Staiger and in [18] by Hoffmann and Staiger, it turned out that for several purposes some other topologies on a space $\Sigma^{\mathbb{N}}$ are useful, for instance for studying fragments of first-order logic over infinite words or for a topological characterisation of random infinite words (see also [17]). In particular, Schwarz and Staiger studied four topologies on the space $\Sigma^{\mathbb{N}}$ of infinite words over a finite alphabet Σ which



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are all related to automata, and refine the Cantor topology on $\Sigma^{\mathbb{N}}$: the Büchi topology, the automatic topology, the alphabetic topology, and the strong alphabetic topology.

Recall that a topological space is Polish iff it is separable, i.e. contains a countable dense subset, and its topology is induced by a complete metric. Classical descriptive set theory is about the topological complexity of definable subsets of Polish topological spaces, as well as the study of hierarchies of complexity (see [19, 30] for basic notions). The analytic sets, which are the projections of Borel sets, are of particular importance. Similar hierarchies of complexity are studied in effective descriptive set theory, which is based on the theory of recursive functions (see [28] for basic notions). The effective analytic subsets of the Cantor space $(2^{\mathbb{N}}, \tau_C)$ are highly related to theoretical computer science, in the sense that they coincide with the sets recognized by some special kind of Turing machines (see [38]).

In [31], Schwarz and Staiger prove that the Büchi topology, which is generated by the regular ω -languages, is metrizable. It is separable, by definition, because there are only countably many regular ω -languages. It remains to see that it is completely metrizable to see that it is Polish. This is one of the main results proved in this paper.

We now give some more details about the topologies studied by Schwarz and Staiger in [31] that we investigate in this paper. Let Σ be a finite alphabet with at least two symbols. We will consider the topology τ_δ on $\Sigma^{\mathbb{N}}$ generated by the set \mathbb{B}_δ of sets accepted by an unambiguous Büchi Turing machine. Let Σ^* be the set of finite sequences of elements of Σ . The following topologies on $\Sigma^{\mathbb{N}}$, related to automata, are considered in [31].

- the *Büchi topology* τ_B , generated by the set \mathbb{B}_B of ω -regular languages,
- the *automatic topology* τ_A , generated by the set \mathbb{B}_A of τ_C -closed ω -regular languages (this topology is remarkable because all τ_C -closed ω -regular languages are accepted by deterministic Büchi automata),
- the *alphabetic topology* τ_α , generated by the set \mathbb{B}_α of sets of the form $B_{w,A} := \{w\sigma \mid \sigma \in A^{\mathbb{N}}\}$, where $w \in \Sigma^*$ and $A \subseteq \Sigma$ (this topology is useful for investigations in restricted first-order theories for infinite words),
- the *strong alphabetic topology* τ_s , generated by the set \mathbb{B}_s of sets of the form

$$S_{w,A} := \{w\sigma \mid \sigma \in A^{\mathbb{N}} \wedge \forall a \in A \ \forall k \in \mathbb{N} \ \exists i \geq k \ \sigma(i) = a\},$$

where $w \in \Sigma^*$ and $A \subseteq \Sigma$ (this topology is derived from τ_α , and considered in [6], together with τ_α).

In [31], Schwarz and Staiger prove that these topologies are metrizable. We improve this result:

► **Theorem 1.** *Let $z \in \{C, \delta, B, A, \alpha, s\}$. Then τ_z is Polish.*

From this result, it is already possible to infer many properties of the space $\Sigma^{\mathbb{N}}$, where Σ is a finite alphabet, equipped with the Büchi topology. In particular, we first get some results about the σ -algebra generated by the ω -regular languages. It is stratified in a hierarchy of length ω_1 (the first uncountable ordinal) and there are universal sets at each level of this hierarchy. Notice that this σ -algebra coincides with the σ -algebra of Borel sets for the Cantor topology, and that a set is Borel for the Cantor topology if and only if it is Borel for the Büchi topology, but the levels of the Borel hierarchy differ for the two topologies. For instance an ω -regular set which is non- Π_2^0 for the Cantor topology is clopen (i.e., Δ_1^0) for the Büchi topology. Therefore the results about the existence of universal sets at each level of the σ -algebra generated by the ω -regular languages are really new and interesting. We derive many other properties from the polishness of the Büchi topology.

We also investigate, following a suggestion of H. Michalewski, whether it is possible to extend these results in the case of a space T_Σ^ω of infinite binary trees labelled with letters of the alphabet Σ . On one hand, the automatic and the alphabetic topologies can be proved to be Polish in a similar way. But on the other hand, we show that the Büchi topology (generated by regular tree languages accepted by Büchi tree automata) and the Muller topology (generated by regular tree languages accepted by Muller tree automata) are both non-Polish. However we prove that these two topologies have quite different properties: in particular, the first one is strong Choquet but not metrizable while the second one is metrizable but not strong Choquet.

This paper is an extended version of a paper which appeared in the proceedings of the 26th EACSL Annual Conference on Computer Science and Logic, CSL 2017, [3]. The main addition with regard to the conference paper consists in the study of the Büchi topology and of the Muller topology in the case of a space of trees, which now forms Section 3.4.

2 Background

We first recall the notions required to understand fully the introduction and the sequel (see for example [30, 37, 19, 28]).

2.1 Theoretical computer science

A *Büchi automaton* is a tuple $\mathcal{A} = (Q, \Sigma, \delta, Q_i, Q_f)$, where Σ is the input alphabet, Q is the finite state set, δ is the transition relation, Q_i and Q_f are the sets of initial and final states. The transition relation δ is a subset of $Q \times \Sigma \times Q$.

A *run* on some sequence $\sigma \in \Sigma^\mathbb{N}$ is a sequence $(q_n)_{i \in \mathbb{N}} \in Q^\mathbb{N}$ of states such that q_0 is initial ($q_0 \in Q_i$) and $(q_i, \sigma(i), q_{i+1})$ is a transition in δ for each $i \geq 0$. It is accepting if it visits infinitely often final states, that is, $q_i \in Q_f$ for infinitely many i . An input sequence σ is accepted if there exists an accepting run on α . The set of accepted inputs is denoted $L(\mathcal{A})$. A set of infinite words is called *ω -regular* if it is equal to $L(\mathcal{A})$ for some automaton \mathcal{A} .

A Büchi automaton is actually similar to a classical finite automaton. A finite word w of length n is accepted by an automaton \mathcal{A} if there is sequence $(q_i)_{i \leq n}$ of $n + 1$ states such that q_0 is initial ($q_0 \in Q_i$), q_n is final ($q_n \in Q_f$) and $(q_i, \sigma(i), q_{i+1})$ is a transition in δ for each $0 \leq i < n$. The set of accepted finite words is denoted by $U(\mathcal{A})$. A set of finite words is called *regular* if it is equal to $U(\mathcal{A})$ for some automaton \mathcal{A} .

Let us recall that the ω -power of a set U of finite words is defined by

$$U^\omega = \{\sigma \in \Sigma^\mathbb{N} \mid \exists (w_i)_{i \in \mathbb{N}} \in U^\mathbb{N} \text{ s.t. } \sigma = w_0 w_1 w_2 \dots\}.$$

The ω -powers play a crucial role in the characterization of ω -regular languages (see [2]):

► **Theorem 2** (Büchi). *Let Σ be a finite alphabet, and $L \subseteq \Sigma^\mathbb{N}$. The following statements are equivalent:*

1. L is ω -regular,
2. there are $2n$ regular languages $(U_i)_{i < n}$ and $(V_i)_{i < n}$ such that $L = \bigcup_{i < n} U_i V_i^\omega$.

The closure properties of the class of ω -regular languages mentioned in the introduction are the following (see [30] and [31]). If Σ is a set and $w \in \Sigma^*$, then w defines the usual basic clopen set $N_w := \{\sigma \in \Sigma^\mathbb{N} \mid w \text{ is a prefix of } \sigma\}$ of the Cantor topology τ_C (so $\mathbb{B}_C := \{\emptyset\} \cup \{N_w \mid w \in \Sigma^*\}$ is a basis for τ_C).

► **Theorem 3** (Büchi). *The class of ω -regular languages contains the usual basic clopen sets and is closed under finite unions and intersections, taking complements, and projections.*

We now turn to the study of Turing machines (see [5, 37]).

A *Büchi Turing machine* is a tuple $\mathcal{M} = (\Sigma, Q, \Gamma, \delta, q_0, Q_f)$, where Σ and Γ are the input and tape alphabets satisfying $\Sigma \subseteq \Gamma$, Q is the finite state set, δ is the transition relation, q_0 is the initial state and Q_f is the set of final states. The relation δ is a subset of $(Q \times \Gamma) \times (Q \times \Gamma \times \{-1, 0, 1\})$.

A *configuration* of \mathcal{M} is a triple (q, γ, j) where $q \in Q$ is the current state, $\gamma \in \Gamma^{\mathbb{N}}$ is the content of the tape and the non-negative integer $j \in \mathbb{N}$ is the position of the head on the tape.

Two configurations (q, γ, j) and (q', γ', j') of \mathcal{M} are *consecutive* if there exists a transition $(q, a, q', b, d) \in \delta$ such that the following conditions are met:

1. $\gamma(j) = a$, $\gamma'(j) = b$ and $\gamma(i) = \gamma'(i)$ for each $i \neq j$. This means that the symbol a is replaced by symbol b at position j and that all other symbols on the tape remain unchanged.
2. the two positions j and j' satisfy the equality $j' = j + d$.

A *run* of the machine \mathcal{M} on some input $\sigma \in \Sigma^{\mathbb{N}}$ is a sequence $(p_i, \gamma_i, j_i)_{i \in \mathbb{N}}$ of consecutive configurations such that $p_0 = q_0$, $\gamma_0 = \sigma$ and $j_0 = 0$. It is *complete* if the head visits all positions of the tape. This means that for each integer N , there exists an integer i such that $j_i \geq N$. The run is *accepting* if it visits infinitely often final states, that is, $p_i \in Q_f$ for infinitely many integers i . The ω -language accepted by \mathcal{M} is the set of inputs σ such that there exists an accepting and complete run on σ .

Notice that other accepting conditions have been considered for the acceptance of infinite words by Turing machines, like the 1' or Muller ones (the latter one was firstly called 3-acceptance), see [5, 37]. Moreover several types of required behaviour on the input tape have been considered in the literature, see [38, 12, 10].

A *Büchi automaton* \mathcal{A} is in fact a Büchi Turing machine whose head only move forwards. This means that each of its transition has the form (p, a, q, b, d) where $d = 1$. Note that the written symbol b does not matter since each position of the tape is just visited once and the symbol b is never read.

2.2 Descriptive set theory

Classical descriptive set theory takes place in Polish topological spaces.

We first recall that if d is a distance on a set X , and $(x_n)_{n \in \mathbb{N}}$ is a sequence of elements of X , then the sequence $(x_n)_{n \in \mathbb{N}}$ is called a Cauchy sequence if

$$\forall k \in \mathbb{N} \quad \exists N \in \mathbb{N} \quad \forall p, p' \geq N \quad d(x_p, x_{p'}) < \frac{1}{2^k}.$$

In a topological space X whose topology is induced by a distance d , the distance d and the metric space (X, d) are said to be *complete* if every Cauchy sequence in X is convergent.

► **Definition 2.1.** *A topological space X is a Polish space if it is*

1. *separable (there is a countable dense sequence (x_n) in X),*
2. *completely metrizable (there is a complete distance d on X which is compatible with the topology of X).*

Effective descriptive set theory is based on the notion of recursive function. A function from \mathbb{N}^k to \mathbb{N}^l is said to be *recursive* if it is total and computable. By extension, a relation is called *recursive* if its characteristic function is recursive.

► **Definition 2.2.** A recursive presentation of a Polish space X is a pair $((x_n)_{n \in \mathbb{N}}, d)$ such that

1. $(x_n)_{n \in \mathbb{N}}$ is dense in X ,
2. d is a compatible complete distance on X such that the following relations P and Q are recursive:

$$P(i, j, m, k) \iff d(x_i, x_j) \leq \frac{m}{k+1},$$

$$Q(i, j, m, k) \iff d(x_i, x_j) < \frac{m}{k+1}.$$

A Polish space X is *recursively presented* if there is a recursive presentation of it.

Note that the formula $(p, q) \mapsto 2^p(2q+1) - 1$ defines a recursive bijection $\mathbb{N}^2 \rightarrow \mathbb{N}$. One can check that the coordinates of the inverse map are also recursive. They will be denoted $n \mapsto (n)_0$ and $n \mapsto (n)_1$ in the sequel. These maps will help us to define some of the basic effective classes.

► **Definition 2.3.** Let $((x_n)_{n \in \mathbb{N}}, d)$ be a recursive presentation of a Polish space X .

1. We fix a countable basis of X : $B(X, n)$ is the open ball $B_d(x_{(n)_0}, \frac{((n)_1)_0}{((n)_1)_1+1})$.
2. A subset S of X is *semirecursive*, or *effectively open* (denoted $S \in \Sigma_1^0$) if

$$S = \bigcup_{n \in \mathbb{N}} B(X, f(n)),$$

for some recursive function f .

3. A subset S of X is *effectively closed* (denoted $S \in \Pi_1^0$) if its complement $\neg S$ is *semirecursive*.
4. One can check that a product of two recursively presented Polish spaces has a recursive presentation, and that the Baire space $\mathbb{N}^{\mathbb{N}}$ has a recursive presentation. A subset S of X is *effectively analytic* (denoted $S \in \Sigma_1^1$) if there is a Π_1^0 subset C of $X \times \mathbb{N}^{\mathbb{N}}$ such that

$$S = \pi_0[C] := \{x \in X \mid \exists \alpha \in \mathbb{N}^{\mathbb{N}} (x, \alpha) \in C\}.$$

5. A subset S of X is *effectively co-analytic* (denoted $S \in \Pi_1^1$) if its complement $\neg S$ is *effectively analytic*, and *effectively Borel* if it is in Σ_1^1 and Π_1^1 (denoted $S \in \Delta_1^1$).
6. We will also use the following relativized classes: if X, Y are recursively presented Polish spaces and $y \in Y$, then we say that $A \subseteq X$ is in $\Sigma_1^1(y)$ if there is $S \in \Sigma_1^1(Y \times X)$ such that $A = S_y := \{x \in X \mid (y, x) \in S\}$. The class $\Pi_1^1(y)$ is defined similarly. We also set $\Delta_1^1(y) := \Sigma_1^1(y) \cap \Pi_1^1(y)$.

The crucial link between the effective classes and the classical corresponding classes is as follows: the class of analytic (resp., co-analytic, Borel) subsets of Y is equal to $\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \Sigma_1^1(\alpha)$ (resp., $\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \Pi_1^1(\alpha)$, $\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \Delta_1^1(\alpha)$). This allows to use effective descriptive set theory to prove results of classical type. In the sequel, when we consider an effective class in some $\Sigma^{\mathbb{N}}$ with Σ finite, we will always use a fixed recursive presentation associated with the Cantor topology. The following result is proved in [38], see also [10]:

► **Theorem 4.** Let Σ be a finite alphabet, and $L \subseteq \Sigma^{\mathbb{N}}$. The following statements are equivalent:

1. $L = L(\mathcal{M})$ for some Büchi Turing machine \mathcal{M} ,
2. $L \in \Sigma_1^1$.

XX:6 Polishness of some topologies related to automata

We now recall the *Choquet game* played by two players on a topological space X . Players 1 and 2 play alternatively. At each turn i , Player 1 plays by choosing an open subset U_i and a point $x_i \in U_i$ such that $U_i \subseteq V_{i-1}$, where V_{i-1} has been chosen by Player 2 at the previous turn. Player 2, plays by choosing an open subset V_i such that $x_i \in V_i$ and $V_i \subseteq U_i$. Player 2 wins the game if $\bigcap_{i \in \mathbb{N}} V_i \neq \emptyset$. We now recall some classical notions of topology.

- **Definition 2.4.** *A topological space X is said to be*
- T_1 if every singleton of X is closed,
 - regular if for every point of X and every open neighborhood U of x , there is an open neighborhood V of x with $\overline{V} \subseteq U$,
 - second countable if its topology has a countable basis,
 - zero-dimensional if there is a basis made of clopen sets,
 - strong Choquet if X is not empty and Player 2 has a winning strategy in the Choquet game.

Note that every zero-dimensional space is regular. The following result is 8.18 in [19].

- **Theorem 5 (Choquet).** *A nonempty, second countable topological space is Polish if and only if it is T_1 , regular, and strong Choquet.*

Let X be a nonempty recursively presented Polish space. The *Gandy-Harrington topology* on X is generated by the Σ_1^1 subsets of X , and denoted Σ_X . By Theorem 4, this topology is also related to automata and Turing machines. As there are some effectively analytic sets whose complement is not analytic, the Gandy-Harrington topology is not metrizable (in fact not regular) in general (see [3E.9] in [28]). In particular, it is not Polish.

3 Proof of the main result

The proof of Theorem 1 is organized as follows. We provide below four properties which ensure that a given topological space is strong Choquet. Then we use Theorem 5 to prove that the considered spaces are indeed Polish.

Let Σ be a countable alphabet. The set $\Sigma^{\mathbb{N}}$ is equipped with the product topology of the discrete topology on Σ , unless another topology is specified. This topology is induced by a natural metric, called the *prefix metric* which is defined as follows. For $\sigma \neq \sigma' \in \Sigma^{\mathbb{N}}$, the distance d is given by

$$d(\sigma, \sigma') = 2^{-r} \quad \text{where } r = \min\{n \mid \sigma(n) \neq \sigma'(n)\}.$$

When Σ is finite this topology is the classical Cantor topology. When Σ is countably infinite the topological space is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$.

Let Σ and Γ be two alphabets. The function which maps each pair $(\sigma, \gamma) \in \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ to the element $(\sigma(0), \gamma(0)), (\sigma(1), \gamma(1)), \dots$ of $(\Sigma \times \Gamma)^{\mathbb{N}}$ is a homeomorphism between $\Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ and $(\Sigma \times \Gamma)^{\mathbb{N}}$ allowing us to identify these two spaces.

If Σ is a set, $\sigma \in \Sigma^{\mathbb{N}}$ and $l \in \mathbb{N}$, then $\sigma|l$ is the prefix of σ of length l .

We set $2 := \{0, 1\}$ and $\mathbb{P}_{\infty} := \{\alpha \in 2^{\mathbb{N}} \mid \forall k \in \mathbb{N} \exists i \geq k \alpha(i) = 1\}$. This latter set is simply the set of infinite words over the alphabet $2 := \{0, 1\}$ having infinitely many symbols 1.

We will work in the spaces of the form $\Sigma^{\mathbb{N}}$, where Σ is a finite set with at least two elements. We will fix a topology τ_{Σ} on $\Sigma^{\mathbb{N}}$, and a basis \mathbb{B}_{Σ} for τ_{Σ} . We consider the following properties of the family $(\tau_{\Sigma}, \mathbb{B}_{\Sigma})_{\Sigma}$, using the previous identification:

- (P1) \mathbb{B}_Σ contains the usual basic clopen sets N_w ,
- (P2) \mathbb{B}_Σ is closed under finite unions and intersections,
- (P3) \mathbb{B}_Σ is closed under projections, in the sense that if Γ is a finite set with at least two elements and $L \in \mathbb{B}_{\Sigma \times \Gamma}$, then $\pi_0[L] \in \mathbb{B}_\Sigma$,
- (P4) for each $L \in \mathbb{B}_\Sigma$ there is a closed subset C of $\Sigma^\mathbb{N} \times \mathbb{P}_\infty$ (i.e. C is the intersection of a closed subset of the Cantor space $\Sigma^\mathbb{N} \times 2^\mathbb{N}$ with $\Sigma^\mathbb{N} \times \mathbb{P}_\infty$) which is in $\mathbb{B}_{\Sigma \times 2}$, and such that $L = \pi_0[C]$.

► **Theorem 6.** *Assume that the family $(\tau_\Sigma, \mathbb{B}_\Sigma)_\Sigma$ satisfies the properties (P1)-(P4). Then the topologies τ_Σ are strong Choquet.*

Proof. We first describe a strategy τ for Player 2. Player 1 first plays $\sigma_0 \in \Sigma^\mathbb{N}$ and a τ_Σ -open neighborhood U_0 of σ_0 . Let L_0 in \mathbb{B}_Σ with $\sigma_0 \in L_0 \subseteq U_0$. Property (P4) gives C_0 with $L_0 = \pi_0[C_0]$. This gives $\alpha_0 \in \mathbb{P}_\infty$ such that $(\sigma_0, \alpha_0) \in C_0$. We choose $l_0^0 \in \mathbb{N}$ big enough to ensure that if $s_0^0 := \alpha_0|l_0^0$, then s_0^0 has at least a coordinate equal to 1. We set $w_0 := \sigma_0|1$ and $V_0 := \pi_0[C_0 \cap (N_{w_0} \times N_{s_0^0})]$. By properties (P1)-(P3), V_0 is in \mathbb{B}_Σ and thus τ_Σ -open. Moreover, $\sigma_0 \in V_0 \subseteq L_0 \subseteq U_0$, so that Player 2 respects the rules of the game if he plays V_0 .

Now Player 1 plays $\sigma_1 \in V_0$ and a τ_Σ -open neighborhood U_1 of σ_1 contained in V_0 . Let L_1 in \mathbb{B}_Σ with $\sigma_1 \in L_1 \subseteq U_1$. Property (P4) gives C_1 with $L_1 = \pi_0[C_1]$. This gives $\alpha_1 \in \mathbb{P}_\infty$ such that $(\sigma_1, \alpha_1) \in C_1$. We choose $l_1^0 \in \mathbb{N}$ big enough to ensure that if $s_1^0 := \alpha_1|l_1^0$, then s_1^0 has at least one coordinate equal to 1. As $\sigma_1 \in V_0$, there is $\alpha'_0 \in \mathbb{P}_\infty$ such that $(\sigma_1, \alpha'_0) \in C_0 \cap (N_{w_0} \times N_{s_0^0})$. We choose $l_1^0 > l_0^0$ big enough to ensure that if $s_1^0 := \alpha'_0|l_1^0$, then s_1^0 has at least two coordinates equal to 1. We set $w_1 := \sigma_1|2$ and $V_1 := \pi_0[C_0 \cap (N_{w_1} \times N_{s_0^0})] \cap \pi_0[C_1 \cap (N_{w_0} \times N_{s_1^0})]$. Here again, V_1 is τ_Σ -open. Moreover, $\sigma_1 \in V_1 \subseteq U_1$ and Player 2 can play V_1 .

Next, Player 1 plays $\sigma_2 \in V_1$ and a τ_Σ -open neighborhood U_2 of σ_2 contained in V_1 . Let L_2 in \mathbb{B}_Σ with $\sigma_2 \in L_2 \subseteq U_2$. Property (P4) gives C_2 with $L_2 = \pi_0[C_2]$. This gives $\alpha_2 \in \mathbb{P}_\infty$ such that $(\sigma_2, \alpha_2) \in C_2$. We choose $l_2^0 \in \mathbb{N}$ big enough to ensure that if $s_2^0 := \alpha_2|l_2^0$, then s_2^0 has at least one coordinate equal to 1. As $\sigma_2 \in V_1$, there is $\alpha'_1 \in \mathbb{P}_\infty$ such that $(\sigma_2, \alpha'_1) \in C_1 \cap (N_{w_0} \times N_{s_1^0})$. We choose $l_2^0 > l_1^0$ big enough to ensure that if $s_2^0 := \alpha'_1|l_2^0$, then s_2^0 has at least two coordinates equal to 1. As $\sigma_2 \in V_1$, there is $\alpha''_0 \in \mathbb{P}_\infty$ such that $(\sigma_2, \alpha''_0) \in C_0 \cap (N_{w_1} \times N_{s_1^0})$. We choose $l_2^0 > l_1^0$ big enough to ensure that if $s_2^0 := \alpha''_0|l_2^0$, then s_2^0 has at least three coordinates equal to 1. We set $w_2 := \sigma_2|3$ and $V_2 := \pi_0[C_0 \cap (N_{w_2} \times N_{s_2^0})] \cap \pi_0[C_1 \cap (N_{w_1} \times N_{s_1^0})] \cap \pi_0[C_2 \cap (N_{w_0} \times N_{s_2^0})]$. Here again, V_2 is τ_Σ -open. Moreover, $\sigma_2 \in V_2 \subseteq U_2$ and Player 2 can play V_2 .

If we go on like this, we build $w_l \in \Sigma^{l+1}$ and $s_l^n \in 2^*$ such that $w_0 \subseteq w_1 \subseteq \dots$ and $s_0^n \subsetneq s_1^n \subsetneq \dots$. This allows us to define $\sigma := \lim_{l \rightarrow \infty} w_l \in \Sigma^\mathbb{N}$ and, for each $n \in \mathbb{N}$, $\alpha_n := \lim_{l \rightarrow \infty} s_l^n \in 2^\mathbb{N}$. Note that $\alpha_n \in \mathbb{P}_\infty$ since s_l^n has at least $l+1$ coordinates equal to 1. As (σ, α_n) is the limit of (w_l, s_l^n) as l goes to infinity and $N_{w_l} \times N_{s_l^n}$ meets C_n (which is closed in $\Sigma^\mathbb{N} \times \mathbb{P}_\infty$), $(\sigma, \alpha_n) \in C_n$. Thus

$$\sigma \in \bigcap_{n \in \mathbb{N}} \pi_0[C_n] = \bigcap_{n \in \mathbb{N}} L_n \subseteq \bigcap_{n \in \mathbb{N}} U_n \subseteq \bigcap_{n \in \mathbb{N}} V_n,$$

so that τ is winning for Player 2. ◀

3.1 The Gandy-Harrington topology

We have already mentioned that the Gandy-Harrington topology is not Polish in general. However, it is almost Polish as it fulfills the four properties (P1)-(P4).

Let Σ be a finite alphabet with at least two elements and let X be the space $\Sigma^{\mathbb{N}}$ equipped with the topology $\tau_{\Sigma} := \Sigma_X$ generated by the family \mathbb{B}_{Σ} of Σ_1^1 subsets of X . Note that the assumption of Theorem 6 are satisfied. Indeed, (P1)-(P3) come from 3E.2 in [28]. For (P4), let F be a Π_1^0 subset of $X \times \mathbb{N}^{\mathbb{N}}$ such that $L = \pi_0[F]$. Let φ be the function from $\mathbb{N}^{\mathbb{N}}$ to $2^{\mathbb{N}}$ defined by $\varphi(\beta) = 0^{\beta(0)}10^{\beta(1)}1\dots$. Note that φ is a homeomorphism from $\mathbb{N}^{\mathbb{N}}$ onto \mathbb{P}_{∞} , and recursive (which means that the relation $\varphi(\beta) \in N(2^{\mathbb{N}}, n)$ is semirecursive in β and n). This implies that $C := (\text{Id} \times \varphi)[F]$ is suitable (see 3E.2 in [28]).

Note that τ_{Σ} is second countable since there are only countably many Σ_1^1 subsets of X (see 3F.6 in [28]), T_1 since it is finer than the usual topology by the property (P1), and strong Choquet by Theorem 6.

One can show that there is a dense basic open subset Ω_X of (X, τ_{Σ}) such that $S \cap \Omega_X$ is a clopen subset of $(\Omega_X, \tau_{\Sigma})$ for each Σ_1^1 subset S of X (see [22]). In particular, $(\Omega_X, \tau_{\Sigma})$ is zero-dimensional, and regular. As it is, just like (X, τ_{Σ}) , second countable, T_1 and strong Choquet, $(\Omega_X, \tau_{\Sigma})$ is a Polish space, by Theorem 5.

3.2 The Büchi topology

Let Σ be a finite alphabet with at least two symbols, and X be the space $\Sigma^{\mathbb{N}}$ equipped with the Büchi topology τ_B generated by the family \mathbb{B}_B of ω -regular languages in X . Theorem 29 in [31] shows that τ_B is metrizable. We now give a distance which is compatible with τ_B . This metric was used in [17] (Theorem 2 and Lemma 21 and several corollaries following Lemma 21). A similar argument for subword metrics is in [[18], Section 4]. If \mathcal{A} is a Büchi automaton, then we denote $|\mathcal{A}|$ the number of states of \mathcal{A} . We say that a Büchi automaton **separates** x and y iff

$$(x \in L(\mathcal{A}) \wedge y \notin L(\mathcal{A})) \vee (y \in L(\mathcal{A}) \wedge x \notin L(\mathcal{A})).$$

The distance δ on $\Sigma^{\mathbb{N}}$ is then defined as follows: for $x, y \in \Sigma^{\mathbb{N}}$, $\delta(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-n} & \text{if } x \neq y, \end{cases}$ where $n := \min\{|\mathcal{A}| \mid \mathcal{A} \text{ is a Büchi automaton which separates } x \text{ and } y\}$. We now describe some properties of the map δ . This is the occasion to illustrate the notion of complete metric.

- **Proposition 3.1.** *1. the map δ defines a distance on $\Sigma^{\mathbb{N}}$,*
- 2. the distance δ is compatible with τ_B ,*
- 3. the distance δ is not complete.*

Proof. 1. If $x, y \in \Sigma^{\mathbb{N}}$, then $\delta(x, y) = \delta(y, x)$, by definition of δ . Let $x, y, z \in \Sigma^{\mathbb{N}}$, and assume that $\delta(x, y) + \delta(y, z) < \delta(x, z) = 2^{-n}$. Then $\delta(x, y) < 2^{-n}$ and $\delta(y, z) < 2^{-n}$ hold. In particular, if \mathcal{A} is a Büchi automaton with n states then it does not separate x and y and similarly it does not separate y and z . Thus either $x, y, z \in L(\mathcal{A})$ or $x, y, z \notin L(\mathcal{A})$. This implies that the Büchi automaton \mathcal{A} does not separate x and z . But this holds for every Büchi automaton with n states and then $\delta(x, z) < 2^{-n}$. This leads to a contradiction and thus $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$ for all $x, y, z \in \Sigma^{\mathbb{N}}$. This shows that δ is a distance on $\Sigma^{\mathbb{N}}$.

2. Recall that an open set for this topology is a union of ω -languages accepted by Büchi automata. Let then $L(\mathcal{A})$ be an ω -language accepted by a Büchi automaton \mathcal{A} having n states, and $x \in L(\mathcal{A})$. We now show that the open ball $B(x, 2^{-(n+1)})$ with center x and δ -radius $2^{-(n+1)}$ is a subset of $L(\mathcal{A})$. Indeed, if $\delta(x, y) < 2^{-(n+1)} < 2^{-n}$, then x and y cannot be separated by any Büchi automaton with n states, and thus $y \in L(\mathcal{A})$. This shows that $L(\mathcal{A})$ (and therefore any open set for τ_B) is open for the topology induced by the distance

δ . Conversely, let $B(x, r)$ be an open ball for the distance δ , where $r > 0$ is a positive real. It is clear from the definition of the distance δ that we may only consider the case $r = 2^{-n}$ for some natural number n . Then $y \in B(x, 2^{-n})$ if and only if x and y cannot be separated by any Büchi automaton with $p \leq n$ states. Therefore the open ball $B(x, 2^{-n})$ is the intersection of the regular ω -languages $L(\mathcal{A}_i)$ for Büchi automata \mathcal{A}_i having $p \leq n$ states and such that $x \in L(\mathcal{A}_i)$ and of the regular ω -languages $\Sigma^{\mathbb{N}} \setminus L(\mathcal{B}_i)$ for Büchi automata \mathcal{B}_i having $p \leq n$ states and such that $x \notin L(\mathcal{B}_i)$. The class of regular ω -languages being closed under complementation and finite intersection, the open ball $B(x, 2^{-n})$ is actually a regular ω -language and thus an open set for τ_B .

3. Without loss of generality, we set $\Sigma = \{0, 1\}$ and we consider, for a natural number $n \geq 1$, the ω -word $X_n = 0^{n!} \cdot 1 \cdot 0^\omega$ over the alphabet $\{0, 1\}$ having only one symbol 1 after $n!$ symbols 0, where $n! := n \times (n-1) \times \cdots \times 2 \times 1$. Let now $m > n > k$ and \mathcal{A} be a Büchi automaton with k states. Using a classical pumping argument, we can see that the automaton \mathcal{A} cannot separate X_n and X_m . Indeed, assume first that $X_n \in L(\mathcal{A})$. Then, when reading the first k symbols 0 of X_n , the automaton enters at least twice in a same state q . This implies that there is a sequence of symbols 0 of length $p \leq k$ which can be added several times to the word X_n so that the resulting word will still be accepted by \mathcal{A} . Formally, any word $0^{n!+lp} \cdot 1 \cdot 0^\omega$, for a natural number $l \geq 1$, will be accepted by \mathcal{A} . In particular $m! = n! \times (n+1) \times \cdots \times m = n! + n! \times ((n+1) \times \cdots \times m) - 1$ is of this form and thus $X_m \in L(\mathcal{A})$. A very similar pumping argument shows that if $X_m \in L(\mathcal{A})$, then $X_n \in L(\mathcal{A})$. This shows that $\delta(X_n, X_m) < 2^{-k}$ and finally that the sequence (X_n) is a Cauchy sequence for the distance δ . On the other hand if this sequence was converging to an ω -word x then x should be the word 0^ω because τ_B is finer than τ_C . But 0^ω is an ultimately periodic word and thus it is an isolated point for τ_B . This would lead to a contradiction, and thus the distance δ is not complete because the sequence (X_n) is a Cauchy sequence which is not convergent. \blacktriangleleft

Proposition 3.1 gives a motivation for deriving Theorem 1 from Theorem 6. Note that the assumption of Theorem 6 are satisfied. Indeed, (P1)-(P3) come from Theorem 3. We now check (P4).

► **Lemma 3.2.** *Let Σ be a finite set with at least two elements, and $L \subseteq \Sigma^{\mathbb{N}}$ be an ω -regular language. Then there is a closed subset C of $\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty$, which is ω -regular as a subset of $(\Sigma \times 2)^{\mathbb{N}}$ identified with $\Sigma^{\mathbb{N}} \times 2^{\mathbb{N}}$, and such that $L = \pi_0[C]$.*

Proof. Let $\mathcal{A} = (\Sigma, Q, \delta, Q_i, Q_f)$ be a Büchi automaton and let $L = L(\mathcal{A})$ be its set of accepted words. Let χ_f be the characteristic function of the final states. It maps each state q to 1 if $q \in Q_f$ and to 0 otherwise. The function χ_f is extended to $Q^{\mathbb{N}}$ by setting $\alpha = \chi_f((q_n)_{n \in \mathbb{N}})$ where $\alpha(n) = \chi_f(q_n)$. Note that a run ρ of \mathcal{A} is accepting if and only if $\chi_f(\rho) \in \mathbb{P}_\infty$.

Let C be the subset of $\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty$ defined by

$$C := \{(\sigma, \alpha) \in \Sigma^{\mathbb{N}} \times \mathbb{P}_\infty \mid \exists \rho \text{ run of } \mathcal{A} \text{ on } \sigma \text{ s. t. } \alpha = \chi_f(\rho)\}.$$

By definition of C , $L = \pi_0[C]$. Let K be the subset of $\Sigma^{\mathbb{N}} \times 2^{\mathbb{N}} \times Q^{\mathbb{N}}$ be defined by

$$K := \{(\sigma, \alpha, \rho) \in \Sigma^{\mathbb{N}} \times 2^{\mathbb{N}} \times Q^{\mathbb{N}} \mid \rho \text{ is a run of } \mathcal{A} \text{ on } \sigma \text{ s. t. } \alpha = \chi_f(\rho)\}.$$

Since K is compact as a closed subset of a compact space and $C = \pi_{\Sigma^{\mathbb{N}} \times 2^{\mathbb{N}}}[K] \cap (\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty)$, the subset C is a closed subset of $\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty$. It remains to show that C is indeed ω -regular.

XX:10 Polishness of some topologies related to automata

Let Δ be defined by

$$\Delta := \{(p, (a, \varepsilon), q) \in Q \times (\Sigma \times 2) \times Q \mid (p, a, q) \in \delta \wedge (\varepsilon = 1 \iff p \in Q_f)\}.$$

This allows us to define a Büchi automaton by $\mathcal{A}' := (\Sigma \times 2, Q, \Delta, Q_i, Q_f)$. Note that

$$\begin{aligned} (\sigma, \alpha) \in L(\mathcal{A}') &\iff \exists (s_i)_{i \in \mathbb{N}} \in Q^{\mathbb{N}} \left(s_0 \in Q_i \wedge \forall i \in \mathbb{N} \left(s_i, (\sigma(i), \alpha(i)), s_{i+1} \right) \in \Delta \right) \wedge \\ &\quad \forall k \in \mathbb{N} \exists i \geq k \ s_i \in Q_f \\ &\iff \alpha \in \mathbb{P}_\infty \wedge \exists (s_i)_{i \in \mathbb{N}} \in Q^{\mathbb{N}} \left(s_0 \in Q_i \wedge \forall i \in \mathbb{N} \left(s_i, \sigma(i), s_{i+1} \right) \in \delta \wedge \right. \\ &\quad \left. (\alpha(i) = 1 \iff s_i \in Q_f) \right) \\ &\iff (\sigma, \alpha) \in C. \end{aligned}$$

Thus $C = L(\mathcal{A}')$ is ω -regular. ◀

► **Corollary 3.3.** *Let Σ be a finite set with at least two elements. Then the Büchi topology τ_B is zero-dimensional and Polish.*

Proof. As there are only countably many possible automata (up to identifications), \mathbb{B}_B is countable. This shows that τ_B is second countable. It is T_1 since it is finer than the usual topology by the property (P1), and strong Choquet by Theorem 6. Moreover, it is zero-dimensional since the class of ω -regular languages is closed under taking complements (see Theorem 3). It remains to apply Theorem 5. ◀

3.3 The other topologies

Proof of the main result. It is well known that $(\Sigma^{\mathbb{N}}, \tau_C)$ is metrizable and compact, and thus Polish.

- By Theorem 3.4 in [26], the implication (iii) \Rightarrow (i), $\Delta_1^1(\Sigma^{\mathbb{N}})$ is a basis for a zero-dimensional Polish topology on $\Sigma^{\mathbb{N}}$. Recall that a Büchi Turing machine is **unambiguous** if every ω -word $\sigma \in \Sigma^{\mathbb{N}}$ has at most one accepting run. By Theorem 3.6 in [10], a subset of $\Sigma^{\mathbb{N}}$ is Δ_1^1 if and only if it is accepted by an unambiguous Büchi Turing machine. Therefore $\mathbb{B}_\delta = \Delta_1^1(\Sigma^{\mathbb{N}})$ is a basis for the zero-dimensional Polish topology τ_δ .

- Corollary 3.3 gives the result for the Büchi topology.

- Let (X, τ) be a Polish space, and $(C_n)_{n \in \mathbb{N}}$ be a sequence of closed subsets of (X, τ) . By 13.2 in [19], the topology τ_n generated by $\tau \cup \{C_n\}$ is Polish. By 13.3 in [19], the topology τ_∞ generated by $\bigcup_{n \in \mathbb{N}} \tau_n$ is Polish. Thus the topology generated by $\tau \cup \{C_n \mid n \in \mathbb{N}\}$, which is τ_∞ , is Polish. This shows that the automatic topology and the alphabetic topology are Polish since they refine the usual product topology on $\Sigma^{\mathbb{N}}$.

- Note that $S_{w,A}$ is a G_δ subset of $\Sigma^{\mathbb{N}}$, and thus a Polish subspace of $\Sigma^{\mathbb{N}}$, by 3.11 in [19]. Note also that $\Sigma^{\mathbb{N}} = \bigcup_{\emptyset \neq A \subseteq \Sigma} S_{\emptyset,A} \cup \bigcup_{\emptyset \neq A \subseteq \Sigma, w \in \Sigma^*, b \in \Sigma \setminus A} S_{wb,A}$, and that this union is disjoint. As all the sets in this union are open for the strong alphabetic topology, they are also clopen for this topology. This shows that this topology is the countable sum of its restrictions to the sets in this union. As the strong alphabetic topology coincides with the usual topology on each of these sets, it is Polish, by 3.3 in [19]. ◀

3.4 The Büchi and Muller topologies on a space of trees

The notion of Büchi automaton has been extended to the case of a Büchi tree automaton reading infinite binary trees whose nodes are labelled by letters of a finite alphabet. We now recall this notion and related ones.

A node of an infinite binary tree is represented by a finite word over the alphabet $\{l, r\}$ where r means “right” and l means “left”. Then an infinite binary tree whose nodes are labelled in Σ is identified with a function $t : \{l, r\}^* \rightarrow \Sigma$. The set of infinite binary trees labelled in Σ will be denoted T_Σ^ω .

A finite binary tree is like an “initial finite subtree” of an infinite binary tree. Thus it can be represented by a function $s : S \subseteq \{l, r\}^* \rightarrow \Sigma$, where S is a finite subset of $\{l, r\}^*$ which is closed under prefix. If $t \in T_\Sigma^\omega$ is an infinite binary tree, and $n \geq 0$ is an integer, then we denote $t|n$ the initial finite subtree of t whose domain is equal to $\{l, r\}^{\leq n}$ where $\{l, r\}^{\leq n}$ is the set of finite words over the alphabet $\{l, r\}$ of length smaller than or equal to n .

Let t be an infinite binary tree. A branch B of t is a subset of the set of nodes of t which is linearly ordered by the tree partial order \sqsubseteq and which is closed under prefix relation, i.e. if x and y are nodes of t such that $y \in B$ and $x \sqsubseteq y$ then $x \in B$.

A branch B of a tree is said to be maximal iff there is no other branch of t which strictly contains B .

Let t be an infinite binary tree in T_Σ^ω . If B is a maximal branch of t , then this branch is infinite. Let $(u_i)_{i \geq 0}$ be the enumeration of the nodes in B which is strictly increasing for the prefix order. The infinite sequence of labels of the nodes of such a maximal branch B , i.e. $t(u_0)t(u_1) \cdots t(u_n) \cdots$ is called a path. It is an ω -word over the alphabet Σ .

Let then $L \subseteq \Sigma^\omega$ be an ω -language over Σ . Then we denote $\exists\text{Path}(L)$ the set of infinite trees t in T_Σ^ω such that t has (at least) one path in L .

We now define tree automata and recognizable tree languages.

► **Definition 3.4.** A (nondeterministic topdown) tree automaton is a quadruple $\mathcal{A} = (K, \Sigma, \Delta, q_0)$, where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state and $\Delta \subseteq K \times \Sigma \times K \times K$ is the transition relation. The tree automaton \mathcal{A} is said to be deterministic if the relation Δ is a functional one, i.e. if for each $(q, a) \in K \times \Sigma$ there is at most one pair of states (q', q'') such that $(q, a, q', q'') \in \Delta$.

A run of the tree automaton \mathcal{A} on an infinite binary tree $t \in T_\Sigma^\omega$ is a infinite binary tree $\rho \in T_K^\omega$ such that:

(a) $\rho(\lambda) = q_0$ and (b) for each $u \in \{l, r\}^*$, $(\rho(u), t(u), \rho(u.l), \rho(u.r)) \in \Delta$.

► **Definition 3.5.** A Büchi (nondeterministic topdown) tree automaton is a 5-tuple $\mathcal{A} = (K, \Sigma, \Delta, q_0, F)$, where (K, Σ, Δ, q_0) is a tree automaton and $F \subseteq K$ is the set of accepting states.

A run ρ of the Büchi tree automaton \mathcal{A} on an infinite binary tree $t \in T_\Sigma^\omega$ is said to be accepting if for each path of ρ there is some accepting state appearing infinitely often on this path.

The tree language $L(\mathcal{A})$ accepted by the Büchi tree automaton \mathcal{A} is the set of infinite binary trees $t \in T_\Sigma^\omega$ such that there is (at least) one accepting run of \mathcal{A} on t .

► **Definition 3.6.** A Muller (nondeterministic topdown) tree automaton is a 5-tuple $\mathcal{A} = (K, \Sigma, \Delta, q_0, \mathcal{F})$, where (K, Σ, Δ, q_0) is a tree automaton and $\mathcal{F} \subseteq 2^K$ is the collection of designated state sets.

A run ρ of the Muller tree automaton \mathcal{A} on an infinite binary tree $t \in T_\Sigma^\omega$ is said to be accepting if for each path p of ρ , the set of states appearing infinitely often on this path is

in \mathcal{F} .

The tree language $L(\mathcal{A})$ accepted by the Muller tree automaton \mathcal{A} is the set of infinite binary trees $t \in T_\Sigma^\omega$ such that there is (at least) one accepting run of \mathcal{A} on t .

The class *REG* of regular, or recognizable, tree languages is the class of tree languages accepted by some Muller automaton.

► **Remark 3.7.** Each tree language accepted by some (deterministic) Büchi automaton is also accepted by some (deterministic) Muller automaton. A tree language is accepted by a Muller tree automaton iff it is accepted by some Rabin tree automaton. We refer for instance to [39, 30] for the definition of a Rabin tree automaton.

► **Example 3.8.** Let $L \subseteq \Sigma^\omega$ be a regular ω -language (see [30] about regular ω -languages which are the ω -languages accepted by Büchi or Muller automata). Then the set $\exists\text{Path}(L) \subseteq T_\Sigma^\omega$ is accepted by a Büchi tree automaton, hence also by a Muller tree automaton. The set of infinite binary trees $t \in T_\Sigma^\omega$ having all their paths in L , denoted $\forall\text{Path}(L)$, is accepted by a deterministic Muller tree automaton. It is in fact the complement of the set $\exists\text{Path}(\Sigma^\omega - L)$.

There is also a natural topology on the set T_Σ^ω [27, 24, 19]. It is defined by the following distance. Let t and s be two distinct infinite trees in T_Σ^ω . Then the distance between t and s is $\frac{1}{2^n}$ where n is the smallest integer such that $t(x) \neq s(x)$ for some word $x \in \{l, r\}^*$ of length n .

The open sets are then in the form $T_0 \cdot T_\Sigma^\omega$ where T_0 is a set of finite labelled trees. $T_0 \cdot T_\Sigma^\omega$ is the set of infinite binary trees which extend some finite labelled binary tree $t_0 \in T_0$, t_0 is here a sort of prefix, an “initial subtree” of a tree in $t_0 \cdot T_\Sigma^\omega$.

It is well known that the set T_Σ^ω , equipped with this topology, is homeomorphic to the Cantor set 2^ω , hence also to the topological spaces Σ^ω , where Σ is a finite alphabet having at least two letters.

We are going to use some similar notations as in the case of the space Σ^ω . First if t is a finite binary tree labelled in Σ , we shall denote N_t the clopen set $t \cdot T_\Sigma^\omega$. Notice that it is easy to see that one can take as a restricted basis for the Cantor topology on T_Σ^ω , the clopen sets of the form $t_0 \cdot T_\Sigma^\omega$ where t_0 is a finite labelled binary tree whose domain is of the special form $\{l, r\}^{\leq n}$.

The Borel hierarchy and the projective hierarchy on T_Σ^ω are defined from open sets in the same manner as in the case of the topological space Σ^ω .

The ω -language $\mathcal{R} = (0^* \cdot 1)^\omega$ is a well known example of Π_2^0 -complete subset of $\{0, 1\}^\omega$. It is the set of ω -words over $\{0, 1\}$ having infinitely many occurrences of the letter 1. Its complement $\{0, 1\}^\omega - (0^* \cdot 1)^\omega$ is a Σ_2^0 -complete subset of $\{0, 1\}^\omega$.

The set of infinite trees in T_Σ^ω , where $\Sigma = \{0, 1\}$, having at least one path in the ω -language $\mathcal{R} = (0^* \cdot 1)^\omega$ is Σ_1^1 -complete. Its complement is the set of trees in T_Σ^ω having all their paths in $\{0, 1\}^\omega - (0^* \cdot 1)^\omega$; it is Π_1^1 -complete.

It follows from the definition of the Büchi acceptance condition for infinite trees that each tree language recognized by a (non deterministic) Büchi tree automaton is an analytic set. Niwinski showed that some Büchi recognized tree languages are actually Σ_1^1 -complete sets. An example is any tree language $T \subseteq T_\Sigma^\omega$ in the form $\exists\text{Path}(L)$, where $L \subseteq \Sigma^\omega$ is a

regular ω -language which is a $\mathbf{\Pi}_2^0$ -complete subset of Σ^ω . In particular, the tree language $\mathcal{L} = \exists\text{Path}(\mathcal{R})$, where $\mathcal{R} = (0^* \cdot 1)^\omega$, is Σ_1^1 -complete hence non Borel [29, 30, 36]. Notice that its complement $\mathcal{L}^- = \forall\text{Path}(\{0, 1\}^\omega - (0^* \cdot 1)^\omega)$ is a $\mathbf{\Pi}_1^1$ -complete set. It cannot be accepted by any Büchi tree automaton because it is not a Σ_1^1 -set. On the other hand, it can be easily seen that it is accepted by a deterministic Muller tree automaton.

We now consider the topology on a space T_Σ^ω which is generated by the regular languages of trees accepted by Büchi tree automata.

We are firstly going to prove an analogue of Theorem 6 as a first step towards the proof that the Büchi topology on a space T_Σ^ω is strong Choquet.

We set $\mathbb{T}_\infty := \{t \in T_2^\omega \mid \text{for every path } p \text{ of } t \ \forall k \geq 0 \ \exists i \geq k \ p(i) = 1\}$. This set is simply the set of infinite trees over the alphabet $2 := \{0, 1\}$ having infinitely many letters 1 on every (infinite) path.

We will work in the spaces of the form T_Σ^ω , where Σ is a finite alphabet with at least two elements. We will fix a topology τ_Σ on T_Σ^ω , and a basis \mathbb{B}_Σ for τ_Σ . We consider the following properties of the family $(\tau_\Sigma, \mathbb{B}_\Sigma)_\Sigma$, using the previous identification:

- (P1) \mathbb{B}_Σ contains the usual basic clopen sets N_w ,
- (P2) \mathbb{B}_Σ is closed under finite unions and intersections,
- (P3) \mathbb{B}_Σ is closed under projections, in the sense that if Γ is a finite set with at least two elements and $L \in \mathbb{B}_{\Sigma \times \Gamma}$, then $\pi_0[L] \in \mathbb{B}_\Sigma$,
- (P4) for each $L \in \mathbb{B}_\Sigma$ there is a closed subset C of $T_\Sigma^\omega \times \mathbb{T}_\infty$ (i.e. C is the intersection of a closed subset of the Cantor space $T_\Sigma^\omega \times T_2^\omega$ with $T_\Sigma^\omega \times \mathbb{T}_\infty$) which is in $\mathbb{B}_{\Sigma \times 2}$, and such that $L = \pi_0[C]$.

Consider now the set of trees \mathbb{T}_∞ . It is easy to see that \mathbb{T}_∞ is accepted by a *deterministic* Büchi tree automaton. On the other hand it is well known that tree languages accepted by *deterministic* Büchi tree automata are Borel $\mathbf{\Pi}_2^0$ -sets, see [1]. Thus the set \mathbb{T}_∞ is actually a $\mathbf{\Pi}_2^0$ -set, i.e. it is the intersection of countably many open sets O_i . We may assume, without loss of generality, that the sequence $(O_i)_{i \geq 0}$ is decreasing with regard to the inclusion relation. Moreover each open set O_i is a countable union of basic clopen sets $N_{t_{i,j}}$, $j \geq 0$, and we may also assume without loss of generality that for all integers $i \geq 0$, and all $j \geq 0$, it holds that the finite tree $t_{i,j} \subseteq \{l, r\}^*$ has a domain of the form $\{l, r\}^{\leq n}$ for some integer n greater than i .

We now state the following result, which is an analogue of Theorem 6 in the case of trees.

► **Theorem 7.** *Assume that the family $(\tau_\Sigma, \mathbb{B}_\Sigma)_\Sigma$ satisfies the properties (P1)-(P4). Then the topologies τ_Σ are strong Choquet.*

Proof. We first describe a strategy τ for Player 2. Player 1 first plays $t_0 \in T_\Sigma^\omega$ and a τ_Σ -open neighborhood U_0 of t_0 . Let L_0 in \mathbb{B}_Σ with $t_0 \in L_0 \subseteq U_0$. Property (P4) gives C_0 with $L_0 = \pi_0[C_0]$. This gives $\alpha_0 \in \mathbb{T}_\infty$ such that $(t_0, \alpha_0) \in C_0$. We choose $l_0^0 \in \mathbb{N}$ big enough to ensure that if $s_0^0 := \alpha_0|l_0^0$, then $N_{s_0^0}$ is included in the open set O_1 . We set $w_0 := t_0|1$ and $V_0 := \pi_0[C_0 \cap (N_{w_0} \times N_{s_0^0})]$. By properties (P1)-(P3), V_0 is in \mathbb{B}_Σ and thus τ_Σ -open. Moreover, $t_0 \in V_0 \subseteq L_0 \subseteq U_0$, so that Player 2 respects the rules of the game if he plays V_0 .

Now Player 1 plays $t_1 \in V_0$ and a τ_Σ -open neighborhood U_1 of t_1 contained in V_0 . Let L_1 in \mathbb{B}_Σ with $t_1 \in L_1 \subseteq U_1$. Property (P4) gives C_1 with $L_1 = \pi_0[C_1]$. This gives $\alpha_1 \in \mathbb{T}_\infty$ such that $(t_1, \alpha_1) \in C_1$. We choose $l_0^1 \in \mathbb{N}$ big enough to ensure that if $s_0^1 := \alpha_1 | l_0^1$, then $N_{s_0^1}$ is included in the open set O_1 . As $t_1 \in V_0$, there is $\alpha'_0 \in \mathbb{T}_\infty$ such that $(t_1, \alpha'_0) \in C_0 \cap (N_{w_0} \times N_{s_0^0})$. We choose $l_1^0 > l_0^0$ big enough to ensure that if $s_1^0 := \alpha'_0 | l_1^0$, then s_1^0 is such that $N_{s_1^0}$ is included in the open set O_2 . We set $w_1 := t_1 | 2$ and $V_1 := \pi_0[C_0 \cap (N_{w_1} \times N_{s_1^0})] \cap \pi_0[C_1 \cap (N_{w_0} \times N_{s_1^1})]$. Here again, V_1 is τ_Σ -open. Moreover, $t_1 \in V_1 \subseteq U_1$ and Player 2 can play V_1 .

Next, Player 1 plays $t_2 \in V_1$ and a τ_Σ -open neighborhood U_2 of t_2 contained in V_1 . Let L_2 in \mathbb{B}_Σ with $t_2 \in L_2 \subseteq U_2$. Property (P4) gives C_2 with $L_2 = \pi_0[C_2]$. This gives $\alpha_2 \in \mathbb{T}_\infty$ such that $(t_2, \alpha_2) \in C_2$. We choose $l_0^2 \in \mathbb{N}$ big enough to ensure that if $s_0^2 := \alpha_2 | l_0^2$, then the basic open set $N_{s_0^2}$ is included in the open set O_1 . As $t_2 \in V_1$, there is $\alpha'_1 \in \mathbb{T}_\infty$ such that $(t_2, \alpha'_1) \in C_1 \cap (N_{w_0} \times N_{s_1^0})$. We choose $l_1^1 > l_0^1$ big enough to ensure that if $s_1^1 := \alpha'_1 | l_1^1$, then the basic open set $N_{s_1^1}$ is included in the open set O_2 . As $t_2 \in V_1$, there is $\alpha''_0 \in \mathbb{T}_\infty$ such that $(t_2, \alpha''_0) \in C_0 \cap (N_{w_1} \times N_{s_1^0})$. We choose $l_2^0 > l_1^0$ big enough to ensure that if $s_2^0 := \alpha''_0 | l_2^0$, then the basic open set $N_{s_2^0}$ is included in the open set O_3 . We set $w_2 := t_2 | 3$ and $V_2 := \pi_0[C_0 \cap (N_{w_2} \times N_{s_2^0})] \cap \pi_0[C_1 \cap (N_{w_1} \times N_{s_1^1})] \cap \pi_0[C_2 \cap (N_{w_0} \times N_{s_2^2})]$. Here again, V_2 is τ_Σ -open. Moreover, $t_2 \in V_2 \subseteq U_2$ and Player 2 can play V_2 .

If we go on like this, we build $w_k \in \Sigma^{\{l,r\}^{\leq k+1}}$ and $s_l^n \in 2^{\{l,r\}^*}$ such that $w_0 \subseteq w_1 \subseteq \dots$ and $s_0^n \subseteq s_1^n \subseteq \dots$. This allows us to define $\sigma := \lim_{l \rightarrow \infty} w_l \in T_\Sigma^\omega$ and, for each $n \in \mathbb{N}$, $\beta_n := \lim_{l \rightarrow \infty} s_l^n \in T_2^\omega$. Note that $\beta_n \in \mathbb{T}_\infty$ since the basic open set $N_{s_l^n}$ is included in the open set O_{l+1} . (σ, β_n) is the limit of (w_l, s_l^n) as l goes to infinity and $N_{w_l} \times N_{s_l^n}$ meets C_n (which is closed in $T_\Sigma^\omega \times \mathbb{T}_\infty$), $(\sigma, \beta_n) \in C_n$. Thus

$$\sigma \in \bigcap_{n \in \mathbb{N}} \pi_0[C_n] = \bigcap_{n \in \mathbb{N}} L_n \subseteq \bigcap_{n \in \mathbb{N}} U_n \subseteq \bigcap_{n \in \mathbb{N}} V_n,$$

so that τ is winning for Player 2. ◀

We now check that the Büchi topology on a space T_Σ^ω satisfies the properties (P1)-(P4).

(P1) It is very easy to see that for each finite tree w labelled in Σ , there exists a Büchi tree automaton accepting the usual basic clopen set N_w ,

(P2) \mathbb{B}_Σ is closed under finite unions, because the basic open sets in the Büchi topology are accepted by *non-deterministic* Büchi tree automata. Moreover one can easily show, using a classical product construction, that tree languages accepted by Büchi tree automata are closed under finite intersections. Thus \mathbb{B}_Σ is closed under finite intersections,

(P3) It follows easily, from the fact that the basic open sets in the Büchi topology are accepted by *non-deterministic* Büchi tree automata, that \mathbb{B}_Σ is closed under projections, in the sense that if Γ is a finite set with at least two elements and $L \in \mathbb{B}_{\Sigma \times \Gamma}$, then $\pi_0[L] \in \mathbb{B}_\Sigma$,

(P4) This property follows from the following lemma which is very similar to the above Lemma 3.2.

► **Lemma 3.9.** *Let Σ be a finite set with at least two elements, and $L \subseteq T_\Sigma^\omega$ be a regular tree language accepted by a Büchi tree automaton. Then there is a closed subset C of $T_\Sigma^\omega \times \mathbb{T}_\infty$, which is accepted by a Büchi tree automaton as a subset of $T_{(\Sigma \times 2)}^\omega$ identified with $T_\Sigma^\omega \times T_2^\omega$, and such that $L = \pi_0[C]$.*

Proof. Let $\mathcal{A} = (K, \Sigma, \Delta, q_0, F)$ be a Büchi tree automaton and let $L = L(\mathcal{A})$ be its set of accepted trees. Let χ_F be the characteristic function of the final states. It maps each state q to 1 if $q \in F$ and to 0 otherwise. The function χ_F is extended to T_Q^ω by setting $t' = \chi_F(t)$ where $t'(s) = \chi_F(t(s))$. Note that a run ρ of \mathcal{A} is accepting if and only if $\chi_F(\rho) \in \mathbb{T}_\infty$.

Let C be the subset of $T_\Sigma^\omega \times \mathbb{T}_\infty$ defined by

$$C := \{(t, t') \in T_\Sigma^\omega \times \mathbb{T}_\infty \mid \exists \rho \text{ run of } \mathcal{A} \text{ on } t \text{ s. t. } t' = \chi_F(\rho)\}.$$

By definition of C , $L = \pi_0[C]$. Let K be the subset of $T_\Sigma^\omega \times T_2^\omega \times T_Q^\omega$ be defined by

$$K := \{(t, t', \rho) \in T_\Sigma^\omega \times T_2^\omega \times T_Q^\omega \mid \rho \text{ is a run of } \mathcal{A} \text{ on } t \text{ s. t. } t' = \chi_F(\rho)\}.$$

Since K is compact as a closed subset of the compact space $T_\Sigma^\omega \times T_2^\omega \times T_Q^\omega$ and $C = \pi_{T_\Sigma^\omega \times T_2^\omega}[K] \cap (T_\Sigma^\omega \times \mathbb{T}_\infty)$, the subset C is a closed subset of $T_\Sigma^\omega \times \mathbb{T}_\infty$. Moreover it is easy to construct a Büchi tree automaton accepting the tree language C . ◀

► **Corollary 3.10.** *Let Σ be a finite set with at least two elements. Then the Büchi topology on T_Σ^ω is strong Choquet.*

Proof. It follows now from the fact that the Büchi topology on T_Σ^ω satisfies properties (P1)-(P4), and from Theorem 7. ◀

On the other hand, as in the case of the Büchi topology on Σ^ω , the Büchi topology on T_Σ^ω is second countable since there are only countably many possible Büchi tree automata (up to identifications), and it is T_1 since it is finer than the usual Cantor topology by the property (P1). However we cannot pursue by showing that the Büchi topology on T_Σ^ω is Polish, as shown by the following result.

► **Theorem 8.** *Let Σ be a finite set with at least two elements. Then the Büchi topology on T_Σ^ω is not metrizable and thus not Polish.*

Proof. Recall that in a metrizable topological space, every closed set is a countable intersection of open sets. We now show that the Büchi topology on T_Σ^ω does not satisfy this property. We have already recalled that the set \mathcal{L} of infinite trees in T_Σ^ω , where $\Sigma = \{0, 1\}$, having at least one path in the ω -language $\mathcal{R} = (0^* \cdot 1)^\omega$ is Σ_1^1 -complete for the usual topology, and it is open for the Büchi topology since it is accepted by a Büchi tree automaton. Its complement \mathcal{L}^- is the set of trees in T_Σ^ω having all their paths in $\{0, 1\}^\omega \setminus (0^* \cdot 1)^\omega$; it is Π_1^1 -complete for the usual topology and closed for the Büchi topology.

On the other hand every tree language accepted by a Büchi tree automaton is an effective analytic set, i.e. a Σ_1^1 -set, and thus also a (boldface) Σ_1^1 -set (for the usual Cantor topology). Moreover every open set for the Büchi topology on T_Σ^ω is a countable union of basic open sets, and thus a countable union of Σ_1^1 -sets. But the class Σ_1^1 is closed under countable unions (see [19]) therefore every open set for the Büchi topology is a Σ_1^1 -set for the usual topology.

Towards a contradiction, assume now that the set \mathcal{L}^- is a countable intersection of open sets for the Büchi topology. Then it is a countable intersection of Σ_1^1 -sets for the usual topology. But the class Σ_1^1 in a Polish space is closed under countable intersections and thus \mathcal{L}^- would be also a Σ_1^1 -set for the usual topology. But \mathcal{L}^- is Π_1^1 -complete and thus in $\Pi_1^1 \setminus \Sigma_1^1$, see [19], and this leads to a contradiction. ◀

► **Remark 3.11.** *One can infer, from the previous results on the Büchi topology on the space T_Σ^ω and from the above Choquet's Theorem 5, that the Büchi topology on the space T_Σ^ω is not regular.*

XX:16 Polishness of some topologies related to automata

► **Remark 3.12.** *The automatic topology and the alphabetic topology on the space T_Σ^ω , which can be defined in a very similar way, as in the case of the space Σ^ω , are Polish, and this can be proved in a very similar way as in the case of the space Σ^ω .*

We now consider the topology on a space T_Σ^ω which is generated by the class of all regular languages of trees accepted by Muller tree automata. We shall call this topology the Muller topology on T_Σ^ω .

This topology is clearly T_1 since it is finer than the usual topology on T_Σ^ω . It is second countable since there exist only countably many Muller automata. It is zero-dimensional, and thus also regular, because the class of regular tree languages over the alphabet Σ is closed under complementation. We now recall the following Urysohn Theorem.

► **Theorem 9** (Urysohn Metrization Theorem, see [19]). *Let X be a second-countable topological space. Then X is metrizable iff its topology is T_1 and regular.*

This implies that the Muller topology is metrizable. Notice that one can define a distance compatible with this topology in a very similar way as we have defined a distance compatible with the Büchi topology on Σ^ω .

On the other hand we recall the following Becker Theorem.

► **Theorem 10** (see Theorem 4.2.6 in [14]). *Let τ be a Polish topology on X and σ a second-countable strong Choquet topology on X finer than τ . Then every σ -open set is Σ_1^1 in τ .*

This implies that the Muller topology is not strong Choquet because there exist some Π_1^1 -complete (for the usual topology), and hence non- Σ_1^1 (for the usual topology), regular set of trees. Such a regular set of trees is open for the Muller topology, which is finer than the usual topology, but is not Σ_1^1 for the usual topology.

We now summarize the results of this section.

► **Theorem 11.** *Let Σ be a finite alphabet having at least two letters.*

1. *The Büchi topology on T_Σ^ω is strong Choquet, but it is not regular (and hence not zero-dimensional) and not metrizable.*
2. *The Muller topology on T_Σ^ω is zero-dimensional, regular and metrizable, but it is not strong Choquet.*

In particular, the Büchi topology and the Muller topology on T_Σ^ω are not Polish.

Notice that we have proved that the Büchi topology and the Muller topology on T_Σ^ω are not Polish but we have also shown that they have quite different properties, like being strong Choquet or not.

On the other hand if we are just interested in the non-polishness of these topologies, we can reason in a more direct way as follows. We first prove the following proposition.

► **Proposition 3.13.** *Let (X, τ) be a topological space, where τ is a Polish topology on X , and let σ be another Polish topology on X which is finer than the topology τ . Then the two topologies τ and σ have the same Borel sets.*

Proof. Let $Id : (X, \sigma) \rightarrow (X, \tau)$ be the identity function on X , where the domain X is equipped with the topology σ and the range X is equipped with the topology τ . This

function is continuous since the topology σ is finer than the topology τ . Notice that this implies, by an easy induction on the rank of a Borel set, that the preimage of any Borel set of (X, τ) is a Borel set of (X, σ) , i.e. that every Borel set of (X, τ) is a Borel set of (X, σ) . On the other hand, it follows from Lusin-Suslin's Theorem, see [19, Theorem 15.1], that the (injective) image by the function Id of any Borel set of (X, σ) is a Borel set of (X, τ) , i.e. that every Borel set of (X, σ) is a Borel set of (X, τ) . ◀

► **Corollary 3.14.** *Let Σ be a finite set with at least two elements. Then the Büchi topology and the Muller topology on T_Σ^ω are not Polish.*

Proof. The Büchi topology and the Muller topology on T_Σ^ω are finer than the usual Cantor topology on T_Σ^ω . On the other hand both the Büchi topology and the Muller topology on T_Σ^ω contain some open sets which are Σ_1^1 -complete and hence non Borel (for the usual topology), like, in the case $\Sigma = \{0, 1\}$, the set $\mathcal{L} = \exists\text{Path}(\mathcal{R})$, of infinite trees in T_Σ^ω , having at least one path in the ω -language $\mathcal{R} = (0^* \cdot 1)^\omega$. The conclusion now follows from Proposition 3.13. ◀

4 Consequences for our topologies

4.1 Consequences not directly related to the polishness, concerning isolated points

4.1.1 Case of a space of infinite words

Notation. If $z \in \{C, \delta, B, A, \alpha, s\}$, then the space $(\Sigma^\mathbb{N}, \tau_z)$ is denoted \mathcal{S}_z . The set of ultimately periodic ω -words on Σ is denoted $\text{Ult} := \{u \cdot v^\omega \mid u, v \in \Sigma^* \setminus \{\emptyset\}\}$, and $P := \Sigma^\mathbb{N} \setminus \text{Ult}$. (1) As noted in [31], Ult is the set of *isolated points* of \mathcal{S}_B and \mathcal{S}_A (recall that a point $\sigma \in \Sigma^\mathbb{N}$ is isolated if $\{\sigma\}$ is an open set). Indeed, each singleton $\{u \cdot v^\omega\}$ formed by an ultimately periodic ω -word is an ω -regular language, and thus each ultimately periodic ω -word is an isolated point of \mathcal{S}_A . Conversely, if $\{\sigma\}$ is τ_B -open, then it is ω -regular and then the ω -word σ is ultimately periodic (because any countable ω -regular language contains only ultimately periodic ω -words, see [2, 30, 37]).

(2) Every nonempty ω -regular set contains an ultimately periodic ω -word, [2, 30, 37]. In particular, the set Ult of isolated points of \mathcal{S}_B and \mathcal{S}_A is *dense*, and a subset of \mathcal{S}_B or \mathcal{S}_A is dense if and only if it contains Ult .

(3) Let X be a topological space in which the set \mathbb{I} of isolated points is dense, for example \mathcal{S}_B by (2).

(a) A subset of X is nowhere dense (i.e., its closure has empty interior) if and only if it is *meager* (i.e., it is a countable union of nowhere dense sets). indeed, a meager set does not meet \mathbb{I} .

(b) Recall that a subset L of a topological space Y has the *Baire property* if there is an open subset O of Y such that the symmetric difference $L \Delta O := (L \setminus O) \cup (O \setminus L)$ is meager. This is equivalent to say that $L = G \cup M$, where G is G_δ (i.e. a countable intersection of open sets) and M is meager (see 8.23 in [19]). Every subset of X has the Baire property (which is very uncommon, see for exemple 8.24 in [19]). Indeed, we can even say that every subset of X can be written $L = O \cup N$, where O is open and N is nowhere dense (O is $L \cap \mathbb{I}$, and N is $L \setminus \mathbb{I}$).

(c) Recall that a topological space is a *Baire space* if the intersection of countably many dense open sets is dense. By the Baire category theorem, every completely metrizable space is Baire. The space X is Baire in a very strong way: the intersection of any family of dense

sets is dense, since a subset of X is dense if and only if it contains \mathbb{I} . This is very unusual, since for example we can find two disjoint countable dense sets in \mathbb{R} .

(d) A consequence of (b), (c) and 8.38 in [19] is that any map from X into a second countable Polish space is *continuous on a dense G_δ subset of X* . Note that if X is \mathcal{S}_B or \mathcal{S}_A , then any map from X into a topological space is continuous on the dense open set Ult of ultimately periodic ω -words.

(e) There is a strong version of the *Kuratowski-Ulam theorem* (see 8.41 in [19]): assume that X and Y are topological spaces in which the set of isolated points is dense. Then this property also holds in the product $X \times Y$, so that a subset of $X \times Y$ is meager if and only if it contains no isolated point, which is also equivalent to the fact that for each isolated point $x \in X$ (resp., $y \in Y$), the vertical (resp., horizontal) section at x (resp., y) contains no isolated point. An interesting example is the case of an infinitary rational relation $R(\mathcal{A}) \subseteq \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ accepted by a 2-tape Büchi automaton \mathcal{A} which may be synchronous or asynchronous. Indeed if $\Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ is equipped with the product topology of the Büchi topologies on $\Sigma^{\mathbb{N}}$ and $\Gamma^{\mathbb{N}}$ then a non-empty rational relation is always non-meager. This follows easily from the fact that $\text{Dom}(R(\mathcal{A})) = \{x \in \Sigma^{\mathbb{N}} \mid \exists y \in \Gamma^{\mathbb{N}} (x, y) \in R(\mathcal{A})\}$ is a regular ω -language. Thus $\text{Dom}(R(\mathcal{A}))$ contains an ultimately periodic word x and then $\{y \in \Gamma^{\mathbb{N}} \mid (x, y) \in R(\mathcal{A})\}$ is also a non-empty regular ω -language and so it contains also an ultimately periodic word y .

4.1.2 Case of a space of infinite trees

Recall that a tree $t \in T_\Sigma^\omega$ is regular iff for each $a \in \Sigma$ the set $\{u \in \{l, r\}^* \mid t(u) = a\}$ is a regular set of finite words over the alphabet $\{l, r\}$. For each regular tree $t \in T_\Sigma^\omega$, the singleton $\{t\}$ is a (closed) regular tree language accepted by a Büchi tree automaton. Moreover a regular tree language accepted by a Muller or Rabin tree automaton is non-empty iff it contains a regular tree, see [39].

Therefore we can state analogue properties to those stated in the case of a space of words in the preceding section 4.1.1. For the automatic, the Büchi or the Muller topologies on a space T_Σ^ω , the set of isolated points is the set *Reg-trees* of regular trees, and this set is dense. It is left here to the reader to see how properties (1)-(3) of preceding section are extended to the case of a space of trees in a very similar way, except that for (3)(e) we consider only here synchronous tree automata reading pairs of infinite trees.

4.2 Consequences of the polishness

Here we concentrate on the case of a space of infinite words.

4.2.1 Consequences related to Cantor-Bendixson Theorem

We consider our topologies on $\Sigma^{\mathbb{N}}$, where Σ is a finite set with at least two elements. We give a (non exhaustive) list of results. We often refer to [19] when classical descriptive set theory is involved. The reader may read this book to see many other results.

(4).(a) The union $P \cup \text{Ult}$ is the *Cantor-Bendixson decomposition* of \mathcal{S}_B and \mathcal{S}_A (see 6.4 in [19]). This means that P is perfect (i.e., closed without isolated points) and Ult is countable open. Let us check that P is perfect. We argue by contradiction, so that we can find $\sigma \in P$ and an ω -regular language L such that $\{\sigma\} = L \setminus \text{Ult}$. Note that $L \subseteq \{\sigma\} \cup \text{Ult}$ is countable. But a countable regular ω -language contains only ultimately periodic words (see [30]), and thus $L \subseteq \text{Ult}$, which is absurd.

(b) The closed subspace (P, τ_B) of \mathcal{S}_B is *homeomorphic to the Baire space* $\mathbb{N}^{\mathbb{N}}$. Indeed, it is not empty since Ult is countable and $\Sigma^{\mathbb{N}}$ is not, zero-dimensional and Polish as a closed subspace of the zero-dimensional Polish space \mathcal{S}_B . By 7.7 in [19], it is enough to prove that every compact subset of (P, τ_B) has empty interior. We argue by contradiction, which gives a compact set K . Note that there is an ω -regular language L such that $P \cap L$ is a nonempty compact subset of K , so that we may assume that $K = P \cap L$. Theorem 2 gives $(U_i)_{i < n}$ and $(V_i)_{i < n}$ with $L = \bigcup_{i < n} U_i \cdot V_i^\omega$. On the other hand, L is not countable since every countable regular ω -language contains only ultimately periodic words and $K = P \cap L$ is non-empty. Thus $n > 0$ and, for example, $U_0 V_0^\omega$ is not countable.

This implies that we can find $v_0, v_1 \in V_0$ which are not powers of the same word. Indeed, we argue by contradiction. Let $v_0 \in V_0 \setminus \{\emptyset\}$, and $v \in \Sigma^*$ of minimal length such that v_0 is a power of v . We can find $w_0, w_1, \dots \in V_0 \setminus \{\emptyset\}$ such that $\sigma := w_0 w_1 \dots \neq v^\omega$. Fix a natural number i . Then w_i and v_0 are powers of the same word w . By Corollary 6.2.5 in [25], v and w are powers of the same word u , and v_0 too. By minimality, $u = v$, and w_i is a power of v . Thus $\sigma = v^\omega$, which is absurd.

Let $u_0 \in U_0$, and $L' := \{u_0\} \cdot \{v_0, v_1\}^\omega$. Note that $P \cap L'$ is a τ_B -closed subset of K , so that it is τ_B -compact. As the identity map from $(P \cap L', \tau_B)$ onto $(P \cap L', \tau_C)$, (where τ_C is the Cantor topology), is continuous, $P \cap L'$ is also τ_C -compact. But the map $\alpha \mapsto u_0 v_{\alpha(0)} v_{\alpha(1)} \dots$ is a homeomorphism from the Cantor space onto L' , by Corollaries 6.2.5 and 6.2.6 in [25]. Thus $P \cap L' = L' \setminus \text{Ult}$ is a dense closed subset of L' . Thus $P \cap L' = L'$, which is absurd since $u_0 \cdot v_0^\omega \in L' \setminus P$.

(c) By (b) and 7.9 in [19], for each Polish space Y , we can find a τ_B -closed set $F \subseteq P$ (an intersection of ω -regular sets) and a continuous bijection from F onto Y . In particular, if Y is not empty, then there is a *continuous surjection* from P onto Y extending the previous bijection. By 7.15 in [19], if moreover Y is perfect, then there is a continuous bijection from P onto Y .

(d) A consequence of (b), Corollary 3.3, and 7.10 in [19], is that the space \mathcal{S}_B is **not** K_σ (i.e., countable union of compact sets).

4.2.2 Consequences related to Borel sets

(5).(a) The σ -algebra \mathcal{A}_z generated by \mathbb{B}_z is exactly the σ -algebra of Borel subsets of \mathcal{S}_C (generated by the open subsets of \mathcal{S}_C). Indeed, the identity map from \mathcal{S}_z onto \mathcal{S}_C is a continuous bijection. By 15.2 in [19], the inverse map is Borel, so that the two Borel structures coincide. It remains to note that the σ -algebra generated by \mathbb{B}_z is exactly the σ -algebra of Borel subsets of \mathcal{S}_z .

(b) We can define a natural *hierarchy* in \mathcal{A}_z : let $\Sigma_1^z := (\mathbb{B}_z)_\sigma$ be the set of countable unions of elements of \mathbb{B}_z , and, inductively on $1 \leq \xi < \omega_1$, $\Pi_\xi^z := \{\neg L \mid L \in \Sigma_\xi^z\}$ and $\Sigma_\xi^z := (\bigcup_{\eta < \xi} \Pi_\eta^z)_\sigma$. This hierarchy is actually the Borel hierarchy in the family of Borel subsets of \mathcal{S}_z , in the sense that $\Sigma_\xi^z = \Sigma_\xi^0(\mathcal{S}_z)$ and $\Pi_\xi^z = \Pi_\xi^0(\mathcal{S}_z)$ (see 11.B in [19]; we will also consider $\Delta_\xi^z := \Sigma_\xi^z \cap \Pi_\xi^z = \Delta_\xi^0(\mathcal{S}_z)$ and $\Pi_0^0(\mathcal{S}_z) := \mathbb{B}_z$). By the main result and 22.4 in [19], this hierarchy is strict and of length ω_1 (the first uncountable ordinal). This comes from the existence of universal sets for these classes (see 22.3 in [19]).

Recall that a subset \mathcal{U} of $2^{\mathbb{N}} \times \Sigma^{\mathbb{N}}$ is **universal** for a class Γ of subsets of $\Sigma^{\mathbb{N}}$ if it is in Γ and the Γ subsets of $\Sigma^{\mathbb{N}}$ coincide with the vertical sections $\mathcal{U}_x = \{y \in \Sigma^{\mathbb{N}} \mid (x, y) \in \mathcal{U}\}$ of \mathcal{U} .

Note that a set is Borel for τ_C if and only if it is Borel for any topology τ_z , but the levels of the Borel hierarchy may differ for the two topologies. For instance any singleton associated with an ultimately constant word is not open for the Cantor topology and is actually open

for the δ , Büchi, automatic, alphabetic and strong alphabetic topologies. Therefore the existence of universal sets for each level of the Borel hierarchy of \mathcal{S}_z is different from the existence of universal sets for each level of the Borel hierarchy of \mathcal{S}_C if $z \neq C$, and could lead to other consequences or new applications to other domains of theoretical computer science.

(c) The *Wadge theorem* holds (see 22.10 in [19]): let $z \in \{\delta, B\}$, $\xi \geq 1$ be a countable ordinal. A subset L of $\Sigma^{\mathbb{N}}$ is in $\Sigma_{\xi}^z \setminus \Pi_{\xi}^z$ if and only if it is in Σ_{ξ}^z , and for each zero-dimensional Polish space X and any Σ_{ξ}^0 subset B of X there is $f: X \rightarrow \mathcal{S}_z$ continuous with $B = f^{-1}(L)$ (we can exchange Σ_{ξ}^z and Π_{ξ}^z).

(d) The *Saint Raymond theorem* holds (see 35.45 in [19]): let Σ, Γ be finite sets with at least two elements, $L \subseteq \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ be in \mathcal{A}_z with Σ_2^z vertical sections. Then $L = \bigcup_{n \in \mathbb{N}} L_n$, where L_n is in \mathcal{A}_z and has Π_1^z vertical sections.

(e) By 4.(b) and 13.7 in [19], for each L in \mathcal{A}_B , we can find a τ_B -closed set $F \subseteq P$ (an intersection of ω -regular sets) and a continuous bijection from F onto L . In particular, if L is not empty, then there is a *continuous surjection* from P onto L extending the previous bijection.

4.2.3 Consequences related to analytic sets

(6) Recall that the analytic sets are the projections of the elements of \mathcal{A}_z . We saw in (5).(a) that τ_z and τ_C have the same Borel sets. This implies that τ_z and τ_C have the same *analytic* sets. Then the following items (a)-(d) are not new, but we cite them as interesting results about the topology τ_z .

(a) The class of analytic sets contains strictly \mathcal{A}_z because of the existence of universal sets (see 14.2 in [19]).

(b) The *Lusin separation theorem* holds (see 14.7 in [19]): assume that $L, M \subseteq \Sigma^{\mathbb{N}}$ are disjoint analytic sets. Then there is S in \mathcal{A}_z such that $L \subseteq S \subseteq \neg M$ (we say that S *separates* L from M).

(c) The *Souslin theorem* holds (see 14.11 in [19]): the elements of \mathcal{A}_z are exactly the analytic sets whose complement is also analytic.

(d) By 14.13 in [19], an analytic set L has the *perfect set property* : either L is countable, or L contains a copy of \mathcal{S}_C (this copy has size continuum and is compact).

(e) The *parametrization theorem for Borel sets* holds (see 35.5 in [19]). We say that a set is *co-analytic* if its complement is analytic. We can find a co-analytic subset \mathcal{D} of $2^{\mathbb{N}}$, an analytic subset \mathcal{S} of $2^{\mathbb{N}} \times \Sigma^{\mathbb{N}}$, and a co-analytic subset \mathcal{P} of $2^{\mathbb{N}} \times \Sigma^{\mathbb{N}}$ such that the vertical sections \mathcal{S}_{α} and \mathcal{P}_{α} of \mathcal{S} and \mathcal{P} at any point α of \mathcal{D} are equal, and the elements of \mathcal{A}_z are exactly the sets \mathcal{S}_{α} for some α in \mathcal{D} .

4.2.4 Consequences related to uniformization problems

(7) Let X, Y be sets, $L \subseteq X \times Y$, and $f: X \rightarrow Y$ be a partial map. We say that f *uniformizes* L if the domain of f is $\pi_0[L]$ and the graph of f is contained in L . In the sequel, Σ and Γ will be finite sets with at least two elements.

(a) Recall that if $L \subseteq \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ is an infinitary rational relation, then L can be uniformized by a function whose graph is ω -rational (see Theorem 5 in [4]). Moreover, the inverse image of an ω -regular language by such a function is itself ω -regular, and thus the function is continuous for τ_B .

(b) The *Arsenin-Kunugui theorem* holds (see 18.18 in [19]): if $L \subseteq \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ is in \mathcal{A}_z and has K_σ vertical sections, then L can be uniformized by a function whose graph is in \mathcal{A}_z (a K_σ set is a countable union of compact sets).

(c) Let $\xi \geq 1$ be a countable ordinal. The class Σ_ξ^0 has the *number uniformization property* : if X is a zero-dimensional Polish space and $L \subseteq X \times \mathbb{N}$ is Σ_ξ^0 , then L can be uniformized by a function whose graph is Σ_ξ^0 . Consequently, for $z \in \{\delta, B\}$,

- Σ_ξ^z has the *generalized reduction property* : if $(L_n)_{n \in \mathbb{N}}$ is a sequence of Σ_ξ^z subsets of $\Sigma^{\mathbb{N}}$, then there is a sequence $(B_n)_{n \in \mathbb{N}}$ of pairwise disjoint Σ_ξ^z subsets of $\Sigma^{\mathbb{N}}$ with $\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} L_n$ and $B_n \subseteq L_n$,

- Π_ξ^z has the *generalized separation property* : if $(L_n)_{n \in \mathbb{N}}$ is a sequence of Π_ξ^z subsets of $\Sigma^{\mathbb{N}}$ with empty intersection, then there is a sequence $(B_n)_{n \in \mathbb{N}}$ of Δ_ξ^z subsets of $\Sigma^{\mathbb{N}}$ with empty intersection and $B_n \supseteq L_n$ (see 22.16 in [19]).

Moreover,

- the class of co-analytic sets has the number uniformization property and the generalized reduction property,

- the class of analytic sets has the generalized separation property (see 35.1 in [19]).

(d) The *Novikov-Kondo theorem* holds (see 36.14 in [19]): if $L \subseteq \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ is co-analytic, then L can be uniformized by a function whose graph is co-analytic.

5 Concluding Remarks

We have obtained in this paper new links and interactions between descriptive set theory and theoretical computer science, showing that four topologies considered in [31] are Polish, and providing many consequences of these results.

Notice that this paper is also motivated by the fact that the Gandy-Harrington topology, generated by the effective analytic subsets of a recursively presented Polish space, is an extremely powerful tool in descriptive set theory. In particular, this topology is used to prove some results of classical type (without reference to effective descriptive set theory in their statement). Among these results, let us mention the dichotomy theorems in [16, 20, 22, 23]. Sometimes, no other proof is known. Part of the power of this technique comes from the nice closure properties of the class Σ_1^1 of effective analytic sets (in particular the closure under projections).

The class of ω -regular languages has even stronger closure properties. So our hope is that the study of the Büchi topology, generated by the ω -regular languages, will help to prove some automatic versions of known descriptive results in the context of theoretical computer science.

From the main result, we know that there is a complete **distance** which is compatible with τ_z . It would be interesting to have a natural complete distance compatible with τ_B . We leave this as an open question for further study.

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