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To cite this version:

Olivier Carton, Olivier Finkel, Dominique Lecomte. Polishness of some topologies related to word or tree automata. Logical Methods in Computer Science, 2019, 10.23638/LMCS-15(2:9)2019 . hal-01593160v3

HAL Id: hal-01593160

https://hal.science/hal-01593160v3

Submitted on 10 May 2019

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Logical Methods in Computer Science
Volume 15, Issue 2, 2019, pp. 9:1–9:21
https://lmcs.episciences.org/

Submitted Oct. 27, 2017
Published May 08, 2019

POLISHNESS OF SOME TOPOLOGIES RELATED TO WORD OR TREE AUTOMATA

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Abstract. We prove that the Büchi topology and the automatic topology are Polish. We also show that this cannot be fully extended to the case of the space of infinite labelled binary trees; in particular the Büchi and the Muller topologies are not Polish in this case.

1. Introduction

This paper is a contribution to the study of the interactions between descriptive set theory and theoretical computer science. These interactions have already been the subject of many studies, see for instance [21, 45, 24, 42, 36, 33, 34, 39, 8, 38, 9, 13, 31, 11, 7, 41, 4, 28].

In particular, the theory of automata reading infinite words, which is closely related to infinite games, is now a rich theory which is used for the specification and the verification of non-terminating systems, see [15, 34]. The space $\Sigma^{\mathbb{N}}$ of infinite words over a finite alphabet Σ, equipped with the usual Cantor topology τ_C, is a natural place to study the topological complexity of the ω-languages accepted by various kinds of automata. In particular, it is interesting to locate them with respect to the Borel and the projective hierarchies.

However, as noticed in [35] by Schwarz and Staiger and in [18] by Hoffmann and Staiger, it turns out that for several purposes some other topologies on the space $\Sigma^{\mathbb{N}}$ are useful, for instance for studying fragments of the first-order logic over infinite words, or for a topological characterization of the random infinite words (see also [17]). In particular, Schwarz and

Key words and phrases: Automata and formal languages; logic in computer science; infinite words; Büchi automaton; regular ω-language; Cantor space; finer topologies; Büchi topology; automatic topology; Polish topology; space of infinite labelled binary trees; Büchi tree automaton; Muller tree automaton.

1998 ACM Subject Classification: F.1.1 Models of Computation; F.1.3 Complexity Measures and Classes; F.4.1 Mathematical Logic; F.4.3 Formal Languages.

LOGICAL METHODS IN COMPUTER SCIENCE
DOI:10.23638/LMCS-15(2:9)2019
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Staiger studied four topologies on the space $\Sigma^{\mathbb{N}}$ of infinite words over a finite alphabet Σ, which are all related to automata, and refine the Cantor topology on $\Sigma^{\mathbb{N}}$: the Büchi topology, the automatic topology, the alphabetic topology, and the strong alphabetic topology.

Recall that a topological space is Polish if and only if it is separable, i.e., contains a countable dense subset, and its topology is induced by a complete metric. Classical descriptive set theory is about the topological complexity of the definable subsets of the Polish topological spaces, as well as the study of some hierarchies of topological complexity (see [19, 34] for the basic notions). The analytic sets, which are the projections of the Borel sets, are of particular importance. Similar hierarchies of complexity are studied in effective descriptive set theory, which is based on the theory of recursive functions (see [30] for the basic notions). The effective analytic subsets of the Cantor space $(2^{\mathbb{N}}, \tau_C)$ are highly related to theoretical computer science, in the sense that they coincide with the sets recognized by some special kind of Turing machine (see [43]).

We now give details about some topologies that we investigate in this paper. Let Σ be a finite alphabet with at least two symbols. We consider the following topologies on $\Sigma^{\mathbb{N}}$.

- the *Büchi topology* τ_B, generated by the set $\mathbb{B}_B$ of ω-regular languages,
- the *automatic topology* τ_A, generated by the set $\mathbb{B}_A$ of τ_C-closed ω-regular languages (this topology is remarkable because any τ_C-closed (or even τ_C-$\mathbf{\Pi}^0_2$) ω-regular language is accepted by some deterministic Büchi automaton, [34]),
- the *topology* τ_δ, generated by the set $\mathbb{B}_\delta$ of languages accepted by some unambiguous Büchi Turing machine,
- the *Gandy-Harrington topology* τ_{GH}, generated by the set $\mathbb{B}_{GH}$ of languages accepted by some Büchi Turing machine.

In [35], Schwarz and Staiger prove that τ_B and τ_A are metrizable. The topology τ_{GH} is second countable, T_1 and strong Choquet, but it is not regular and thus not metrizable and not Polish. However, there is a dense basic open set on which τ_{GH} is Polish and zero dimensional, which is sufficient in many applications. The topology τ_B is separable, by definition, because there are only countably many regular ω-languages. It remains to see that it is completely metrizable to see that it is Polish. This is one of the main results proved in this paper.

Theorem 1.1. *Let $z \in \{C, B, A, \delta\}$. Then τ_z is Polish and zero-dimensional.*

From this result, it is already possible to infer many properties of the space $\Sigma^{\mathbb{N}}$, equipped with the Büchi topology (see [3] for an extended list). In particular, we get some results about the σ-algebra generated by the ω-regular languages. It is stratified in a hierarchy of length ω_1 (the first uncountable ordinal) and there are universal sets at each level of this hierarchy. Notice that this σ-algebra coincides with the σ-algebra of Borel sets for the Cantor topology. However, the levels of the Borel hierarchy differ for the two topologies. For instance, an ω-regular set which is non-$\mathbf{\Pi}^0_2$ for the Cantor topology is clopen (i.e., $\mathbf{\Delta}^0_1$) for the Büchi topology. Therefore the results about the existence of universal sets at each level of the σ-algebra generated by the ω-regular languages are really new and interesting.

We also investigate, following a suggestion of H. Michalewski, whether it is possible to extend these results to the case of a space T^ω_Σ of infinite binary trees labelled with letters of the alphabet Σ. On the one hand, the automatic topology can be proved to be Polish in a similar way. On the other hand, we show that the Büchi topology (generated by the set of regular tree languages accepted by some Büchi tree automaton) and the Muller topology

(generated by the set of regular tree languages accepted by some Muller tree automaton) are both non-Polish. However we prove that these two topologies have quite different properties: the first one is strong Choquet but not metrizable while the second one is metrizable but not strong Choquet.

2. Background

We first recall the notions required to understand fully the introduction and the sequel (see for example [34, 42, 19, 30]).

2.1. Theoretical computer science.

A *Büchi automaton* is a tuple $\mathcal{A} = (\Sigma, Q, Q_i, Q_f, \delta)$, where Σ is the input alphabet, Q is the finite set of states, Q_i and Q_f are the sets of initial and final states, and δ is the transition relation. The transition relation δ is a subset of $Q \times \Sigma \times Q$.

A *run* on some sequence $\sigma \in \Sigma^{\mathbb{N}}$ is a sequence $(q_n)_{i\in\mathbb{N}} \in Q^{\mathbb{N}}$ of states such that q_0 is initial ($q_0 \in Q_i$) and $\big(q_i, \sigma(i), q_{i+1}\big)$ is a transition in δ for each $i \geq 0$. It is *accepting* if it visits infinitely often final states, i.e., $q_i \in Q_f$ for infinitely many i's. An input sequence σ is accepted if there exists an accepting run on α. The set of accepted inputs is denoted $L(\mathcal{A})$. A set of infinite words is called *ω-regular* if it is equal to $L(\mathcal{A})$ for some automaton $\mathcal{A}$ (see [34] for the basic notions about regular ω-languages, which are the ω-languages accepted by some Büchi or Muller automaton).

A Büchi automaton is actually similar to a *classical finite automaton.* A finite word w of length n is accepted by some automaton $\mathcal{A}$ if there is sequence $(q_i)_{i\leq n}$ of $n+1$ states such that q_0 is initial ($q_0 \in Q_i$), q_n is final ($q_n \in Q_f$) and $(q_i, \sigma(i), q_{i+1})$ is a transition in δ for each $0 \leq i < n$. The set of accepted finite words is denoted by $U(\mathcal{A})$. A set of finite words is called *regular* if it is equal to $U(\mathcal{A})$ for some automaton $\mathcal{A}$.

Let U be a set of finite words, and V be a set of finite or infinite words. We recall that

$$U \cdot V = \{u \cdot v \mid u \in U \wedge v \in V\}.$$

An infinite word is *ultimately periodic* if it is of the form $u \cdot v^\omega$, where u, v are finite words. The *ω-power* of a set U of finite words is defined by

$$U^\omega = \{\sigma \in \Sigma^{\mathbb{N}} \mid \exists\, (w_i)_{i\in\mathbb{N}} \in U^{\mathbb{N}} \text{ such that } \sigma = w_0 \cdot w_1 \cdot w_2 \cdots\}.$$

The ω-powers play a crucial role in the characterization of ω-regular languages (see [2]).

Theorem 2.1 (Büchi)**.** *Let Σ be a finite alphabet, and $L \subseteq \Sigma^{\mathbb{N}}$. The following are equivalent:*
(1) *L is ω-regular,*
(2) *there are $2n$ regular languages $(U_i)_{i<n}$ and $(V_i)_{i<n}$ such that $L = \bigcup_{i<n} U_i \cdot V_i^\omega$.*

In particular, each singleton $\{uv^\omega\}$ formed by an ultimately periodic ω-word is an ω-regular language. We now recall some important properties of the class of ω-regular languages (see [34] and [35]). Let Σ be a set, and Σ^* be the set of finite sequences of elements of Σ. If $w \in \Sigma^*$, then w defines the usual basic clopen (i.e., closed and open) set $N_w := \{\sigma \in \Sigma^{\mathbb{N}} \mid w \text{ is a prefix of } \sigma\}$ of the Cantor topology τ_C (so $\mathbb{B}_C := \{\emptyset\} \cup \{N_w \mid w \in \Sigma^*\}$ is a basis for τ_C).

Theorem 2.2 (Büchi)**.** *The class of ω-regular languages contains the usual basic clopen sets and is closed under finite unions and intersections, taking complements, and projections (from a product alphabet onto one of its coordinates).*

We now turn to the study of Turing machines (see [6, 42]). A *Büchi Turing machine* is a tuple $\mathcal{M} = (\Sigma, \Gamma, Q, q_0, Q_f, \delta)$, where Σ and Γ are the input and tape alphabets satisfying $\Sigma \subseteq \Gamma$, Q is the finite set of states, q_0 is the initial state, Q_f is the set of final states, and δ is the transition relation. The relation δ is a subset of $(Q \times \Gamma) \times (Q \times \Gamma \times \{-1, 0, 1\})$.

A *configuration* of $\mathcal{M}$ is a triple (q, γ, j) where $q \in Q$ is the current state, $\gamma \in \Gamma^{\mathbb{N}}$ is the content of the tape and the non-negative integer $j \in \mathbb{N}$ is the position of the head on the tape.

Two configurations (q, γ, j) and (q', γ', j') of $\mathcal{M}$ are *consecutive* if there exists a transition $(q, a, q', b, d) \in \delta$ such that the following conditions are met.

(1) $\gamma(j) = a$, $\gamma'(j) = b$ and $\gamma(i) = \gamma'(i)$ for each $i \neq j$. This means that the symbol a is replaced by the symbol b at the position j and that all the other symbols on the tape remain unchanged.
(2) The two positions j and j' satisfy the equality $j' = j + d$.

A *run* of the machine $\mathcal{M}$ on some input $\sigma \in \Sigma^{\mathbb{N}}$ is a sequence $(p_i, \gamma_i, j_i)_{i \in \mathbb{N}}$ of consecutive configurations such that $p_0 = q_0$, $\gamma_0 = \sigma$ and $j_0 = 0$. The run is *accepting* if it visits infinitely often the final states, i.e., $p_i \in Q_f$ for infinitely many i's. The ω-language accepted by $\mathcal{M}$ is the set of inputs σ such that there exists an accepting run on σ.

Notice that some other accepting conditions have been considered for the acceptance of infinite words by Turing machines, like the 1' or Muller ones (the latter one was firstly called 3-acceptance), see [6, 42]. Moreover, several types of required behaviour on the input tape have been considered in the literature, see [43, 12, 10].

A *Büchi automaton* $\mathcal{A}$ is in fact a Büchi Turing machine whose head only moves forwards. This means that each of its transitions has the form (p, a, q, b, d) where $d = 1$. Note that the written symbol b is never read.

2.2. **Descriptive set theory.**

Classical descriptive set theory takes place in Polish topological spaces. We first recall that if d is a distance on a set X, and $(x_n)_{n \in \mathbb{N}}$ is a sequence of elements of X, then the sequence $(x_n)_{n \in \mathbb{N}}$ is called a *Cauchy sequence* if

$$\forall k \in \mathbb{N}\ \exists N \in \mathbb{N}\ \forall p, p' \geq N\ d(x_p, x_{p'}) < \frac{1}{2^k}.$$

In a topological space X whose topology is induced by a distance d, the distance d and the metric space (X, d) are said to be *complete* if every Cauchy sequence in X is convergent.

Definition 2.3. A topological space X is a *Polish space* if it is

(1) separable (there is a countable dense sequence $(x_n)_{n \in \mathbb{N}}$ in X),
(2) completely metrizable (there is a complete distance d on X which is compatible with the topology of X).

The most classical hierarchy of topological complexity in descriptive set theory is the one given by the Borel classes. If $\mathbf{\Gamma}$ is a class of sets in metrizable spaces, then $\check{\mathbf{\Gamma}} := \{\neg S \mid S \in \mathbf{\Gamma}\}$, and $(\mathbf{\Gamma})_\sigma$ is the class of countable unions of sets in $\mathbf{\Gamma}$. Recall that the *Borel hierarchy* is the inclusion from left to right in the following picture.

$$\begin{array}{ccccccc}
 & \mathbf{\Sigma}^0_1 = \text{open} & & \mathbf{\Sigma}^0_2 = (\mathbf{\Pi}^0_1)_\sigma & & & \mathbf{\Sigma}^0_\xi = (\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta)_\sigma \\
\mathbf{\Delta}^0_1 = \text{clopen} & & \mathbf{\Delta}^0_2 = \mathbf{\Sigma}^0_2 \cap \mathbf{\Pi}^0_2 & & \cdots & \mathbf{\Delta}^0_\xi = \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi & \cdots \\
 & \mathbf{\Pi}^0_1 = \text{closed} & & \mathbf{\Pi}^0_2 = \check{\mathbf{\Sigma}}^0_2 & & & \mathbf{\Pi}^0_\xi = \check{\mathbf{\Sigma}}^0_\xi
\end{array}$$

Above the Borel hierarchy sits the *projective hierarchy*, which is the inclusion from left to right in the following picture.

$$\begin{array}{ccc}
\mathbf{\Sigma}^1_1 = \text{analytic} & \mathbf{\Sigma}^1_2 = \text{Projections of } \mathbf{\Pi}^1_1 \text{ sets} & \mathbf{\Sigma}^1_{n+1} = \text{Projections of } \mathbf{\Pi}^1_n \text{ sets} \\
\mathbf{\Pi}^1_1 = \check{\mathbf{\Sigma}}^1_1 & \mathbf{\Pi}^1_2 = \check{\mathbf{\Sigma}}^1_2 & \mathbf{\Pi}^1_{n+1} = \check{\mathbf{\Sigma}}^1_{n+1}
\end{array}$$

Effective descriptive set theory is based on the notion of a recursive function. A function from $\mathbb{N}^k$ to $\mathbb{N}^l$ is said to be *recursive* if it is total and computable. By extension, a relation is called *recursive* if its characteristic function is recursive.

Definition 2.4. A *recursive presentation* of a Polish space X is a pair $\big((x_n)_{n\in\mathbb{N}}, d\big)$ such that

(1) $(x_n)_{n\in\mathbb{N}}$ is dense in X,
(2) d is a compatible complete distance on X such that the following relations P and Q are recursive:

$$P(i,j,m,k) \iff d(x_i,x_j) \leq \frac{m}{k+1},$$
$$Q(i,j,m,k) \iff d(x_i,x_j) < \frac{m}{k+1}.$$

A Polish space X is *recursively presented* if there is a recursive presentation of it.

Note that the formula $(p,q) \mapsto 2^p(2q+1)-1$ defines a recursive bijection $\mathbb{N}^2 \to \mathbb{N}$. One can check that the coordinates of the inverse map are also recursive. They will be denoted $n \mapsto (n)_0$ and $n \mapsto (n)_1$ in the sequel. These maps will help us to define some of the basic effective classes.

Definition 2.5. Let $\big((x_n)_{n\in\mathbb{N}}, d\big)$ be a recursive presentation of a Polish space X.

(1) We fix a countable basis of X: $B(X,n)$ is the open ball $B_d(x_{(n)_0}, \frac{((n)_1)_0}{((n)_1)_1+1})$.
(2) A subset S of X is *semirecursive*, or *effectively open* (denoted $S \in \Sigma^0_1$) if
$$S = \bigcup_{n\in\mathbb{N}} B\big(X, f(n)\big),$$
for some recursive function f.
(3) A subset S of X is *effectively closed* (denoted $S \in \Pi^0_1$) if its complement $\neg S$ is semirecursive.
(4) One can check that a product of two recursively presented Polish spaces has a recursive presentation, and that the Baire space $\mathbb{N}^\mathbb{N}$ has a recursive presentation. A subset S of X is *effectively analytic* (denoted $S \in \Sigma^1_1$) if there is a Π^0_1 subset C of $X \times \mathbb{N}^\mathbb{N}$ such that
$$S = \pi_0[C] := \{x \in X \mid \exists \alpha \in \mathbb{N}^\mathbb{N} \ \ (x,\alpha) \in C\}.$$

(5) A subset S of X is *effectively co-analytic* (denoted $S \in \Pi^1_1$) if its complement $\neg S$ is effectively analytic, and *effectively Borel* if it is in Σ^1_1 and Π^1_1 (denoted $S \in \Delta^1_1$).
(6) We will also use the following *relativized classes*: if X, Y are recursively presented Polish spaces and $y \in Y$, then we say that $A \subseteq X$ is in $\Sigma^1_1(y)$ if there is $S \in \Sigma^1_1(Y \times X)$ such that $A = S_y := \{x \in X \mid (y, x) \in S\}$. The class $\Pi^1_1(y)$ is defined similarly. We also set $\Delta^1_1(y) := \Sigma^1_1(y) \cap \Pi^1_1(y)$.

The crucial link between the effective classes and the classical corresponding classes is as follows: the class of analytic (resp., co-analytic, Borel) subsets of Y is equal to $\bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} \Sigma^1_1(\alpha)$ (resp., $\bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} \Pi^1_1(\alpha)$, $\bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} \Delta^1_1(\alpha)$). This allows to use effective descriptive set theory to prove results of classical type. In the sequel, when we consider an effective class in some $\Sigma^\mathbb{N}$ with Σ finite, we will always use a fixed recursive presentation associated with the Cantor topology. The following result is proved in [43], see also [10].

Theorem 2.6. *Let Σ be a finite alphabet, and $L \subseteq \Sigma^\mathbb{N}$. The following statements are equivalent:*
(1) *$L = L(\mathcal{M})$ for some Büchi Turing machine $\mathcal{M}$,*
(2) *$L \in \Sigma^1_1$.*

We now recall the *strong Choquet game* played by two players on a topological space X. Players 1 and 2 play alternatively. At each turn i, Player 1 plays by choosing an open subset U_i and a point $x_i \in U_i$ such that $U_i \subseteq V_{i-1}$, where V_{i-1} has been chosen by Player 2 at the previous turn. Player 2, plays by choosing an open subset V_i such that $x_i \in V_i$ and $V_i \subseteq U_i$. Player 2 wins the game if $\bigcap_{i \in \mathbb{N}} V_i \neq \emptyset$. We now recall some classical notions of topology.

Definition 2.7. A topological space X is said to be
- T_1 if every singleton of X is closed,
- *regular* if for every point of X and every open neighborhood U of x, there is an open neighborhood V of x with $\overline{V} \subseteq U$,
- *second countable* if its topology has a countable basis,
- *zero-dimensional* if there is a basis made of clopen sets,
- *strong Choquet* if X is not empty and Player 2 has a winning strategy in the strong Choquet game.

Note that every zero-dimensional space is regular. The following result is Theorem 8.18 in [19].

Theorem 2.8 (Choquet)**.** *A nonempty, second countable topological space is Polish if and only if it is T_1, regular, and strong Choquet.*

Let X be a nonempty recursively presented Polish space. The *Gandy-Harrington topology* on X is generated by the Σ^1_1 subsets of X, and denoted τ^X_{GH}. By Theorem 2.6, this topology is also related to automata and Turing machines. As there are some effectively analytic sets whose complement is not analytic, the Gandy-Harrington topology is not metrizable (in fact not regular) in general (see 3E.9 in [30]). In particular, it is not Polish.

Let $\mathbf{\Gamma}$ be a class of sets in Polish spaces. If Y is a Polish space, then we say that $A \in \mathbf{\Gamma}(Y)$ is $\mathbf{\Gamma}$-*complete* if, for each zero-dimensional Polish space X and each $B \in \mathbf{\Gamma}(X)$, there is $f : X \to Y$ continuous such that $B = f^{-1}(A)$. By Section 22.B in [19], if $\mathbf{\Gamma}$ is of the form $\mathbf{\Sigma}$ or $\mathbf{\Pi}$ in the Borel or the projective hierarchy, and if A is $\mathbf{\Gamma}$-complete, then A is not in $\check{\mathbf{\Gamma}}$. Theorem 22.10 in [19] gives a converse in the Borel hierarchy.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is organized as follows. We provide below four properties which ensure that a given topological space is strong Choquet. Then we use Theorem 2.8 to prove that the considered spaces are indeed Polish.

Let Σ be a countable alphabet. The set $\Sigma^{\mathbb{N}}$ is equipped with the product topology of the discrete topology on Σ, unless another topology is specified. This topology is induced by a natural metric, called the *prefix metric* which is defined as follows. For $\sigma \neq \sigma' \in \Sigma^{\mathbb{N}}$, the distance d is given by

$$d(\sigma, \sigma') = \frac{1}{2^r} \quad \text{, where} \quad r = \min\{n \in \mathbb{N} \mid \sigma(n) \neq \sigma'(n)\}.$$

When Σ is finite this topology is the classical Cantor topology. When Σ is countably infinite the topological space is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$.

Let Σ and Γ be two alphabets. The function which maps each pair $(\sigma, \gamma) \in \Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ to the element $\big(\sigma(0), \gamma(0)\big), \big(\sigma(1), \gamma(1)\big), \ldots$ of $(\Sigma \times \Gamma)^{\mathbb{N}}$ is a homeomorphism between $\Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ and $(\Sigma \times \Gamma)^{\mathbb{N}}$ allowing us to identify these two spaces.

If Σ is a set, $\sigma \in \Sigma^{\mathbb{N}}$ and $l \in \mathbb{N}$, then $\sigma|l$ is the prefix of σ of length l.

We set $2 := \{0, 1\}$ and $\mathbb{P}_\infty := \{\alpha \in 2^{\mathbb{N}} \mid \forall k \in \mathbb{N} \ \ \exists i \geq k \ \ \alpha(i) = 1\}$. This latter set is simply the set of infinite words over the alphabet 2 having infinitely many 1's.

We will work in the spaces of the form $\Sigma^{\mathbb{N}}$, where Σ is a finite set with at least two elements. We consider a topology τ_Σ on $\Sigma^{\mathbb{N}}$, and a basis $\mathbb{B}_\Sigma$ for τ_Σ. We consider the following properties of the family $(\tau_\Sigma, \mathbb{B}_\Sigma)_\Sigma$, using the previous identification of $\Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ and $(\Sigma \times \Gamma)^{\mathbb{N}}$:

(P1) $\mathbb{B}_\Sigma$ contains the usual basic clopen sets N_w,

(P2) $\mathbb{B}_\Sigma$ is closed under finite unions and intersections,

(P3) $\mathbb{B}_\Sigma$ is closed under projections, in the sense that if Γ is a finite set with at least two elements and $L \in \mathbb{B}_{\Sigma \times \Gamma}$, then $\pi_0[L] \in \mathbb{B}_\Sigma$,

(P4) for each $L \in \mathbb{B}_\Sigma$ there is a closed subset C of $\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty$ (i.e., C is the intersection of a τ_C-closed subset of the Cantor space $\Sigma^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty$), which is in $\mathbb{B}_{\Sigma \times 2}$, and such that $L = \pi_0[C]$.

Theorem 3.1. *Assume that a family $(\tau_\Sigma, \mathbb{B}_\Sigma)_\Sigma$ satisfies Properties (P1)-(P4). Then the topologies τ_Σ are strong Choquet.*

Proof. We first describe a strategy τ for Player 2. Player 1 first plays $\sigma_0 \in \Sigma^{\mathbb{N}}$ and a τ_Σ-open neighborhood U_0 of σ_0. Let L_0 in $\mathbb{B}_\Sigma$ with $\sigma_0 \in L_0 \subseteq U_0$. Property (P4) gives C_0 with $L_0 = \pi_0[C_0]$. This gives $\alpha_0 \in \mathbb{P}_\infty$ such that $(\sigma_0, \alpha_0) \in C_0$. We choose $l_0^0 \in \mathbb{N}$ big enough to ensure that if

$$s_0^0 := \alpha_0 | l_0^0,$$

then s_0^0 has at least a coordinate equal to 1. We set $w_0 := \sigma_0|1$ and $V_0 := \pi_0[C_0 \cap (N_{w_0} \times N_{s_0^0})]$. By Properties (P1)-(P3), V_0 is in $\mathbb{B}_\Sigma$ and thus τ_Σ-open. Moreover, $\sigma_0 \in V_0 \subseteq L_0 \subseteq U_0$, so that Player 2 respects the rules of the game if he plays V_0.

Now Player 1 plays $\sigma_1 \in V_0$ and a τ_Σ-open neighborhood U_1 of σ_1 contained in V_0. Let L_1 in $\mathbb{B}_\Sigma$ with $\sigma_1 \in L_1 \subseteq U_1$. Property (P4) gives C_1 with $L_1 = \pi_0[C_1]$. This gives $\alpha_1 \in \mathbb{P}_\infty$ such that $(\sigma_1, \alpha_1) \in C_1$. We choose $l_0^1 \in \mathbb{N}$ big enough to ensure that if $s_0^1 := \alpha_1 | l_0^1$, then s_0^1 has at least one coordinate equal to 1. As $\sigma_1 \in V_0$, there is $\alpha_0' \in \mathbb{P}_\infty$ such that $(\sigma_1, \alpha_0') \in C_0 \cap (N_{w_0} \times N_{s_0^0})$. We choose $l_1^0 > l_0^0$ big enough to ensure that if $s_1^0 := \alpha_0' | l_1^0$, then s_1^0 has at least two coordinates equal to 1. We set $w_1 := \sigma_1|2$ and

$V_1 := \pi_0[C_0 \cap (N_{w_1} \times N_{s_1^0})] \cap \pi_0[C_1 \cap (N_{w_0} \times N_{s_0^1})]$. Here again, V_1 is τ_Σ-open. Moreover, $\sigma_1 \in V_1 \subseteq U_1$ and Player 2 can play V_1.

Next, Player 1 plays $\sigma_2 \in V_1$ and a τ_Σ-open neighborhood U_2 of σ_2 contained in V_1. Let L_2 in $\mathbb{B}_\Sigma$ with $\sigma_2 \in L_2 \subseteq U_2$. Property (P4) gives C_2 with $L_2 = \pi_0[C_2]$. This gives $\alpha_2 \in \mathbb{P}_\infty$ such that $(\sigma_2, \alpha_2) \in C_2$. We choose $l_0^2 \in \mathbb{N}$ big enough to ensure that if $s_0^2 := \alpha_2|l_0^2$, then s_0^2 has at least one coordinate equal to 1. As $\sigma_2 \in V_1$, there is $\alpha_1' \in \mathbb{P}_\infty$ such that $(\sigma_2, \alpha_1') \in C_1 \cap (N_{w_0} \times N_{s_0^1})$. We choose $l_1^1 > l_0^1$ big enough to ensure that if $s_1^1 := \alpha_1'|l_1^1$, then s_1^1 has at least two coordinates equal to 1. As $\sigma_2 \in V_1$, there is $\alpha_0'' \in \mathbb{P}_\infty$ such that $(\sigma_2, \alpha_0'') \in C_0 \cap (N_{w_1} \times N_{s_1^0})$. We choose $l_2^0 > l_1^0$ big enough to ensure that if $s_2^0 := \alpha_0''|l_2^0$, then s_2^0 has at least three coordinates equal to 1. We set $w_2 := \sigma_2|3$ and $V_2 := \pi_0[C_0 \cap (N_{w_2} \times N_{s_2^0})] \cap \pi_0[C_1 \cap (N_{w_1} \times N_{s_1^1})] \cap \pi_0[C_2 \cap (N_{w_0} \times N_{s_0^2})]$. Here again, V_2 is τ_Σ-open. Moreover, $\sigma_2 \in V_2 \subseteq U_2$ and Player 2 can play V_2.

If we go on like this, we build $w_l \in \Sigma^{l+1}$ and $s_l^n \in 2^*$ such that $w_0 \subseteq w_1 \subseteq \ldots$ and $s_0^n \subsetneqq s_1^n \subsetneqq \ldots$ This allows us to define $\sigma := \lim_{l\to\infty} w_l \in \Sigma^\mathbb{N}$ and, for each $n \in \mathbb{N}$, $\alpha_n := \lim_{l\to\infty} s_l^n \in 2^\mathbb{N}$. Note that $\alpha_n \in \mathbb{P}_\infty$ since s_l^n has at least $l+1$ coordinates equal to 1. As (σ, α_n) is the limit of (w_l, s_l^n) as l goes to infinity and $N_{w_l} \times N_{s_l^n}$ meets C_n (which is closed in $\Sigma^\mathbb{N} \times \mathbb{P}_\infty$ in the sense of Property (P4)), $(\sigma, \alpha_n) \in C_n$. Thus

$$\sigma \in \bigcap_{n\in\mathbb{N}} \pi_0[C_n] = \bigcap_{n\in\mathbb{N}} L_n \subseteq \bigcap_{n\in\mathbb{N}} U_n \subseteq \bigcap_{n\in\mathbb{N}} V_n,$$

so that τ is winning for Player 2. □

3.1. **The Gandy-Harrington topology.**

We have already mentioned the fact that the Gandy-Harrington topology is not Polish in general. However, it is almost Polish since it fulfills Properties (P1)-(P4).

Let Σ be a finite alphabet with at least two elements and X be the space $\Sigma^\mathbb{N}$ equipped with the topology $\tau_\Sigma := \tau_{GH}^X$ generated by the family $\mathbb{B}_\Sigma$ of Σ_1^1 subsets of X. Note that the assumption of Theorem 3.1 are satisfied. Indeed, (P1)-(P3) come from 3E.2 in [30]. For (P4), let F be a Π_1^0 subset of $X \times \mathbb{N}^\mathbb{N}$ such that $L = \pi_0[F]$. Let φ be the function from $\mathbb{N}^\mathbb{N}$ to $2^\mathbb{N}$ defined by

$$\varphi(\beta) = 0^{\beta(0)}10^{\beta(1)}1\ldots$$

Note that φ is a homeomorphism from $\mathbb{N}^\mathbb{N}$ onto $\mathbb{P}_\infty$, and recursive (which means that the relation $\varphi(\beta) \in N(2^\mathbb{N}, n)$ is semirecursive in β and n). This implies that $C := (\mathrm{Id} \times \varphi)[F]$ is suitable (see 3E.2 in [30]).

Note that τ_Σ is second countable since there are only countably many Σ_1^1 subsets of X (see 3F.6 in [30]), T_1 since it is finer than the usual topology by the property (P1), and strong Choquet by Theorem 3.1.

One can show that there is a dense basic open subset Ω_X of (X, τ_Σ) such that $S \cap \Omega_X$ is a clopen subset of (Ω_X, τ_Σ) for each Σ_1^1 subset S of X (see [22]). In particular, (Ω_X, τ_Σ) is zero-dimensional, and regular. As it is, just like (X, τ_Σ), second countable, T_1 and strong Choquet, (Ω_X, τ_Σ) is a Polish space, by Theorem 2.8.

3.2. **The Büchi topology.**

Let Σ be a finite alphabet with at least two symbols, and X be the space $\Sigma^{\mathbb{N}}$ equipped with the Büchi topology τ_B generated by the family $\mathbb{B}_B$ of ω-regular languages in X. Theorem 29 in [35] shows that τ_B is metrizable. We now give a distance which is compatible with τ_B. This metric was used in [17] (Theorem 2 and Lemma 21 and several corollaries following Lemma 21). A similar argument for subword metrics is in Section 4 in [18]. If $\mathcal{A}$ is a Büchi automaton, then we denote $|\mathcal{A}|$ the number of states of $\mathcal{A}$. We say that a Büchi automaton *separates* x and y if and only if

$$\big(x \in L(\mathcal{A}) \wedge y \notin L(\mathcal{A})\big) \vee \big(y \in L(\mathcal{A}) \wedge x \notin L(\mathcal{A})\big).$$

The distance δ on $\Sigma^{\mathbb{N}}$ is then defined as follows:

$$\delta(x,y) = \begin{cases} 0 \text{ if } x = y, \\ \frac{1}{2^n} \text{ if } x \neq y, \end{cases} \qquad \text{for } x, y \in \Sigma^{\mathbb{N}}.$$

where $n := \min\{|\mathcal{A}| \mid \mathcal{A} \text{ is a Büchi automaton which separates } x \text{ and } y\}$. We now describe some properties of the map δ. This is the occasion to illustrate the notion of a complete metric.

Proposition 3.2. *The following properties of δ hold:*

(1) *the map δ defines a distance on $\Sigma^{\mathbb{N}}$,*
(2) *the distance δ is compatible with τ_B,*
(3) *the distance δ is not complete.*

Proof. 1. If $x, y \in \Sigma^{\mathbb{N}}$, then $\delta(x,y) = \delta(y,x)$, by the definition of δ. Let $x, y, z \in \Sigma^{\mathbb{N}}$, and assume that $\delta(x,y) + \delta(y,z) < \delta(x,z) = \frac{1}{2^n}$. Then $\delta(x,y) < \frac{1}{2^n}$ and $\delta(y,z) < \frac{1}{2^n}$ hold. In particular, if $\mathcal{A}$ is a Büchi automaton with n states then it does not separate x and y and similarly it does not separate y and z. Thus either $x, y, z \in L(\mathcal{A})$ or $x, y, z \notin L(\mathcal{A})$. This implies that the Büchi automaton $\mathcal{A}$ does not separate x and z. As this holds for every Büchi automaton with n states, $\delta(x,z) < \frac{1}{2^n}$. This leads to a contradiction and thus $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$ for all $x, y, z \in \Sigma^{\mathbb{N}}$. This shows that δ is a distance on $\Sigma^{\mathbb{N}}$.

2. Recall that an open set for this topology is a union of ω-languages accepted by some Büchi automaton. Let then $L(\mathcal{A})$ be an ω-language accepted by some Büchi automaton $\mathcal{A}$ having n states, and $x \in L(\mathcal{A})$. We now show that the open ball $B(x, \frac{1}{2^{n+1}})$ with center x and δ-radius $\frac{1}{2^{n+1}}$ is a subset of $L(\mathcal{A})$. Indeed, if $\delta(x,y) < \frac{1}{2^{n+1}} < \frac{1}{2^n}$, then x and y cannot be separated by any Büchi automaton with n states, and thus $y \in L(\mathcal{A})$. This shows that $L(\mathcal{A})$ (and therefore any open set for τ_B) is open for the topology induced by the distance δ. Conversely, let $B(x,r)$ be an open ball for the distance δ, where $r > 0$ is a positive real. It is clear from the definition of the distance δ that we may only consider the case $r = \frac{1}{2^n}$ for some natural number n. Then $y \in B(x, \frac{1}{2^n})$ if and only if x and y cannot be separated by any Büchi automaton with $p \leq n$ states. Therefore the open ball $B(x, \frac{1}{2^n})$ is the intersection of the regular ω-languages $L(\mathcal{A}_i)$ for some Büchi automata $\mathcal{A}_i$ having $p \leq n$ states and such that $x \in L(\mathcal{A}_i)$, and of the regular ω-languages $\Sigma^{\mathbb{N}} \setminus L(\mathcal{B}_i)$ for some Büchi automata $\mathcal{B}_i$ having $p \leq n$ states and such that $x \notin L(\mathcal{B}_i)$. The class of regular ω-languages being closed under taking complements and finite intersections, the open ball $B(x, \frac{1}{2^n})$ is actually a regular ω-language and thus an open set for τ_B.

3. Without loss of generality, we set $\Sigma = 2$ and we consider, for a natural number $n \geq 1$, the ω-word $X_n = 0^{n!} \cdot 1 \cdot 0^{\omega}$ over the alphabet 2 having only one symbol 1 after $n!$ symbols 0, where $n! := n \times (n-1) \times \cdots \times 2 \times 1$. Let now $m > n > k$ and $\mathcal{A}$ be a Büchi automaton with

k states. Using a classical pumping argument, we can see that the automaton $\mathcal{A}$ cannot separate X_n and X_m. Indeed, assume first that $X_n \in L(\mathcal{A})$. Then, when reading the first k symbols 0 of X_n, the automaton enters at least twice in a same state q. This implies that: $(\exists p \leq k)\ (\forall l \geq 1)\ 0^{n!+lp} \cdot 1 \cdot 0^\omega \in L(\mathcal{A})$.
In particular $m! = n! \times (n+1) \times \cdots \times m = n! + n! \times \Big(\big((n+1) \times \cdots \times m\big) - 1\Big)$ is of this form and thus $X_m \in L(\mathcal{A})$. A very similar pumping argument shows that if $X_m \in L(\mathcal{A})$, then $X_n \in L(\mathcal{A})$. This shows that $\delta(X_n, X_m) < \frac{1}{2^k}$ and finally that the sequence (X_n) is a Cauchy sequence for the distance δ. On the other hand if this sequence was converging to an ω-word x then x should be the word 0^ω because τ_B is finer than τ_C. But 0^ω is an ultimately periodic word and thus it is an isolated point for τ_B. This leads to a contradiction, and thus the distance δ is not complete because the sequence (X_n) is a Cauchy sequence which is not convergent. □

Proposition 3.2 gives a motivation for deriving Theorem 1.1 from Theorem 3.1. Note that the assumption of Theorem 3.1 are satisfied. Indeed, (P1)-(P3) come from Theorem 2.2. We now check (P4).

Lemma 3.3. *Let Σ be a finite set with at least two elements, and $L \subseteq \Sigma^{\mathbb{N}}$ be an ω-regular language. Then there is a closed subset C of $\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty$, which is ω-regular as a subset of $(\Sigma \times 2)^{\mathbb{N}}$ identified with $\Sigma^{\mathbb{N}} \times 2^{\mathbb{N}}$, and such that $L = \pi_0[C]$.*

Proof. Let $\mathcal{A} = (\Sigma, Q, \delta, Q_i, Q_f)$ be a Büchi automaton and let $L = L(\mathcal{A})$ be its set of accepted words. Let χ_f be the characteristic function of Q_f. It maps the state q to 1 if $q \in Q_f$, and to 0 otherwise. The function χ_f is extended to $Q^{\mathbb{N}}$ by setting $\alpha = \chi_f((q_n)_{n\in\mathbb{N}})$ where $\alpha(n) = \chi_f(q_n)$. Note that a run ρ of $\mathcal{A}$ is accepting if and only if $\chi_f(\rho) \in \mathbb{P}_\infty$. Let C be the subset of $\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty$ defined by

$$C := \big\{(\sigma, \alpha) \in \Sigma^{\mathbb{N}} \times \mathbb{P}_\infty \mid \exists \rho \text{ run of } \mathcal{A} \text{ on } \sigma \text{ such that } \alpha = \chi_f(\rho)\big\}.$$

By the definition of C, $L = \pi_0[C]$. Let K be the subset of $\Sigma^{\mathbb{N}} \times 2^{\mathbb{N}} \times Q^{\mathbb{N}}$ defined by

$$K := \big\{(\sigma, \alpha, \rho) \in \Sigma^{\mathbb{N}} \times 2^{\mathbb{N}} \times Q^{\mathbb{N}} \mid \rho \text{ is a run of } \mathcal{A} \text{ on } \sigma \text{ such that } \alpha = \chi_f(\rho)\big\}.$$

As K is compact as a closed subset of a compact space and $C = \pi_{\Sigma^{\mathbb{N}} \times 2^{\mathbb{N}}}[K] \cap (\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty)$, C is a closed subset of $\Sigma^{\mathbb{N}} \times \mathbb{P}_\infty$. It remains to show that C is indeed ω-regular. Let Δ be defined by

$$\Delta := \big\{\big(p, (a, \varepsilon), q\big) \in Q \times (\Sigma \times 2) \times Q \mid (p, a, q) \in \delta \wedge (\varepsilon = 1 \iff p \in Q_f)\big\}.$$

This allows us to define a Büchi automaton by $\mathcal{A}' := (\Sigma \times 2, Q, \Delta, Q_i, Q_f)$. Note that

$$\begin{aligned}
(\sigma, \alpha) \in L(\mathcal{A}') &\Leftrightarrow \exists (s_i)_{i\in\mathbb{N}} \in Q^{\mathbb{N}}\ \big(s_0 \in Q_i \ \wedge\ \forall i \in \mathbb{N}\ (s_i, \big(\sigma(i), \alpha(i)\big), s_{i+1}) \in \Delta\big) \wedge \\
&\qquad\qquad \forall k \in \mathbb{N}\ \exists i \geq k\ s_i \in Q_f) \\
&\Leftrightarrow \alpha \in \mathbb{P}_\infty \ \wedge\ \exists (s_i)_{i\in\mathbb{N}} \in Q^{\mathbb{N}}\ \big(s_0 \in Q_i \ \wedge\ \forall i \in \mathbb{N}\ (s_i, \sigma(i), s_{i+1}) \in \delta \wedge \\
&\qquad\qquad (\alpha(i) = 1 \Leftrightarrow s_i \in Q_f)\big) \\
&\Leftrightarrow (\sigma, \alpha) \in C.
\end{aligned}$$

Thus $C = L(\mathcal{A}')$ is ω-regular. □

Corollary 3.4. *Let Σ be a finite set with at least two elements. Then the Büchi topology τ_B is zero dimensional and Polish.*

Proof. As there are only countably many possible automata (up to identifications), $\mathbb{B}_B$ is countable. This shows that τ_B is second countable. It is T_1 since it is finer than the usual topology by Property (P1), and strong Choquet by Theorem 3.1. Moreover, it is zero-dimensional since the class of ω-regular languages is closed under taking complements (see Theorem 2.2). It remains to apply Theorem 2.8. □

3.3. **The other topologies.**

Lemma 3.5. *Let (X, τ) be a Polish space, and $(C_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of (X, τ). Then the topology generated by $\tau \cup \{C_n \mid n \in \mathbb{N}\}$ is Polish.*

Proof. By Lemma 13.2 in [19], the topology τ_n generated by $\tau \cup \{C_n\}$ is Polish. By Lemma 13.3 in [19], the topology τ_∞ generated by $\bigcup_{n\in\mathbb{N}} \tau_n$ is Polish. Thus the topology generated by $\tau \cup \{C_n \mid n \in \mathbb{N}\}$, which is τ_∞, is Polish. □

Proof of Theorem 1.1. It is well known that $(\Sigma^{\mathbb{N}}, \tau_C)$ is metrizable and compact, and thus Polish, and zero-dimensional.

- By Theorem 3.4 in [26], the implication (iii) $\Rightarrow$ (i), $\Delta^1_1(\Sigma^{\mathbb{N}})$ is a basis for a zero-dimensional Polish topology on $\Sigma^{\mathbb{N}}$. Recall that a Büchi Turing machine is *unambiguous* if every ω-word $\sigma \in \Sigma^{\mathbb{N}}$ has at most one accepting run. By Theorem 3.6 in [10], a subset of $\Sigma^{\mathbb{N}}$ is Δ^1_1 if and only if it is accepted by some unambiguous Büchi Turing machine. Therefore $\mathbb{B}_\delta = \Delta^1_1(\Sigma^{\mathbb{N}})$ is a basis for the zero-dimensional Polish topology τ_δ.
- Corollary 3.4 gives the result for the Büchi topology.
- Lemma 3.5 shows that the automatic topology is Polish since it refines the usual product topology on $\Sigma^{\mathbb{N}}$. For this reason also, it is zero-dimensional. □

4. The Büchi and Muller topologies on a space of trees

The notion of a Büchi automaton has been extended to the case of a Büchi tree automaton reading infinite binary trees whose nodes are labelled by letters of a finite alphabet. We now recall this notion and some related ones.

A node of an infinite binary tree is represented by a finite word over the alphabet $\{l, r\}$ where l means "left" and r means "right". An infinite binary tree whose nodes are labelled in Σ is identified with a function $t : \{l, r\}^\star \to \Sigma$. The set of infinite binary trees labelled in Σ will be denoted T^ω_Σ.

A finite binary tree is like an "initial finite subtree" of an infinite binary tree. Thus it can be represented by a function $s : S \subseteq \{l, r\}^\star \to \Sigma$, where S is a finite subset of $\{l, r\}^\star$ which is closed under prefix. If $t \in T^\omega_\Sigma$ is an infinite binary tree, and $n \geq 0$ is an integer, then we denote by $t|n$ the initial finite subtree of t whose domain is equal to $\{l, r\}^{\leq n}$, where $\{l, r\}^{\leq n}$ is the set of finite words over the alphabet $\{l, r\}$ of length smaller than or equal to n.

Let t be an infinite binary tree. A branch B of t is a subset of the set of nodes of t which is linearly ordered by the tree partial order $\sqsubseteq$ and which is closed under prefix relation (i.e., if x and y are nodes of t such that $y \in B$ and $x \sqsubseteq y$, then $x \in B$). A branch B of a tree is said to be maximal if and only if there is no other branch of t which strictly contains B. Let t be an infinite binary tree in T^ω_Σ. If B is a maximal branch of t, then this branch is infinite. Let $(u_i)_{i\geq 0}$ be the enumeration of the nodes in B which is strictly increasing for

the prefix order. The infinite sequence of the labels of the nodes of such a maximal branch B, i.e., $t(u_0)t(u_1)\cdots t(u_n)\cdots$, is called a path. It is an ω-word over the alphabet Σ.

Let then $L \subseteq \Sigma^\omega$ be an ω-language over Σ. We denote $\exists\mathrm{Path}(L)$ the set of infinite trees t in T_Σ^ω such that t has (at least) one path in L.

We now define the tree automata and the recognizable tree languages.

Definition 4.1. A (non deterministic) tree automaton is a quadruple $\mathcal{A} = (\Sigma, Q, q_0, \Delta)$, where Σ is the finite input alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state and $\Delta \subseteq Q \times \Sigma \times Q \times Q$ is the transition relation. The tree automaton $\mathcal{A}$ is said to be deterministic if the relation Δ is a functional one, i.e., if for each $(q, a) \in Q \times \Sigma$ there is at most one pair of states (q', q'') such that $(q, a, q', q'') \in \Delta$.

A run of the tree automaton $\mathcal{A}$ on an infinite binary tree $t \in T_\Sigma^\omega$ is an infinite binary tree $\rho \in T_Q^\omega$ such that

(a) $\rho(\lambda) = q_0$,
(b) for each $u \in \{l, r\}^\star$, $\big(\rho(u), t(u), \rho(u.l), \rho(u.r)\big) \in \Delta$.

Definition 4.2. A Büchi (non deterministic) tree automaton is a tuple

$$\mathcal{A} = (\Sigma, Q, q_0, Q_f, \Delta),$$

where (Σ, Q, q_0, Δ) is a tree automaton and $Q_f \subseteq Q$ is the set of accepting states.

A run ρ of the Büchi tree automaton $\mathcal{A}$ on an infinite binary tree $t \in T_\Sigma^\omega$ is said to be accepting if for each path of ρ there is some accepting state appearing infinitely often on this path.

The tree language $L(\mathcal{A})$ accepted by the Büchi tree automaton $\mathcal{A}$ is the set of infinite binary trees $t \in T_\Sigma^\omega$ such that there is (at least) one accepting run of $\mathcal{A}$ on t.

Definition 4.3. A Muller (non deterministic) tree automaton is a tuple

$$\mathcal{A} = (\Sigma, Q, q_0, Q_f, \Delta),$$

where (Σ, Q, q_0, Δ) is a tree automaton and $\mathcal{F} \subseteq 2^Q$ is the collection of designated state sets.

A run ρ of the Muller tree automaton $\mathcal{A}$ on an infinite binary tree $t \in T_\Sigma^\omega$ is said to be accepting if for each path p of ρ, the set of states appearing infinitely often on this path is in $\mathcal{F}$.

The tree language $L(\mathcal{A})$ accepted by the Muller tree automaton $\mathcal{A}$ is the set of infinite binary trees $t \in T_\Sigma^\omega$ such that there is (at least) one accepting run of $\mathcal{A}$ on t.

The class REG of regular, or recognizable, tree languages is the class of tree languages accepted by some Muller automaton.

Remark 4.4. *Each tree language accepted by some (deterministic) Büchi automaton is also accepted by some (deterministic) Muller automaton. A tree language is accepted by some Muller tree automaton if and only if it is accepted by some Rabin tree automaton. We refer for instance to* [44, 34] *for the definition of a Rabin tree automaton.*

Example 4.5. *Let $L \subseteq \Sigma^\omega$ be a regular ω-language. Then the set $\exists\mathrm{Path}(L) \subseteq T_\Sigma^\omega$ is accepted by some Büchi tree automaton, hence also by some Muller tree automaton.*

The set of infinite binary trees $t \in T_\Sigma^\omega$ having all their paths in L, denoted $\forall\mathrm{Path}(L)$, is accepted by some deterministic Muller tree automaton. It is in fact the complement of the set $\exists\mathrm{Path}(\Sigma^\omega - L)$.

There is a natural topology on the set T_Σ^ω [29, 24, 19]. It is defined by the following distance. Let t and s be two distinct infinite trees in T_Σ^ω. Then the distance between t and

s is $\frac{1}{2^n}$, where n is the smallest integer such that $t(x) \neq s(x)$ for some word $x \in \{l, r\}^\star$ of length n.

Let T_0 be a set of finite labelled trees, and $T_0 \cdot T_\Sigma^\omega$ be the set of infinite binary trees which extend some finite labelled binary tree $t_0 \in T_0$. Here, t_0 is here a sort of prefix, an "initial subtree" of a tree in $t_0 \cdot T_\Sigma^\omega$. The open sets are then of the form $T_0 \cdot T_\Sigma^\omega$.

It is well known that the set T_Σ^ω, equipped with this topology, is homeomorphic to the Cantor set 2^ω, hence also to the topological spaces Σ^ω, where Σ is a finite alphabet having at least two letters.

We are going to use some notation similar to the one used in the case of the space Σ^ω. First, if t is a finite binary tree labelled in Σ, we shall denote by N_t the clopen set $t \cdot T_\Sigma^\omega$. Notice that it is easy to see that one can take, as a restricted basis for the Cantor topology on T_Σ^ω, the clopen sets of the form $t_0 \cdot T_\Sigma^\omega$, where t_0 is a finite labelled binary tree whose domain is of the special form $\{l, r\}^{\leq n}$.

The Borel hierarchy and the projective hierarchy on T_Σ^ω are defined in the same manner as in the case of the topological space Σ^ω.

The ω-language $\mathbb{P}_\infty = (0^\star \cdot 1)^\omega$ is a well known example of $\mathbf{\Pi}_2^0$-complete subset of 2^ω (see Exercise 23.1 in [19]). It is the set of ω-words over 2 having infinitely many occurrences of the letter 1. Its complement $2^\omega - (0^\star \cdot 1)^\omega$ is a $\mathbf{\Sigma}_2^0$-complete subset of 2^ω.

It follows from the definition of the Büchi acceptance condition for infinite trees that each tree language recognized by some (non deterministic) Büchi tree automaton is an analytic set.

Niwiński showed that some Büchi recognized tree languages are actually $\mathbf{\Sigma}_1^1$-complete sets, [32]. An example is any tree language $T \subseteq T_\Sigma^\omega$ of the form $\exists\mathrm{Path}(L)$, where $L \subseteq \Sigma^\omega$ is a regular ω-language which is a $\mathbf{\Pi}_2^0$-complete subset of Σ^ω. In particular, for $\Sigma = 2$, the tree language $\mathcal{L} = \exists\mathrm{Path}(\mathbb{P}_\infty)$ is $\mathbf{\Sigma}_1^1$-complete and hence non Borel [32, 34, 40].

Notice that its complement $\mathcal{L}^- = \forall\mathrm{Path}(2^\omega - (0^\star \cdot 1)^\omega)$ is a $\mathbf{\Pi}_1^1$-complete set. It cannot be accepted by some Büchi tree automaton because it is not a $\mathbf{\Sigma}_1^1$ set. On the other hand, it can be easily seen that it is accepted by some deterministic Muller tree automaton.

We now consider the topology on the space T_Σ^ω generated by the regular languages of trees accepted by some Büchi tree automaton.

We prove a version of Theorem 3.1 as a first step towards the proof that the Büchi topology on a space T_Σ^ω is strong Choquet. We set

$$\mathbb{T}_\infty := \{t \in T_2^\omega \mid \text{ for every path } p \text{ of } t \ \ \forall k \geq 0 \ \ \exists i \geq k \ \ p(i) = 1\}.$$

This set is simply the set of infinite trees over the alphabet 2 having infinitely many letters 1 on every (infinite) path. We will work in the spaces of the form T_Σ^ω, where Σ is a finite alphabet with at least two elements. We consider a topology τ_Σ on T_Σ^ω, and a basis $\mathbb{B}_\Sigma$ for τ_Σ. We consider the following properties of the family $(\tau_\Sigma, \mathbb{B}_\Sigma)_\Sigma$, using the previous identification:

(P1) $\mathbb{B}_\Sigma$ contains the usual basic clopen sets N_t,

(P2) $\mathbb{B}_\Sigma$ is closed under finite unions and intersections,

(P3) $\mathbb{B}_\Sigma$ is closed under projections, in the sense that if Γ is a finite set with at least two elements and $L \in \mathbb{B}_{\Sigma\times\Gamma}$, then $\pi_0[L] \in \mathbb{B}_\Sigma$,

(P4) for each $L \in \mathbb{B}_\Sigma$ there is a closed subset C of $T_\Sigma^\omega \times \mathbb{T}_\infty$ (i.e., C is the intersection of a closed subset of the Cantor space $T_\Sigma^\omega \times T_2^\omega$ with $T_\Sigma^\omega \times \mathbb{T}_\infty$), which is in $\mathbb{B}_{\Sigma\times 2}$, and such that $L = \pi_0[C]$.

Consider now the set of trees $\mathbb{T}_\infty$. It is easy to see that $\mathbb{T}_\infty$ is accepted by some *deterministic* Büchi tree automaton. On the other hand it is well known that the tree languages accepted by some *deterministic* Büchi tree automaton are $\mathbf{\Pi}^0_2$ sets, see [1]. Thus the set $\mathbb{T}_\infty$ is actually a $\mathbf{\Pi}^0_2$ set, it is the intersection of a countable sequence $(O_i)_{i\in\mathbb{N}}$ of open sets. We may assume, without loss of generality, that the sequence $(O_i)_{i\in\mathbb{N}}$ is decreasing with respect to the inclusion relation. Moreover, each open set O_i is a countable union of basic clopen sets $N_{t_{i,j}}$, $j \geq 0$, and we may also assume, without loss of generality, that for all integers $i \geq 0$, and all $j \geq 0$, the finite tree $t_{i,j} \subseteq \{l,r\}^\star$ has a domain of the form $\{l,r\}^{\leq n}$ for some integer n greater than i. We now state the following result, which is a version of Theorem 3.1 in the case of trees.

Theorem 4.6. *Assume that the family $(\tau_\Sigma, \mathbb{B}_\Sigma)_\Sigma$ satisfies Properties (P1)-(P4). Then the topologies τ_Σ are strong Choquet.*

Proof. We first describe a strategy τ for Player 2. Player 1 first plays $t_0 \in T^\omega_\Sigma$ and a τ_Σ-open neighborhood U_0 of t_0. Let L_0 in $\mathbb{B}_\Sigma$ with $t_0 \in L_0 \subseteq U_0$. Property (P4) gives C_0 with $L_0 = \pi_0[C_0]$. This gives $\alpha_0 \in \mathbb{T}_\infty$ such that $(t_0, \alpha_0) \in C_0$. We choose $l^0_0 \in \mathbb{N}$ big enough to ensure that if

$$s^0_0 := \alpha_0|l^0_0,$$

then $N_{s^0_0}$ is included in the open set O_1. We set $w_0 := t_0|1$ and $V_0 := \pi_0[C_0 \cap (N_{w_0} \times N_{s^0_0})]$. By Properties (P1)-(P3), V_0 is in $\mathbb{B}_\Sigma$ and thus τ_Σ-open. Moreover, $t_0 \in V_0 \subseteq L_0 \subseteq U_0$, so that Player 2 respects the rules of the game if he plays V_0.

Now Player 1 plays $t_1 \in V_0$ and a τ_Σ-open neighborhood U_1 of t_1 contained in V_0. Let L_1 in $\mathbb{B}_\Sigma$ with $t_1 \in L_1 \subseteq U_1$. Property (P4) gives C_1 with $L_1 = \pi_0[C_1]$. This gives $\alpha_1 \in \mathbb{T}_\infty$ such that $(t_1, \alpha_1) \in C_1$. We choose $l^1_0 \in \mathbb{N}$ big enough to ensure that if $s^1_0 := \alpha_1|l^1_0$, then $N_{s^1_0}$ is included in the open set O_1. As $t_1 \in V_0$, there is $\alpha'_0 \in \mathbb{T}_\infty$ such that $(t_1, \alpha'_0) \in C_0 \cap (N_{w_0} \times N_{s^0_0})$. We choose $l^0_1 > l^0_0$ big enough to ensure that if $s^0_1 := \alpha'_0|l^0_1$, then s^0_1 is such that $N_{s^0_1}$ is included in the open set O_2. We set $w_1 := t_1|2$ and $V_1 := \pi_0[C_0 \cap (N_{w_1} \times N_{s^0_1})] \cap \pi_0[C_1 \cap (N_{w_0} \times N_{s^1_0})]$. Here again, V_1 is τ_Σ-open. Moreover, $t_1 \in V_1 \subseteq U_1$ and Player 2 can play V_1.

Next, Player 1 plays $t_2 \in V_1$ and a τ_Σ-open neighborhood U_2 of t_2 contained in V_1. Let L_2 in $\mathbb{B}_\Sigma$ with $t_2 \in L_2 \subseteq U_2$. Property (P4) gives C_2 with $L_2 = \pi_0[C_2]$. This gives $\alpha_2 \in \mathbb{T}_\infty$ such that $(t_2, \alpha_2) \in C_2$. We choose $l^2_0 \in \mathbb{N}$ big enough to ensure that if $s^2_0 := \alpha_2|l^2_0$, then the basic open set $N_{s^2_0}$ is included in the open set O_1. As $t_2 \in V_1$, there is $\alpha'_1 \in \mathbb{T}_\infty$ such that $(t_2, \alpha'_1) \in C_1 \cap (N_{w_0} \times N_{s^1_0})$. We choose $l^1_1 > l^1_0$ big enough to ensure that if $s^1_1 := \alpha'_1|l^1_1$, then the basic open set $N_{s^1_1}$ is included in the open set O_2. As $t_2 \in V_1$, there is $\alpha''_0 \in \mathbb{P}_\infty$ such that $(\sigma_2, \alpha''_0) \in C_0 \cap (N_{w_1} \times N_{s^0_1})$. We choose $l^0_2 > l^0_1$ big enough to ensure that if $s^0_2 := \alpha''_0|l^0_2$, then the basic open set $N_{s^0_2}$ is included in the open set O_3. We set $w_2 := t_2|3$ and $V_2 := \pi_0[C_0 \cap (N_{w_2} \times N_{s^0_2})] \cap \pi_0[C_1 \cap (N_{w_1} \times N_{s^1_1})] \cap \pi_0[C_2 \cap (N_{w_0} \times N_{s^2_0})]$. Here again, V_2 is τ_Σ-open. Moreover, $t_2 \in V_2 \subseteq U_2$ and Player 2 can play V_2.

If we go on like this, we build $w_k \in \Sigma^{\{l,r\}^{\leq k+1}}$ and $s^n_l \in 2^{\{l,r\}^\star}$ such that $w_0 \subseteq w_1 \subseteq ...$ and

$$s^n_0 \subsetneq s^n_1 \subsetneq ...$$

This allows us to define $\sigma := \lim_{l\to\infty} w_l \in T^\omega_\Sigma$ and, for each $n \in \mathbb{N}$, $\beta_n := \lim_{l\to\infty} s^n_l \in T^\omega_2$. Note that $\beta_n \in \mathbb{T}_\infty$ since the basic open set $N_{s^n_l}$ is included in the open set O_{l+1}. (σ, β_n) is the limit of (w_l, s^n_l) as l goes to infinity and $N_{w_l} \times N_{s^n_l}$ meets C_n (which is closed in

$T_\Sigma^\omega \times \mathbb{T}_\infty$), $(\sigma, \beta_n) \in C_n$. Thus

$$\sigma \in \bigcap_{n\in\mathbb{N}} \pi_0[C_n] = \bigcap_{n\in\mathbb{N}} L_n \subseteq \bigcap_{n\in\mathbb{N}} U_n \subseteq \bigcap_{n\in\mathbb{N}} V_n,$$

so that τ is winning for Player 2. □

We now check that the Büchi topology on a space T_Σ^ω satisfies Properties (P1)-(P4).

(P1) It is very easy to see that for each finite tree t labelled in Σ, there exists a Büchi tree automaton accepting the usual basic clopen set N_t.

(P2) $\mathbb{B}_\Sigma$ is closed under finite unions, because any basic open set in the Büchi topology is accepted by some *non-deterministic* Büchi tree automaton. Moreover one can easily show, using a classical product construction, that the class of tree languages accepted by some Büchi tree automaton is closed under finite intersections. Thus $\mathbb{B}_\Sigma$ is closed under finite intersections.

(P3) It follows easily, from the fact that any basic open set in the Büchi topology is accepted by some *non-deterministic* Büchi tree automaton, that $\mathbb{B}_\Sigma$ is closed under projections.

(P4) This property follows from the following lemma, which is very similar to Lemma 3.3 above.

Lemma 4.7. *Let Σ be a finite set with at least two elements, and $L \subseteq T_\Sigma^\omega$ be a regular tree language accepted by some Büchi tree automaton. Then there is a closed subset C of $T_\Sigma^\omega \times \mathbb{T}_\infty$, which is accepted by some Büchi tree automaton as a subset of $T_{(\Sigma\times 2)}^\omega$ identified with $T_\Sigma^\omega \times T_2^\omega$, and such that $L = \pi_0[C]$.*

Proof. Let $\mathcal{A} = (\Sigma, Q, q_0, Q_f, \Delta)$ be a Büchi tree automaton, and $L = L(\mathcal{A})$ be its set of accepted trees. We call χ_f the characteristic function of Q_f. It maps the state q to 1 if $q \in Q_f$, and to 0 otherwise. The function χ_f is extended to T_Q^ω by setting $t' = \chi_f(t)$ and $t'(s) = \chi_f\big(t(s)\big)$. Note that a run ρ of $\mathcal{A}$ is accepting if and only if $\chi_f(\rho) \in \mathbb{T}_\infty$. Let C be the subset of $T_\Sigma^\omega \times \mathbb{T}_\infty$ defined by

$$C := \big\{(t, t') \in T_\Sigma^\omega \times \mathbb{T}_\infty \mid \exists \rho \text{ run of } \mathcal{A} \text{ on } t \text{ such that } t' = \chi_F(\rho)\big\}.$$

By definition of C, $L = \pi_0[C]$. Let K be the subset of $T_\Sigma^\omega \times T_2^\omega \times T_Q^\omega$ defined by

$$K := \big\{(t, t', \rho) \in T_\Sigma^\omega \times T_2^\omega \times T_Q^\omega \mid \rho \text{ is a run of } \mathcal{A} \text{ on } t \text{ such that } t' = \chi_F(\rho)\big\}.$$

As K is compact as a closed subset of the compact space $T_\Sigma^\omega \times T_2^\omega \times T_Q^\omega$ and

$$C = \pi_{T_\Sigma^\omega \times T_2^\omega}[K] \cap (T_\Sigma^\omega \times \mathbb{T}_\infty),$$

the subset C is a closed subset of $T_\Sigma^\omega \times \mathbb{T}_\infty$. Moreover, it is easy to construct a Büchi tree automaton accepting the tree language C. □

Corollary 4.8. *Let Σ be a finite set with at least two elements. Then the Büchi topology on T_Σ^ω is strong Choquet.*

Proof. This follows from the fact that the Büchi topology on T_Σ^ω satisfies Properties (P1)-(P4), and from Theorem 4.6. □

On the other hand, as in the case of the Büchi topology on Σ^ω, the Büchi topology on T_Σ^ω is second countable since there are only countably many possible Büchi tree automata (up to identifications), and it is T_1 since it is finer than the usual Cantor topology by Property (P1). However, the Büchi topology on T_Σ^ω is not Polish, by the following result.

Theorem 4.9. *Let Σ be a finite set with at least two elements. Then the Büchi topology on T_Σ^ω is not metrizable and thus not Polish.*

Proof. Recall that in a metrizable topological space, every closed set is a countable intersection of open sets. We now show that the Büchi topology on T_Σ^ω does not satisfy this property. We have already recalled that the set $\mathcal{L} = \exists\mathrm{Path}(\mathbb{P}_\infty)$ is $\mathbf{\Sigma}^1_1$-complete for the usual topology, and it is open for the Büchi topology since it is accepted by some Büchi tree automaton. Its complement $\mathcal{L}^-$ is the set of trees in T_Σ^ω having all their paths in $2^\omega \setminus (0^\star \cdot 1)^\omega$; it is $\mathbf{\Pi}^1_1$-complete for the usual topology and closed for the Büchi topology.

On the other hand every tree language accepted by some Büchi tree automaton is an effective analytic set, i.e., a Σ^1_1 set, and thus also a (boldface) $\mathbf{\Sigma}^1_1$ set (for the usual Cantor topology). Moreover, every open set for the Büchi topology on T_Σ^ω is a countable union of basic open sets, and thus a countable union of $\mathbf{\Sigma}^1_1$ sets. But the class $\mathbf{\Sigma}^1_1$ is closed under countable unions (see [19]). Therefore, every open set for the Büchi topology is a $\mathbf{\Sigma}^1_1$ set for the usual topology.

Towards a contradiction, assume now that the set $\mathcal{L}^-$ is a countable intersection of open sets for the Büchi topology. Then it is a countable intersection of $\mathbf{\Sigma}^1_1$ sets for the usual topology. But the class $\mathbf{\Sigma}^1_1$ is closed under countable intersections and thus $\mathcal{L}^-$ would be also a $\mathbf{\Sigma}^1_1$ set for the usual topology. But $\mathcal{L}^-$ is $\mathbf{\Pi}^1_1$-complete and thus in $\mathbf{\Pi}^1_1 \setminus \mathbf{\Sigma}^1_1$ (see [19]), which is absurd. □

Remark 4.10. *One can infer, from the previous results on the Büchi topology on the space T_Σ^ω and from Theorem 2.8, that the Büchi topology on the space T_Σ^ω is not regular.*

Remark 4.11. *The automatic topology on the space T_Σ^ω, which can be defined as in the case of the space Σ^ω, is Polish, and the proof of this fact is very similar to the one in the case of the space Σ^ω (see Lemma 3.5).*

We now consider the topology on the space T_Σ^ω generated by the class of all regular languages of trees accepted by some Muller tree automaton. We shall call this topology the Muller topology on T_Σ^ω. This topology is clearly T_1 since it is finer than the usual topology on T_Σ^ω. It is second countable since there are only countably many Muller automata. It is zero-dimensional, and thus also regular, because the class of regular tree languages over the alphabet Σ is closed under taking complements. We now recall the following Urysohn metrization theorem (see [19]).

Theorem 4.12. *Let X be a second-countable topological space. Then X is metrizable if and only if its topology is T_1 and regular.*

This implies that the Muller topology is metrizable. Notice that one can define a distance compatible with this topology in a way very similar to the way we defined a distance compatible with the Büchi topology on Σ^ω. On the other hand we recall the following Becker theorem (see Theorem 4.2.6 in [14]).

Theorem 4.13. *Let τ be a Polish topology on X and τ' be a second-countable strong Choquet topology on X finer than τ. Then every τ'-open set is $\mathbf{\Sigma}^1_1$ in τ.*

This implies that the Muller topology is not strong Choquet because there exist some $\mathbf{\Pi}^1_1$-complete (for the usual topology), and hence non-$\mathbf{\Sigma}^1_1$ (for the usual topology), regular set of trees. Such a regular set of trees is open for the Muller topology, which is finer than the usual topology, but is not $\mathbf{\Sigma}^1_1$ for the usual topology. We now summarize the results in this section.

Theorem 4.14. *Let Σ be a finite alphabet having at least two letters.*

(1) *The Büchi topology on the space T_Σ^ω is strong Choquet, but it is not regular (and hence not zero-dimensional) and not metrizable.*
(2) *The Muller topology on the space T_Σ^ω is zero-dimensional, regular and metrizable, but it is not strong Choquet.*

In particular, the Büchi topology and the Muller topology on T_Σ^ω are not Polish.

If we are just interested in the non-polishness of these topologies, we can argue in a more direct way as follows. We first prove the following proposition.

Proposition 4.15. *Let (X, τ) be a Polish topological space, and let τ' be another Polish topology on X finer than τ. Then the two topologies τ and τ' have the same Borel sets.*

Proof. Let $Id : (X, \sigma) \to (X, \tau)$ be the identity function on X, where the domain X is equipped with the topology σ and the range X is equipped with the topology τ. This function is continuous since the topology σ is finer than the topology τ. Notice that this implies, by an easy induction on the rank of a Borel set, that the preimage of any Borel set of (X, τ) is a Borel set of (X, σ), i.e. that every Borel set of (X, τ) is a Borel set of (X, σ). On the other hand, it follows from Lusin-Suslin's Theorem, see [19, Theorem 15.1], that the (injective) image by the function Id of any Borel set of (X, σ) is a Borel set of (X, τ), i.e. that every Borel set of (X, σ) is a Borel set of (X, τ). □

Corollary 4.16. *Let Σ be a finite set with at least two elements. Then the Büchi topology and the Muller topology on T_Σ^ω are not Polish.*

Proof. The Büchi topology and the Muller topology on T_Σ^ω are finer than the usual Cantor topology on T_Σ^ω. On the other hand both the Büchi topology and the Muller topology on T_Σ^ω contain some open sets which are $\mathbf{\Sigma}_1^1$-complete and hence non Borel (for the usual topology), like, in the case $\Sigma = 2$, the set $\mathcal{L} = \exists\mathrm{Path}(\mathbb{P}_\infty)$ of infinite trees in T_Σ^ω having at least one path in the ω-language $\mathbb{P}_\infty = (0^\star \cdot 1)^\omega$. The conclusion now follows from Proposition 4.15.□

5. Consequences for our topologies

5.1. Consequences not directly related to the polishness, concerning isolated points.

5.1.1. *The case of a space of infinite words.*

Notation. If $z \in \{C, B, A, \delta\}$, then the space $(\Sigma^\mathbb{N}, \tau_z)$ is denoted $\mathcal{S}_z$. The set of ultimately periodic ω-words on Σ is denoted $\mathrm{Ult} := \{u \cdot v^\omega \mid u, v \in \Sigma^* \setminus \{\emptyset\}\}$, and $P := \Sigma^\mathbb{N} \setminus \mathrm{Ult}$.

(1) As noted in [35], Ult is the set of *isolated points* of $\mathcal{S}_B$ and $\mathcal{S}_A$ (recall that a point $\sigma \in \Sigma^\mathbb{N}$ is isolated if $\{\sigma\}$ is an open set). Indeed, each singleton $\{u \cdot v^\omega\}$ formed by an ultimately periodic ω-word is an ω-regular language, and thus each ultimately periodic ω-word is a $\mathcal{S}_A$-isolated point. Conversely, if $\{\sigma\}$ is τ_B-open, then it is ω-regular and then the ω-word σ is ultimately periodic (because any countable ω-regular language contains only ultimately periodic ω-words, see [2, 34, 42]).

(2) Every nonempty ω-regular set contains an ultimately periodic ω-word, [2, 34, 42]. In particular, the set Ult of isolated points of $\mathcal{S}_B$ and $\mathcal{S}_A$ is *dense*, and a subset of $\mathcal{S}_B$ or $\mathcal{S}_A$ is dense if and only if it contains Ult.

5.1.2. *The case of a space of infinite trees.*

Recall that a tree $t \in T_\Sigma^\omega$ is regular if and only if for each $a \in \Sigma$ the set $\{u \in \{l,r\}^\star \mid t(u) = a\}$ is a regular set of finite words over the alphabet $\{l,r\}$. For each regular tree $t \in T_\Sigma^\omega$, the singleton $\{t\}$ is a (closed) regular tree language accepted by some Büchi tree automaton. Moreover, a regular tree language accepted by some Muller or Rabin tree automaton is non-empty if and only if it contains a regular tree, see [44].

Therefore we can state properties similar to those stated in the case of a space of words in Section 5.1.1. For the automatic, the Büchi or the Muller topologies on a space T_Σ^ω, the set of isolated points is the set *Reg-trees* of regular trees, and this set is dense. It is left here to the reader to see how Properties (1)-(2) of the preceding section are extended to the case of a space of trees (in a very similar way).

5.2. **Consequences of the polishness.**

Here we concentrate on the case of a space of infinite words. We consider our topologies on $\Sigma^\mathbb{N}$, where Σ is a finite set with at least two elements. We refer to [19] when classical descriptive set theory is involved.

(a) The union $P \cup \mathrm{Ult}$ is the *Cantor-Bendixson decomposition* of $\mathcal{S}_B$ and $\mathcal{S}_A$ (see Theorem 6.4 in [19]). This means that P is perfect (i.e., closed without isolated points) and Ult is countable open. Let us check that P is perfect. We argue by contradiction, so that we can find $\sigma \in P$ and an ω-regular language L such that $\{\sigma\} = L \setminus \mathrm{Ult}$. Note that $L \subseteq \{\sigma\} \cup \mathrm{Ult}$ is countable. But a countable regular ω-language contains only ultimately periodic words (see [34]), and thus $L \subseteq \mathrm{Ult}$, which is absurd.

(b) The closed subspace (P, τ_B) of $\mathcal{S}_B$ is *homeomorphic to the Baire space* $\mathbb{N}^\mathbb{N}$. Indeed, it is not empty since Ult is countable and $\Sigma^\mathbb{N}$ is not, zero-dimensional and Polish as a closed subspace of the zero-dimensional Polish space $\mathcal{S}_B$. By Theorem 7.7 in [19], it is enough to prove that every compact subset of (P, τ_B) has empty interior. We argue by contradiction, which gives a compact set K. Note that there is an ω-regular language L such that $P \cap L$ is a nonempty compact subset of K, so that we may assume that $K = P \cap L$. Theorem 2.1 gives $(U_i)_{i<n}$ and $(V_i)_{i<n}$ with $L = \bigcup_{i<n} U_i \cdot V_i^\omega$. On the other hand, L is not countable since every countable regular ω-language contains only ultimately periodic words and $K = P \cap L$ is non-empty. Thus $n > 0$ and, for example, $U_0 \cdot V_0^\omega$ is not countable.

This implies that we can find $v_0, v_1 \in V_0$ which are not powers of the same word (which means that we cannot find a finite word u and integers m, n with $v_0 = u^m$ and $v_1 = u^n$). Indeed, we argue by contradiction. Let $v_0 \in V_0 \setminus \{\emptyset\}$, and $v \in \Sigma^*$ of minimal length such that v_0 is a power of v. We can find $w_0, w_1, ... \in V_0 \setminus \{\emptyset\}$ such that $\sigma := w_0 \cdot w_1 ... \neq v^\omega$. Fix a natural number i. Then w_i and v_0 are powers of the same word w. By Corollary 6.2.5 in [25], v and w are powers of the same word u, and v_0 too. By minimality, $u = v$, and w_i is a power of v. Thus $\sigma = v^\omega$, which is absurd.

Let $u_0 \in U_0$, and $L' := \{u_0\} \cdot \{v_0, v_1\}^\omega$. Note that $P \cap L'$ is a τ_B-closed subset of K, so that it is τ_B-compact. As the identity map from $(P \cap L', \tau_B)$ onto $(P \cap L', \tau_C)$ is continuous, $P \cap L'$ is also τ_C-compact. But the map $\alpha \mapsto u_0 \cdot v_{\alpha(0)} \cdot v_{\alpha(1)} \dots$ is a homeomorphism from the Cantor space onto L', by Corollaries 6.2.5 and 6.2.6 in [25]. Thus $P \cap L' = L' \setminus \mathrm{Ult}$ is a dense closed subset of L'. Thus $P \cap L' = L'$, which is absurd since $u_0 \cdot v_0^\omega \in L' \setminus P$.

6. Concluding remarks

We obtained in this paper new links and interactions between descriptive set theory and theoretical computer science, showing that two topologies considered in [35] are Polish.

Notice that this paper is also motivated by the fact that the Gandy-Harrington topology, generated by the effective analytic subsets of a recursively presented Polish space, is an extremely powerful tool in descriptive set theory. In particular, this topology is used to prove some results of classical type (without reference to effective descriptive set theory in their statement). Among these results, let us mention the dichotomy theorems in [16, 20, 22, 23]. Sometimes, no other proof is known. Part of the power of this technique comes from the nice closure properties of the class Σ^1_1 of effective analytic sets (in particular the closure under projections).

The class of ω-regular languages has even stronger closure properties. So our hope is that the study of the Büchi topology, generated by the ω-regular languages, will help to prove some automatic versions of known descriptive results in the context of theoretical computer science. For instance, more precisely, let Σ, Γ be finite sets with at least two elements, and L be a subset of $\Sigma^{\mathbb{N}} \times \Gamma^{\mathbb{N}}$ which is ω-regular and also a countable union of Borel rectangles. It would be very interesting to know whether L is open for the product topology $\tau_B \times \tau_B$. Indeed, this would give a version of the $\mathcal{G}_0$-dichotomy for ω-regular languages, and thus a very serious hope to get versions of many difficult dichotomy results of descriptive set theory for ω-regular languages (see [27]).

From Theorem 1.1, we know that there is a complete distance which is compatible with τ_z. It would be interesting to have a natural complete distance compatible with τ_B. We leave this as an open question for further study.

Acknowledgements. We are very grateful to the anonymous referees for their careful reading and their useful comments on a preliminary version of our paper that led to a great improvement of the presentation of the paper. The second author thanks very much Henryk Michalewski who suggested to study the analogue of the Büchi topology in the case of a space of infinite labelled trees.

References

[1] A. Arnold, J. Duparc, F. Murlak, and D. Niwiński. On the topological complexity of tree languages. In J. Flum, E. Grädel, and T. Wilke, editors, *Logic and Automata: History and Perspectives*, pages 9–28. Amsterdam University Press, 2007.

[2] J. R. Büchi. On a decision method in restricted second order arithmetic. In Stanford University Press, editor, *Proceedings of the 1960 International Congress on Logic Methodology and Philosophy of Science*, pages 1–11. Stanford University Press, 1962.

[3] O. Carton, O. Finkel, and D. Lecomte. Polishness of Some Topologies Related to Automata. In V. Goranko and M. Dam, editors, *26th EACSL Annual Conference on Computer Science Logic (CSL 2017)*, volume 82 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 22:1–22:16, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[4] F. Cavallari, H. Michalewski, and M. Skrzypczak. A characterisation of Π^0_2 regular tree languages. In K. G. Larsen, H. L. Bodlaender, and J.-F. Raskin, editors, *42nd International Symposium on Mathematical Foundations of Computer Science, MFCS 2017*, volume 83 of *LIPIcs*, pages 56:1–56:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017.

[5] C. Choffrut and S. Grigorieff. Uniformization of rational relations. In Juhani Karhumäki, Hermann A. Maurer, G. Paun, and G. Rozenberg, editors, *Jewels are Forever, Contributions on Theoretical Computer Science in Honor of Arto Salomaa*, pages 59–71. Springer, 1999.

[6] R. S. Cohen and A. Y. Gold. ω-computations on Turing machines. *Theoretical Computer Science*, 6:1–23, 1978.

[7] J. Duparc, O. Finkel, and J.-P. Ressayre. The Wadge hierarchy of Petri nets ω-languages. In V. Brattka, H. Diener, and D. Spreen, editors, *Logic, Computation, Hierarchies*, volume 4 of *Ontos Mathematical Logic, collection of papers published in Honor of Victor Selivanov at the occasion of his sixtieth birthday*, pages 109–138. Ontos-Verlag, 2014.

[8] O. Finkel. Borel ranks and Wadge degrees of omega context free languages. *Mathematical Structures in Computer Science*, 16(5):813–840, 2006.

[9] O. Finkel. The complexity of infinite computations in models of set theory. *Logical Methods in Computer Science*, 5(4:4):1–19, 2009.

[10] O. Finkel. Ambiguity of ω-languages of Turing machines. *Logical Methods in Computer Science*, 10(3:12):1–18, 2014.

[11] O. Finkel and D. Lecomte. Classical and effective descriptive complexities of ω-powers. *Annals of Pure and Applied Logic*, 160(2):163–191, 2009.

[12] O. Finkel and D. Lecomte. Decision problems for Turing machines. *Information Processing Letters*, 109:1223–1226, 2009.

[13] O. Finkel and P. Simonnet. Topology and ambiguity in omega context free languages. *Bulletin of the Belgian Mathematical Society*, 10(5):707–722, 2003.

[14] S. Gao. *Invariant descriptive set theory*, volume 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009.

[15] E. Grädel, W. Thomas, and W. Wilke, editors. *Automata, Logics, and Infinite Games: A Guide to Current Research [outcome of a Dagstuhl seminar, February 2001]*, volume 2500 of *Lecture Notes in Computer Science*. Springer, 2002.

[16] L. A. Harrington, A. S. Kechris, and A. Louveau. A Glimm-Effros dichotomy for Borel equivalence relations. *Journal of the American Mathematical Society*, 3:903–928, 1990.

[17] S. Hoffmann, S. Schwarz, and L. Staiger. Shift-invariant topologies for the Cantor space X^ω. *Theoretical Computer Science*, 679:145161, 2017.

[18] S. Hoffmann and L. Staiger. Subword metrics for infinite words. In F. Drewes, editor, *Implementation and Application of Automata - 20th International Conference, CIAA 2015, Umeå, Sweden, August 18-21, 2015, Proceedings*, volume 9223 of *Lecture Notes in Computer Science*, pages 165–175. Springer, 2015.

[19] A. S. Kechris. *Classical descriptive set theory*. Springer-Verlag, New York, 1995.

[20] A. S. Kechris, S. Solecki, and S. Todorčević. Borel chromatic numbers. *Advances in Mathematics*, 141(1):1–44, 1999.

[21] L. H. Landweber. Decision problems for ω-automata. *Mathematical Systems Theory*, 3(4):376–384, 1969.

[22] D. Lecomte. A dichotomy characterizing analytic digraphs of uncountable Borel chromatic number in any dimension. *Transactions of the American Mathematical Society*, 361(8):4181–4193, 2009.

[23] D. Lecomte. Potential Wadge classes. *Memoirs of the American Mathematical Society*, 221(1038):vi+83, 2013.

[24] H. Lescow and W. Thomas. Logical specifications of infinite computations. In J. W. de Bakker, W. P. de Roever, and G. Rozenberg, editors, *A Decade of Concurrency*, volume 803 of *Lecture Notes in Computer Science*, pages 583–621. Springer, 1994.

[25] M. Lothaire. *Algebraic combinatorics on words*, volume 90 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002.

[26] A. Louveau. *Ensembles analytiques et boréliens dans les espaces produit*, volume 78. Astérisque (SMF), 1980.

[27] B. D. Miller. The graph-theoretic approach to descriptive set theory. *Bulletin of Symbolic Logic*, 18(4):554–575, 2012.

[28] H. Michalewski, M. Mio, and M. Skrzypczak. Monadic second order logic with measure and category quantifiers. *Logical Methods in Computer Science*, 14(2):Paper No. 2, 29, 2018.

[29] Y. N. Moschovakis. *Descriptive set theory*. North-Holland Publishing Co., Amsterdam, 1980.

[30] Y. N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2009.

[31] F. Murlak. The Wadge hierarchy of deterministic tree languages. *Logical Methods in Computer Science*, 4(4), 2008.

[32] D. Niwiński. An example of non Borel set of infinite trees recognizable by a Rabin automaton. 1985. in Polish, manuscript.

[33] D. Niwiński and I. Walukiewicz. A gap property of deterministic tree languages. *Theoretical Computer Science*, 303(1):215–231, 2003. Logic and complexity in computer science (Créteil, 2001).

[34] D. Perrin and J.-É. Pin. *Infinite words, automata, semigroups, logic and games*, volume 141 of *Pure and Applied Mathematics*. Elsevier, 2004.

[35] S. Schwarz and L. Staiger. Topologies refining the Cantor topology on X^ω. In C. S. Calude and V. Sassone, editors, *Theoretical Computer Science - 6th IFIP TC 1/WG 2.2 International Conference, TCS 2010, Held as Part of WCC 2010, Brisbane, Australia, September 20-23, 2010. Proceedings*, volume 323 of *IFIP Advances in Information and Communication Technology*, pages 271–285. Springer, 2010.

[36] V. L. Selivanov. Wadge degrees of ω-languages of deterministic Turing machines. *RAIRO-Theoretical Informatics and Applications*, 37(1):67–83, 2003.

[37] V. L. Selivanov. Wadge degrees of ω-languages of deterministic Turing machines. In *Proceedings of the International Conference STACS 2003, 20th Annual Symposium on Theoretical Aspects of Computer Science, Berlin, Germany*, volume 2607 of *Lecture Notes in Computer Science*, pages 97–108. Springer, 2003.

[38] V. L. Selivanov. Wadge reducibility and infinite computations. *Mathematics in Computer Science*, 2(1):5–36, 2008.

[39] O. Serre. Games with winning conditions of high Borel complexity. *Theoretical Computer Science*, 350(2-3):345–372, 2006.

[40] P. Simonnet. *Automates et théorie descriptive*. PhD thesis, Université Paris VII, 1992.

[41] M. Skrzypczak. *Descriptive Set Theoretic Methods in Automata Theory - Decidability and Topological Complexity*, volume 9802 of *Lecture Notes in Computer Science*. Springer, 2016.

[42] L. Staiger. ω-languages. In *Handbook of formal languages, Vol. 3*, pages 339–387. Springer, Berlin, 1997.

[43] L. Staiger. On the power of reading the whole infinite input tape. *Grammars*, 2 (3):247–257, 1999.

[44] W. Thomas. Automata on infinite objects. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, Formal models and semantics, pages 135–191. Elsevier, 1990.

[45] K. Wagner. On ω-regular sets. *Information and Control*, 43(2):123–177, 1979.

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