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Abstract

For a simple Lie algebra $\mathfrak{g}$ and an irreducible faithful representation π of $\mathfrak{g}$, we introduce the Schur polynomials of $(\mathfrak{g}, \pi)$-type. We then derive the Sato–Zhou type formula for tau functions of the Drinfeld–Sokolov (DS) hierarchy of $\mathfrak{g}$-type. Namely, we show that the tau functions are linear combinations of the Schur polynomials of $(\mathfrak{g}, \pi)$-type with the coefficients being the Plücker coordinates. As an application, we provide a way of computing polynomial tau functions for the DS hierarchy. For $\mathfrak{g}$ of low rank, we give several examples of polynomial tau functions, and use them to detect bilinear equations for the DS hierarchy.

1 Introduction

Given a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, Drinfeld and Sokolov in [14] explained how to associate to it a family of commuting bi–hamiltonian PDEs known as the Drinfeld–Sokolov (DS) hierarchy of $\mathfrak{g}$–type. Nowadays, Drinfeld–Sokolov hierarchies are certainly among the most studied examples of integrable systems; one of their remarkable properties is that they are tau–symmetric [19, 18, 36, 7], meaning that they admit the so-called tau function of an arbitrary solution to the hierarchy. For the case $\mathfrak{g} = \mathrm{sl}_n(\mathbb{C})$ the DS hierarchy of $\mathfrak{g}$-type coincides (under a particular choice of the DS gauge [14, 2]) with the Gelfand–Dickey hierarchy, and so, in particular, for $n = 2$, with the celebrated Korteweg–de Vries (KdV) hierarchy. It is known that tau functions of the Gelfand–Dickey hierarchies can be expressed as linear combinations of Schur polynomials with the coefficients being Plücker coordinates[1] [32, 38, 3, 30]. In this short paper we aim to generalize this fact to an arbitrary given Lie algebra $\mathfrak{g}$. The generalization will depend on matrix realizations of $\mathfrak{g}$ (note that the tau function itself is independent of the realizations of $\mathfrak{g}$ [6]!). Indeed, one of our main observations is that the generalization of Schur polynomials are associated to faithful representations.

[1] Indeed, more generally this is true for the KP hierarchy, of which the Gelfand–Dickey hierarchies are reductions.

As an application of our result, we describe a systematic way of finding simple solutions (i.e. solutions whose tau function is a polynomial or a fractional power of it) of the DS hierarchy of $\mathfrak{g}$-type. Of course, in the case of the hierarchies of type A_n, we recover the well-known results, since polynomial tau functions of these hierarchies (more generally of the KP hierarchy) had been studied for many years, due to their relations with Bäcklund transformations [1] and the dynamical systems of Calogero type (see for instance [35] and the references therein). Moreover, it had been proved that the polynomial tau functions of the so–called BKP hierarchy can be written in terms of the projective representations of the symmetric group [37] and this hierarchy, moreover, contains as reductions some of the DS hierarchies of D_n-type, as explained in [12]. Nevertheless, it seems to us that a systematic approach to the study of polynomial tau functions associated to the general case (i.e. for an arbitrary Lie algebra) is still missing, and this paper gives a first result in this direction. The polynomial tau functions we obtain are, actually, quite non–trivial, and can also be used to give some explicit information about the structure of the bilinear equations for the hierarchy.

In order to state precisely our results, we need to fix some notations about finite dimensional Lie algebras, loop algebras and Toeplitz determinants. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank n, and $h, h^\vee$ the Coxeter and dual Coxeter numbers, respectively. Fix $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Take $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ a set of simple roots, and let $\triangle \subset \mathfrak{h}^*$ be the root system. We know that $\mathfrak{g}$ has the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \triangle} \mathfrak{g}_\alpha.$$

Let θ denote the highest root with respect to Π, and $(\cdot|\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the *normalized* Cartan–Killing form, i.e. $(\theta|\theta) = 2$. For a root $\alpha \in \triangle$, denote by H_α the unique vector in $\mathfrak{h}$ satisfying $(H_\alpha|H_\beta) = (\alpha|\beta)$, $\forall\, \beta \in \triangle$.

Let $E_i \in \mathfrak{g}_{\alpha_i}$, $F_i \in \mathfrak{g}_{-\alpha_i}$, $H_i = 2H_{\alpha_i}/(\alpha_i|\alpha_i)$ be a set of Weyl generators of $\mathfrak{g}$. They satisfy

$$[E_i, F_i] = H_i, \quad [H_i, E_j] = A_{ij}\, E_j, \quad [H_i, F_j] = -\, A_{ij}\, F_j, \qquad 1 \le i, j \le n,$$

where $(A_{ij})_{i,j=1}^n$ is the Cartan matrix of $\mathfrak{g}$. Choose $E_{-\theta} \in \mathfrak{g}_{-\theta}$, $E_\theta \in \mathfrak{g}_\theta$, normalized by the conditions $(E_\theta \,|\, E_{-\theta}) = 1$ and $\omega(E_{-\theta}) = -E_\theta$, where $\omega : \mathfrak{g} \to \mathfrak{g}$ is the Chevalley involution. Let $I_+ := \sum_{i=1}^n E_i$ be a principal nilpotent element of $\mathfrak{g}$. Denote by $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ the loop algebra of $\mathfrak{g}$. On $L(\mathfrak{g})$ there is the *principal gradation* defined by assigning

$$\deg E_i = 1, \;\; \deg H_i = 0, \;\; \deg F_i = -1, \;\; i = 1, \dots, n, \quad \deg \lambda = h$$

such that $L(\mathfrak{g})$ decomposes into homogeneous subspaces

$$L(\mathfrak{g}) = \bigoplus_{j \in \mathbb{Z}} L(\mathfrak{g})^j.$$

Here, elements in $L(\mathfrak{g})^j$ have degree j. Define $\Lambda \in L(\mathfrak{g})$ by

$$\Lambda = I_+ + \lambda\, E_{-\theta}. \tag{1}$$

Clearly, Λ is homogeneous of degree 1. Denote by $L(\mathfrak{g})^{<0}$ elements in $L(\mathfrak{g})$ with negative degrees, similarly, by $L(\mathfrak{g})^{\le 0}$ elements with non-positive degrees.

It was shown in [26, 29] that $\operatorname{Ker}\operatorname{ad}_\Lambda \subset L(\mathfrak{g})$ has the following decomposition

$$\operatorname{Ker}\operatorname{ad}_\Lambda = \bigoplus_{\ell\in E} \mathbb{C}\Lambda_\ell, \qquad \deg\Lambda_\ell = \ell \in E := \bigsqcup_{i=1}^{n} (m_i + h\mathbb{Z})$$

where the integers $m_1, \dots, m_n$ are the exponents of $\mathfrak{g}$, and E is called the set of exponents of $L(\mathfrak{g})$. We use E_+ to denote the set of positive exponents. The elements Λ_i commute pairwise

$$[\Lambda_i, \Lambda_j] = 0, \qquad \forall\, i, j \in E. \tag{2}$$

They can be normalized by

$$\begin{aligned} &\Lambda_{m_a+kh} = \Lambda_{m_a}\,\lambda^k, \qquad k \in \mathbb{Z}, \\ &(\Lambda_{m_a} |\, \Lambda_{m_b}) = h\,\lambda\,\delta_{a+b,n+1}. \end{aligned}$$

In particular, we can choose $\Lambda_1 = \Lambda$.

Let us now take

$$\pi :\, \mathfrak{g} \to \mathrm{gl}(m, \mathbb{C}) \tag{3}$$

an irreducible faithful representation. When no confusion can arise, for $b \in \mathfrak{g}$, we write $\pi(b)$ simply as b. Our generalization will be based on the infinite Grassmannian approach [32, 33] and the related Plucker coordinates.

<u>*Notations*</u>*:*
a) For $M = \sum_{k\in\mathbb{Z}} M_k\,\lambda^k$ with $M_k \in \mathrm{gl}(m, \mathbb{C})$, define the *Laurent matrix* $\mathbb{L}(M)$ associated with M by

$$[\mathbb{L}(M)]_{IJ} = M_{I-J}, \qquad I, J \in \mathbb{Z}. \tag{4}$$

Here, capital-letter indices $I, J, K, \dots$ are used for block row/column coordinates, and small-letter indices are for ordinary row/column coordinates.
b) $\mathbb{Y}$ will denote the set of all partitions; for $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) \in \mathbb{Y}$, $\ell(\lambda)$ denotes the length of λ, $|\lambda|$ the weight of λ; denote by $\lambda = (\,k_1, \dots, k_d \,|\, l_1, \dots, l_d\,)$ be the Frobenius notation of λ with d being the Frobenius rank.

Definition 1.1. *Let $\xi := \sum_{\ell\in\mathbb{E}_+} t_\ell\,\Lambda_\ell$ with t_ℓ, $\ell \in E_+$ being indeterminates and let s denote the Laurent matrix associated with e^ξ, namely,*

$$s := \mathbb{L}(e^\xi). \tag{5}$$

The Schur polynomials of $(\mathfrak{g}, \pi)$-type are labelled by partitions and defined by

$$\begin{aligned} &s_\lambda := \det\left(s_{i-1,\,j-\lambda_j-1}\right)_{i,j=1}^{\ell(\lambda)}, \qquad \lambda \in \mathbb{Y} - \emptyset, \\ &s_\emptyset := 1. \end{aligned} \tag{6}$$

Definition 1.2. *In the case π is taken as the adjoint representation of $\mathfrak{g}$, we call s_λ, $\lambda \in \mathbb{Y}$ the intrinsic Schur polynomials of $\mathfrak{g}$-type.*

Remark 1.3. *In the case $\mathfrak{g} = A_n$. Take $\pi(\mathfrak{g})$ the well-known matrix realization of $\mathfrak{g}$, i.e. $\pi(\mathfrak{g}) = \mathrm{sl}_{n+1}(\mathbb{C})$. We have $\Lambda = \sum_{i=1}^n E_{i,i-1} + \lambda E_{1,n+1}$. The Schur polynomials of $(\mathfrak{g}, \pi)$-type then coincide with the Schur polynomials [30] under the restriction $t_{(n+1)k} \equiv 0$, $k = 1, 2, 3, \dots$.*

Definition 1.4. *$\forall\, X \in \lambda^{-1}\mathfrak{g}[[\lambda^{-1}]]$, denote by r_X the Laurent matrix associated with e^X, i.e.*

$$r_X := \mathbb{L}(e^X). \tag{7}$$

For $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \in \mathbb{Y}$, define

$$r_{X,\lambda} := \det\left(r_{X,i-\lambda_i-1,j-1}\right)_{i,j=1}^{\ell(\lambda)}.$$

Definition 1.5. *For $\xi = \sum_{\ell\in\mathbb{E}_+} t_\ell\,\Lambda_\ell$ (as above), and for any $X \in \lambda^{-1}\mathfrak{g}[[\lambda^{-1}]]$, define matrices D_{IJ} and $Z_{X,IJ}$ $(I, J \geq 0)$ by*

$$\frac{I - e^{\xi(\lambda)}\, e^{-\xi(\mu)}}{\lambda - \mu} = \sum_{I,J=0}^{\infty} D_{IJ}\,\lambda^{I+1}\mu^{J+1}, \tag{8}$$

$$\frac{I - e^{X(\lambda)}\, e^{-X(\mu)}}{\lambda - \mu} = \sum_{I,J=0}^{\infty} Z_{X,\,IJ}\,\lambda^{-I-1}\mu^{-J-1}. \tag{9}$$

Define $s_{(i|j)}$, $r_{(i|j)}$ $(i, j \geq 0)$ via

$$(D_{IJ})_{ab} = s_{m\cdot I+a-1,\,m\cdot J+m-b},$$
$$(Z_{X,\,IJ})_{ab} = r_{m\cdot I+m-a,\,m\cdot J+b-1}$$

where $a, b = 1, \ldots, m$. We call $Z_{X,\,IJ}$ the matrix-valued affine coordinates and $r_{X,\,(i|j)}$ the affine coordinates.

Remark 1.6. *The matrix-valued affine coordinates $Z_{X,\,IJ}$ and their generating formula* (9) *were introduced in [3] by F. Balogh and one of the authors of the present paper for the $\mathrm{sl}_2(\mathbb{C})$ case.*

The following theorem is the main result of the paper. Denote by κ the constant such that

$$(a|b) = \kappa\,\mathrm{Tr}(\pi(a)\pi(b))\ \forall a, b \in \mathfrak{g}. \tag{10}$$

Theorem 1.7. *For any $X \in \lambda^{-1}\mathfrak{g}[[\lambda^{-1}]]$, the formal series τ defined by*

$$\tau := \left(\sum_{\nu\in\mathbb{Y}} r_{X,\nu}\, s_\nu\right)^{\kappa} \tag{11}$$

is a tau function of the Drinfeld–Sokolov hierarchy of $\mathfrak{g}$-type. Moreover, s_ν and $r_{X,\nu}$ have the following expressions

$$s_\nu = \det\left(s_{(k_i|l_j)}\right)_{i,j=1}^{d}, \tag{12}$$

$$r_{X,\nu} = (-1)^{l_1+\cdots+l_d}\det\left(r_{X,(k_i|l_j)}\right)_{i,j=1}^{d}. \tag{13}$$

We refer to (11)–(13) as the Sato–Zhou type formula for tau functions of the DS hierarchy.

Remark 1.8. *As the reader might already have noticed, here the terminology is very similar to the one used to deal with the KP hierarchy in the Sato's approach. However, it is worth mentioning that tau functions of the DS hierarchies of $\mathfrak{g}$-type in general are* not *KP tau functions (except for $\mathfrak{g} = \mathrm{sl}_{n+1}(\mathbb{C})$). One way to see it (which is close to the spirit of this paper) is that the generalized Schur polynomials s_ν of $(\mathfrak{g}, \pi)$-type we defined are "reductions" (in the sense of the Remark 1.3) of the usual ones [30] just in the A_n case.*

Remark 1.9. *The formula* (11) *is* **intrinsic** *when π is taken as the adjoint representation of $\mathfrak{g}$. We will study the intrinsic Schur polynomials associated to $\mathfrak{g}$ in a future publication.*

Remark 1.10. *For the ABCD cases, a result similar to Theorem 1.7 was obtained in [39] where a different method was used; see also in [4] for more details for the A_n case.*

Organization of the paper In Section 2 we review the Drinfeld–Sokolov hierarchies and their tau functions. In Section 3 we prove Theorem 1.7. Some explicit examples and applications are given in Section 4. A list of first few Schur polynomials of $(\mathfrak{g}, \pi)$-type for $\mathfrak{g}$ of low ranks and particular choices of π are given in the Appendix.

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2 Review of the Grassmannian approach to the DS hierarchy

Denote by $\mathfrak{b}$ the Borel subalgebra of $\mathfrak{g}$, i.e. $\mathfrak{b} := \mathfrak{g}^{\leq 0}$, and by $\mathfrak{n}$ the nilpotent subalgebra $\mathfrak{n} := \mathfrak{g}^{<0}$. Define a linear operator $\mathcal{L}$ by

$$\mathcal{L} := \partial_x + \Lambda + q(x) \tag{14}$$

where $q(x) \in \mathfrak{b}$. It is proved by V. G. Drinfeld and V. V. Sokolov [14] that there exists a unique smooth function $U(x) \in \mathfrak{g}((\lambda^{-1}))^{<0} \cap \operatorname{Im} \mathrm{ad}_\Lambda$ such that

$$e^{-\mathrm{ad}_{U(x)}} \mathcal{L} = \partial_x + \Lambda + H(x), \qquad H(x) \in \operatorname{Ker} \mathrm{ad}_\Lambda.$$

The following commuting system of PDEs

$$\frac{\partial \mathcal{L}}{\partial t_\ell} = -\Big[\big(e^{\mathrm{ad}_U} \Lambda_\ell\big)_{\geq 0}\, , \mathcal{L} \Big], \qquad \ell \in E_+ \tag{15}$$

are called the pre-DS hierarchy of $\mathfrak{g}$-type.

Gauge transformations. For any smooth function $N(x) \in \mathfrak{n}$, the map

$$\mathcal{L} \mapsto \widetilde{\mathcal{L}} = e^{\mathrm{ad}_N} \mathcal{L} = \partial_x + \Lambda + \tilde{q}$$

is called a gauge transformation. A vector space $V \subset \mathfrak{g}$ is called a *DS gauge* if it satisfies

$$[I_+, \mathfrak{n}] \oplus V = \mathfrak{b}. \tag{16}$$

Below we fix V a DS gauge. It was observed in [14] that the flows (15) can be reduced to gauge equivalent classes; moreover, for any $q(x) \in \mathfrak{b}$, there exists a unique $N(x)$ such that $\tilde{q}(x) \in V$. Let us denote

$$\mathcal{L}^{\mathrm{can}} := \partial_x + \Lambda + q^{\mathrm{can}}(x), \qquad q^{\mathrm{can}}(x) \in V.$$

Take $v_1, \dots, v_n$ a homogeneous basis of V, namely $\deg v_i = -m_i$, and write

$$q^{\mathrm{can}}(x) = \sum_{i=1}^{n} u^i(x)\, v_i.$$

The DS hierarchy of $\mathfrak{g}$-type is defined as the system of the pre-DS flows for the complete set of representatives (aka gauge invariants) $u^1, \dots, u^n$. Clearly, the precise form[2] of this integrable hierarchy depends on the choice of the DS gauge V; the hierarchies under different choices of V are Miura equivalent [?, 24, 25, 19, 6]. We remark that a unified algorithm of writing the DS hierarchy of $\mathfrak{g}$-type for an arbitrary choice V was obtained recently in [6]; it has the form

$$\frac{\partial u^i}{\partial t_\ell} = a^i_\ell[u^1, \dots, u^n], \qquad \ell \in E_+ \tag{17}$$

where $a_{i,\ell}[u^1, \dots, u^n]$ are differential polynomials of $u^1, \dots, u^n$. It should also be noted that for the DS hierarchy of $\mathfrak{g}$-type the time variable t_1 can be identified with $-x$.

The hierarchy (17) is known to be Hamiltonian and tau-symmetric [19, 24, 36, 7]. Therefore, for an arbitrary solution q^{can} of (17), there exists a tau function $\tau(\mathbf{t})$ of q^{can}. The tau function is determined up to a multiplicative factor of the form

$$\exp\Big(\sum_{\ell \in E_+} c_\ell\, t_\ell\Big)$$

where c_ℓ are arbitrary constants. We review in this subsection the Grassmannian approach to tau functions.

Denote $E = \mathbb{C}^m$ where m is defined in (3). Let $H := E((\lambda^{-1}))$ be the linear space of E-valued formal series in z with finitely many positive powers and let $H_+ := E[z]$. Denote by Gr the Sato–Segal–Wilson Grassmannian [32, 33]. A point $W \in \mathrm{Gr}$ is a subspace of H. Here we are interested in the *big cell* $\mathrm{Gr}^{(0)} \subset \mathrm{Gr}$ which consists of points W of the form

$$W = \mathrm{Span}_{\mathbb{C}}\Big\{ e_i\, \lambda^\ell + \sum_{k \geq 0} A_{k,\ell,i}\, e_i\, \lambda^{-k-1} \Big\}_{i=1,\dots,m,\, \ell \geq 0}.$$

Here $A_{k,\ell,i}$ are called the affine coordinates [20] of W.

[2] It also depends on scalings of the basis v_i which gives rise to scalings of u^i. Such a coordinate change is trivial (In the case $\mathfrak{g} = D_{\mathrm{even}}$ other linear transformation of u_i needs to be considered but is again trivial).

Definition 2.1. *Define* $\mathrm{Gr}_{\mathfrak{g}}^{(0)}$ *as the following subset of the big cell* $\mathrm{Gr}^{(0)}$

$$\mathrm{Gr}_{\mathfrak{g}}^{(0)} = \Big\{ e^a H_+ \,|\, a \in \lambda^{-1}\mathfrak{g}[[\lambda^{-1}]] \Big\}.$$

We call $\mathrm{Gr}_{\mathfrak{g}}^{(0)}$ *the embedded big cell of* $\mathfrak{g}$*-type.*

For $a \in \lambda^{-1}\mathfrak{g}[[\lambda^{-1}]]$, write $G = e^a = \sum_{k\geq 0} G_k \lambda^{-k}$. The matrices $G_0, G_1, \ldots$ serve as the matrix-valued coordinates for the point W corresponding to a; see Fig. 1. Clearly, $G_0 = I$.

$$\begin{array}{c|c|c}
\vdots & \vdots & \iddots \\
\hline
G_2 & G_3 & \cdots \\
\hline
G_1 & G_2 & \cdots \\
\hline
G_0 & G_1 & \cdots \\
\hline
 & G_0 & \cdots \\
 & & \ddots
\end{array}$$

Figure 1: Matrix-valued coordinates in Sato–Segal–Wilson Grassmannian

Definition 2.2. $\forall\, M = \sum_{k\in\mathbb{Z}} M_k\, \lambda^k$, $M_k \in \mathrm{gl}(m, \mathbb{C})$, *the* N*-th* ($N \geq 0$) *block Toeplitz matrix associated to* M *is defined by*

$$T_N(M) = (M_{I-J})_{I,J=0}^{N}.$$

The following theorem comes from the results obtained in [9, 10].

Theorem A. (Cafasso–Wu [9, 10]) *For any* $X \in \lambda^{-1}\mathfrak{g}[[\lambda^{-1}]]$, *let* $\gamma = e^{\xi} e^{X}$. *Define* $\tau = \tau(\mathbf{t})$ *by*

$$\tau = \Big[\lim_{N\to\infty} \det\, T_N(\gamma) \Big]^{\kappa}, \tag{18}$$

where κ *is defined in* (10). *Then* τ *is a tau function of the DS hierarchy associated to* $\mathfrak{g}$.

Remark 2.3. *The stabilization proved in [22] for the case of the Witten–Kontsevich tau function and extended in [10] for the general cases ensures that the limit in* (18) *is meaningful.*

3 Proof of Theorem 1.7

Define $\gamma = e^{\xi} e^{X}$, where we recall that X is the given element in $\lambda^{-1}\mathfrak{g}[[\lambda^{-1}]]$, and $\xi = \sum_{\ell\in E_+} t_\ell\, \Lambda_\ell$. We have

$$\mathbb{L}(\gamma) = \mathbb{L}(e^{\xi})\, \mathbb{L}(e^{X}) = s\, r_X$$

where s, r_X are defined in (5), (7), respectively. For any $N \geq 1$, define two matrices

$$s_N = (s_{N,ij})_{i\in\{0,\dots,N\},\, j\in\{-N-1,\dots,N\}}$$

and

$$r_N = (r_{N,ij})_{i\in\{-N-1,\dots,N\},\, j\in\{0,\dots,N\}}$$

by

$$s_{N,ij} := \mathbb{L}(e^{\xi})_{ij}, \qquad r_{N,ij} := \mathbb{L}(e^{X})_{ij}.$$

Then we have

$$\lim_{N\to\infty} \det T_N(\gamma) = \lim_{N\to\infty} \det(s_N\, r_N).$$

By using the well-known Cauchy–Binet formula (see for instance [21]) we obtain [32, 20] from Theorem A that

$$\tau^{1/\kappa} = \sum_{\lambda\in\mathbb{Y}} r_{X,\lambda}\, s_\lambda$$

where we recall that $r_{X,\lambda}$ and s_λ are defined by

$$r_{X,\lambda} = \det\left(r_{i-\lambda_i-1,j-1}\right)_{i,j=1}^{\ell(\lambda)}$$

and

$$s_\lambda = \det\left(s_{i-1,j-\lambda_j-1}\right)_{i,j=1}^{\ell(\lambda)}.$$

As explained in [3], formulae (8) and (9) give the Gaussian eliminations and formulae (12) and (13) are due to the Giambelli-type formula [20, 30, 3]. The theorem is proved. □

4 Polynomial tau functions and bilinear equations

Theorem 1.7 gives a simple procedure to compute the tau function τ when $\tau^{1/\kappa}$ is a polynomial. Indeed, let us fix the Lie algebra $\mathfrak{g}$ and take a faithful representation π. Choosing $X \in \lambda^{-1}\mathfrak{g}[[\lambda^{-1}]]$ such that $\pi(X)$ is a nilpotent matrix, the infinite series in (11) becomes finite, as it is easy to verify that only finitely many Plücker coordinates $\{r_\mu,\ \mu \in \mathbb{Y}\}$ are non zero. Consequently, $\tau^{1/\kappa}$ is polynomial. This simple idea was used for example in [3] for the KdV hierarchy. If $\kappa = 1$, then the tau function itself is a polynomial. Interestingly enough, in the computations we will perform, even when $\kappa = 1/2$, we obtain some polynomial tau functions: in other words, the finite sum in (11) is a perfect square. Even if this result has not been proved in general, we expect that our procedure gives a systematic way to compute all the polynomials tau functions (up to a shift of the time variables $\{t_i,\ i \in E_+\}$) of the DS hierarchy of $\mathfrak{g}$-type. As stated in the introduction of [28], this is an interesting open problem.

In what follows we compute the first few polynomial tau functions of the DS hierarchy of $\mathfrak{g}$-type for $\mathfrak{g} = A_1,\ A_2,\ B_2$ and D_4. We use these particular tau functions to deduce possible bilinear equations of small degrees. Note that each Drinfeld–Sokolov hierarchy has infinitely many solutions. The usual question is to find particular solutions to the DS hierarchy (solve all PDEs in this hierarchy together). Here we consider the inverse:

Deduce possible PDEs from particular solutions.

Sometimes, one particular solution already contains all the information of an equation and of the whole hierarchy. For example, the "topological solution" was used by B. Dubrovin and Y. Zhang to construct the integrable hierarchy of topological type [19, 17]. However, a polynomial tau function τ_{poly} of the DS hierarchy contains less information, namely, if τ_{poly} satisfies some PDE, it will not guarantee directly that other tau functions of the DS hierarchy satisfy this PDE. Nevertheless, if τ_{poly} does not satisfy a PDE, then the PDE cannot belong to the DS hierarchy.

4.1 Bilinear derivatives

Given two smooth functions $f(\mathbf{x}),\ g(\mathbf{x})$ with independent variables $\mathbf{x} = (x_i)_{i\in I}$, where I denotes an index set. The bilinear derivatives $D_{i_1}\cdots D_{i_k}$ are operators defined via the identity

$$e^{\sum_{i\in I} h_i\, D_i}(f,g) \equiv f(\mathbf{x}+\mathbf{h})\, g(\mathbf{x}-\mathbf{h}), \quad \forall\, \mathbf{h}.$$

It means that, expanding both sides of this identity in $\mathbf{h}$

$$e^{\sum_{i\in I} h_i\, D_i}(f,g) = (f,g) + \sum_{i\in I} h_i\, D_i(f,g) + \sum_{i,j\in I} \frac{h_i h_j}{2} D_i D_j(f,g) + \cdots,$$

$$f(\mathbf{x}+\mathbf{h})\, g(\mathbf{x}-\mathbf{h}) = f(x)g(x) + \sum_{i\in I} h_i \left(\frac{\partial f}{\partial x_i} g - f\frac{\partial g}{\partial x_i}\right) + \cdots$$

and comparing the coefficients of monomials of $\mathbf{h}$, we obtain, for example,

$$D_i(f,g) = \frac{\partial f}{\partial x_i} g - f\frac{\partial g}{\partial x_i},$$

$$D_i D_j(f,g) = \frac{\partial^2 f}{\partial x_i \partial x_j} g + f\frac{\partial^2 g}{\partial x_i \partial x_j} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}.$$

For the Drinfeld–Sokolov hierarchy of $\mathfrak{g}$-type, we take $I := E_+$. There is a natural gradation for the bilinear derivatives, defined by assigning $\deg D_i = i$ for $i \in E_+$. Denote by $\mathcal{H}_{\mathfrak{g}}$ the linear space of bilinear equations satisfied by the Drinfeld–Sokolov hierarchies of $\mathfrak{g}$-type, which decomposes into homogeneous subspaces

$$\mathcal{H}_{\mathfrak{g}} = \bigoplus_i \mathcal{H}_{\mathfrak{g}}^{[i]}.$$

The gradation allows us to list all possible bilinear equations up to certain degree.

4.2 Examples of polynomial tau functions

4.2.1 The A_1 case

Let us chose the standard matrix realization $\mathfrak{g} = \mathrm{sl}(2;\mathbb{C})$. Consider the following two elements in $\lambda^{-1}\mathfrak{g}[[\lambda^{-1}]]$

$$\frac{1}{\lambda}F = \frac{1}{\lambda}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \frac{1}{\lambda}E = \frac{1}{\lambda}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{19}$$

The associated polynomial tau functions are

$$\tau_1 = 1 + t_1, \qquad \tau_2 = 1 + t_3 - \frac{t_1^3}{3} \tag{20}$$

respectively. Similarly, one computes polynomial tau functions corresponding to elements of the form $\lambda^{-k}F$, $\lambda^{-k}E$, $k \geq 2$. For example, for $k = 2$, we obtain

$$\tau_3 = 1 + 2t_3 - t_5t_1 + t_3^2 + \frac{t_1^3}{3} + \frac{1}{3}t_3t_1^3 - \frac{1}{45}t_1^6, \tag{21}$$

$$\tau_4 = 1 - t_3t_7 + 2t_5 + t_5^2 + t_3^3t_1 - t_3t_5t_1^2 - t_3t_1^2 + \frac{1}{3}t_7t_1^3 - \frac{t_1^5}{15} - \frac{1}{15}t_5t_1^5 + \frac{1}{105}t_3t_1^7 - \frac{t_1^{10}}{4725}, \tag{22}$$

corresponding to $\lambda^{-2}F$ and $\lambda^{-2}E$, respectively.

Now consider all bilinear equations up to degree 4:

$$(\beta + \alpha_0 D_1^2 + \alpha_1 D_1^4 + \alpha_2 D_1 D_3)(\tau, \tau) = 0 \tag{23}$$

where $\beta, \alpha_0, \alpha_1, \alpha_2$ are complex constants. Requiring that τ_1, τ_2 satisfy the above ansatz (23), we find that up to a multiplicative constant there is only one possible choice of coefficients:

$$(D_1^4 - 4D_1D_3)(\tau, \tau) = 0. \tag{24}$$

Similarly up to degree 6, we find out only two more possible linearly independent bilinear equations that are satisfied by $\tau_1, \tau_2, \tau_3, \tau_4$

$$(D_1^6 + 20D_1^3D_3 - 96D_1D_5)(\tau, \tau) = 0, \tag{25}$$

$$(D_1^3D_3 + 2D_3^2 - 6D_1D_5)(\tau, \tau) = 0, \tag{26}$$

which are well known to belong to the hierarchy of A_1-type, that is the KdV hierarchy. Consequently, we have shown that

$$\dim_{\mathbb{C}} \mathcal{H}_{A_1}^{[\deg \leq 6]} \leq 3.$$

Moreover, (24)–(26) are the three only possible choices of homogeneous basis (up to constant factors) of $\mathcal{H}_{\mathfrak{g}}^{[\deg \leq 6]}$.

Relation with the Adler–Moser polynomials. An alternative way of computing polynomial tau functions for the KdV hierarchy was given by Adler and Moser [1]. Define a family of polynomials $\theta_k(x = q_1, q_3, q_5, \ldots, q_{2k-1})$, $k \geq 0$ recursively by

$$\theta_0 = 1,\ \theta_1 = x,$$
$$\theta'_{k+1}\theta_{k-1} + \theta_{k+1}\theta'_{k-1} = (2k-1)\theta_k^2, \quad \forall\, k \geq 2,$$

where the prime denotes the x-derivative and for each $k \geq 2$ the integration constant is chosen to be q_{2k-1}. The polynomials θ_k are known as the Adler–Moser polynomials. It was also proven in [1] that there exists a unique change of variables $\mathbf{q} \to \mathbf{t}$ that transforms the Adler–Moser polynomials into the polynomial tau functions of the KdV hierarychy. In [15], one of the authors of the present paper proved that the desired change of variables is given by $q_1 = t_1 = x$ and

$$\sum_{i \geq 2} \frac{q_{2i-1}}{\alpha_{2i-1}} z^{2i-1} = \tanh\left(\sum_{i \geq 2} t_{2i-1} z^{2i-1}\right),$$

where $\alpha_{2i-1} := (-1)^{i-1} 3^2 \cdots (2i-3)^2(2i-1)$. Up to a shift and renormalisation of the times, we recover in particular the polynomials given in equations (20)–(22).

4.2.2 The A_2 case

We still chose the standard matrix realization $\mathfrak{g} = \mathrm{sl}(3;\mathbb{C})$. Consider for example the following two elements in $\lambda^{-1}\mathfrak{g}[[\lambda^{-1}]]$:

$$X_1 = \frac{1}{\lambda}\begin{pmatrix} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ a_2 & a_3 & 0 \end{pmatrix}, \quad X_2 = \frac{1}{\lambda}\begin{pmatrix} 0 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix}, \tag{27}$$

where a_1, a_2, a_3 are arbitrary constants. The corresponding polynomial tau functions will be denoted by τ_1, τ_2, respectively. We have

$$\begin{aligned}
\tau_1 \;=\; & 1 + a_2 t_1 + \frac{1}{2}a_1 t_1^2 - \frac{1}{2}a_3 t_1^2 + \frac{1}{8}a_1 a_3 t_1^4 - \frac{1}{160}a_1^2 a_3 t_1^6 + \frac{1}{160}a_1 a_3^2 t_1^6 - \frac{a_1^2 a_3^2 t_1^8}{1792} + a_1 t_2 + a_3 t_2 + \frac{1}{16}a_1^2 a_3 t_1^4 t_2 \\
& + \frac{1}{16}a_1 a_3^2 t_1^4 t_2 + \frac{3}{2}a_1 a_3 t_2^2 - \frac{1}{8}a_1^2 a_3 t_1^2 t_2^2 + \frac{1}{8}a_1 a_3^2 t_1^2 t_2^2 + \frac{1}{32}a_1^2 a_3^2 t_1^4 t_2^2 + \frac{1}{4}a_1^2 a_3 t_2^3 + \frac{1}{4}a_1 a_3^2 t_2^3 + \frac{1}{16}a_1^2 a_3^2 t_2^4 - \frac{1}{4}a_1^2 a_3 t_1^2 t_4 \\
& - \frac{1}{4}a_1 a_3^2 t_1^2 t_4 - \frac{1}{2}a_1^2 a_3 t_2 t_4 + \frac{1}{2}a_1 a_3^2 t_2 t_4 - \frac{1}{4}a_1^2 a_3^2 t_1^2 t_2 t_4 - \frac{1}{4}a_1^2 a_3^2 t_4^2 + \frac{1}{2}a_1^2 a_3 t_1 t_5 - \frac{1}{2}a_1 a_3^2 t_1 t_5 + \frac{1}{4}a_1^2 a_3^2 t_1 t_7 \,,
\end{aligned}$$

$$\begin{aligned}
\tau_2 \;=\; & 1 - \frac{1}{8}a_1 t_1^4 + \frac{1}{8}a_3 t_1^4 + \frac{1}{20}a_2 t_1^5 + \frac{1}{640}a_1 a_3 t_1^8 - \frac{a_1^2 a_3 t_1^{12}}{358400} + \frac{a_1 a_3^2 t_1^{12}}{358400} - \frac{a_1^2 a_3^2 t_1^{16}}{90112000} - \frac{1}{2}a_1 t_1^2 t_2 - \frac{1}{2}a_3 t_1^2 t_2 - \frac{a_1^2 a_3 t_1^{10} t_2}{12800} - \frac{a_1 a_3^2 t_1^{10} t_2}{12800} + \frac{1}{2}a_1 t_2^2 - \frac{1}{2}a_3 t_2^2 \\
& - a_2 t_1 t_2^2 + \frac{1}{16}a_1 a_3 t_1^4 t_2^2 - \frac{13 a_1^2 a_3 t_1^8 t_2^2}{17920} + \frac{13 a_1 a_3^2 t_1^8 t_2^2}{17920} + \frac{3 a_1^2 a_3^2 t_1^{12} t_2^2}{1126400} - \frac{1}{320}a_1^2 a_3 t_1^6 t_2^3 - \frac{1}{320}a_1 a_3^2 t_1^6 t_2^3 - \frac{3}{8}a_1 a_3 t_2^4 - \frac{1}{128}a_1^2 a_3 t_1^4 t_2^4 + \frac{1}{128}a_1 a_3^2 t_1^4 t_2^4 \\
& - \frac{a_1^2 a_3^2 t_1^8 t_2^4}{10240} - \frac{1}{32}a_1^2 a_3 t_1^2 t_2^5 - \frac{1}{32}a_1 a_3^2 t_1^2 t_2^5 - \frac{1}{32}a_1^2 a_3 t_2^6 + \frac{1}{32}a_1 a_3^2 t_2^6 + \frac{1}{256}a_1^2 a_3^2 t_1^4 t_2^6 + \frac{1}{256}a_1^2 a_3^2 t_2^8 + a_1 t_4 + a_3 t_4 + \frac{a_1^2 a_3 t_1^8 t_4}{1280} + \frac{a_1 a_3^2 t_1^8 t_4}{1280} \\
& - \frac{3}{2}a_1 a_3 t_1^2 t_2 t_4 + \frac{1}{160}a_1^2 a_3 t_1^6 t_2 t_4 - \frac{1}{160}a_1 a_3^2 t_1^6 t_2 t_4 - \frac{a_1^2 a_3^2 t_1^{10} t_2 t_4}{12800} + \frac{1}{32}a_1^2 a_3 t_1^4 t_2^2 t_4 + \frac{1}{32}a_1 a_3^2 t_1^4 t_2^2 t_4 - \frac{1}{8}a_1^2 a_3 t_1^2 t_2^3 t_4 + \frac{1}{8}a_1 a_3^2 t_1^2 t_2^3 t_4 - \frac{1}{320}a_1^2 a_3^2 t_1^6 t_2^3 t_4 \\
& - \frac{3}{16}a_1^2 a_3 t_2^4 t_4 - \frac{3}{16}a_1 a_3^2 t_2^4 t_4 - \frac{1}{32}a_1^2 a_3^2 t_1^2 t_2^5 t_4 + \frac{3}{2}a_1 a_3 t_4^2 + \frac{1}{32}a_1^2 a_3 t_1^4 t_4^2 - \frac{1}{32}a_1 a_3^2 t_1^4 t_4^2 + \frac{a_1^2 a_3^2 t_1^8 t_4^2}{2560} - \frac{3}{8}a_1^2 a_3 t_1^2 t_2 t_4^2 - \frac{3}{8}a_1 a_3^2 t_1^2 t_2 t_4^2 - \frac{1}{8}a_1^2 a_3 t_2^2 t_4^2 \\
& + \frac{1}{8}a_1 a_3^2 t_2^2 t_4^2 + \frac{1}{64}a_1^2 a_3^2 t_1^4 t_2^2 t_4^2 - \frac{3}{32}a_1^2 a_3^2 t_2^4 t_4^2 + \frac{1}{4}a_1^2 a_3 t_4^3 + \frac{1}{4}a_1 a_3^2 t_4^3 - \frac{1}{8}a_1^2 a_3^2 t_1^2 t_2 t_4^3 + \frac{1}{16}a_1^2 a_3^2 t_4^4 + a_2 t_5 + \frac{1}{2}a_1 a_3 t_1^3 t_5 + \frac{3 a_1^2 a_3 t_1^7 t_5}{1120} - \frac{3 a_1 a_3^2 t_1^7 t_5}{1120} \\
& + \frac{a_1^2 a_3^2 t_1^{11} t_5}{140800} + \frac{1}{80}a_1^2 a_3 t_1^5 t_2 t_5 + \frac{1}{80}a_1 a_3^2 t_1^5 t_2 t_5 + \frac{1}{8}a_1^2 a_3 t_1^3 t_2^2 t_5 - \frac{1}{8}a_1 a_3^2 t_1^3 t_2^2 t_5 + \frac{1}{320}a_1^2 a_3^2 t_1^7 t_2^2 t_5 + \frac{1}{4}a_1^2 a_3 t_1 t_2^3 t_5 + \frac{1}{4}a_1 a_3^2 t_1 t_2^3 t_5 - \frac{1}{32}a_1^2 a_3^2 t_1^3 t_2^4 t_5 \\
& + \frac{1}{4}a_1^2 a_3 t_1^3 t_4 t_5 + \frac{1}{4}a_1 a_3^2 t_1^3 t_4 t_5 - \frac{1}{2}a_1^2 a_3 t_1 t_2 t_4 t_5 + \frac{1}{2}a_1 a_3^2 t_1 t_2 t_4 t_5 + \frac{1}{80}a_1^2 a_3^2 t_1^5 t_2 t_4 t_5 + \frac{1}{4}a_1^2 a_3^2 t_1 t_2^3 t_4 t_5 + \frac{1}{8}a_1^2 a_3^2 t_1^3 t_4^2 t_5 + \frac{1}{4}a_1^2 a_3 t_1^2 t_5^2 - \frac{1}{4}a_1 a_3^2 t_1^2 t_5^2 \\
& - \frac{1}{160}a_1^2 a_3^2 t_1^6 t_5^2 - \frac{1}{2}a_1^2 a_3 t_2 t_5^2 - \frac{1}{2}a_1 a_3^2 t_2 t_5^2 - \frac{1}{8}a_1^2 a_3^2 t_1^2 t_2^2 t_5^2 - \frac{1}{2}a_1^2 a_3^2 t_2 t_4 t_5^2 - \frac{1}{4}a_1^2 a_3^2 t_1 t_5^3 - \frac{1}{40}a_1^2 a_3 t_1^5 t_7 + \frac{1}{40}a_1 a_3^2 t_1^5 t_7 + \frac{1}{2}a_1^2 a_3 t_1 t_2^2 t_7 - \frac{1}{2}a_1 a_3^2 t_1 t_2^2 t_7 \\
& - \frac{1}{2}a_1^2 a_3 t_5 t_7 + \frac{1}{2}a_1 a_3^2 t_5 t_7 - \frac{1}{16}a_1^2 a_3 t_1^4 t_8 - \frac{1}{16}a_1 a_3^2 t_1^4 t_8 - \frac{1}{4}a_1^2 a_3 t_1^2 t_2 t_8 + \frac{1}{4}a_1 a_3^2 t_1^2 t_2 t_8 - \frac{1}{160}a_1^2 a_3^2 t_1^6 t_2 t_8 + \frac{1}{4}a_1^2 a_3 t_2^2 t_8 + \frac{1}{4}a_1 a_3^2 t_2^2 t_8 + \frac{1}{8}a_1^2 a_3^2 t_1^2 t_2^3 t_8 \\
& + \frac{1}{2}a_1^2 a_3 t_4 t_8 - \frac{1}{2}a_1 a_3^2 t_4 t_8 - \frac{1}{16}a_1^2 a_3^2 t_1^4 t_4 t_8 + \frac{1}{4}a_1^2 a_3^2 t_2^2 t_4 t_8 + \frac{1}{2}a_1^2 a_3^2 t_1 t_2 t_5 t_8 - \frac{1}{4}a_1^2 a_3^2 t_8^2 + \frac{1}{80}a_1^2 a_3^2 t_1^5 t_{11} - \frac{1}{4}a_1^2 a_3^2 t_1 t_2^2 t_{11} + \frac{1}{4}a_1^2 a_3^2 t_5 t_{11} \,.
\end{aligned}$$

Consider all possible bilinear equations of degree 4:

$$(\alpha_1 D_1^4 + \alpha_2 D_2^2)(\tau, \tau) = 0,$$

Requiring τ_1 satisfies this ansatz we find that there is only one possible choice:

$$(D_1^4 + 3D_2^2)(\tau, \tau) = 0.$$

Similarly, requiring that τ_1 and τ_2 to both satisfy the ansatz of bilinear equation of degree 6, we find that there are only two linearly independent bilinear equations of degree 6:

$$\begin{aligned}
(D_1^6 + 45D_1^2D_2^2 + 90D_2D_4 - 216D_1D_5)(\tau, \tau) = 0, \\
(D_1^6 + 15D_1^2D_2^2 + 60D_2D_4 - 96D_1D_5)(\tau, \tau) = 0,
\end{aligned}$$

which are well known to belong to the hierarchy of A_2-type (i.e. the Boussinesq hierarchy).

4.2.3 The B_2 case

We chose the matrix realization of the B_2 simple Lie algebra as in [14]. We consider two explicit examples given respectively by the following matrices[3]

$$X_1 = \frac{1}{\lambda}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_5 & 0 & 0 & 0 \\ a_4 & 0 & a_5 & 0 & 0 \\ 0 & a_4 & -a_3 & a_2 & 0 \end{pmatrix}, \qquad X_2 = \frac{1}{\lambda}\begin{pmatrix} 0 & 0 & a_3 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 0 & -a_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The associated tau functions will be denoted by τ_1 and τ_2. They have the expressions

$$\begin{aligned}
\tau_1 = {} & 1 + \frac{1}{2}a_4t_1 + \frac{1}{4}a_3t_1^2 + \frac{1}{12}a_2t_1^3 - \frac{1}{12}a_5t_1^3 - \frac{1}{192}a_3^2t_1^4 + \frac{1}{96}a_2a_4t_1^4 + \frac{1}{192}a_3a_5t_1^5 + \frac{a_2a_5t_1^6}{1920} - \frac{1}{720}a_5^2t_1^6 - \frac{a_2a_4a_5t_1^7}{11520} - \frac{a_2a_3a_5t_1^8}{53760} - \frac{a_2^2a_5t_1^9}{483840} \\
& + \frac{a_2a_5^2t_1^9}{1088640} + \frac{a_2a_4a_5^2t_1^{10}}{3110400} + \frac{11a_2a_3a_5^2t_1^{11}}{87091200} + \frac{43a_2^2a_5^2t_1^{12}}{2090188800} - \frac{79a_2a_5^3t_1^{12}}{2874009600} - \frac{37a_2^2a_5^3t_1^{15}}{1931334451200} + \frac{a_2^2a_5^4t_1^{18}}{115880067072000} + \frac{1}{2}a_2t_3 + a_5t_3 - \frac{1}{8}a_3^2t_1t_3 \\
& + \frac{1}{4}a_2a_4t_1t_3 + \frac{1}{8}a_3a_5t_1^2t_3 + \frac{1}{16}a_2a_5t_1^3t_3 - \frac{1}{24}a_5^2t_1^3t_3 + \frac{1}{192}a_2a_4a_5t_1^4t_3 + \frac{1}{640}a_2a_3a_5t_1^5t_3 + \frac{a_2^2a_5t_1^6t_3}{3840} - \frac{a_2a_5^2t_1^6t_3}{1080} - \frac{a_2a_4a_5^2t_1^7t_3}{34560} - \frac{a_2a_3a_5^2t_1^8t_3}{193536} \\
& - \frac{31a_2^2a_5^2t_1^9t_3}{17418240} + \frac{11a_2a_5^3t_1^9t_3}{8709120} + \frac{101a_2^2a_5^3t_1^{12}t_3}{22992076800} - \frac{29a_2^2a_5^4t_1^{15}t_3}{6437781504000} + \frac{3}{4}a_2a_5t_3^2 + \frac{1}{4}a_5^2t_3^2 + \frac{1}{16}a_2a_4a_5t_1t_3^2 + \frac{5}{288}a_2a_5^2t_1^3t_3^2 + \frac{a_2a_4a_5^2t_1^4t_3^2}{1152} + \frac{7a_2a_3a_5^2t_1^5t_3^2}{23040} \\
& + \frac{a_2^2a_5^2t_1^6t_3^2}{138240} - \frac{17a_2a_5^3t_1^6t_3^2}{69120} - \frac{13a_2^2a_5^3t_1^9t_3^2}{34836480} + \frac{a_2^2a_5^4t_1^{12}t_3^2}{2043740160} + \frac{1}{16}a_2^2a_5t_3^3 + \frac{11}{36}a_2a_5^2t_3^3 + \frac{1}{144}a_2a_4a_5^2t_1t_3^3 - \frac{1}{576}a_2a_3a_5^2t_1^2t_3^3 + \frac{a_2^2a_5^2t_1^3t_3^3}{3456} + \frac{7a_2a_5^3t_1^3t_3^3}{1728} \\
& - \frac{11a_2^2a_5^3t_1^6t_3^3}{414720} - \frac{a_2^2a_5^4t_1^9t_3^3}{34836480} + \frac{25}{576}a_2^2a_5^2t_3^4 + \frac{5}{144}a_2a_5^3t_3^4 + \frac{a_2^2a_5^3t_1^3t_3^4}{3456} - \frac{a_2^2a_5^4t_1^6t_3^4}{552960} + \frac{5}{576}a_2^2a_5^3t_3^5 + \frac{a_2^2a_5^4t_3^6}{2304} - \frac{1}{4}a_2a_5t_1t_5 + \frac{1}{4}a_5^2t_1t_5 - \frac{1}{16}a_2a_4a_5t_1^2t_5 \\
& - \frac{1}{48}a_2a_3a_5t_1^3t_5 - \frac{1}{192}a_2^2a_5t_1^4t_5 + \frac{1}{72}a_2a_5^2t_1^4t_5 + \frac{a_2a_4a_5^2t_1^5t_5}{5760} - \frac{11a_2a_3a_5^2t_1^6t_5}{69120} + \frac{a_2^2a_5^2t_1^7t_5}{483840} + \frac{41a_2a_5^3t_1^7t_5}{483840} - \frac{a_2^2a_5^3t_1^{10}t_5}{17418240} + \frac{a_2^2a_5^4t_1^{13}t_5}{4379443200} \\
& - \frac{1}{8}a_2a_3a_5t_3t_5 - \frac{1}{8}a_2^2a_5t_1t_3t_5 - \frac{1}{24}a_2a_5^2t_1t_3t_5 - \frac{1}{48}a_2a_4a_5^2t_1^2t_3t_5 - \frac{5}{576}a_2a_3a_5^2t_1^3t_3t_5 - \frac{a_2^2a_5^2t_1^4t_3t_5}{1152} + \frac{7a_2a_5^3t_1^4t_3t_5}{1152} + \frac{29a_2^2a_5^3t_1^7t_3t_5}{967680} \\
& - \frac{a_2^2a_5^4t_1^{10}t_3t_5}{116121600} - \frac{1}{24}a_2a_3a_5^2t_3^2t_5 - \frac{7}{96}a_2^2a_5^2t_1t_3^2t_5 - \frac{1}{48}a_2a_5^3t_1t_3^2t_5 + \frac{a_2^2a_5^3t_1^4t_3^2t_5}{1152} + \frac{a_2^2a_5^4t_1^7t_3^2t_5}{258048} - \frac{11}{576}a_2^2a_5^3t_1t_3^3t_5 + \frac{a_2^2a_5^4t_1^4t_3^3t_5}{9216} - \frac{1}{768}a_2^2a_5^4t_1t_3^4t_5 \\
& - \frac{1}{24}a_2a_4a_5^2t_5^2 - \frac{1}{32}a_2a_3a_5^2t_1t_5^2 + \frac{1}{64}a_2^2a_5^2t_1^2t_5^2 - \frac{1}{96}a_2a_5^3t_1^2t_5^2 - \frac{a_2^2a_5^3t_1^5t_5^2}{2304} - \frac{a_2^2a_5^4t_1^8t_5^2}{1290240} + \frac{1}{192}a_2^2a_5^3t_1^2t_3t_5^2 - \frac{a_2^2a_5^4t_1^5t_3t_5^2}{7680} + \frac{a_2^2a_5^4t_1^3t_5^3}{2304} - \frac{1}{192}a_2^2a_5^4t_3t_5^3 \\
& + \frac{1}{8}a_2a_3a_5t_1t_7 + \frac{1}{16}a_2^2a_5t_1^2t_7 - \frac{1}{8}a_2a_5^2t_1^2t_7 + \frac{1}{288}a_2a_3a_5^2t_1^4t_7 + \frac{a_2^2a_5^2t_1^5t_7}{1440} - \frac{1}{640}a_2a_5^3t_1^5t_7 - \frac{a_2^2a_5^3t_1^8t_7}{967680} - \frac{a_2^2a_5^4t_1^{11}t_7}{87091200} + \frac{1}{12}a_2a_3a_5^2t_1t_3t_7 \\
& + \frac{1}{24}a_2^2a_5^2t_1^2t_3t_7 - \frac{1}{16}a_2a_5^3t_1^2t_3t_7 - \frac{a_2^2a_5^3t_1^5t_3t_7}{11520} - \frac{a_2^2a_5^4t_1^8t_3t_7}{1935360} + \frac{1}{192}a_2^2a_5^3t_1^2t_3^2t_7 - \frac{a_2^2a_5^4t_1^5t_3^2t_7}{46080} + \frac{a_2^2a_5^4t_1^2t_3^3t_7}{1152} + \frac{1}{12}a_2^2a_5^2t_5t_7 - \frac{1}{12}a_2a_5^3t_5t_7 \\
& + \frac{1}{576}a_2^2a_5^3t_1^3t_5t_7 + \frac{7a_2^2a_5^4t_1^6t_5t_7}{138240} + \frac{1}{24}a_2^2a_5^3t_3t_5t_7 + \frac{a_2^2a_5^4t_1^3t_3t_5t_7}{1152} + \frac{1}{96}a_2^2a_5^4t_3^2t_5t_7 + \frac{1}{192}a_2^2a_5^4t_1t_5^2t_7 - \frac{1}{48}a_2^2a_5^3t_1t_7^2 - \frac{a_2^2a_5^4t_1^4t_7^2}{2304} - \frac{1}{96}a_2^2a_5^4t_1t_3t_7^2,
\end{aligned}$$

$$\begin{aligned}
\tau_2 = {} & 1 + \frac{1}{288}a_3t_1^6 - \frac{a_4t_1^7}{2016} - \frac{a_3^2t_1^{12}}{11612160} - \frac{1}{12}a_3t_1^3t_3 + \frac{1}{48}a_4t_1^4t_3 - \frac{a_3^2t_1^9t_3}{69120} + \frac{1}{2}a_3t_3^2 - \frac{1}{2}a_4t_1t_3^2 - \frac{a_3^2t_1^6t_3^2}{1920} \\
& - \frac{1}{48}a_3^2t_1^3t_3^3 + \frac{1}{16}a_3^2t_3^4 + \frac{a_3^2t_1^7t_5}{4032} - \frac{1}{96}a_3^2t_1^4t_3t_5 + \frac{1}{4}a_3^2t_1t_3^2t_5 + a_4t_7 + \frac{1}{160}a_3^2t_1^5t_7 - \frac{1}{2}a_3^2t_5t_7.
\end{aligned}$$

Consider all bilinear equations up to degree 4

$$(\alpha_0 + \alpha_1 D_1^2 + \alpha_2 D_1^4 + \alpha_3 D_1 D_3)(\tau, \tau) = 0$$

where $\alpha_0, \ldots, \alpha_3$ are constants. Requiring that τ_1 satisfies this ansatz of bilinear equations we find that there is no solution. Similarly, up to degree 8, we find that there are only two possible homogeneous equations (one is of degree 6 and the other is of degree 8). We arrive at

[3] X_2 is not the most general upper triangular element of homogeneous degree -1, as in this case the tau function is too big to be written.

Proposition 4.1. *The following dimension estimates hold true*

$$\dim_{\mathbb{C}} \mathcal{H}_{B_2}^{[\deg\leq 4]} = 0, \quad \dim_{\mathbb{C}} \mathcal{H}_{B_2}^{[\deg\leq 6]} \leq 1, \quad \dim_{\mathbb{C}} \mathcal{H}_{B_2}^{[\deg\leq 8]} \leq 2.$$

Moreover, the only possible elements in $\mathcal{H}_{B_2}^{[\deg\leq 8]}$ *are linear combinations of*

$$(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5)(\tau,\tau) = 0,$$

and

$$(D_1^8 + 7D_1^5 D_3 - 35D_1^2 D_3^2 - 21D_1^3 D_5 - 42D_3 D_5 + 90D_1 D_7)(\tau,\tau) = 0.$$

Remark 4.2. *As far as we know, explicit bilinear equations for the DS hierarchy of B_2-type are not pointed out in the literature, except that there is a super-variable version given in [28]. However, the relationship between the super bilinear equations of Kac–Wakimoto [28] and the DS hierarchy of B_2-type is not known. Finding explicit generating series of bilinear equations for the DS hierarchy of B_2-type remains an open question. It is also interesting to remark that the very same equations are contained in [13], as the first two equations of the BKP hierarchy.*

4.2.4 The D_4 case

Take the matrix realization of $\mathfrak{g}$ as in [14, 5]. Consider the particular point of the Sato Grassmannian of D_4-type given by

$$\gamma = 1 + \lambda\, E_\theta.$$

We put $t_{11} = 0$. It follows from Theorem 1.7 that the corresponding tau function is given by

$$\tau = \Big(1 - \frac{1}{2}\, s_{(7|6)} - \frac{1}{2}\, s_{(6|7)} - \frac{1}{4}\, s_{(7,6|7,6)}\Big)^{\frac{1}{2}}$$

where $s_{(7,6|7,6)} = s_{(7|7)} s_{(6|6)} - s_{(7|6)} s_{(6|7)}$, $s_{(6|6)} = s_{(7|7)} = 0$, and

$$\begin{aligned} s_{(6|7)} = s_{(7|6)} = {} & \frac{t_1^{11}}{1900800} - \frac{1}{480} t_5 t_1^6 + \frac{1}{160} t_3^2 t_1^5 + \frac{1}{120} t_{3'}^2 t_1^5 + \frac{1}{80} t_3 t_{3'} t_1^5 \\ & - \frac{1}{8} t_3^3 t_1^2 - \frac{1}{4} t_3 t_{3'}^2 t_1^2 - \frac{3}{8} t_3^2 t_{3'} t_1^2 + \frac{1}{2} t_5^2 t_1 + \frac{3}{4} t_3^2 t_5 + t_{3'}^2 t_5 + \frac{3}{2} t_3 t_{3'} t_5. \end{aligned} \tag{28}$$

Hence we have

$$\begin{aligned} \tau \;=\; 1 - \frac{1}{2} s_{(7|6)} \;= {} & 1 - \frac{t_1^{11}}{3801600} + \frac{1}{960} t_5 t_1^6 - \frac{1}{320} t_3^2 t_1^5 - \frac{1}{240} t_{3'}^2 t_1^5 - \frac{1}{160} t_3 t_{3'} t_1^5 \\ & + \frac{1}{16} t_3^3 t_1^2 + \frac{1}{8} t_3 t_{3'}^2 t_1^2 + \frac{3}{16} t_3^2 t_{3'} t_1^2 - \frac{1}{4} t_5^2 t_1 - \frac{3}{8} t_3^2 t_5 - \frac{1}{2} t_{3'}^2 t_5 - \frac{3}{4} t_3 t_{3'} t_5. \end{aligned}$$

Proposition 4.3. *The following dimension estimates hold true*

$$\dim_{\mathbb{C}} \mathcal{H}_{D_4}^{[\deg\leq 4]} = 0, \quad \dim_{\mathbb{C}} \mathcal{H}_{D_4}^{[\deg\leq 6]} \leq 3.$$

Moreover, the only possible elements in $\mathcal{H}_{D_4}^{[\deg\leq 6]}$ *are linear combinations of*

$$(2D_1^3 D_{3'} \,+\, 4D_3\, D_{3'} \,-\, 3D_{3'}^2)(\tau,\tau) = 0, \tag{29}$$

$$(D_1^3 D_3 \,-\, D_1^3 D_{3'} \,+\, D_3 D_{3'} \,-\, D_3^2)(\tau,\tau) = 0, \tag{30}$$

$$(D_1^6 \,+\, 9D_1 D_5 \,-\, 10D_1^3 D_3 \,+\, 5D_1^3 D_{3'} \,-\, 5D_3 D_{3'})(\tau,\tau) = 0. \tag{31}$$

Our last remark is that under the following linear change of time variables

$$\begin{aligned}
\partial_{t_1} &\mapsto 2^{-1/6}\partial_{T_1}, \\
\partial_{t_3} &\mapsto 2^{1/2}\partial_{T_3}, \\
\partial_{t_{3'}} &\mapsto 2^{1/2}\partial_{T_3} + 2^{1/2}3^{-1/2}\partial_{T_{3'}}, \\
\partial_{t_5} &\mapsto 2^{7/6}\partial_{T_5}
\end{aligned}$$

the bilinear equations (29)–(31) in the new time variables $T_1, T_3, T_{3'}, T_5$ coincide with those of Kac and Wakimoto [28]. Essentially speaking such a change of times is simply a renormalization of flows.

A List of generalized Schur polynomials of $(\mathfrak{g}, \pi)$-type

Take π as in [14, 10]. We list in Table 1 the first several Schur polynomials of $(\mathfrak{g}, \pi)$-type for simple Lie algebras of low ranks.

$\mathfrak{g}$	A_1	A_2	B_2	B_3	C_2	D_4
s_1	t_1	t_1	0	0	t_1	0
s_2	$\frac{1}{2}t_1^2$	$\frac{1}{2}t_1^2 + t_2$	$\frac{1}{2}t_1$	$\frac{1}{2}t_1$	$\frac{1}{2}t_1^2$	$\frac{1}{2}t_1$
s_{1^2}	$\frac{1}{2}t_1^2$	$\frac{1}{2}t_1^2 - t_2$	$-\frac{1}{2}t_1$	$-\frac{1}{2}t_1$	$\frac{1}{2}t_1^2$	$-\frac{1}{2}t_1$
s_3	$\frac{1}{6}t_1^3 + t_3$	$\frac{1}{6}t_1^3 + t_1t_2$	$\frac{1}{4}t_1^2$	$\frac{1}{4}t_1^2$	$\frac{1}{3}t_1^3 + 2t_3$	$\frac{1}{4}t_1^2$
s_{21}	$\frac{1}{3}t_1^3 - t_3$	$\frac{1}{3}t_1^3$	0	0	$\frac{1}{3}t_1^3 - t_3$	0
s_{1^3}	$\frac{1}{6}t_1^3 + t_3$	$\frac{1}{6}t_1^3 - t_1t_2$	$-\frac{1}{4}t_1^2$	$-\frac{1}{4}t_1^2$	$\frac{1}{3}t_1^3 + 2t_3$	$-\frac{1}{4}t_1^2$
s_4	$\frac{1}{24}t_1^4 + t_3t_1$	$\frac{1}{24}t_1^4 + \frac{1}{2}t_2t_1^2 + \frac{1}{2}t_2^2 + t_4$	$\frac{1}{12}t_1^3 + \frac{1}{2}t_3$	$\frac{1}{12}t_1^3 + \frac{1}{2}t_3$	$\frac{1}{12}t_1^4 + 2t_1t_3$	$\frac{1}{12}t_1^3 + \frac{1}{2}t_3 + t_{3'}$
s_{31}	$\frac{1}{8}t_1^4$	$\frac{1}{8}t_1^4 + \frac{1}{2}t_1^2t_2 - \frac{1}{2}t_2^2 - t_4$	$\frac{1}{12}t_1^3 - t_3$	$\frac{1}{12}t_1^3 - t_3$	$\frac{1}{4}t_1^4$	$\frac{1}{12}t_1^3 - t_3 - t_{3'}$
s_{2^2}	$\frac{1}{12}t_1^4 - t_1t_3$	$\frac{1}{12}t_1^4 + t_2^2$	$\frac{1}{4}t_1^2$	$\frac{1}{4}t_1^2$	$\frac{1}{12}t_1^4 - t_1t_3$	$\frac{1}{4}t_1^2$
s_{21^2}	$\frac{1}{8}t_1^4$	$\frac{1}{8}t_1^4 - \frac{1}{2}t_1^2t_2 - \frac{1}{2}t_2^2 + t_4$	$-\frac{1}{12}t_1^3 + t_3$	$-\frac{1}{12}t_1^3 + t_3$	$\frac{1}{4}t_1^4$	$-\frac{1}{12}t_1^3 + t_3 + t_{3'}$
s_{1^4}	$\frac{1}{24}t_1^4 + t_3t_1$	$\frac{1}{24}t_1^4 - \frac{1}{2}t_2t_1^2 + \frac{1}{2}t_2^2 - t_4$	$-\frac{1}{12}t_1^3 - \frac{1}{2}t_3$	$-\frac{1}{12}t_1^3 - \frac{1}{2}t_3$	$\frac{1}{12}t_1^4 + 2t_1t_3$	$-\frac{1}{12}t_1^3 - \frac{1}{2}t_3 - t_{3'}$

Table 1: Simple Lie algebras and Schur polynomials of $(\mathfrak{g}, \pi)$-type

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