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LARGE TIME BEHAVIOR OF UNBOUNDED SOLUTIONS OF FIRST-ORDER HAMILTON-JACOBI EQUATIONS IN THE WHOLE SPACE

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Abstract. We study the large time behavior of solutions of first-order convex Hamilton-Jacobi Equations of Eikonal type set in the whole space. We assume that the solutions may have arbitrary growth. A complete study of the structure of solutions of the ergodic problem is provided : contrarily to the periodic setting, the ergodic constant is not anymore unique, leading to different large time behavior for the solutions. We establish the ergodic behavior of the solutions of the Cauchy problem (i) when starting with a bounded from below initial condition and (ii) for some particular unbounded from below initial condition, two cases for which we have different ergodic constants which play a role. When the solution is not bounded from below, an example showing that the convergence may fail in general is provided.

1. Introduction

This work is concerned with the large time behavior for unbounded solutions of the first-order Hamilton-Jacobi equation

$$\begin{cases} u_t(x,t) + H(x, Du(x,t)) = l(x), & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(\cdot,0) = u_0(\cdot) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $H \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^N)$ satisfies

(1.2) There exists $\nu \in C(\mathbb{R}^N)$, $\nu > 0$ such that $H(x,p) \geq \nu(x)|p|$,

(1.3) $0 = H(x,0) < H(x,p) \quad$ for $p \neq 0$,

(1.4) $H(x,\cdot)$ is convex,

There exist a constant $C_H > 0$ and, for all $R > 0$, a constant k_R such that

(1.5) $|H(x,p) - H(y,q)| \leq k_R(1+|p|)|x-y| + C_H|p-q|$,

for all $|x|, |y| \leq R$, $p, q \in \mathbb{R}^N$.

We always assume $u_0, l \in C(\mathbb{R}^N)$ and

$$l \geq 0 \quad \text{in } \mathbb{R}^N . \tag{1.6}$$

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These assumptions are those used in the so-called Namah-Roquejoffre case introduced in [25] in the periodic case, and in Barles-Roquejoffre [4] in the unbounded case. They are not the most general but, for simplicity, we choose to state as above since they are well-designed to encompass the classical Eikonal equation

$$u_t(x,t) + a(x)|Du(x,t)| = l(x), \quad \text{in } \mathbb{R}^N \times (0,+\infty), \tag{1.7}$$

where $a(\cdot)$ is a locally Lipschitz, bounded function such that $a(x) > 0$ in $\mathbb{R}^N$. The assumption (1.2) is a coercivity assumption, which may be replaced by (2.2). We also may replace (1.6) by l is bounded from below up to assume that $H(x,0) - \inf_{\mathbb{R}^N} l = 0$ in (1.3).

Our goal is to prove that, under suitable additional assumptions, there exists a unique viscosity solution u of (1.1) and that this solution satisfies

$$u(x,t) + ct \to v(x) \ \text{ in } C(\mathbb{R}^N) \text{ as } t \to +\infty,$$

where $(c,v) \in \mathbb{R}_+ \times C(\mathbb{R}^N)$ is a solution to the ergodic problem

$$H(x, Dv(x)) = l(x) + c \qquad \text{in } \mathbb{R}^N. \tag{1.8}$$

This problem has not been widely studied comparing to the periodic case [13, 25, 5, 14, 12, 6, 1] and references therein. The main works in the unbounded setting are Barles-Roquejoffre [4] which extends the well-known periodic result of Namah-Roquejoffre [25], the works of Ishii [21] and Ichihara-Ishii [18]. A very interesting reference is the review of Ishii [22]. We will compare more precisely our results with the existing ones below but let us mention that our main goal is to make more precise the large time behavior for the Eikonal Equation (1.7) in a setting where the equation is well-posed for solutions with arbitrary growth, which brings delicate issues. Most of our results were already obtained or are close to results of [4, 18] but we use pure PDE arguments to prove them without using Weak KAM methods and making a priori assumptions on the structure of solutions or subsolutions of (1.8).

Changing $u(x,t)$ in $u(x,t) - \inf_{\mathbb{R}^N}\{l\}t$ allows to reduce to the case when $\inf_{\mathbb{R}^N} l = 0$ and we are going to actually reduce to that case to simplify the exposure. Taking this into account, we use below the assumption

$$\liminf_{|x|\to+\infty} l(x) > \inf_{\mathbb{R}^N}(l) = 0. \tag{1.9}$$

which is a compactness assumption in the sense that it implies

$$\mathcal{A} := \operatorname{argmin} l = \{x \in \mathbb{R}^N : l(x) = 0\} \ \text{ is a nonempty compact subset of } \mathbb{R}^N.$$

This subset corresponds to the Aubry set in the framework of Weak KAM theory.

Our first main result collects all the properties we obtain for the solutions of (1.8).

Theorem 1.1. *(Ergodic problem)*
Assume that $0 \le l \in C(\mathbb{R}^N)$ *and* $H \in C(\mathbb{R}^N \times \mathbb{R}^N)$.
(i) If H *satisfies* (1.2) *and* $H(x,0) = 0$ *for all* $x \in \mathbb{R}^N$ *then, for all* $c \ge 0$, *there exists a solution* $(c,v) \in \mathbb{R}_+ \times W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ *of* (1.8).
(ii) Assume that (1.1) *satisfies a comparison principle in* $C(\mathbb{R}^N \times [0,+\infty))$. *If* (c,v) *and* (d,w) *are solutions of* (1.8) *with* $\sup_{\mathbb{R}^N} |v-w| < \infty$, *then* $c = d$.
(iii) If $\mathcal{A} \neq \emptyset$ *and* H *satisfies* (1.2) *and* (1.3), *then there exists a solution* $(c,v) \in \mathbb{R}_+ \times W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ *of* (1.8) *with* $c = 0$ *and* $v \ge 0$. *If, in addition,*

$$H(x,p) \le m(|p|) \text{ for some increasing function } m \in C(\mathbb{R}_+, \mathbb{R}_+), \tag{1.10}$$

and $\mathcal{A}$ *satisfies* (1.9), *then* $v(x) \to +\infty$ *as* $|x| \to \infty$.
(iv) Let $c > 0$. *If* H *satisfies* (1.10) *then any solution* (c,v) *of* (1.8) *is unbounded from below. If* H *satisfies* $H(x,0) = 0$ *and* (1.4) *and* (c,v), (c,w) *are two solutions of* (1.8) *with* $v(x) - w(x) \to 0$ *as* $|x| \to \infty$, *then* $v = w$.

The situation is completely different with respect to the periodic setting where there is a unique ergodic constant (or critical value) for which (1.8) has a solution (e.g., Lions-Papanicolaou-Varadhan [24] or Fathi-Siconolfi [15]). We recover some results of Barles-Roquejoffre [4] and Fathi-Maderna [16], see Remark 2.3 for a discussion. As far as the case of unbounded solutions of elliptic equations is concerned, let us mention the recent work of Barles-Meireles [7] and the references therein.

Coming back to (1.1), when H satisfies (1.5), we have a comparison principle by a "finite speed of propagation" type argument, which allows to compare sub- and supersolutions without growth condition ([19, 23] and Theorem A.1). It follows that there exists a unique continuous solution defined for all time as soon as there exist a sub- and supersolution.

Proposition 1.2. *Assume that* $l \ge 0$ *and* H *satisfies* (1.2) *and* (1.5). *Let* $u_0 \in C(\mathbb{R}^N)$ *and* $c \ge 0$.
(i) There exists a smooth supersolution (c, v^+) *of* (1.8) *satisfying* $u_0 \le v^+$ *in* $\mathbb{R}^N$.
(ii) If there exists a subsolution (c, v^-) *of* (1.8) *satisfying* $v^- \le u_0$ *in* $\mathbb{R}^N$, *then there exists a unique viscosity solution* $u \in C(\mathbb{R}^N \times [0,+\infty))$ *of* (1.1) *such that*

$$v^-(x) \le u(x,t) + ct \le v^+(x) \qquad \text{for all } (x,t) \in \mathbb{R}^N \times [0,+\infty). \tag{1.11}$$

Notice that the existence of a subsolution is given by (1.3) for instance.

We give two convergence results depending on the critical value $c = 0$ or $c > 0$.

Theorem 1.3. *(Large time behavior starting with bounded from below initial data) Assume* (1.2)-(1.3)-(1.4)-(1.5), $l \ge 0$, *and* (1.9). *Then, for every bounded from below initial data* u_0, *the unique viscosity solution* u *of* (1.1) *satisfies*

$$u(x,t) \underset{t\to+\infty}{\to} v(x) \quad \text{locally uniformly in } \mathbb{R}^N, \tag{1.12}$$

where $(0, v)$ *is a solution to* (1.8).

Theorem 1.4. *(Large time behavior starting from particular unbounded from below initial data) Assume* (1.2)-(1.3)-(1.4)-(1.5)*,* $l \geq 0$ *and let* (c, v) *be a solution of* (1.8) *with* $c > 0$*. If there exists a subsolution* $(0, \psi)$ *of* (1.8) *such that the initial data* u_0 *satisfies*

$$\min\{\psi(x), u_0(x)\} - \min\{\psi(x), v(x)\} \to 0 \quad \text{as } |x| \to +\infty, \tag{1.13}$$

then there exists a unique viscosity solution u *of* (1.1) *and* $u(x,t) + ct \to v(x)$ *locally uniformly in* $\mathbb{R}^N$ *as* $t \to +\infty$*.*

Let us comment these results. The first convergence result means that, starting from any bounded from below initial condition (with arbitrary growth from above), the unique viscosity solution of (1.1) converges to a solution (c, v) of the ergodic problem (1.8), which is given by Theorem 1.1(iii), i.e., with $c = 0$ and $v \geq 0$, $v \to +\infty$ at infinity. When u_0 is not bounded from below, even if it is close to a solution of the ergodic problem, we give an example showing that the convergence may fail, see Section 5, where several examples and interpretations in terms of the underlying optimal control problem are given.

To describe the second convergence result, suppose that (1.13) holds with the particular constant subsolution $(0, M)$ for some constant M. In this case, (1.13) is equivalent to $(u_0 - v)(x) \to 0$ when $v(x) \to -\infty$. Since, for $c > 0$, any solution (c, v) of the ergodic problem is necessarily unbounded from below (by Theorem 1.1(iv)), Condition (1.13) may only happen for unbounded from below initial condition u_0. In this sense, Theorem 1.4 sheds a new light on the picture of the asymptotic behavior for (1.1), bringing a positive result for some particular unbounded from below initial data.

Theorem 1.3 and Theorem 1.4 generalize and make more precise [4, Theorem 4.1] and [4, Theorem 4.2] respectively. In [4], H is bounded uniformly continuous in $\mathbb{R}^N \times B(0, R)$ for any $R > 0$ and u_0 is bounded from below and Lipschitz continuous. Our results are also close to [18, Theorem 6.2] as far as Theorem 1.3 is concerned and [18, Theorem 5.3] is very close to Theorem 1.4, see Remark 4.6. In [18], H may have arbitrary growth with respect to p ((1.5) is not required) and the initial condition is bounded from below with possible arbitrary growth from above. The results apply to more general equations than ours. The counterpart is that the unique solvability of (1.1) is not ensured by the assumptions so *the* solution of (1.1) is the one given by the representation formula in the optimal control framework. The assumptions are given in terms of existence of particular sub or supersolutions of (1.8), which may be difficult to check in some cases. Finally, let us point out that the proofs of [4, 18] use in a crucial way the interpretation of (1.1)-(1.8) in terms of control problems and need some arguments of Weak KAM theory. In this work, we give pure PDE proofs, which are interesting by themselves. Finally, let us underline that in the arbitrary unbounded setting, we do not have in hands local Lipschitz bounds, i.e. bounds on $|u_t|, |Du| \leq C$, with C independent of t. These bounds are easy consequences of the coercivity of H in the periodic setting and in the Lipschitz setting of [4]. In the general unbounded case, such bounds require additional restrictive assumptions. Instead, we provide a more involved proof without further assumptions, see the proof of Theorem 1.3.

Let us also mention that several other convergence results are established in [21] and [18] in the case of *strictly* convex Hamiltonian H and [17] is devoted to a precise study in the one dimensional case. We refer again the reader to the review [22] for details and many examples.

The paper is organized as follows. We start by solving the ergodic problem (1.8), see Section 2. Then, we consider the evolution problem (1.1) in Section 3. Section 4 is devoted to the proofs of the theorems of convergence. Finally, Section 5 provides several examples based both on the Hamilton-Jacobi equations (1.1)-(1.8) and on the associated optimal control problem.

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2. The Ergodic problem

Before giving the proof of Theorem 1.1, we start with a lemma based on the coercivity of H.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset and H satisfies* (1.2)*. For every subsolution $(c, v) \in \mathbb{R}_+ \times USC(\overline{\Omega})$ of* (1.8)*, we have $v \in W^{1,\infty}(\Omega)$ and*

$$|Dv(x)| \leq \max_{y \in \overline{\Omega}} \left\{ \frac{l(y) + c}{\nu(y)} \right\} \quad \text{for a.e. } x \in \Omega. \tag{2.1}$$

Remark 2.2. Assumption (1.2) was stated in that way having in mind the Eikonal Equation (1.7) but it can be replaced by the classical assumption of coercivity

$$\lim_{|p| \to +\infty} \inf_{x \in B(0,R)} H(x, p) = +\infty \quad \text{for all } R > 0. \tag{2.2}$$

Proof of Lemma 2.1. Let $B(x_0, R)$ be any ball contained in Ω. Since v is a viscosity subsolution of

$$|Dv(x)| \leq \max_{y \in \overline{\Omega}} \left\{ \frac{l(y) + c}{\nu(y)} \right\} \quad \text{in } \Omega,$$

we see from [1, Proposition 1.14, p.140] that v is Lipschitz continuous in $B(x_0, R)$ with the Lipschitz constant $\max_{\overline{\Omega}} \{\frac{l+c}{\nu}\}$, which implies together with Rademacher theorem

$$|Dv(x)| \leq \max_{y \in \overline{\Omega}} \left\{ \frac{l(y)+c}{\nu(y)} \right\} \quad \text{for a.e. } x \in B(x_0, R).$$

□

We are now able to give the proof of Theorem 1.1.

Proof of Theorem 1.1.

(i) We follow some arguments of the proof of [4, Theorem 2.1]. Fix $c \geq 0$, noticing that $l(x) + c \geq 0$ for every $x \in \mathbb{R}^N$ and recalling that $H(x, 0) = 0$, we infer that 0 is a subsolution of (1.8). For $R > 0$, we consider the Dirichlet problem

$$H(x, Dv) = l(x) + c \;\text{ in } B(0, R), \quad v = 0 \;\text{ on } \partial B(0, R). \tag{2.3}$$

If $p_R \in \mathbb{R}^N$ and $C_R > 0$, $|p_R|$ are big enough, then, using (1.2), $C_R + \langle p_R, x\rangle$ is a supersolution of (2.3). By Perron's method up to the boundary ([11, Theorem 6.1]), the function

$$\begin{aligned} V_R(x) := \sup\{v \in USC(\overline{B}(0, R)) \text{ subsolution of (2.3)} :& \\ 0 \leq v(x) \leq C_R + \langle p_R, x\rangle \text{ for } x \in \overline{B}(0, R)\},& \end{aligned}$$

is a discontinuous viscosity solution of (2.3). Recall that the boundary conditions are satisfied in the viscosity sense meaning that either the viscosity inequality or the boundary condition for the semicontinuous envelopes holds at the boundary. We claim that $V_R \in W^{1,\infty}(\overline{B}(0, R))$ and $V_R(x) = 0$ for every $x \in \partial B(0, R)$, i.e., the boundary conditions are satisfied in the classical sense. At first, from Lemma 2.1, $V_R \in W^{1,\infty}(B(0, R))$. By definition, $V_R \geq 0$ in $\overline{B}(0, R)$, so $(V_R)_* \geq 0$ on $\partial B(0, R)$ and the boundary condition holds in the classical sense for the supersolution. It remains to check that $(V_R)^* \leq 0$ on $\partial B(0, R)$. We argue by contradiction assuming there exists $\hat{x} \in \partial B(0, R)$ such that $(V_R)^*(\hat{x}) > 0$. It follows that the viscosity inequality for subsolutions holds at $\hat{x}$, i.e., for every $\varphi \in C^1(\overline{B}(0, R))$ such that $\varphi \geq (V_R)^*$ over $\overline{B}(0, R)$ with $(V_R)^*(\hat{x}) = \varphi(\hat{x})$, we have $H(\hat{x}, D\varphi(\hat{x})) \leq l(\hat{x}) + c$ and there exists at least one such φ. Consider, for $K > 0$, $\tilde{\varphi}(x) := \varphi(x) - K\langle \frac{\hat{x}}{|\hat{x}|}, x - \hat{x}\rangle$. We still have $\tilde{\varphi} \geq (V_R)^*$ over $\overline{B}(0, R)$ and $(V_R)^*(\hat{x}) = \tilde{\varphi}(\hat{x})$. Therefore $H(\hat{x}, D\varphi(\hat{x}) - K\frac{\hat{x}}{|\hat{x}|}) \leq l(\hat{x}) + c$ for every $K > 0$, which is absurd for large K by (1.2). It ends the proof of the claim.

We set $v_R(x) = V_R(x) - V_R(0)$. By Lemma 2.1, for every $R > R'$, we have

$$|Dv_R(x)| = |DV_R(x)| \leq C_{R'} := \max_{\overline{B}(0,R')} \left\{ \frac{l+c}{\nu} \right\} \quad \text{a.e. } x \in B(0, R'), \tag{2.4}$$

$$|v_R(x)| = |V_R(x) - V_R(0)| \leq C_{R'}R' \quad \text{for } x \in B(0, R'). \tag{2.5}$$

Up to an extraction, by Ascoli's Theorem and a diagonal process, v_R converges in $C(\mathbb{R}^N)$ to a function v as $R \to +\infty$, which still satisfies (2.4)-(2.5). By stability of viscosity solutions, (c, v) is a solution of (1.8).

(ii) Let (c, v) and (d, w) be two solutions of (1.8) and set

$$\begin{aligned} V(x,t) &= v(x) - ct \\ W(x,t) &= w(x) - dt. \end{aligned}$$

To show that $c = d$, we argue by contradiction, assuming that $c < d$. Obviously, V is a viscosity solution of (1.1) with $u_0 = v$ and W is a viscosity solution of (1.1) with $u_0 = w$. Using the comparison principle for (1.1), we get that

$$V(x,t) - W(x,t) \leq \sup_{\mathbb{R}^N}\{v - w\} \qquad \text{for all } (x,t) \in \mathbb{R}^N \times [0, \infty).$$

This means that

$$(d-c)t + v(x) - w(x) \leq \sup_{\mathbb{R}^N}\{v - w\} \qquad \text{for all } (x,t) \in \mathbb{R}^N \times [0, \infty).$$

Recalling that $\sup_{\mathbb{R}^N} |v - w| < \infty$, we get a contradiction for t large enough. By exchanging the roles of v, w, we conclude that $c = d$.

(iii) Let $c = 0$. We apply the Perron's method using in a crucial way $\mathcal{A} \neq \emptyset$. Let $\mathcal{S} = \{w \in USC(\mathbb{R}^N)$ subsolution of (1.8) : $0 \leq w$ and $w = 0$ on $\mathcal{A}\}$ and set

$$v(x) := \sup_{w \in \mathcal{S}} w(x).$$

Noticing that $l + c \geq 0$ and since $H(x, 0) = 0$, we have $0 \in \mathcal{S}$. Let $x \in \mathbb{R}^N$ and $R > 0$ large enough such that $x \in B(0, R)$ and there exists $x_{\mathcal{A}} \in B(0, R) \cap \mathcal{A}$. For all $w \in \mathcal{S}$, by Lemma 2.1, we have

$$0 \leq w(x) \leq w(x_{\mathcal{A}}) + \max_{\overline{B}(0,R)} \left\{\frac{l+c}{\nu}\right\} |x - x_{\mathcal{A}}| \leq 2R \max_{\overline{B}(0,R)} \left\{\frac{l+c}{\nu}\right\},$$

since $w(x_{\mathcal{A}}) = 0$. The above upper-bound does not depend on $w \in \mathcal{S}$, so we deduce that $0 \leq v(x) < +\infty$ for every $x \in \mathbb{R}^N$.

We claim that v is a solution of (1.8). At first, by classical arguments ([3]), v is still a subsolution of (1.8) satisfying $v \geq 0$ in $\mathbb{R}^N$ and $v = 0$ on $\mathcal{A}$. By Lemma 2.1, $v \in W^{1,\infty}_{\rm loc}(\mathbb{R}^N)$. To prove that v is a supersolution, we argue as usual by contradiction assuming that there exists $\hat{x}$ and $\varphi \in C^1(\mathbb{R}^N)$ such that $\varphi \leq v$, $v(\hat{x}) = \varphi(\hat{x})$ and the viscosity supersolution inequality does not hold, i.e., $H(\hat{x}, D\varphi(\hat{x})) < l(\hat{x}) + c$. To reach a contradiction, one slightly modify v near $\hat{x}$ in order to build a new subsolution $\hat{v}$ in $\mathcal{S}$, which is strictly bigger than v near $\hat{x}$. To be able to proceed as in the classical proof, it is enough to check that $\hat{x} \notin \mathcal{A}$; otherwise $\hat{v}$ will not be 0 on $\mathcal{A}$ leading to $\hat{v} \notin \mathcal{S}$. If $\hat{x} \in \mathcal{A}$, then $l(\hat{x}) + c = 0$. By (1.3), we obtain $0 \leq H(\hat{x}, D\varphi(\hat{x})) < l(\hat{x}) + c = 0$, which is not possible. It ends the proof of the claim.

From (1.9), there exists $\epsilon_{\mathcal{A}}, R_{\mathcal{A}} > 0$ such that $l(x) > \min_{\mathbb{R}^N} l + \epsilon_{\mathcal{A}}$ for all $x \in \mathbb{R}^N \setminus B(0, R_{\mathcal{A}})$. By (1.10), v satisfies, in the viscosity sense

$$m(|Dv|) \geq H(x, Dv) \geq l(x) + c \geq \epsilon_{\mathcal{A}} \quad \text{in } \mathbb{R}^N \setminus B(0, R_{\mathcal{A}}).$$

Therefore, for all $x \in \mathbb{R}^N$ and every p in the viscosity subdifferential $D^- v(x)$ of v at x, we have $|p| \geq m^{-1}(\epsilon_{\mathcal{A}}) > 0$. By the viscosity decrease principle [23, Lemma 4.1], for all $B(x, R) \subset \mathbb{R}^N \setminus B(0, R_{\mathcal{A}})$, we obtain

$$\inf_{B(x,R)} v \leq v(x) - m^{-1}(\epsilon_{\mathcal{A}}) R.$$

Since $v \geq 0$, for any $R > 0$ and x such that $|x| > R_{\mathcal{A}} + R$, we conclude $v(x) \geq m^{-1}(\epsilon_{\mathcal{A}}) R$, which proves that $v(x) \to +\infty$ as $|x| \to +\infty$.

(iv) Since $c > 0$, there exists $\alpha > 0$ such that $l(x) + c \geq \alpha$ for all $x \in \mathbb{R}^N$.

To prove that v is unbounded from below, we use again the viscosity decrease principle [23, Lemma 4.1]. By (1.10), v satisfies, in the viscosity sense

$$m(|Dv|) \geq H(x, Dv) \geq \alpha \quad \text{in } \mathbb{R}^N,$$

which implies, for all $R > 0$,

$$\inf_{B(0,R)} v \leq v(0) - m^{-1}(\alpha) R$$

and so v cannot be bounded from below.

For the second part of the result, we argue by contradiction assuming that $v \neq w$. Without loss of generality, there exists $\eta > 0$ and $\hat{x} \in \mathbb{R}^N$ such that $(v - w)(\hat{x}) > 3\eta$. Since $(v - w)(x) \to 0$ as $|x| \to +\infty$, there exists $R > 0$ such that $|(v - w)(x)| < \eta$ when $|x| \geq R$. Up to choose $0 < \mu < 1$ sufficiently close to 1, we have $|(\mu v - w)(\hat{x})| > 2\eta$ and, by compactness of $\partial B(0, R)$, $|(\mu v - w)(x)| < 2\eta$ for all $x \in \partial B(0, R)$. It follows that $M := \max_{\overline{B}(0,R)} \mu v - w$ cannot be achieved at the boundary of $\overline{B}(0, R)$. Consider

$$M_\varepsilon := \max_{x,y \in \overline{B}(0,R)} \left\{ \mu v(x) - w(y) - \frac{|x - y|^2}{\varepsilon^2} \right\},$$

which is achieved at some $(\bar{x}, \bar{y})$. By classical properties ([2, 3]), up to extract some subsequences $\epsilon \to 0$,

$$\begin{aligned} &\frac{|\bar{x} - \bar{y}|^2}{\epsilon^2} \to 0, \\ &\bar{x}, \bar{y} \to x_0 \quad \text{for some } x_0 \in \overline{B}(0, R), \\ &M_\varepsilon \to M. \end{aligned}$$

It follows that $M = (\mu v - w)(x_0)$ and therefore, for ε small enough, neither $\bar{x}$ nor $\bar{y}$ is on the boundary of $\overline{B}(0, R)$. We can write the viscosity inequalities for v

subsolution at $\bar{x}$ and w supersolution at $\bar{y}$ for small ε leading to

$$H(\bar{x}, \frac{\bar{p}}{\mu}) \leq l(\bar{x}) + c,$$
$$H(\bar{y}, \bar{p}) \geq l(\bar{y}) + c,$$

where we set $\bar{p} = 2\frac{(\bar{x}-\bar{y})}{\varepsilon^2}$. Noticing that

$$\mu v(\bar{x}) - w(\bar{x}) \leq \mu v(\bar{x}) - w(\bar{y}) - \frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}$$

and using that w is Lipschitz continuous with some constant C_R in $\overline{B}(0,R)$ by Lemma 2.1, we obtain $|\bar{p}| \leq C_R$. Therefore, up to extract a subsequence $\varepsilon \to 0$, we have $\bar{p} \to p_0$. By the convexity of H,

$$H(\bar{x}, p) = H(\bar{x}, \mu\frac{\bar{p}}{\mu} + (1-\mu)0) \leq \mu H(\bar{x}, \frac{\bar{p}}{\mu}) + (1-\mu)H(\bar{x}, 0).$$

Using $H(\bar{x}, 0) = 0$, we get

$$0 \leq \mu H(\bar{x}, \frac{\bar{p}}{\mu}) - H(\bar{x}, \bar{p}).$$

Subtracting the viscosity inequalities and using the above estimates, we obtain

$$0 \leq \mu H(\bar{x}, \frac{\bar{p}}{\mu}) - H(\bar{x}, \bar{p}) \leq H(\bar{y}, \bar{p}) - H(\bar{x}, \bar{p}) + \mu(l(\bar{x}) + c) - (l(\bar{y}) + c).$$

Sending $\varepsilon \to 0$, we reach $0 \leq (\mu - 1)(l(x_0) + c) \leq (\mu - 1)\alpha < 0$, which is a contradiction. It ends the proof of the theorem. $\square$

Remark 2.3.
(i) In the periodic setting, there is a unique $c = 0$ such that (1.8) has a solution. It is not anymore the case in the unbounded setting where there exist solutions for all $c \geq 0$. The proof is adapted from [4, Theorem 2.1]. Similar issues are studied in [16]. Notice that, when $c < 0$, there is no subsolution (thus no solution) because of (1.3).
(ii) In the periodic setting, the classical proof of existence of a solution to (1.8) ([24]) uses the auxiliary approximate equation

$$\lambda v^\lambda + H(x, Dv^\lambda) = l(x) \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

In our case, it gives only the existence of a solution (c, v) with $c = 0$ but not for all $c \geq 0$.
(iii) Neither the proof using (2.6), nor the proof of Theorem 1.1(i) using the Dirichlet problem (2.3) yields a nonnegative (or bounded from below) solution v of (1.8) for $c = 0$. See Section 5.1 for an explicit computation of the solution of (2.3). It is why we need another proof to construct such a solution. See [7] for the same result in the viscous case.
(iv) For $c = 0$, bounded solutions to the ergodic problem may exist, e.g., when l

is periodic ([24] and the example in Remark 4.5). If $\mathcal{A}$ is bounded, we can prove with similar arguments as in the proof of the theorem that all solutions of the ergodic problem are unbounded.
(v) When $c > 0$, there is no bounded solution to (1.8) even if l is periodic or bounded.
(vi) Theorem 1.1 does not require H to satisfy (1.5) so it applies to more general equations than (1.7), for instance with quadratic Hamiltonians.
(vii) The assumption that a comparison principle in $C(\mathbb{R}^N \times [0, +\infty))$ holds for (1.1) in Theorem 1.1(ii) may seem to be a strong assumption but it is true for the Eikonal equation, i.e., when H satisfies (1.5), see Theorem A.1. In this case, H automatically satisfies (1.10) with $m(r) = C_H r$.

3. The Cauchy problem

In this section we study the Cauchy problem (1.1). We start with some comments about Proposition 1.2 and then we prove it.

Existence and uniqueness are based on the comparison Theorem A.1 without growth condition, which holds when (1.5) is satisfied thanks to the finite speed of propagation. When u_0 is bounded from below and (1.3) holds, $\inf_{\mathbb{R}^N} u_0$ is a subsolution of (1.1) and (1.11) takes the simpler form

$$\inf_{\mathbb{R}^N} u_0 \le u(x,t) + ct \le v^+(x).$$

Proof of Proposition 1.2.
(i) Let

$$v^+(x) := f_0(|x|) + \int_0^{|x|} f_1(s)ds,$$

where

$$\begin{cases} f_0 : \mathbb{R}_+ \to \mathbb{R}_+ \ C^1 \text{ nondecreasing, } f_0'(0) = 0 \text{ and } f_0(|x|) \ge u_0(x) \\ f_1 : \mathbb{R}_+ \to \mathbb{R}_+ \text{ continuous, } f_1(0) = 0 \text{ and } f_1(|x|) \ge \dfrac{l(x) + c}{\nu(x)}, \end{cases}$$

where ν appears in (1.2).

The existence of such functions f_0, f_1 is classical (see [3, Proof of Theorem 2.2] for instance). It is straightforward to see that $v^+ \in C^1(\mathbb{R}^N)$, $v^+ \ge u_0$ and (c, v^+) is a supersolution of (1.8) thanks to (1.2).
(ii) It is obvious that (c, v) is a solution (respectively a subsolution, supersolution) of (1.8) if and only if $V(x,t) = v(x) - ct$ is a solution (respectively a subsolution, supersolution) of (1.1) with initial data $V(x, 0) = v(x)$. We have $v^- \le u_0 \le v^+$, where v^- is the subsolution given by assumption and v^+ is the supersolution built in (i). Using Perron's method and Theorem A.1, which holds thanks to (1.5),

we conclude that there exists a unique viscosity solution $u \in C(\mathbb{R}^N \times [0,+\infty))$ of (1.1) such that

$$v^-(x) - ct \le u(x,t) \le v^+(x) - ct.$$

□

4. Large time behavior of solutions

4.1. **Proof of Theorem 1.3.** We first consider the case when u_0 is bounded. Recalling that $c = 0$ and u is solution of (1.1), we see by Proposition 1.2 that

$$\inf_{\mathbb{R}^N} u_0 \le u(x,t) \le v^+(x), \tag{4.1}$$

where $(0, v^+)$ is a supersolution of (1.8) satisfying $v^+ \ge u_0$.

The first step is to obtain better estimates for the large time behavior of u. To do so, we consider $(0, v_1)$ and $(0, v_2)$ two solutions of (1.8). Such solutions exist from Theorem 1.1(iii) with $c = 0$ and $\mathcal{A} \ne \emptyset$. Moreover, $v_1(x), v_2(x) \to +\infty$ as $|x| \to +\infty$ since $\mathcal{A}$ is supposed to be compact and (1.10) holds because of Assumptions (1.3) and (1.5).

We have

Lemma 4.1. *There exist two constants $k_1, k_2 \ge 0$ such that*

$$v_1(x) - k_1 \le \liminf_{t\to+\infty} u(x,t) \le \limsup_{t\to+\infty} u(x,t) \le v_2(x) + k_2 \quad \text{in } \mathbb{R}^N.$$

As a consequence, for any solutions $(0, v_1)$ and $(0, v_2)$ of (1.8), $v_1 - v_2$ is bounded.

Proof of Lemma 4.1. The proof of third inequality in Lemma 4.1 is obvious : since u_0 is bounded and $v_2(x) \to +\infty$ as $|x| \to +\infty$, there exists k_2 such that $u_0 \le v_2 + k_2$ in $\mathbb{R}^N$. Then, by comparison (Theorem A.1)

$$u(x,t) \le v_2(x) + k_2 \quad \text{in } \mathbb{R}^N \times [0,+\infty) \ ,$$

which implies the lim sup-inequality.

The lim inf-one is less standard. Let $R_{\mathcal{A}} > 0$ be such that $\mathcal{A} \subset B(0, R_{\mathcal{A}}/2)$ and set

$$C_1 = C_1(\mathcal{A}, v_1) := \sup_{\overline{B}(0,R_{\mathcal{A}})} v_1 + 1.$$

Notice that, by definition of $\mathcal{A}$ and (1.9), there exists $\eta_{\mathcal{A}} > 0$ such that

$$l(x) \ge \eta_{\mathcal{A}} > 0 \quad \text{for all } x \in \mathbb{R}^N \setminus B(0, R_{\mathcal{A}}). \tag{4.2}$$

Using that $\min\{v_1, C_1\}$ is bounded from above and u_0 is bounded, there exists $k_1 = k_1(\mathcal{A}, v_1, u_0)$ such that

$$\min\{v_1, C_1\} - k_1 \le u_0 \quad \text{in } \mathbb{R}^N. \tag{4.3}$$

Next we have to examine the large time behavior of the solution associated to the initial condition $\min\{v_1, C_1\} - k_1$ and to do so, we use the following result of Barron and Jensen (see Appendix).

Lemma 4.2. [8] *Assume* (1.4) *and let* $u, \tilde{u}$ *be locally Lipschitz subsolutions (resp. solutions) of* (1.1)*. Then* $\min\{u, \tilde{u}\}$ *is still a subsolution (resp. a solution) of* (1.1)*.*

To use it, we remark that the function $w^-(x,t) := C_1 + \eta_{\mathcal{A}} t$ is a smooth subsolution of (1.1) in $(\mathbb{R}^N \setminus \overline{B}(0, R_{\mathcal{A}})) \times (0, +\infty)$. Indeed, for all $|x| > R_{\mathcal{A}}$, $t > 0$,

$$w_t^- + H(x, Dw^-) = \eta_{\mathcal{A}} + H(x, 0) = \eta_{\mathcal{A}} \leq l(x)$$

using (1.3) and (4.2). Since v_1 is a locally Lipschitz continuous subsolution of (1.1) in $\mathbb{R}^N \times (0, +\infty)$, we can use Lemma 4.2 in $(\mathbb{R}^N \setminus \overline{B}(0, R_{\mathcal{A}})) \times (0, +\infty)$ to conclude that $\min\{v_1, w^-\} - k_1$ is a subsolution, while in a neighborhood of $\overline{B}(0, R_{\mathcal{A}}) \times (0, +\infty)$, we have $\min\{v_1, w^-\} - k_1 = v_1 - k_1$ by definition of C_1.

Then, by comparison (Theorem A.1)

$$\min\{v_1(x), C_1 + \eta_{\mathcal{A}} t\} - k_1 \leq u(x,t) \quad \text{in } \mathbb{R}^N \times [0, +\infty) ,$$

and one concludes easily.

The last assertion of Lemma 4.1 is obvious since v_1, v_2 are arbitrary solutions of (1.8) and we can exchange their roles. □

The next step of the proof of Theorem 1.3 consists in introducing the half-relaxed limits [9, 3]

$$\underline{u}(x) = \liminf_{t \to +\infty} {}_* \, u(x,t), \qquad \overline{u}(x) = \limsup_{t \to +\infty} {}^* \, u(x,t).$$

They are well-defined for all $x \in \mathbb{R}^N$ thanks to (4.1) or Lemma 4.1. We recall that $\underline{u} \leq \overline{u}$ by definition and $\underline{u} = \overline{u}$ if and only if $u(x,t)$ converges locally uniformly in $\mathbb{R}^N$ as $t \to +\infty$. Therefore, to prove (1.12), it is enough to prove $\overline{u} \leq \underline{u}$ in $\mathbb{R}^N$.

A formal direct proof of this inequality is easy: $\overline{u}$ is a subsolution of (1.8), while $\underline{u}$ is a supersolution of (1.8); by Lemma 4.2, for any constant $C > 0$, $\min\{\overline{u}, C\}$ is still a subsolution of (1.8) and Lemma 4.1 shows that $\min\{\overline{u}, C\} - \underline{u} \to -\infty$ as $|x| \to +\infty$. Moreover 0 is a strict subsolution of (1.8) outside $\mathcal{A}$, therefore by comparison arguments of Ishii [20]

$$\max_{\mathbb{R}^N} \{\min\{\overline{u}, C\} - \underline{u}\} \leq \max_{\mathcal{A}} \{\min\{\overline{u}, C\} - \underline{u}\} ,$$

and letting C tend to $+\infty$ gives $\overline{u} - \underline{u} \leq \max_{\mathcal{A}} \{\overline{u} - \underline{u}\}$. But the right-hand side is 0 since $u(x,t)$ is decreasing in t on $\mathcal{A}$ using $H(x,p) \geq 0$ and $l(x) = 0$ if $x \in \mathcal{A}$. This gives the result.

This formal proof, although almost correct, is not correct since we do not have a local uniform convergence of u in a neighborhood of $\mathcal{A}$, in particular because we do not have equicontinuity of the family $\{u(\cdot,t), t \geq 0\}$. To overcome this difficulty, we use some approximations by inf- and sup-convolutions.

For all $\varepsilon > 0$, we introduce

$$u_\varepsilon(x,t) = \inf_{s\in(0,+\infty)} \{u(x,s) + \frac{|t-s|^2}{\varepsilon^2}\},$$
$$u^\varepsilon(x,t) = \sup_{s\in(0,+\infty)} \{u(x,s) - \frac{|t-s|^2}{\varepsilon^2}\}.$$

By (4.1), they are well-defined for all $(x,t) \in \mathbb{R}^N \times [0,+\infty)$ and we have

$$\inf_{\mathbb{R}^N} u_0 \leq u_\varepsilon(x,t) \leq u(x,t) \leq u^\varepsilon(x,t) \leq v^+(x). \tag{4.4}$$

Notice that the infimum and the supremum are achieved in $u_\varepsilon(x,t)$ and $u^\varepsilon(x,t)$ respectively. Moreover Lemma 4.1 still holds for u_ε and u^ε. Taking in the same way the half-relaxed limits for u_ε and u^ε, we obtain (with obvious notations)

$$\inf_{\mathbb{R}^N} u_0 \leq \underline{u_\varepsilon} \leq \underline{u} \leq \overline{u} \leq \overline{u^\varepsilon} \leq v^+ \quad \text{in } \mathbb{R}^N.$$

To prove the convergence result (1.12), it is therefore sufficient to establish

$$\overline{u^\varepsilon} \leq \underline{u_\varepsilon} \quad \text{in } \mathbb{R}^N, \tag{4.5}$$

which is our purpose from now on.

The following lemma, the proof of which is standard and left to the reader, collects some useful properties of u_ε and u^ε.

Lemma 4.3.
(i) The functions u_ε and u^ε converge locally uniformly to u in $\mathbb{R}^N \times [0,+\infty)$ as $\varepsilon \to 0$.

(ii) The functions u_ε and u^ε are Lipschitz continuous with respect to t locally uniformly in space, i.e., for all $R > 0$, there exists $C_{\varepsilon,R} > 0$ such that, for all $x \in B(0,R)$, $t,t' \geq 0$,

$$|u_\varepsilon(x,t) - u_\varepsilon(x,t')|, |u^\varepsilon(x,t) - u^\varepsilon(x,t')| \leq C_{\varepsilon,R}|t-t'|. \tag{4.6}$$

(iii) For all open bounded subset $\Omega \subset \mathbb{R}^N$, here exists $t_{\varepsilon,\Omega} > 0$ with $t_{\varepsilon,\Omega} \to 0$ as $\varepsilon \to 0$, such that u_ε is solution of (1.1) and u^ε is subsolution of (1.1) in $\Omega \times (t_{\varepsilon,\Omega},+\infty)$.

(iv) For all $R > 0$, there exists $C_{\varepsilon,R}, t_{\varepsilon,R} > 0$ such that, for all $t > t_{\varepsilon,R}$, $u_\varepsilon(\cdot,t)$ and $u^\varepsilon(\cdot,t)$ are subsolutions of

$$H(x,Dw(x)) \leq l(x) + c + 2C_{\varepsilon,R}, \qquad \text{in } B(0,R). \tag{4.7}$$

Therefore, $u_\varepsilon(\cdot,t)$ and $u^\varepsilon(\cdot,t)$ are locally Lipschitz continuous in space with a Lipschitz constant independent of t.

We are now ready to prove that $u_\varepsilon(\cdot,t)$ and $u^\varepsilon(\cdot,t)$ converge uniformly on $\mathcal{A}$ as $t\to+\infty$. We follow the arguments of [25] (or alternatively, one may use [10, Theorem I.14]). We fix $R>0$ such that $\mathcal{A}\subset B(0,R)$ and consider $t_{\varepsilon,R}>0$ given by Lemma 4.3. Since $w=u_\varepsilon$ or $w=u^\varepsilon$ is a locally Lipschitz continuous subsolution of (1.1) in $B(0,R)\times(t_{\varepsilon,R},+\infty)$, we have

$$w_t(x,t)\le w_t(x,t)+H(x,Dw(x,t))\le l(x),\quad \text{a.e. } (x,t), \tag{4.8}$$

since $H\ge 0$ by (1.3). Let $x\in\mathcal{A}$, $t>t_{\varepsilon,R}$, and $h,r>0$. We have

$$\begin{aligned} &\frac{1}{|B(x,r)|}\int_{B(x,r)}(w(y,t+h)-w(y,t))dy\\ =\ &\frac{1}{|B(x,r)|}\int_{B(x,r)}\int_t^{t+h} w_t(y,s)dsdy\\ \le\ &\frac{1}{|B(x,r)|}\int_{B(x,r)}\int_t^{t+h} l(y)dsdy\end{aligned}$$

Using the continuity of w,l and $l(x)=0$, and letting $r\to 0$, we obtain

$$w(x,t+h)\le w(x,t)\qquad \text{for all } x\in\mathcal{A},\ t>t_{\varepsilon,R},\ h\ge 0. \tag{4.9}$$

Therefore $t\mapsto w(x,t)$ is a nonincreasing function on $[t_{\varepsilon,R},\infty)$, Lipschitz continuous in space on the compact subset $\mathcal{A}$ (uniformly in time) and bounded from below according to (4.4). By Dini Theorem, $w(\cdot,t)$ converges uniformly on $\mathcal{A}$ as $t\to+\infty$ to a Lipschitz continuous function. Therefore, there exist Lipschitz continuous functions $\phi_\varepsilon,\phi^\varepsilon:\mathcal{A}\to\mathbb{R}$ with $\phi_\varepsilon\le\phi^\varepsilon$ and

$$u_\varepsilon(x,t)\to\phi_\varepsilon(x),\quad u^\varepsilon(x,t)\to\phi^\varepsilon(x),\quad \text{uniformly on } \mathcal{A} \text{ as } t\to+\infty.$$

We now use the previous results to prove the convergence of u on $\mathcal{A}$. By Lemma 4.3(i), we first obtain that (4.9) holds for u. Therefore $t\mapsto u(x,t)$ is nonincreasing for $x\in\mathcal{A}$, so $u(\cdot,t)$ converges pointwise as $t\to+\infty$ to some function $\phi:\mathcal{A}\to\mathbb{R}$. Notice that we cannot conclude to the uniform convergence at this step since we do not know that ϕ is continuous.

We claim that $u_\varepsilon(x,t),u^\varepsilon(x,t)\to\phi(x)$ as $t\to+\infty$, for all $x\in\mathcal{A}$. The proof is similar in both cases so we only provide it for $u_\varepsilon(x,t)$. Let $x\in\mathcal{A}$ and $\overline{s}>0$ be such that

$$u(x,\overline{s})+\frac{|t-\overline{s}|^2}{\varepsilon^2}=u_\varepsilon(x,t)\le u(x,t). \tag{4.10}$$

By (4.4), we have

$$\frac{|t-\overline{s}|^2}{\varepsilon^2}\le v^+(x)-\inf u_0.$$

It follows that $\overline{s} \to +\infty$ as $t \to +\infty$. Thanks to the pointwise convergence $u(x,s) \to \phi(x)$ as $s \to +\infty$, sending t to $+\infty$ in (4.10), we obtain

$$\phi(x) + \limsup_{t\to+\infty} \frac{|t-\overline{s}|^2}{\varepsilon^2} \leq \phi(x),$$

from which we infer $\lim_{t\to+\infty} \frac{|t-\overline{s}|^2}{\varepsilon^2} = 0$. Therefore, by (4.10), $u_\varepsilon(x,t) \to \phi(x)$. The claim is proved, which implies $\phi_\varepsilon = \phi^\varepsilon = \phi$ on $\mathcal{A}$.

At this stage, we can apply here the above formal argument to the locally Lipschitz continuous functions u^ε and u_ε, noticing that $\overline{u^\varepsilon}$ and $\underline{u_\varepsilon}$ are also locally Lipschitz continuous functions. We deduce that

$$\max_{\mathbb{R}^N}\{\min\{\overline{u^\varepsilon}, C\} - \underline{u_\varepsilon}\} \leq \max_{\mathcal{A}}\{\min\{\overline{u^\varepsilon}, C\} - \underline{u_\varepsilon}\} = \max_{\mathcal{A}}\{\min\{\phi, C\} - \phi\},$$

and therefore letting C tend to $+\infty$ we have $\overline{u^\varepsilon} = \underline{u_\varepsilon}$ in $\mathbb{R}^N$.

Recalling that $\underline{u_\varepsilon} \leq \underline{u} \leq \overline{u} \leq \overline{u^\varepsilon}$ in $\mathbb{R}^N$, we have also $\underline{u} = \overline{u}$ in $\mathbb{R}^N$, and the conclusion follows, completing the proof of the case when u_0 is bounded.

We consider now the case when u_0 is only bounded from below (but not necessarely from above). We set $u_0^C = \min\{u_0, C\}$. If w denotes the solution of (1.1) associated to the initial data 0, then, because of the Barron-Jensen results, the solution associated to u_0^C is $\min\{u, w + C\}$.

But, from the first step, we know that (i) w converges locally uniformly to some solution v_1 of (1.8), (ii) $\min\{u, w + C\}$ converges to some solution v_2^C of (1.8) (depending perhaps on C) and (iii) we have (4.1) for u.

Let $\mathcal{K}$ be any compact subset of $\mathbb{R}^N$. If C is large enough in order to have $v_1 + C > v^+$ on $\mathcal{K}$ (the size of such C depends only on $\mathcal{K}$), then for large t, $\min\{u, w + C\} = u$ on $\mathcal{K}$ by the uniform convergence of w to v_1 on $\mathcal{K}$. It follows that u converges locally uniformly to v_2^C on $\mathcal{K}$, which is independent on C. The proof of Theorem 1.3 is complete. $\square$

To conclude this section, we point out the following result which is a consequence of the comparison argument we used in the proof.

Corollary 4.4. *Assume* (1.2)-(1.3)-(1.4)-(1.5), $l \geq 0$ *and* (1.9). *Then, for all bounded from below solutions* $(0, v_1)$ *and* $(0, v_2)$ *of* (1.8), $v_1, v_2 \to +\infty$ *as* $|x| \to +\infty$ *and*

$$\sup_{\mathbb{R}^N}\{v_1 - v_2\} \leq \max_{\mathcal{A}}\{v_1 - v_2\} < +\infty.$$

Remark 4.5. It is quite surprising that, though a lot of different solutions to (1.8) may exist (see Section 5.1), all the bounded from below solutions associated to $c = -\min l$ have the same growth at infinity. This is not true when $\mathcal{A}$ is not compact, e.g., in the periodic case. Consider for instance

$$|Dv| = |\sin(x)| \quad \text{in } \mathbb{R}.$$

For $c = -\min_{\mathbb{R}} |\sin(x)| = 0$, it is possible to build infinitely many solutions with very different behaviors by gluing some branches of cosine functions, see Figure 1.

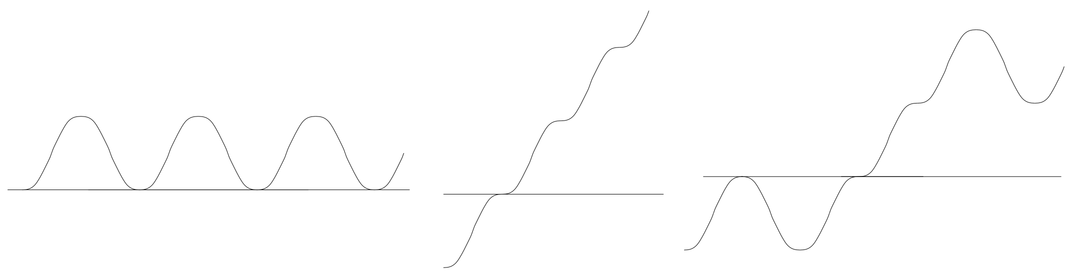

FIGURE 1. Some solutions of $|Dv| = |\sin(x)|$ in $\mathbb{R}$, $\mathcal{A} = \pi\mathbb{Z}$

4.2. **Proof of Theorem 1.4.** By (1.13), there exists a subsolution $(0, \psi)$ of (1.8) such that, for every $\epsilon > 0$, there exists $R_\varepsilon > 0$ such that, for all $|x| \geq R_\varepsilon$,

$$\min\{\psi(x), v(x)\} - \varepsilon \leq \min\{\psi(x), u_0(x)\} \leq \min\{\psi(x), v(x)\} + \varepsilon. \tag{4.11}$$

Let $M_\varepsilon > 0$ be such that

$$-M_\varepsilon \leq u_0(x), v(x) \qquad \text{for all } |x| \leq R_\varepsilon.$$

Setting $\psi_\varepsilon(x) = \min\{\psi(x), -M_\varepsilon\}$, we claim that, for all $x \in \mathbb{R}^N$,

$$\min\{\psi_\varepsilon(x), v(x)\} - \varepsilon \leq \min\{\psi_\varepsilon(x), u_0(x)\} \leq \min\{\psi_\varepsilon(x), v(x)\} + \varepsilon. \tag{4.12}$$

Indeed, this inequality comes from the M-property (4.11) if $|x| \geq R_\varepsilon$, while it is obvious by the choice of M_ε, if $|x| \leq R_\varepsilon$.

From Lemma 4.2, $(0, \psi_\varepsilon)$ is a subsolution of (1.8) as a minimum of subsolutions. Since $c \geq 0$, (c, ψ_ε) is also subsolution of (1.8). Applying again Lemma 4.2 to (c, ψ_ε) and (c, v), we obtain that $(c, \min\{\psi_\varepsilon, v\})$ is a subsolution of (1.8). From (4.12), we have $\min\{\psi_\varepsilon, v\} - \varepsilon \leq u_0$. It follows from Proposition 1.2, that there exists a unique viscosity solution u of (1.1) with initial data u_0 and it satisfies

$$\min\{\psi_\varepsilon, v\} - \varepsilon \leq u(x,t) + ct \leq v^+(x), \tag{4.13}$$

where (c, v^+) is a supersolution of (1.8) such that $u_0 \leq v^+$.

In the same way, there exists unique viscosity solutions w_ε and w of (1.1) associated with initial datas ψ_ε and 0 respectively. Since $\psi_\varepsilon \leq -M_\varepsilon$, by comparison and Proposition 1.2, we have

$$\psi_\varepsilon(x) \leq w_\varepsilon(x,t) \leq -M_\varepsilon + w(x,t) \leq \tilde{v}^+(x) \quad \text{for } (x,t) \in \mathbb{R}^N \times [0, +\infty), \tag{4.14}$$

where $(0, \tilde{v}^+)$ is a supersolution of (1.8) such that $-M_\varepsilon \leq \tilde{v}^+$.

Arguing as at the end of the proof of Theorem 1.3, the solutions of (1.1) associated to the initial datas

$$\min\{\psi_\varepsilon(x), v(x)\} - \varepsilon, \quad \min\{\psi_\varepsilon(x), u_0(x)\} \quad \text{and} \quad \min\{\psi_\varepsilon(x), v(x)\} + \varepsilon$$

are respectively

$\min\{w_\varepsilon(x,t), v(x)-ct\}-\varepsilon,\ \min\{w_\varepsilon(x,t), u(x,t)\}$ and $\min\{w_\varepsilon(x,t), v(x)-ct\}+\varepsilon.$

By comparison, we have, in $\mathbb{R}^N \times (0,+\infty)$

$$\min\{w_\varepsilon(x,t), v(x) - ct\} - \varepsilon \le \min\{w_\varepsilon(x,t), u(x,t)\} ,$$

and

$$\min\{w_\varepsilon(x,t), u(x,t)\} \le \min\{w_\varepsilon(x,t), v(x) - ct\} + \varepsilon.$$

Recalling that c is positive and using (4.14), if $\mathcal{K}$ is a compact subset of $\mathbb{R}^N$, then for t large enough and $x \in \mathcal{K}$

$$\min\{w_\varepsilon(x,t), v(x) - ct\} = v(x) - ct ,$$

leading to the inequality

$$v(x) - ct - \varepsilon \le \min\{w_\varepsilon(x,t), u(x,t)\} \le v(x) - ct + \varepsilon.$$

From (4.13) and (4.14), t can be chosen large enough to have $w_\varepsilon(x,t) + ct > u(x,t) + ct$ so we end up with

$$v(x) - ct - \varepsilon \le u(x,t) \le v(x) - ct + \varepsilon,$$

for t large enough and x in $\mathcal{K}$. Since ε is arbitrary, the conclusion follows. □

Remark 4.6. Theorem 1.4 is very close to [18, Theorem 5.3]. In the latter paper, the authors obtain the convergence assuming that $\inf_{\mathbb{R}^N}\{u_0 - \min\{\psi, v\}\} > -\infty$ and

$$\lim_{r\to+\infty} \{|(u_0 - v)(x)| : \psi(x) > v(x) + r\} = 0. \tag{4.15}$$

We do not know if this assumption is equivalent to ours. But in both assumptions, the point is that $u_0(x)$ must be close to $v(x)$ when $v(x)$ is "far below" $\psi(x)$, which means $\psi(x) > v(x) + r$ for large r in (4.15) and $\min\{\psi(x), -r\} > v(x)$ for large r in our case. This situation occurs for instance if v, u_0 are unbounded from below and close when $v(x) \to -\infty$.

5. Optimal control problem and examples

Consider the one-dimensional Hamilton-Jacobi Equation

$$\begin{cases} u_t(x,t) + |Du(x,t)| = 1 + |x| & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x), \end{cases} \tag{5.1}$$

where $l(x) = |x|+1 \geq 0$, $\min_{\mathbb{R}^N} l = 1$, and $\mathcal{A} = \operatorname{argmin} l = \{0\}$ satisfies (1.9). We can come back to our framework by looking at $\tilde{u}(x,t) = u(x,t) - t$ which solves

$$\tilde{u}_t(x,t) + |D\tilde{u}(x,t)| = |x| \qquad \text{in } \mathbb{R} \times (0,\infty) ,$$

where $\tilde{l}(x) := |x|$ satisfies the assumptions of our results.

There exists a unique continuous solution u of (5.1) for every continuous u_0 satisfying $|u_0(x)| \leq C(1+|x|^2)$ (use Theorem A.1 and the fact that $\pm K e^{Kt}(1+|x|^2)$ are super- and subsolution for large K).

We can represent u as the value function of the following associated deterministic optimal control problem. Consider the controlled ordinary differential equation

$$\begin{cases} \dot{X}(s) = \alpha(s), \\ X(0) = x \in \mathbb{R}, \end{cases} \tag{5.2}$$

where the control $\alpha(\cdot) \in L^\infty([0,+\infty);[-1,1])$ (i.e., $|\alpha(t)| \leq 1$ a.e. $t \geq 0$). For any given control α, (5.2) has a unique solution $X(t) = X_{x,\alpha(\cdot)} = x + \int_0^t \alpha(s)ds$. We define the cost functional

$$J(x,t,\alpha) = \int_0^t (|X(s)|+1)ds + u_0(X(t)),$$

and the value function

$$V(x,t) = \inf_{\alpha \in L^\infty([0,+\infty);[-1,1])} J(x,t,\alpha).$$

It is classical to check that $V(x,t) = u(x,t)$ is the unique viscosity solution of (5.1), see [3, 2].

5.1. **Solutions to the ergodic problem.** There are infinitely many essentially different solutions with different constants to the associated ergodic problem

$$|Dv(x)| = 1 + |x| + c \qquad \text{in } \mathbb{R}. \tag{5.3}$$

Define $S(x) = \int_0^x |y|dy$. The following pairs (c, v) are solutions.

- $(-1, \frac{1}{2}x^2)$ and $(-1, -\frac{1}{2}x^2)$. They are bounded from below (respectively from above) with $c = -\min l$;
- $(-1, S(x))$ and $(-1, -S(x))$. They are neither bounded from below nor from above and $c = -\min l$;
- $(\lambda - 1, \lambda x + S(x))$ and $(\lambda - 1, -\lambda x - S(x))$ for every $\lambda > 0$. They are neither bounded from below nor from above and $c > -\min l$;
- $(\lambda - 1, -\frac{1}{2}x^2 - \lambda|x|)$ for every $\lambda > 0$. These solutions are nonsmooth (notice that $-v$ is not anymore a viscosity solution), they are not bounded from below. Actually, they are the solutions obtained by the constructive proof of Theorem 1.1(i). Indeed, the unique solution V_R of the Dirichlet

problem (2.3) is $V_R(x) = \frac{R^2-x^2}{2} + \lambda(R - |x|)$ for $x \in [-R, R]$, leading to $v(x) = \lim_{R\to\infty}\{V_R(x) - V_R(0)\} = -\frac{1}{2}x^2 - \lambda|x|$.

- (c, v) where (c, v_1) and (c, v_2) are solutions, $v = \min\{v_1 + C_1, v_2 + C_2\}$ and $C_1, C_2 \in \mathbb{R}$. This is a consequence of Lemma 4.2.

5.2. **Equation** (5.1) **with** $u_0(x) = S(x)$**.** For any solution (c, v) to (5.3), it is obvious that $u(x,t) = -ct + v(x)$ is the unique solution to (5.1) with $u_0(x) = v(x)$ and the convergence holds, i.e., $u(x,t) + ct \to v(x)$ as $t \to +\infty$. In particular, if $u_0(x) = S(x)$, the solution of (5.1) is $u(x,t) = t + S(x)$.

Let us find in another way the solution by computing the value function of the control problem stated above. Let $t > 0$. We compute $V(x,t)$ for any $x \in \mathbb{R}$ by determining the optimal controls and trajectories.

1st case: $x \geq 0$.
There are infinitely many optimal strategies: they consist in going as quickly as possible to 0 (= argmin l), to wait at 0 for a while and to go as quickly as possible towards $-\infty$. For any $0 \leq \tau \leq t - x$, it corresponds to the optimal controls and trajectories

$$\alpha(s) = \begin{cases} -1, & 0 \leq s \leq x, \\ 0, & x \leq s \leq x+\tau, \\ -1, & x+\tau \leq s \leq t, \end{cases} \qquad X(s) = \begin{cases} x - s, & 0 \leq s \leq x, \\ 0, & x \leq s \leq x+\tau, \\ -(s - x - \tau), & x+\tau \leq s \leq t. \end{cases} \tag{5.4}$$

They lead to $V(x,t) = J(x,t,\alpha) = t + S(x)$. Among these optimal strategies, there are two of particular interest:

- The first one is to go as quickly as possible to 0 and to remain there ($\tau = t - x$). This strategy is typical of what happens in the periodic case: the optimal trajectories are attracted by $\mathcal{A} = \operatorname{argmin} l$.
- The second one is to go as quickly as possible towards $-\infty$ during all the available time t ($\tau = 0$). This situation is very different to the periodic case. Due to the unbounded (from below) final cost u_0, some optimal trajectories are not anymore atttracted by argmin l and are unbounded.

2nd case: $x < 0$.
In this case there is not anymore bounded optimal trajectories. The only optimal strategy is to go as quickly as possible towards $-\infty$. The optimal control are $\alpha(s) = -1$, $X(s) = x - s$ for $0 \leq s \leq t$ leading to $V(x,t) = J(x,t,-1) = t + S(x)$.

The analysis of this case in terms of control will help us for the following examples.

5.3. **Equation** (5.1) **with** $u_0(x) = \frac{1}{2}x^2 + b(x)$ **with** b **bounded from below.** To illustrate Theorem 1.3, we choose an initial condition which is a bounded perturbation of a bounded from below solution of the ergodic problem. To simplify the computations, we choose a periodic perturbation b.

For any x, an optimal strategy can be chosen among those described in Example 5.2. More precisely: go as quickly as possible to 0, wait nearly until time t and move a little to reach the minimum of the periodic perturbation. For t large enough (at least $t > x$), we compute the cost with α, X given by (5.4),

$$J(x,t,\alpha) = t + \frac{1}{2}x^2 + b(-t + x + \tau).$$

For every t large enough, there exists $0 \le \tau = \tau_t < t - x$ such that $b(-t+x+\tau_t) = \min b$. It leads to

$$V(x,t) = J(x,t,\alpha) = t + \frac{1}{2}x^2 + \min b.$$

Therefore, we have the convergence as announced in Theorem 1.3.

5.4. **Equation** (5.1) **with** $u_0(x) = S(x) + b(x)$ **with** b **bounded Lipschitz continuous.** We compute the value function as above. Due to the unboundedness from below of u_0 we need to distinguish the cases $x \ge 0$ and $x < 0$ as in Example 5.2.

1st case: $x \ge 0$.
We use the same strategy as in Example 5.3 leading to $V(x,t) = J(x,t,\alpha) = t + \frac{1}{2}x^2 + \min b$.

2nd case: $x < 0$.
In this case, the optimal strategy suggested by Examples 5.2 and 5.3 is to start by waiting a small time τ before going as quickly as possible towards $-\infty$. The waiting time correspond to an attempt to reach a minimum of b at the left end of the trajectory. It corresponds to the control and trajectory

$$\alpha(s) = \begin{cases} 0, & 0 \le s \le \tau, \\ -1, & \tau \le s \le t, \end{cases} \qquad X(s) = \begin{cases} x, & 0 \le s \le \tau, \\ x - (s - \tau), & \tau \le s \le t, \end{cases}$$

leading to

$$J(x,t,\alpha) = t + S(x) + \tau|x| + b(x - t + \tau).$$

Due to the boundedness of b, in order to be optimal, we see that necessarily $\tau = O(1/|x|)$ to keep bounded the positive term $\tau|x|$ in $J(x,t,\alpha)$. So, for large $|x|$, $x < 0$, we have $b(x - t + \tau) \approx b(x - t)$. When b is not constant, $b(x - t)$ has no limit as $t \to +\infty$, the convergence for $V(x,t)$ cannot hold.

In this case, u_0 is a bounded perturbation of a solution $(c, v) = (-1, S(x))$ of the ergodic problem with $c = -\min l$ but v is not bounded from below and the convergence of the value function may not hold. It follows that the assumptions of Theorem 1.3 cannot be weakened easily. In particular, the boundedness from below of the solution of the ergodic problem seems to be crucial.

5.5. **Equation** (5.1) **with** $u_0(x) = S(x) + x + \sin(x)$**.** The solution of (5.1) is $u(x,t) = S(x)+x+\sin(x-t)$. Clearly, we do not have the convergence. In this case, u_0 is a bounded perturbation of the solution $(0, S(x)+x)$ of the ergodic problem with $c > -\min l$ and $S(x)+x-u_0(x) \not\to 0$ as $x \to -\infty$ (where $S(x)+x \to -\infty$). This example shows that the convergence in Theorem 1.4 may fail when (1.13) does not hold.

Appendix A. Comparison principle for the solutions of (1.1)

The comparison result for the unbounded solutions of (1.1) is a consequence of a general comparison result for first-order Hamilton-Jacobi equations which holds without growth conditions at infinity.

Theorem A.1. [19, 23] *Assume that H satisfies* (1.5) *and that $u \in USC(\mathbb{R}^N \times [0,T])$ and $v \in LSC(\mathbb{R}^N \times [0,T])$ are respectively a subsolution of* (1.1) *with initial data $u_0 \in C(\mathbb{R}^N)$ and a supersolution of* (1.1) *with initial data $v_0 \in C(\mathbb{R}^N)$. Then, for every $x_0 \in \mathbb{R}^N$ and $r > 0$,*

$$u(x,t) - v(x,t) \le \sup_{\overline{B}(x_0,r)} \{u_0(y) - v_0(y)\} \quad \textit{for every } (x,t) \in \bar{\mathcal{D}}(x_0, r),$$

where

$$\bar{\mathcal{D}}(x_0, r) = \{(x,t) \in B(x_0,r) \times (0,T) : e^{C_H T}(1 + |x - x_0|) - 1 \le r\}.$$

When $\sup_{\mathbb{R}^N}\{u_0 - v_0\} < +\infty$, a straightforward consequence is

$$u(x,t) - v(x,t) \le \sup_{\mathbb{R}^N}\{u_0 - v_0\} \quad \text{for every } (x,t) \in \mathbb{R}^N \times [0, +\infty).$$

Appendix B. Barron-Jensen solutions of convex HJ equations

Theorem B.1. *Assume that H satisfies* (1.4) *and* (1.5)*. Then $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N \times (0,+\infty))$ is a viscosity solution (respectively subsolution) of* (1.1) *if and ony if it is a Barron-Jensen solution (respectively subsolution) of* (1.1)*, i.e., for every $(x,t) \in \mathbb{R}^N \times (0,+\infty)$ and $\varphi \in C^1(\mathbb{R}^N \times (0,+\infty))$ such that $u - \varphi$ has a local minimum at (x,t), one has*

$$\varphi_t(x,t) + H(x, D\varphi(x,t)) = l(x) \quad \textit{(respectively } \le l(x)\textit{)}.$$

This result is due to Barron and Jensen [8] and we refer to Barles [1, p. 89]. Lemmas 4.3(iii) and 4.2 are consequences of this theorem.

As far as Lemma 4.3(iii) is concerned, the fact that the inf-convolution (respectively the sup-convolution) preserves the supersolution (respectively the subsolution) property is classical ([3, 2]). What is more suprising is the preservation of the subsolution property of the inf-convolution which comes from the convexity of H and the Theorem of Barron-Jensen B.1. For a proof, notice first that U, being a solution of (1.1), is a Barron-Jensen solution of (1.1). We then apply [23, Lemma 3.2] using that H, l are independent of t.

For Lemma 4.2, we refer the reader to [1, Theorem 9.2, p.90].

References

[1] Yves Achdou, Guy Barles, Hitoshi Ishii, and Grigory L. Litvinov. *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, volume 2074 of *Lecture Notes in Mathematics*. Springer, Heidelberg; Fondazione C.I.M.E., Florence, 2013. Lecture Notes from the CIME Summer School held in Cetraro, August 29–September 3, 2011, Edited by Paola Loreti and Nicoletta Anna Tchou, Fondazione CIME/CIME Foundation Subseries.

[2] M. Bardi and I. Capuzzo Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Birkhäuser Boston Inc., Boston, MA, 1997.

[3] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*. Springer-Verlag, Paris, 1994.

[4] G. Barles and J.-M. Roquejoffre. Ergodic type problems and large time behaviour of unbounded solutions of Hamilton-Jacobi equations. *Comm. Partial Differential Equations*, 31(7-9):1209–1225, 2006.

[5] G. Barles and P. E. Souganidis. On the large time behavior of solutions of Hamilton-Jacobi equations. *SIAM J. Math. Anal.*, 31(4):925–939 (electronic), 2000.

[6] Guy Barles, Hitoshi Ishii, and Hiroyoshi Mitake. A new PDE approach to the large time asymptotics of solutions of Hamilton-Jacobi equations. *Bull. Math. Sci.*, 3(3):363–388, 2013.

[7] Guy Barles and Joao Meireles. On unbounded solutions of ergodic problems in $\mathbb{R}^m$ for viscous Hamilton-Jacobi equations. *Comm. Partial Differential Equations*, 41(12):1985–2003, 2016.

[8] E. N. Barron and R. Jensen. Semicontinuous viscosity solutions of Hamilton-Jacobi equations with convex hamiltonians. *Comm. Partial Differential Equations*, 15(12):1713–1740, 1990.

[9] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[10] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277(1):1–42, 1983.

[11] F. Da Lio. Comparison results for quasilinear equations in annular domains and applications. *Comm. Partial Differential Equations*, 27(1-2):283–323, 2002.

[12] A. Davini and A. Siconolfi. A generalized dynamical approach to the large time behavior of solutions of Hamilton-Jacobi equations. *SIAM J. Math. Anal.*, 38(2):478–502 (electronic), 2006.

[13] A. Fathi. Sur la convergence du semi-groupe de Lax-Oleinik. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(3):267–270, 1998.

[14] A. Fathi and A. Siconolfi. Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation. *Invent. Math.*, 155(2):363–388, 2004.

[15] A. Fathi and A. Siconolfi. PDE aspects of Aubry-Mather theory for quasiconvex Hamiltonians. *Calc. Var. Partial Differential Equations*, 22(2):185–228, 2005.

[16] Albert Fathi and Ezequiel Maderna. Weak KAM theorem on non compact manifolds. *NoDEA Nonlinear Differential Equations Appl.*, 14(1-2):1–27, 2007.

[17] Naoyuki Ichihara and Hitoshi Ishii. The large-time behavior of solutions of Hamilton-Jacobi equations on the real line. *Methods Appl. Anal.*, 15(2):223–242, 2008.

[18] Naoyuki Ichihara and Hitoshi Ishii. Long-time behavior of solutions of Hamilton-Jacobi equations with convex and coercive Hamiltonians. *Arch. Ration. Mech. Anal.*, 194(2):383–419, 2009.

[19] H. Ishii. Uniqueness of unbounded viscosity solution of Hamilton-Jacobi equations. *Indiana Univ. Math. J.*, 33(5):721–748, 1984.

[20] H. Ishii. A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of eikonal type. *Proc. Amer. Math. Soc.*, 100(2):247–251, 1987.

[21] H. Ishii. Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean n space. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(2):231–266, 2008.

[22] H. Ishii. Asymptotic solutions of Hamilton-Jacobi equations for large time and related topics. In *ICIAM 07—6th International Congress on Industrial and Applied Mathematics*, pages 193–217. Eur. Math. Soc., Zürich, 2009.

[23] O. Ley. Lower-bound gradient estimates for first-order Hamilton-Jacobi equations and applications to the regularity of propagating fronts. *Adv. Differential Equations*, 6(5):547–576, 2001.

[24] P.-L. Lions, B. Papanicolaou, and S. R. S. Varadhan. Homogenization of Hamilton-Jacobi equations. *Unpublished*, 1986.

[25] G. Namah and J.-M. Roquejoffre. Remarks on the long time behaviour of the solutions of Hamilton-Jacobi equations. *Comm. Partial Differential Equations*, 24(5-6):883–893, 1999.

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