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Colorful paths for 3-chromatic graphs

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Abstract

In this paper, we prove that every 3-chromatic connected graph, except C_7, admits a 3-vertex coloring in which every vertex is the beginning of a 3-chromatic path. It is a special case of a conjecture due to S. Akbari, F. Khaghanpoor, and S. Moazzeni, cited in [P.J. Cameron, Research problems from the BCC22, *Discrete Math.* **311** (2011), 1074–1083], stating that every connected graph G other than C_7 admits a $\chi(G)$-coloring such that every vertex of G is the beginning of a colorful path (i.e. a path of on $\chi(G)$ vertices containing a vertex of each color). We also provide some support for the conjecture in the case of 4-chromatic graphs.

Keywords: vertex coloring, colorful path, rainbow coloring

1 Introduction

In this paper, we deal with oriented and non-oriented graphs. When it is not specified, graphs are supposed to be non-oriented. Notations not given here are consistent with [5]. The vertex set of a graph or an oriented graph G is denoted by $V(G)$ and its edge set (or arc set) by $E(G)$[1] Classically, for a vertex x of a graph G, a vertex y with $\{x, y\} \in E(G)$ is called a *neighbour* of x. The set of all the neighbours of x, denoted by $N_G(x)$, is the *neighbourhood* of x in G. In the oriented case, an *out-neighbour* (resp. *in-neighbour*) of a vertex x of an oriented graph G is a vertex y with $xy \in E(G)$ (resp. $yx \in E(G)$). Similarly, the set of all the out-neighbours (resp. in-neighbours) of x in G, denoted by $N_G^+(x)$ (resp. $N_G^-(x)$) is the *out-neighbourhood* (resp. *out-neighbourhood*) of x in G.
In a graph G, we denote by $x_1 \dots x_{\ell+1}$ the *path* of length ℓ on the distinct vertices $\{x_1, \dots, x_{\ell+1}\}$ with edges $\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_\ell, x_{\ell+1}\}$. We denote also by $x_1 \dots x_\ell x_1$ the *cycle* C_ℓ of length ℓ on the distinct vertices $\{x_1, \dots, x_\ell\}$ with edges $\{x_1, x_2\}, \dots, \{x_{\ell-1}, x_\ell\}, \{x_\ell, x_1\}$. Classically, these notions are extended to oriented graphs, where the arcs $x_i x_{i+1}$ replace the edges $\{x_i, x_{i+1}\}$

[1] Throughout the paper, we use the notation xy to indicate the (oriented) arc from x to y, while $\{x, y\}$ designates the (non-oriented) edge between x and y.

(computed modulo ℓ for the oriented cycle C_ℓ).

A *k-(proper) coloring* of a graph G is a mapping $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ if u and v are adjacent in G. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k for which G admits a k-coloring and thus, we say that G is a *$\chi(G)$-chromatic graph*. For a k-coloring of a graph G, a *rainbow path* of G is a path whose vertices have all distinct colors. Given a $\chi(G)$-coloring of G, a rainbow path on $\chi(G)$ vertices is a *colorful path*. In particular a rainbow path is transversal to the set of colors (*i.e.* it has a non empty intersection with every color class). Finding structures transversal to a partition of the ground set is a general problem in combinatorics. Examples arise from Steiner Triple Systems (see [8]), systems of representatives (see [1]) or extremal graph theory (see [10]). Rainbow and colorful paths have been extensively studied in the last few years, see for instance [2], [3], [7], [11] and [12]. In this paper, we concentrate on a conjecture of S. Akbari, F. Khaghanpoor and S. Moazzeni raised in [2] (also cited in [6]).

Conjecture 1 (S. Akbari, F. Khaghanpoor and S. Moazzeni [2])**.** Every connected graph G other than C_7 admits a $\chi(G)$-coloring such that every vertex of G is the beginning of a colorful path.

Conjecture 1 holds for 1-chromatic graphs and 2-chromatic graphs. Indeed in connected bipartite graphs, every vertex is connected to a vertex of another color. The classical proof of Gallai-Roy Theorem also shows that in any $\chi(G)$-coloring of a graph G, there exists at least one colorful path (see [5], for instance). Furthermore, much more is known concerning this conjecture which, through recent, have already received attention. In [3], S. Akbari, V. Liaghat, and A. Nikzad proved that Conjecture 1 is true for the graphs G having a complete subgraph of size $\chi(G)$. They also proved that every graph G admits a $\chi(G)$-coloring such that every vertex is the beginning of a rainbow path on $\lfloor \frac{\chi(G)}{2} \rfloor$ vertices. This result was improved by M. Alishahi, A. Taherkhani and C. Thomassen in [4], who showed that we can obtain rainbow paths on $\chi(G) - 1$ vertices.

In this paper, we give another evidence for Conjecture 1, and prove it for 3-chromatic graphs.

Theorem 2. *Every connected 3-chromatic graph G other than C_7 admits a 3-coloring such that every vertex of G is the beginning of a colorful path.*

The proof of Theorem 2 uses an auxiliary oriented graph build from a coloring of the instance graph. This oriented graph was already used in [3]. In the next section, we recall its definition and strengthen the results known about it to obtain some useful lemmas. In Section 3, we use these tools to derive the proof of Theorem 2. Finally, in Section 4, we conclude the paper with some remarks and open questions. In particular, we prove that Conjecture 1 is true for 4-chromatic graphs containing a cycle of length four.

2 Preliminaries

In this section, $G = (V, E)$ is a connected graph and c is a proper coloring of G with $\chi(G)$ colors. Here, G is not necessarily 3-chromatic and, for short, we write χ instead of $\chi(G)$. In

the following, we will consider modifications of colors and all these modifications have to be understood modulo χ.
As defined in [3], the oriented graph D_c has vertex set V and ab is an arc of D_c if $\{a, b\}$ is an edge of G and the color of b equals the color of a plus one (this oriented graph was first introduced in [9, 13]). A colorful path starting at the vertex x is called a *certifying path for* x. A colorful path $x_1 \dots x_\chi$ is *forward* (resp. *backward*) if for every $i \in \{1, \dots, \chi - 1\}$ we have $c(x_{i+1}) = c(x_i) + 1 \mod \chi$ (resp. $c(x_{i+1}) = c(x_i) - 1 \mod \chi$). Note that a forward (resp. backward) certifying path for a vertex x is an oriented path in D_c on χ vertices starting (resp. ending) at x.

An *initial section* of D_c is a subset X of V such that there is no arc of D_c entering into X (i.e. from $V(G) \setminus X$ to X). The *initial recoloring* of X consists of reducing the color used on each vertex in X by one. We have the following basic facts (which are mentioned in [3], but we recall here their short proofs for the sake of completeness).

Lemma 3 (S. Akbari et al. [3])**.** *An initial recoloring of an initial section is still a proper coloring.*

Proof. Let c be a coloring of G and X an initial section of D_c. We denote by c' the coloring of G obtained after the initial recoloring of X. Let x and y be two adjacent vertices. If both x and y are not in X, we have $c'(x) = c(x) \neq c(y) = c'(y)$. If both x and y are in X, we have $c'(x) = c(x) - 1 \neq c(y) - 1 = c'(y)$. So, by symmetry we may assume that $x \notin X$ and $y \in X$. Since X is an initial section, there is no arc from x to y in D_c and then we have $c(x) \neq c(y) - 1$. Thus we have $c'(x) = c(x) \neq c(y) - 1 = c'(y)$. □

We will intensively use Lemma 3 to prove Theorem 2, and so, without refereeing it precisely. Notice that when performing an initial recoloring on an initial section X, we remove from D_c all the arcs leaving X and possibly add some arcs entering into X (the arcs xy with $\{x, y\} \in E(G)$, $x \notin X$, $y \in X$ and $c(x) = c(y) - 2$). Moreover, we do not create any arc leaving X. Indeed suppose by contradiction that an arc xy is created with $x \in X$ and $y \notin X$, then in the original coloring c, we must have $c(x) = c(y)$, contradicting c being proper. The other arcs, standing inside or outside X remain unchanged.

Similarly, a subset X of vertices is a *terminal section* of D_c if there is no arc leaving X (i.e. from X to $V(G) \setminus X$). The *terminal recoloring* of X consists in adding one to the color of the vertices of X. As for the initial recoloring, this coloring is still proper. Note also that, when performing a terminal recoloring of X, we remove from D_c all the arcs entering into X and possibly add some arcs leaving X (the arcs xy with $\{x, y\} \in E(G)$, $x \in X$, $y \notin X$ and $c(x) = c(y) - 2$).
Using initial and terminal recolorings, we prove some basic facts on the existence of colorful paths. Two colorings c and c' are *identical* on X if $c(x) = c'(x)$ for all $x \in X$.

Lemma 4 (S. Akbari et al. [3])**.** *Let c be a χ-coloring of G and X be a subset of vertices of G. There exists a χ-coloring c' of G identical to c on X such that every vertex is the beginning of an oriented path of $D_{c'}$ which ends in X.*

Proof. Let c' be a χ-coloring of G identical with c on X. We define $Y_{c'}$ as the set of vertices of G which are the beginning of an oriented path in $D_{c'}$ ending in X. The path can have length

0, *i.e.* X is included in $Y_{c'}$. Now, we choose c' a χ-coloring of G identical with c on X with an associated set $Y_{c'}$ of maximal cardinality. Let us prove that $Y_{c'} = V$. Otherwise, notice that, by definition, $Y_{c'}$ is an initial section of D_c, and so that $V \setminus Y_{c'}$ is a terminal section of D_c. Denote by c_t the terminal recoloring of $V \setminus Y_{c'}$. As $X \subset Y_{c'}$, c_t is also identical to c on X. Moreover, the arcs from $Y_{c'}$ to $V(G) \setminus Y_{c'}$ of $D_{c'}$ are not anymore in D_{c_t} and the only arcs which can be created are arcs from $V(G) \setminus Y_{c'}$ to $Y_{c'}$ in D_{c_t}. If no arc from $V(G) \setminus Y_{c'}$ to $Y_{c'}$ is created, we can repeat the terminal recoloring of $V \setminus Y_{c'}$ until such an arc appears. As G is connected, the process must stop at some step, and at least one arc zz' must appear from $V(G) \setminus Y_{c'}$ to $Y_{c'}$. So, c_t is identical to c on X, and we have $Y_{c'} \cup \{z\} \subseteq Y_{c_t}$ which contradicts the maximality of $Y_{c'}$. $\square$

In particular, Lemma 4 implies that Conjecture 1 holds if D_c contains an oriented cycle. Indeed, if C is such a cycle, then C has length a multiple of χ, and so C has length greater or equal than χ. If we apply Lemma 4 with $X = V(C)$, then we obtain a χ-coloring c' such that every vertex of G is the beginning of an oriented path of $D_{c'}$ ending in $V(C)$. So, extending possibly these paths with some vertices and arcs of C, every vertex of G is the beginning of an oriented path of $D_{c'}$ of length χ. Thus, G satisfies Conjecture 1.
As mentioned in [3], note that if G contains a clique of size $\chi(G)$, then for every χ-coloring c of G, D_c contains an oriented cycle, and so G verifies Conjecture 1.

So, we have to focus on the case where D_c is an oriented acyclic graph. We introduce some notations for this case. Let D be an oriented acyclic graph. The *level partition of* D is the unique partition of $V(D)$ into subsets $(V_1, \cdots, V_k)$ such that V_i consists of all sinks of the oriented acyclic graph induced by D on $V \setminus \cup_{j=1}^{i-1} V_j$. As each V_i is the set of sinks of an acyclic induced oriented subgraph of D, it is in particular an independent set. The *height of a vertex* x, denoted by $h_D(x)$, is the index of the level x belongs to in the level partition of D. And, the *height of the partition* is the maximal height of a vertex (i.e. k in our notations, here).
Now, we introduce notations for an oriented acyclic graph D_c associated with a χ-coloring c of G. Assuming that D_c is acyclic, we denote by $(V_1^c, \cdots, V_k^c)$ its level partition. Note that by construction, if xy is an arc of D_c with $x \in V_i$ and $y \in V_j$ we have $i > j$ and $i - j = 1 \mod \chi$. We define the *height of* c as the height of this level partition. It is also the number of vertices in a longest oriented path of D_c. We denote it by $h(c)$, and to shorten notations, we write $h_c(x)$ instead of $h_{D_c}(x)$ to indicate the height of a vertex in the level partition of D_c.
Finally, a χ-coloring c is a *nice coloring* of G if D_c is an oriented acyclic graph with a unique sink. Given a nice coloring c, every vertex of D_c is the beginning of an oriented path which ends at this unique sink. An *in-branching* is an orientation of a tree in which every vertex has out-degree 1, except one vertex, called *the root* of the in-branching. It is well-known that, for a fixed vertex x of a digraph D, every vertex is the beginning of an oriented path ending at x if, and only if, D has a spanning in-branching rooted at x (see [5] Chap. 4 for instance). Thus, c is nice if, and only if, D_c has a spanning in-branching. So, if we apply Lemma 4 with a set X containing a unique vertex v, we obtain a coloring c where every vertex is the beggining of a path ending in v, or equivalently where D_c has an in-branching rooted at v. If v is not a sink of D_c, then D_c contains an oriented cycle. Otherwise D_c has a unique sink v, and then c is a nice

coloring. Thus we have the following.

Corollary 5. *Either G admits a χ-coloring c such that D_c contains an oriented cycle, or for every vertex v of G, there is a nice χ-coloring of G with v as unique sink.*

Note that, given a nice coloring c of G, the vertices belonging to a same level of the level partition of D_c receive the same color by c. Indeed, V_1^c only contains the unique sink r of D_c, and, as every vertex in V_i^c has an out-neighbour in V_{i-1}^c, an easy induction shows that $c(x) = c(r) - i + 1 \mod \chi$ for every $x \in V_i$.
Now, we can establish the following lower bound on the height of D_c, for a nice coloring c of G.

Lemma 6. *Let c be a nice χ-coloring of G. We have $h(c) \geq 2\chi - 1$.*

Proof. Assume by contradiction that $h(c) \leq 2\chi - 2$ for a nice coloring c of G. Denote by r the unique sink of D_c (which forms the level V_1^c), and consider the set $X = V_\chi^c \cup V_{\chi+1}^c \cup \cdots \cup V_{h(c)}^c$. This set X is not empty (otherwise G would have a partition in less then χ independent sets) and is an initial section of D_c. When performing the initial recoloring of X, the color $c(r) + 1$ disappears. Indeed, only the vertices of V_χ^c used this color before the recoloring, and no vertex of X uses it after the recoloring. So, we obtain a $(\chi - 1)$-proper coloring of G, a contradiction. $\square$

As a consequence, in a nice coloring of G, there exists a backward certifying path for the sink of D_c. As, by Lemma 5, every vertex of G can be the sink of D_c for a nice coloring c, or we find an oriented cycle in D_c, it means that for every vertex x, there exists a coloring of G containing a colorful path with end x, what was already proved in [11].
But, we can be more precise. Let c be a nice coloring of G, we denote by B_c the set of vertices of G which have no certifying path. We have seen that the sink of D_c is not in B_c. Moreover, in the level partition $(V_1^c, \cdots, V_k^c)$ of D_c, every vertex in V_i^c has an out-neighbour in V_{i-1}^c, and then, every vertex in V_i^c with $i \geq \chi$ has a forward certifying path. Then, we obtain the following.

Lemma 7. *Let c be a nice coloring of G. We have $B_c \subseteq V_2^c \cup V_3^c \cup \cdots \cup V_{\chi-1}^c$.*

Now, we pay attention to 3-chromatic graphs.

3 Colorful paths for 3-chromatic graphs

In this section, we focus on the special case $\chi = 3$ and prove Theorem 2. Note that, when considering a 3-coloring c of a 3-chromatic graph G, every edge of G appears as an arc of the oriented graph D_c (indeed, for any edge x, y of G, we have $c(x) - c(y) \in \{-1, 1\}$). Furthermore, when we perform an initial recoloring on an initial section X of D_c, all the arcs leaving X become arcs entering into X.

Lemma 8 (S. Akbari et al. [3])**.** *Conjecture 1 is true for every odd cycle except C_7.*

Proof. For the sake of completeness, we just give the coloring yielding the result. Let $C = v_0 v_1 \ldots v_k v'_k v'_{k-1} \ldots v'_2 v'_1 v_0$ be an odd cycle different from C_7. We define the 3-coloring c of C

by $c(v_0) = 3$, $c(v_i) = c(v'_i) = i \mod 3$ for $1 \leq i \leq k-1$, $c(v_k) = k \mod 3$ and $c(v'_k) = k+1 \mod 3$. Now, it is easy to check that if $k \neq 3$ (i.e. $C \neq C_7$), then every vertex is the beginning of a colorful path. $\square$

In the following we assume by contradiction that Theorem 2 is not true and consider a minimal counter-example G (subject to its number of vertices) distinct from C_7. The only consequence of the minimal cardinality of G we use is given by the following claim.

Claim 8.1. *The graph G does not contain any twins, that is, there is no two vertices x and y in G with $N_G(x) = N_G(y)$.*

Proof. Assume that G has two vertices x and y with $N_G(x) = N_G(y)$. First, notice that $G \setminus \{y\}$ is 3-chromatic, as we can extend every coloring of $G \setminus \{y\}$ to G by coloring y with the color of x. Now, if $G \setminus \{y\} = C_7$, then in the coloring of G given Figure 1 every vertex is the beginning of a colorful path. So, $G \setminus \{y\}$ is different from C_7. Since G is a minimum counterexample,

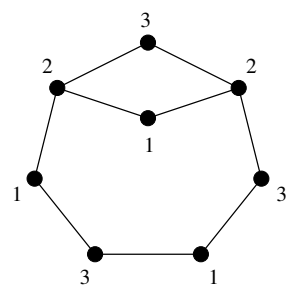

Figure 1: A coloring of the 'twinned C_7' in which every vertex is the beginning of a colorful path.

there is a proper coloring c of $G \setminus \{y\}$ such that every vertex is the beginning of a colorful path. Extending the coloring c to y with $c(y) = c(x)$ provides a certifying path for y since x has one. So, G would not be a counter-example to Theorem 2, a contradiction. $\square$

Now, let c be a nice coloring of G, which exists by Corollary 5. As previously noticed, the associated oriented graph D_c is acyclic since G is a counter-example to Theorem 2. We can describe precisely the structure of D_c as follows. By Lemma 7, the set B_c of vertices which do not have a certifying path is a subset of V_2, the second level in the level partition of D_c. So, if we denote by r_c the unique sink of D_c, every vertex of B_c has a unique out-neighbour which is r_c. Moreover, every vertex b of B_c is not the end of an oriented path of length two in D_c. Thus, either b is a source of D_c or all its in-neighbours are sources of D_c, and by construction of D_c these in-neighbours belong to levels V_i^c with $i = 0 \mod 3$. Finally, D_c has height at least five by Lemma 6 and so, at least one vertex of V_2 is the beginning of a backward certifying path. In particular, we know that B_c is a proper subset of V_2. Figure 2 depicts the situation.

Consider the following special initial recoloring of D_c. For a vertex b of B_c, the previous argument ensures that $\{b\} \cup N^-_{D_c}(b)$ is an initial section of D_c. The *switch recoloring on b* is

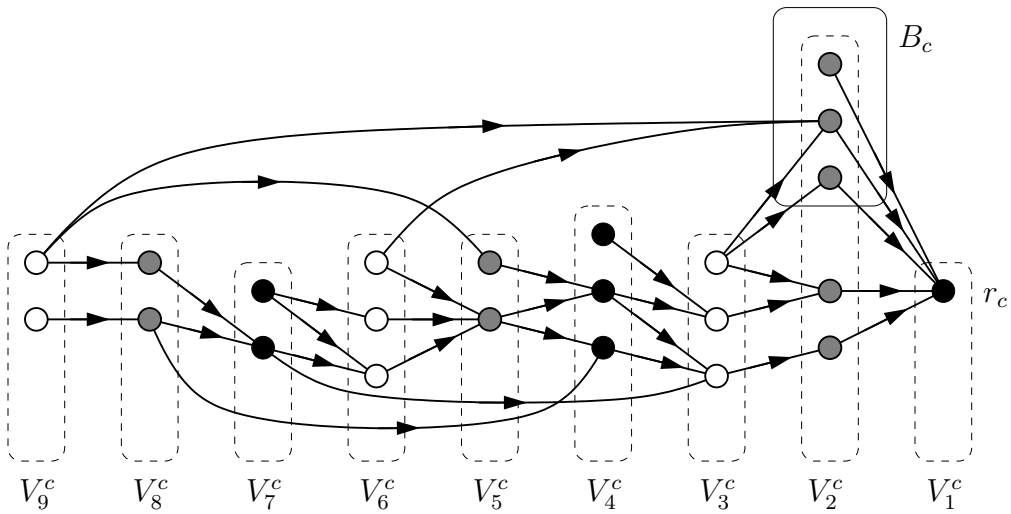

Figure 2: An illustrative example of oriented graph D_c. The vertices colored black, gray and white respectively receive value 3, 2 and 1 by c.

the initial recoloring on $\{b\} \cup N^-_{D_c}(b)$. By Lemma 3, this coloring is proper, and moreover, it satisfies the following properties.

Claim 8.2. *Let b be a vertex of B_c and denote by c' the switch recoloring on b. The oriented graph $D_{c'}$ has the following properties:*

(a) *c' is a nice coloring of G, the unique sink of $D_{c'}$ is b and so $V_1^{c'} = \{b\}$.*

(b) *$V_2^{c'} = \{r_c\} \cup N^-_{D_c}(b)$, $V_3^{c'} = V_2^c \setminus \{b\}$ and for every $i = 4, \ldots, h(c)+1$, if $i = 1 \mod 3$ we have $V_i^{c'} = V_{i-1}^c \setminus N^-_{D_c}(b)$ and if $i \neq 1 \mod 3$ we have $V_i^{c'} = V_{i-1}^c$.*

(c) *If $V_{h(c)}^c \setminus N^-_{D_c}(b) \neq \emptyset$ then $h(c') = h(c)+1$.*

(d) *If $V_{h(c)}^c \subseteq N^-_{D_c}(b)$ then $h(c') = h(c)$.*

(e) *$B_{c'}$ is a subset of $N^-_{D_c}(b)$, i.e. r_c has a certifying path in $D_{c'}$.*

Proof. Denote by N the set $N^-_{D_c}(b)$. The oriented graph $D_{c'}$ is obtained from D_c by reversing all the arcs from $\{b\} \cup N$ to $V \setminus (\{b\} \cup N)$. The vertex r_c is the unique sink of $D_c[V \setminus (\{b\} \cup N)]$ and b is the unique sink of $D_c[\{b\} \cup N]$. Since $r_c b$ is an arc of $D_{c'}$, b is the unique sink of $D_{c'}$ which proves **(a)**.

Now, we will prove that $V_2^{c'} = \{r_c\} \cup N$ and $V_{i-1}^c \setminus (\{b\} \cup N) \subseteq V_i^{c'}$ for $i = 3, \ldots, h(c)+1$. We call $(\star)$ this property. Assuming that $(\star)$ is true, each part of the partition $(\{b\}, \{r_c\} \cup N, V_2^c \setminus (\{b\} \cup N), V_3^c \setminus (\{b\} \cup N), \ldots, V_{h(c)}^c \setminus (\{b\} \cup N))$ of G will be respectively included in the corresponding part of the partition $(V_1^{c'}, \ldots, V_{h(c')}^{c'})$. So, the two partitions will be equal, and using that $N^-_{D_c}(b) \subseteq \bigcup\{V_i^c : i = 0 \mod 3\}$ we have **(b)**. The proof of $(\star)$ runs by induction on i. Let start with the case $i = 2$. As r_c is the unique out-neighbour of b in D_c, we have $d^-_{D_{c'}}(b) = \{r_c\} \cup N$. In $D_{c'}$, the vertex b is the unique out-neighbour of r_c. In $D_{c'}$, the vertex b is also the unique out-neighbour of the vertices of N. Indeed, N is an independent set of G with no in-neighbour in D_c. Moreover, when we perform the switch recoloring on b, we invert

all the arc leaving N except the one with head b. So, we have $V_2^{c'} = \{r_c\} \cup N$. Now, assume that for some integer $i \in \{3, \dots, h(c)+1\}$ the property $(\star)$ is true for all j with $j < i$. Let x be a vertex of $V_{i-1}^c \setminus (\{b\} \cup N)$. The out-neighbours of x in D_c are in $\cup_{k=1}^{i-2} V_k^c$. All these vertices are in $\cup_{k=1}^{i-1} V_k^{c'}$ by induction hypothesis. The other possible out-neighbours of x are vertices of $\{b\} \cup N$ which are in $V_1^{c'} \cup V_2^{c'}$ as previously shown. Therefore, the height of x is at most i in $D_{c'}$. Let y be an out-neighbour of x in D_c such that $y \in V_{i-2}^c$. Since y is not a source in D_c we have $y \notin N$, and as $y \neq b$ (otherwise, x would be in N), we have by induction hypothesis $y \in V_{i-1}^{c'}$. So, the height of x is exactly i and $x \in V_i^{c'}$, which finally proves $(\star)$ and **(b)**.

In particular, **(b)** directly implies **(c)**. It also implies **(d)** easily. Indeed, assume that $V_{h(c)}^c \subseteq N$, then as $V_{h(c)}^c \cap N \neq \emptyset$ we have $h(c) = 0 \mod 3$. Thus, by **(b)**, we have $V_{h(c)+1}^{c'} = V_{h(c)}^c \setminus N = \emptyset$ and $V_{h(c)}^{c'} = V_{h(c)-1}^c \neq \emptyset$. So we obtain $h(c') = h(c)$.

To prove **(e)**, as we know that $B_{c'}$ is a subset of $V_2^{c'}$, which is $\{r_c\} \cup N$ by **(b)**, we just have to check that r_c is the end of a certifying path in $D_{c'}$. Let $P = x_4x_3x_2x_1$ be an oriented path of length 3 in D_c with $x_1 = r_c$ and $x_i \in V_i^c$ for $i = 1, 2, 3, 4$. Such a path exists since the height of c is at least 5 by Lemma 6. As x_3 is not a source in D_c, we have $x_3 \notin N$. As x_2 is certified in c by the path $x_4x_3x_2$, we also have $x_2 \neq b$. So, the oriented path $x_3x_2x_1$ still exists in $D_{c'}$ and is a certifying path in $D_{c'}$ for r_c. So, **(e)** is proved. □

Before going on the main proof, let us establish the following technical result.

Claim 8.3. *Let c be a nice coloring of G and assume that there exists at least one arc from $V_{h(c)}^c$ to B_c in D_c. If $X \subseteq V_{h(c)}^c \cup V_{h(c)-1}^c$ is an initial section of D_c, then every vertex of $D_c \setminus (X \cup B_c)$ has a certifying path lying in $D_c \setminus X$.*

Proof. As X is an initial section of D_c, every forward certifying path starting at a vertex of $D_c \setminus X$ lies in $D_c \setminus X$. Thus every vertex of $D_c \setminus (X \cup V_2^c \cup \{r_c\})$ has a (forward) certifying path in $D_c \setminus X$. We have $X \cap V_3^c = \emptyset$ since $h(c) \geq 5$ and $X \subseteq V_{h(c)}^c \cup V_{h(c)-1}^c$. Thus there exists a path on three vertices starting in V_3^c, ending on r_c and lying in $D_c \setminus X$. This path is a backward certifying path for r_c. To conclude, let z be a vertex of $V_2^c \setminus B_c$. As $z \notin B_c$, z is certified in D_c by a backward path $P = vwz$ on three vertices. By construction we have $h_c(z) = 2 \mod 3$ and then $h_c(w) = 0 \mod 3$ and $h_c(v) = 1 \mod 3$. Note that, since there is an arc from $V_{h(c)}^c$ to B_c, we have $h_c(y) = 0 \mod 3$ for every vertex y of $V_{h(c)}^c$ and then $h_c(y') = 2 \mod 3$ for every vertex y' of $V_{h(c)-1}^c$. Since $h_c(w) = 0$, if $w \in X$ then w must be in $V_{h(c)}^c$, which is impossible since w is not a source. So w is not contained in X. Moreover since $h_c(v) = 1 \mod 3$ and X only contains vertices y satisfying $h_c(y) \in \{0, 2\} \mod 3$, we have $v \notin X$. Thus vwz is a certifying backward path in $D_c \setminus (X \cup B_c)$. □

From now on, we consider a nice coloring of G with maximal height among all the nice colorings of G. This choice implies that if we perform a switch coloring in c, case **(c)** of Claim 8.2 cannot occur. The next claim establishes some properties on the structure of the last levels of D_c for such nice colorings of G.

Claim 8.4. *Let c be a nice coloring of G with maximal height. Then the graph G induces a complete bipartite graph on $V^c_{h(c)} \cup B_c$ with bipartition $(V^c_{h(c)}, B_c)$ and also on $V^c_{h(c)} \cup V^c_{h(c)-1}$ with bipartition $(V^c_{h(c)}, V^c_{h(c)-1})$.*

Proof. We know that B_c, $V^c_{h(c)}$ and $V^c_{h(c)-1}$ form three independent sets of G. By maximality of $h(c)$, for every vertex b of B_c a switch recoloring on b produces case **(d)** of Claim 8.2, so we have $V^c_{h(c)} \subseteq N^-_{D_c}(b)$. Thus G induces a complete bipartite graph on $V^c_{h(c)} \cup B_c$ with bipartition $(V^c_{h(c)}, B_c)$.

Now let us prove that G induces a complete bipartite graph on $V^c_{h(c)} \cup V^c_{h(c)-1}$ with bipartition $(V^c_{h(c)}, V^c_{h(c)-1})$. By contradiction assume that there exist vertices $x \in V^c_{h(c)}$ and $y \in V^c_{h(c)-1}$ such that $\{x, y\} \notin E(G)$. First we will prove that y has an in-neighbour in $V^c_{h(c)}$. Indeed, we consider a vertex b in B_c and remark that y has no out-neighbour which is also an in-neighbour of b. Otherwise let y' be such a vertex and denote by i its level, ie. $y' \in V^c_i$. As $yy' \in E(D_c)$ we would have $h(c) - 1 = i - 1 \mod 3$ and as $y'b \in E(D_c)$ we would have $i = 2 - 1 \mod 3$. So we would get $h(c) = 2 \mod 3$ a contradiction to $h(c) = 0 \mod 3$, previously noticed. So the only neighbours of y which are in-neighbours of b in D_c are in-neighbours of y and lie in $V^c_{h(c)}$. Now we apply a switch recoloring on b to obtain the coloring c'. By assumption G is not a counter-example to Theorem 2, and then $B_{c'}$ contains at least one vertex z. By Claim 8.2 **(e)**, we know that $B_{c'} \subseteq N^-_{D_c}(b)$. Moreover, Claim 8.2 **(b)** ensures that $V^{c'}_{h(c')} = V^c_{h(c)-1}$ (because $h(c) = 0 \mod 3$). Thus y belongs to $V^{c'}_{h(c')}$, and by maximality of $h(c) = h(c')$ the first part of the claim implies that yz is an arc of $D_{c'}$. Thus z was an in-neighbour of b in D_c and is a neighbour of y. By the previous remark we know that $z \in V^c_{h(c)}$ and z is an in-neighbour of y in D_c.

Now, in D_c, we know that y has at least one non neighbour in $V^c_{h(c)}$ (the vertex x) and also at least one in-neighbour in $V^c_{h(c)}$. We denote by Y the set $N^-_{D_c}(y)$, which is contained in $V^c_{h(c)}$, and apply an initial recoloring on the initial section $\{y\} \cup Y$ to obtain the coloring c'. Let us prove that every vertex of G has a certifying path in $D_{c'}$. First, every vertex of $V^c_{h(c)} \cup \{y\} \cup B_c = (V^c_{h(c)} \setminus Y) \cup B_c \cup Y \cup \{y\}$ has a certifying path. Indeed for every vertex $z \in V^c_{h(c)} \setminus Y$ (which is non empty since it contains x), for every vertex $b \in B_c$ and for every vertex z' of Y (which is non empty by the previous paragraph), the oriented path $zbz'y$ exists in $D_{c'}$. Moreover, the oriented graph $D_c \setminus (Y \cup \{y\})$ is unchanged by the recoloring. So it is possible to apply Claim 8.3 with $X = Y \cup \{y\}$ to conclude that every vertex of $D_c \setminus (Y \cup \{y\} \cup B_c)$ has also a certifying path in $D_c \setminus (Y \cup \{y\}) = D_{c'} \setminus (Y \cup \{y\})$.

In all, every vertex of G has a certifying path in $D_{c'}$, a contradiction to the fact that G is a counter-example to Theorem 2. □

The final claim gives more precision on the in-neighbourhood of the vertices of B_c in D_c.

Claim 8.5. *For a coloring c of G with maximal height, every vertex b of B_c satisfies $N^-_{D_c}(b) = V^c_{h(c)}$.*

Proof. Let b be a vertex of B_c. By Claim 8.4 we know that $V^c_{h(c)} \subseteq N^-_G(b)$. So assume that b has an in-neighbour $u \notin V^c_{h(c)}$, and consider the initial recoloring of the initial section $V^c_{h(c)} \cup V^c_{h(c)-1}$

of D_c. We denote by c' the obtained coloring of G, and let us prove that there is a certifying path in c' for every vertex of the graph G. Let z, z', b' be respectively vertices of $V^c_{h(c)}$, $V^c_{h(c)-1}$ and B_c. By Claim 8.4 both $(V^c_{h(c)}, V^c_{h(c)-1})$ and $(B_c, V^c_{h(c)})$ induce complete bipartite graphs in G, and then $b'zz'$ is an oriented path in $D_{c'}$. So, it is a backward certifying path for z' and a forward certifying path of b'. Moreover, ubz is also an oriented path of $D_{c'}$, and then a certifying backward path for z. So every vertex of $B_c \cup V^c_{h(c)} \cup V^c_{h(c)-1}$ has a certifying path in $D_{c'}$. To conclude we notice that the oriented graph $D_c \setminus (V^c_{h(c)} \cup V^c_{h(c)-1})$ is unchanged by the recoloring. So we apply Claim 8.3 with $X = V^c_{h(c)} \cup V^c_{h(c)-1}$ and conclude that every vertex of $D_c \setminus (V^c_{h(c)} \cup V^c_{h(c)-1} \cup B_c)$ has a certifying path in $D_c \setminus (V^c_{h(c)} \cup V^c_{h(c)-1}) = D_{c'} \setminus (V^c_{h(c)} \cup V^c_{h(c)-1})$. In all, every vertex of G has a certifying path in $D_{c'}$, a contradiction to the fact that G is a counter-example to Theorem 2. □

Now, it is possible to conclude the proof of Theorem 2 by applying repeated switch colorings on vertices of B_c. More precisely, let c be a coloring of G with maximal height. By Claim 8.5, the in-neighbourhood of every vertex b of B_c is exactly $V^c_{h(c)}$, and its out-neighbourhood is $\{r_c\}$. So, as G has no twins by Claim 8.1, it means that B_c contains only one vertex which is linked to r_c and to all the vertices of $V^c_{h(c)}$. Then, we consider the following partition of G: $\mathcal{U}_c = (U^c_1, \ldots, U^c_{h(c)+1}) = (V^c_1, B_c, V^c_{h(c)}, V^c_{h(c)-1}, \ldots, V^c_3, V^c_2 \setminus B^c)$. By the previous remark, we have $|U^c_1| = |U^c_2| = 1$ and the neighbourhood of the unique vertex of U^c_2 is included in $U^c_1 \cup U^c_3$. Now, if we apply a switch coloring on the unique vertex of B_c and obtain a coloring c', properties **(b)** and **(d)** of Claim 8.2 imply that $U^{c'}_i = U^c_{i+1}$ for $i = 1, \ldots, h(c)$ and $U^{c'}_{h(c)+1} = U^c_1$. As c' is also a nice coloring of G with maximal height, we also have that $|U^{c'}_2| = |U^c_3| = 1$ and that the neighbourhood of the unique vertex of $U^{c'}_2 = U^c_3$ is $U^{c'}_1 \cup U^{c'}_3 = U^c_4 \cup U^c_2$. By repeating switch colorings, a direct induction shows that, for every i, U^c_i contains exactly one vertex, and the neighbourhood of this vertex is $U^c_{i-1} \cup U^c_{i+1}$. So, the graph G is a cycle, a contradiction to Lemma 8.

4 Concluding remarks

Notice that we can derive a polynomial time algorithm from the proof of Theorem 2. Indeed, one can verify that all the proofs of the lemmas and the claims provide algorithms in order to improve B^c at each step or find a coloring for which every vertex admits a colorful path. So, given a 3-chromatic graph G different from C_7 and a 3-coloring of G, we can find in polynomial time a 3-coloring of G for which every vertex of G is the beginning of a colorful path.

Besides, it seems that the methods used in the proof of Theorem 2 cannot be immediately generalized to graphs with higher chromatic number. We can nevertheless state the following weaker result for 4-chromatic graphs.

Lemma 9. *Every 4-chromatic graph G containing a cycle of length 4 admits a 4-coloring such that every vertex of G is the beginning of a colorful path.*

Proof. Let c be a 4-coloring of a 4-chromatic graph G. Denote by $H = x_1x_2x_3x_4x_1$ a (non necessarily induced) cycle of length 4 of G. Recall that Lemma 4 ensures that the conclusion

holds if D_c contains a circuit. If the four colors appear on H, then up to a permutation on colors, we can assume that $c(x_i) = i$ for $i = 1, 2, 3, 4$. So, H appear as an oriented cycle in D_c and we are done. So we can assume that H is colored with two or three colors. Let us prove that in both cases, either there is a oriented cycle in D_c or the number of colors appearing on H in a 4-coloring of G can be increased.

First assume that only two colors of c appear in H. Up to a permutation on colors, we can assume that x_1, x_3 are colored with 1 and that x_2, x_4 are colored with 2. For a vertex y of G, we denote by $S^-_{D_c}(y)$ the set of vertices of G which are the beginning of an oriented path in D_c with end y. Notice that $S^-_{D_c}(y)$ is always an initial section of D_c. Consider the set $S^-_{D_c}(x_1)$. If it contains x_2 or x_4, it means that D_c has an oriented cycle. So, we assume that $x_2 \notin S^-_{D_c}(x_1)$ and $x_4 \notin S^-_{D_c}(x_1)$. If $x_3 \notin S^-_{D_c}(x_1)$, then we perform an initial recoloring on $S^-_{D_c}(x_1)$ and in the resulting coloring, H receives three colors. Indeed x_2, x_3 and x_4 do not belong to $S^-_{D_c}(x_1)$ and still have the same color (i.e. respectively 2, 1 and 2), and x_1 receive now color 4. Otherwise there is a directed path in D_c from x_3 to x_1. But, by symmetry there also exists a directed path from x_1 to x_3, which provides an oriented cycle in D_c.

Assume now that H receives three colors by c. Up to permutation on colors, we can assume that x_1 is colored with 1, x_2 and x_4 are colored with 2 and x_3 is colored with 3. Consider the set $S^-_{D_c}(x_2)$. If it contains x_3, then D_c has an oriented cycle. So we assume that $x_3 \notin S^-_{D_c}(x_2)$. If $S^-_{D_c}(x_2)$ does not contain x_4, then we apply an initial recoloring on it. As $x_1 \in S^-_{D_c}(x_2)$, x_1, x_2, x_3 and x_4 respectively receive colors 4, 1, 3, 2, and we have a coloring of G with four different colors on H. So, assume that $x_4 \in S^-_{D_c}(x_2)$. It means that there exists an oriented path from x_4 to x_2 in D_c. By symmetry, there exists also a directed path from x_2 to x_4 in D_c and D_c contains an oriented cycle. □

In particular, if a 3-chromatic graph contains a cycle of length three, it appears as an oriented cycle in D_c for any 3-coloring c, and then Lemma 4 ensures that Conjecture 1 holds. If a 4-chromatic graph contains a cycle of length four, then Lemma 9 ensures that Conjecture 1 holds. In both cases the proofs are simple. It raises the following natural question.

Problem 10. Does Conjecture 1 hold for k-chromatic connected graphs containing a cycle of length k?

To conclude, note also that in the case of 2 and 3-chromatic graphs, every colorful path is either forward or backward (recall that forward (resp. backward) means that the color of the i-th vertex is the color of the $(i-1)$-th vertex plus (resp. minus) one). When D_c contains an oriented cycle or in Lemma 9, we obtain certifying paths which are all forward or backward. In [4], M. Alishahi, A. Taherkhani and C. Thomassen provides paths with $\chi - 1$ vertices intersecting $\chi - 1$ colors which are union of at most 2 increasing paths. It raises the following strengthened conjecture.

Conjecture 11. Every k-chromatic connected graph different from C_7 admits a k-coloring such that every vertex is the end of a forward or backward certifying path.

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