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Maslov's canonical operator on a pair of Lagrangian manifolds and asymptotic solutions of stationary equations with localized right-hand sides

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Аннотация

The problem of constructing the asymptotics of the Green's function for the Helmholtz operator $h^2\Delta + n^2(x)$, $x \in \mathbf{R}^n$ with the small positive parameter h and smooth $n^2(x)$ was studied in many papers, e.g. [1, 2, 4]. In the case of variable coefficients, the asymptotics was constructed by matching the asymptotics of the Green's function for the equation with frozen coefficients and WKB–type asymptotics or more generally Maslov's canonical operator. In our work we present a different method for calculating Green's function without assuming any knowledge of exact Green's function for the operator with frozen variables. Our approach also works for a wider class of operators, and also if the right hand side is a smooth localized function rather than δ-function. In particular, the method works for the linearized water waves equation.

1. Introduction. The problem of short-wave asymptotics for Green's function of Helholtz operator in $\mathbf{R}^n$ was studied in several works, e.g. [1], [2, 3], [4] (other references could be fount in book [5]). If the refractive index does not depend on $x \in \mathbf{R}^n$, then the Green's functions can be calculated exactly (e.g. for $n = 2$ it is expressed via Hankel function). If the refractive index is variable, the asympotics for Green's function was constructed in above mentioned papers by pasting the exact solution of the equation with constant refractive index ('frozen' in the point of the source)

with the WKB-asymptotics (see [2]) or more generally with MaslovвЂ™s canonical operator [4], when focal points appear. This approach can not be straightforwardly generalized to different types of operators, because it is necessary to know the exact Green's function for operators with frozen coefficients, and also to have in onevЂ™s disposal effective asymptotic formulas for large $|x|$, and finally be able to paste the latter with WKB-type solutions.

In this problem a geometric object called a 'pair of Lagrangian manifolds' natutally appears. It was first observed by Melrose and Uhlmann [6] (who introduced the term 'Lagrangian intersection') in a close problem of constructing parametrix for operators with real characterstics by methods of Fourier integral operators. In the problem of short-wave asyptotics for right inverse of Helmholtz-type operators the analogous construction was proposed in [7]. However, in that work a Lagrangian manifold with boundary was used rather than pair of Lagrangian manifolds. As a result some terms in their formula have 'parasitic' singularities (which cancel out in sum), making the proposed formula less effecient in concrete calculations.

The problem of Green's function is a limit case of a more physically interesting problem with localized near a point $\xi \in \mathbf{R}^n$ right-hand side instead of DiracвЂ™s δ-function.This problem in turn is a special case of a right-hand side represented with help of Maslov's canonical operator. We construct a formal asymptotic solution of the latter generalized problem, and then study this solution in case of a localized right-hand side. Our approach works for a wide class of operators, and does not appeal to exact solutions of equation with frozen coefficients, and does not lead to 'parasitic' singularities in a formula for solution.

We will use the following terminology. Let $f(x,h)$ be a smooth function of $x = (x_1, \ldots, x_n)$ and parameter $h \in (0,1]$. It will be said that $f = O(h^m)$ in domain $U \subset \mathbf{R}^n$, which may depend on parameter h, if the function f is defined for $x \in U$ for all $h \in (0,1]$, moreover, for any measurable compact

set $K \Subset \mathbf{R}^n$, not depending on h, and all integer $s \geq 0$ it follows that $\|f\|_{s,K\cap U} \leq Ch^m$ for some constant C (depending on K, s, but not h), where

$$\|f\|_{s,\Omega} = \left\{ \sum_{|\alpha|=0}^{s} \left\| h^{|\alpha|} f^{(\alpha)} \right\|^2_{L^2(\Omega)} \right\}^{1/2}.$$

If domain U is not specified explicitly, it is assumed that $U = \mathbf{R}^n$. Let f depend on additional parameters. It is said that $f = O(h^m)$ uniformly with respect to these parameters, if the constants in the above bounds do not depend on the parameters. Finally, we say that $f = O(h^\infty)$, if $f = O(h^m)$ for all m.

2. Statement of the problem. Let $\widehat{\mathcal{H}} = \mathcal{H}(\overset{2}{x}, -ih\overset{1}{\partial}/\partial x, h)$ be an h-pseudo-differential operator with the symbol $\mathcal{H}(x,p,h) \in S^\infty$ being a smooth function of variables $x, p \in \mathbf{R}^n$ and parameter $h \in [0,1]$ that satisfies for some m the bounds

$$\left| \frac{\partial^{|\alpha|+|\beta|+k} \mathcal{H}(x,p,h)}{\partial x^\alpha \partial p^\beta \partial h^k} \right| \leq C_{\alpha\beta k}(1 + |x| + |p|)^m, \qquad |\alpha|, |\beta|, k = 0, 1, 2, \ldots$$

(see [8] for details). Consider in $\mathbf{R}^n$ the inhomogeneous equation:

$$\widehat{\mathcal{H}} u = f(x,h). \tag{1}$$

Definition. A function $u(x,h)$, $x \in \mathbf{R}^n$, $h \in (0,1]$ is called *an asymptotic solution of equation* (1) *with* (*asymptotic*) *limit absorbtion condition*, if there exists a family of functions $u_\varepsilon(x,h)$ depending on parameter $\varepsilon \in (0,1)$ such that $(\widehat{\mathcal{H}} - i\varepsilon)u_\varepsilon = f + O(h^\infty)$ uniformly in ε, and $\|u - u_\varepsilon\|_{s,\Omega} \to 0$ uniformly in h for $\varepsilon \to 0$ for any bounded domain $\Omega \subset \mathbf{R}^n$ and any integer $s \geq 0$.

Remark. The condition stated in this definition is none else than the asymptotic analog of the well-known limit absorbtion principle for Helmholtz equation

We will construct asymptotic solutions of the equation (1) in a sense of the above definition in case, when the right-hand side $f(x,h)$ is represented as follows:

$$f(x,h) = [K^h_{(\Lambda,d\mu)}A](x,h), \tag{2}$$

where $K^h_{(\Lambda,d\mu)}$ — is a Maslov's canonical operator [10], [11] on a Lagrangian manifold $\Lambda \subset \mathbf{R}^{2n}$, satisfying the quantization conditions, and equipped with a smooth measure (volume form) $d\mu$, and $A \in C^\infty(\Lambda)$ is a compactly supported function in Λ. (Recall that [11] the canonical operator is defined uniquely up to arbitrary unimodular factor modulo higher-order terms as $h \to 0$. To define this factor, it is necessary to choose an initial point on Λ and also choose the argument of some Jacobian; we don't reflect these objects in our notation to avoid clumsity.) Particularly, if the Lagrangian manifold and measure are as follows:

$$\Lambda = \{(x,p) \in \mathbf{R}^{2n} \colon x = \xi, p \text{ is arbitrary}\}, \quad d\mu = dp_1 \wedge \cdots \wedge dp_n, \tag{3}$$

the right-hand side as $h \to 0$ near the point $x = \xi$ has a form

$$f(x,h) = \frac{1}{h^{n/2}} V\left(\frac{x-\xi}{h}\right), \text{ where } V(y) = \frac{e^{i\pi n/4}}{(2\pi)^{n/2}} \int\limits_{\mathbf{R}^n} e^{ipy} A(p)\, dp_1 \cdots dp_n \tag{4}$$

is an inverse Fourier transform of the function $A(p)$.

We will describe construction of asymptotic solution for a general right-hand side (2), and then study the equation with right-hand side (4), which is the most important in applications and admits far more constructive solution formula.

3. General case. First of all, we introduce the assumptions:

C0. The principal symbol $H(x,p) = \mathcal{H}(x,p,0)$ of the operator $\widehat{\mathcal{H}}$ is real-valued.

C1. The Hamiltonian vector field $V_H \equiv \frac{d}{dt} = (H_p, -H_x)$ is tranversal to Λ everywhere in $L_0 = \{(x,p) \in \Lambda \colon H(x,p) = 0\}$.

C2. Every trajectory $(X(x_0,p_0,t),P(x_0,p_0,t)) = g_H^t((x_0,p_0))$ of the vector field V_H issueing from $(x_0,p_0)\in L_0$ (a) does not go back to L_0, (b) satisfy the condition $\lim_{t\to T}|X(x_0,p_0,t)| = \infty$, where $[0,T)$, $T\le +\infty$ is the maximal existence interval of the trajectory. (Here the assumption (b) is more essential, sufficient condition for it can be found e.g. in [4]. The condition (P°) has a technical nature, it can be dropped out but theorems would be a bit more complicated.)

Proposition 1. *If conditions* C0–C2 *hold, then:*

1. *The set L_0 is a smooth $(n-1)$-dimensional submanifold in Λ.*

2. *The set $\Lambda_+ = \bigcup_{t\ge 0} g_H^t(L_0)$ is a Lagrangian manifold with boundary L_0 in the space* $\mathbf{R}^{2n}$[1].

3. *The pair (Λ,Λ_+) defines a Lagrangian intersection in sense of* [6]: $T_{(x_0,p_0)}\Lambda\cap T_{(x_0,p_0)}\Lambda_+ = T_{(x_0,p_0)}L_0$ *for* $(x_0,p_0)\in L_0$.

4. *Let $\widetilde{\Lambda}\subset\Lambda$ be sufficiently small neighborhood of the submanifold L_0 in Λ, and $t_0>0$ is sufficiently small. Then for all $t\in[0,t_0]$ the following Lagrangian manifolds $\Lambda_t = g_H^t(\widetilde{\Lambda})$ are well-defined.*

5. *If the manifold Λ meets the quantization conditions, then so do the manifolds Λ_+ and Λ_t, $t\in[0,t_0]$.*

Let us define some other objects that are necessary to construct a canonical operator on Λ_+ and Λ_t.

Without loss of generality, we will assume that the initial point α^* on the manifold Λ is chosen on L_0; let us also choose the same point as initial on Λ_+. Finally, on Λ_t we choose the point $g_H^t(\alpha^*)$. Next, we define a measure on Λ_t as follows: $d\mu_t = (g_H^{t*})^{-1}d\mu$, where $d\mu$ is measure on Λ, and the measure $d\mu_+$ on Λ_+ in the following way. At points of the submanifold $L_0\subset\Lambda$, the measure $d\mu$ may written out as $d\mu = dH\wedge d\sigma$, where $d\sigma$ is some

[1] In case of Green's function for the Helmholtz equation, the manifold Λ_+ was introduced in [4].

(uniquely determinted) measure on L_0. At last, we define the measure $d\mu_+$ on submanifold $g_H^t(L_0) \subset \Lambda_+$ for all $t > 0$, by setting $d\mu_+ = dt \wedge (g_H^{t*})^{-1} d\sigma$.

For trajectories of the vector field V_H, issuing from every point $(x_0, p_0) \in \Lambda$, we determine a function $\widetilde{A}$ as a solution of the transport equation:

$$\frac{d\widetilde{A}}{dt} + H_{sub}(X(x_0, p_0, t), P(x_0, p_0, t))\widetilde{A} = 0, \qquad \widetilde{A}\,\big|_{t=0} = A(x_0, p_0), \tag{5}$$

where $A \in C_0^\infty(\Lambda)$ is the amplitude in formula (2) for the right-hand side, and

$$H_{sub}(x, p) = i\frac{\partial \mathcal{H}}{\partial h}(x, p, 0) - \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2 H}{\partial x_j \partial p_j}(x, p)$$

is a sub-principal symbol of the operator $\widehat{H}$. Let us denote the restrictions of the function $\widetilde{A}$ onto Lagrangian manifolds Λ_+ and Λ_t by A_+ and $A(t)$ respectively. Let $\rho \in C^\infty(\Lambda)$ and $\theta \in C_0^\infty(\mathbf{R}^1)$ be cutoff functions such that

$$\begin{aligned} &\rho(\alpha) = 1 \quad \text{in a neighborhood of } L_0, \quad \operatorname{supp}\rho \subset \Lambda; \\ &\theta(t) = 1 \quad \text{for } |t| \le t_0/2, \qquad \theta(t) = 0 \quad \text{for } |t| \ge t_0. \end{aligned} \tag{6}$$

The function $\theta(t)$ may be interpretted as a function on Λ_+, if we think of t as a time along a trajectory of the field V_H. Similarly, ρ may be thought of as a function on Λ_t, if we assume it to be constant along the trajectories of V_H.

Theorem 1. *If conditions* C0–C2 *are fullfilled, then the equation* (1) *with right-hand side* (2) *has an asymptotic solution* $u = u(x, h)$*, satisfying the asymptotic limit absorbtion principle and* (*if the arguments of Jacobians in definition of canonical operators are chosen appropriately*)[2] *with the*

[2]The choice of arguments of Jacobians will be described in a detailed version of the paper.

principal term as follows:

$$u = K^h_{(\Lambda,d\mu)}\left[\frac{(1-\rho(\alpha))A}{H}\right] + \left(\frac{2\pi}{h}\right)^{1/2} e^{i\pi/4} K^h_{(\Lambda_+,d\mu_+)}\left[(1-\theta(t))A_+\right]$$
$$+ \frac{i}{h}\int\limits_0^{t_0} \exp\left\{\int\limits_{\alpha^*}^{g^t_H(\alpha^*)} p\,dx\right\} K^h_{(\Lambda_t,d\mu_t)}\rho(\alpha)\theta(t)A(t)\,dt + O(h^{1/2}), \tag{7}$$

where the integral in the exponent is calculated along the trajectory of the field V_H.

4. The case of localized right-hand side. Let us study in more detail the particular case of (1), (2), when the right-hand side is the localized function (4). Then the Lagrangian manifold and the measure are (3), and the condition C1 takes the form:

$$H_p(\xi,p) \neq 0 \quad \text{PïCTЭPë} \quad H(\xi,p) = 0.$$

Theorem 2. *If the right-hand side of* (1) *has the form* (4)*, then the solution $u(x,h)$ constructed in Theorem* 1 *$u(x,h)$ has the principle term as follows:*

$$u(x,h) = \frac{e^{i\pi n/4}}{(2\pi h)^{n/2}} \lim_{\varepsilon\to+0} \int\limits_{\mathbf{R}^n} \frac{A(p)e^{\frac{i}{h}p(x-\xi)}}{H(\xi,p)-i\varepsilon}\,dp_1\cdots dp_n + O(h^{2\beta-3/2}) \tag{8}$$

in domain $|x-\xi| < Ch^\beta$, and

$$u(x,h) = \left(\frac{2\pi}{h}\right)^{1/2} e^{i\pi/4}[K^h_{(\Lambda_+,d\mu_+)}(1-\theta(Lt/h^\beta))A_+](x,h) + O(h^{2\beta-3/2}) \tag{9}$$

in domain $|x-\xi| > Ch^\beta$ for all constant $C > 0$, $\beta \in (1/2,1)$, and sufficiently large $L > 0$.

The equality (8) means that the principal term of the asymptotic solution for small $|x-\xi|$ coincides with the solution $u_0(x,h)$ given by the limit

absorbtion principle of the equation (1) with coefficients frozen in the point $x = \xi$. This solution can be calculated without passage to the limit as $\varepsilon \to 0$ (which is difficult from the numerical viewpoint, since the integrand contains a singularity). To do this, one should use the following formula:

$$u_0(x,h) = \frac{e^{i\pi(n+1)/4}}{(2\pi h)^{n/2} h^{1/2}} \int\limits_0^{t_0} \int\limits_{\mathbf{R}^n} e^{\frac{i}{h}(p(x-\xi)-tH(\xi,p))} \theta(t)\rho(p)A(p)\, dp_1 \dots dp_n dt$$
$$+ \left(\frac{i}{2\pi h}\right)^{n/2} \int\limits_{\mathbf{R}^n} \frac{(1-\rho(p))A(p)e^{\frac{i}{h}p(x-\xi)}}{H(\xi,p)}\, dp_1 \cdots dp_n + O(h^{1/2}).$$

Formula (8) describes a part of the solution associated with the Lagrangian manifold Λ, which becomes singular in a limit $A(p) \to 1$ (if it exists). Formula (9) describes rapidly oscillating 'wave' (or 'hyperbolic') part of the solution. One of the key results of our work is that we have an explicit formula for the function (9) in terms of Fourier transform A of the right-hand side (4). This function according to general theory [10], [11] as $h \to +0$ is localized near the projection of the support of A_+ onto $\mathbf{R}^n_x$. The canonical operator admits different representation in different points x, depending on whether they are located on caustics, i.e. a projection onto $\mathbf{R}^n_x$ of the singularity cycle on the manifold Λ_+. Let ϕ be the coordinates on L_0. Then we can choose as coordinates on Λ_+ the pair (τ, ϕ), where $\tau \geq 0$ is the Hamiltonian time along V_H. Thus, the manifold Λ_+ is given by the equations: $(x,p) = (\mathcal{X}(\tau,\phi), \mathcal{P}(\tau,\phi))$. The point (τ,ϕ) on Λ_+ is said to be regular, if the Jacobian $J(\tau,\phi) = \det \frac{\partial \mathcal{X}}{\partial \mu_+}(\tau,\phi)$ does not vanish in it. If the point x belongs to the domain $|x-\xi| > Ch^\beta$, and its pre-image on the manifold Λ_+ is the unique and regular point (τ,ϕ), then the principal

part of the solution (9) near x may be written out as

$$u(x,h)=\frac{\sqrt{2\pi}A_+(\tau,\phi)}{\sqrt{h|J(\tau,\phi)|}}e^{\frac{i}{h}S(\tau,\phi)-\frac{i\pi}{2}(m-\frac{1}{2})},\quad S(\tau,\phi)=\int\limits_0^\tau p\cdot H_p\Big|_{\substack{x=\mathcal{X}(\phi,\eta)\\ p=\mathcal{P}(\phi,\eta)}}d\eta,$$

$$m=\operatorname{ind}\gamma+s_-\left(\begin{pmatrix}0 & H_p^*(\mathcal{X}(0,\phi_0),\xi)\\ H_p(\mathcal{X}(0,\phi_0),\xi) & H_{pp}(\mathcal{X}(0,\phi_0),\xi)\end{pmatrix}\right).$$

Here ϕ_0 is the point on L_0, where the determinant of the described above self-adjoint matrix $\mathcal{A}$ does not vanish. Let $s_-(\mathcal{A})$ be its negative inertia index, and $\operatorname{ind}\gamma$ the Maslov's index [10], [11] of the path γ, connecting in Λ_+ two regular points: $(+0,\phi_0)$ and (τ,ϕ).

If points ϕ_0 with mentioned earlier properties do not exist, then the formula for m becomes more complicated. This formula along with a solution expression near caustics (CŕPj.[11]) will be presented in a detailed paper.

Remark. Under additional assumptions in formulas (8) and (9) one can set $A=1$, then one obtains the Green's function asymptotics. And vice versa, if one knows the asymptotics for Green's function, then one can write out a solution of (1) as a convolution with the right-hand side. However, the direct way of finding the asymptotic formulas from Theorems 1 and 2 in our opinion is simpler and more naturally from the viewpoint of applications.

5. Examples. Let $n=2$. Consider the Helmholtz equation $\mathcal{H}(x,p,h)=H(x,p)=p^2-\mathbf{n}^2(x)$, $\mathbf{n}^2(x)>0$, and the linear water waves equation [12] ($H(x,p)=|p|\tanh(|p|D(x))-\omega^2$ and $H_{sub}(x,p)=0$). Here the smooth function $D(x)$ determines the depth, and $\omega>0$ is the frequency. Let us study two forms of right-hand sides (4), in which

$$V(y)=\exp\left[-\frac{1}{2}\left(\frac{y_1^2}{a^2}+\frac{y_2^2}{b^2}\right)\right]\text{ or }(a_1y_1+b_1y_2)\exp\left[-\frac{1}{2}\left(\frac{y_1^2}{a^2}+\frac{y_2^2}{b^2}\right)\right]$$

and, respectively,

$$A(p)=ab\exp\left[-\frac{a^2p_1^2+b^2p_2^2}{2}\right]\text{ or }iab(a^2a_1p_1+b^2b_1p_2)\exp\left[-\frac{a^2p_1^2+b^2p_2^2}{2}\right].$$

Let $\xi = 0$. Then $L_0 = \{x = 0,\, p = \rho(\cos\phi, \sin\phi)\}$, where $\rho = |\mathbf{n}(0)|$ for Helmholtz equation, and ρ is the root of the equation $\rho \tanh(\rho D(0)) = \omega^2$ for the water waves problem. As coordinates on Λ_+ we choose (ϕ, t), where $t \geq 0$ is the Hamiltonian time along V_H, and ϕ is the initial point on L_0. The amplitude A_+ has the following form:

$$A_+(\phi, t) = abE(\phi), \quad \text{where} \quad E(\phi) = \exp\Big(-\frac{\rho^2(a^2\cos^2\phi + b^2\sin^2\phi}{2}\Big), \quad \text{(a)}$$

$$A_+(\phi, t) = iab\rho(a^2 a_1 \cos\phi + b^2 b_1 \sin\phi)E(\phi) \quad \text{(b)}$$

respectively for the first and second form of the right-hand side. In case (a) the maximal amplitude is attained for smaller values of ρ, and in case (b) this maximum will be at the point $\rho = 1/\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}$, $\tan\phi = \frac{b^2 b_1}{a^2 a_1}$. Thus, by varying the parameters ω, a and b, we can increase or decrease the amplitude in a given domain of the space $\mathbf{R}^2_x$. For the Helmholtz equation with constant refractive index $\mathbf{n}^2(x)$, and $A = 1$, theorem gives the Green's function as a Hankel function, along with its asymptotics for large x. At last, we remark that the Maupertuis-Jacobi principle the initial Hamiltonian for Helmholtz and water waves equation can be replaced with $C(x)|p|$, which simplifies the computer implementation of canonical operator [13].

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