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ON DAVENPORT CONSTANT

BENJAMIN GIRARD

ABSTRACT. We prove that for every fixed integer $r \geq 1$ the Davenport constant $D(C_n^r)$ is asymptotic to rn when n tends to infinity. An extension of this theorem to a wider framework is also provided.

For every integer $n \geq 1$, let C_n be the cyclic group of order n . It is well known that every non-trivial finite Abelian group G can be uniquely decomposed as a direct product of cyclic groups $C_{n_1} \oplus \cdots \oplus C_{n_r}$ such that $1 < n_1 \mid \cdots \mid n_r \in \mathbb{N}$. The integers r and n_r - sometimes also denoted by $\exp(G)$ - appearing in this decomposition are respectively called the rank and exponent of G . For every integer $1 \leq d \mid \exp(G)$, we denote by G_d the subgroup of G consisting of all elements of order dividing d .

Any finite sequence S of ℓ elements of G will be called a sequence over G of length $|S| = \ell$. Also, we denote by $\sigma(S)$ the sum of all elements in S . The sequence S will be referred to as a zero-sum sequence whenever $\sigma(S) = 0$.

By $D(G)$ we denote the smallest integer $t \geq 1$ such that every sequence S over G of length $|S| \geq t$ contains a non-empty zero-sum subsequence. This number, which is called the Davenport constant, drew over the last fifty years an ever growing interest, most notably in additive combinatorics and algebraic number theory. A detailed account on the many aspects of this invariant can be found in [8, 10, 13, 17].

To name but one striking feature, let us recall the Davenport constant has the following arithmetical interpretation. Given the ring of integers $\mathcal{O}_{\mathbf{K}}$ of some number field \mathbf{K} with ideal class group G , the maximum number of prime ideals in the decomposition of an irreducible element of $\mathcal{O}_{\mathbf{K}}$ is $D(G)$ [22]. The importance of this fact is best highlighted by the following generalization of the prime number theorem [17, Theorem 9.15], stating that the number $F(x)$ of pairwise non-associated irreducible elements in $\mathcal{O}_{\mathbf{K}}$ whose norms do not exceed x in absolute value satisfies,

$$F(x) \underset{x \rightarrow +\infty}{\sim} C \frac{x}{\log x} (\log \log x)^{D(G)-1},$$

with a suitable constant $C > 0$ depending solely on G .

We are thus naturally led to the problem of determining the exact value of $D(G)$. The best explicit bounds known so far are

$$(1) \quad \sum_{i=1}^r (n_i - 1) + 1 \leq D(G) \leq n_r \left(1 + \log \frac{|G|}{n_r} \right).$$

The lower bound follows easily from the fact that if (e_1, \dots, e_r) is a basis of G such that $\text{ord}(e_i) = n_i$ for all $i \in \llbracket 1, r \rrbracket$, the sequence S consisting of $n_i - 1$ copies of e_i for each $i \in \llbracket 1, r \rrbracket$ contains no non-empty zero-sum subsequence. The upper bound first appeared in [7, Theorem 7.1] and was rediscovered in [16, Theorem 1]. See also [1, Theorem 1.1] for a reformulation of the proof's original argument as well as an application of the Davenport constant to the study of Carmichael numbers.

$D(G)$ has been proved to match the lower bound in (1) when G is either a p -group [18] or has rank at most 2 [19, Corollary 1.1]. Eventhough there are infinitely many finite Abelian groups whose Davenport constant is known to exceed this lower bound [7, 11, 12, 15], none of the ones identified so far either have rank 3 or the form

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C_n^r . These two types of groups are still conjectured to have a Davenport constant matching the lower bound in (1).

Conjecture 1. *For all integers $n, r \geq 1$,*

$$D(C_n^r) = r(n-1) + 1.$$

Besides the already mentioned results settling Conjecture 1 for all r when n is a prime power and for all n when $r \leq 2$, note that $D(C_n^3)$ is known only when $n = 2p^\alpha$, with p prime and $\alpha \geq 1$ [6, Corollary 4.3], or $n = 2^\alpha 3$ with $\alpha \geq 2$ [7, Corollary 5.1], and satisfies Conjecture 1 in both cases. To the best of our knowledge, the exact value of $D(C_n^r)$ is currently unknown for all pairs (n, r) such that n is not a prime power and $r \geq 4$. In all those remaining cases, the bounds in (1) translate into

$$(2) \quad r(n-1) + 1 \leq D(C_n^r) \leq n(1 + (r-1)\log n),$$

which leaves a substantial gap to be bridged. Conjecture 1 thus remains wide open.

The aim of the present note is to clarify the behavior of $D(C_n^r)$ for any fixed $r \geq 1$ when n goes to infinity. Our main theorem proves Conjecture 1 in the following asymptotic sense.

Theorem 1. *For every integer $r \geq 1$,*

$$D(C_n^r) \underset{n \rightarrow +\infty}{\sim} rn.$$

The proof of Theorem 1 relies on a new upper bound for $D(C_n^r)$, turning out to be a lot sharper than the one in (2) for large values of n . So as to state it properly, we now make the following definition.

For every integer $n \geq 1$, we denote by $P(n)$ the greatest prime power dividing n , with the convention $P(1) = 1$. In what follows, the key point will be that $P(n)$ tends to infinity when n does so. Indeed, denoting by $(a_i)_{i \geq 1}$ the integer sequence of prime powers, let $(b_n)_{n \geq 1}$ be the integer sequence defined by $b_n = \prod_{a_i \leq n} a_i$ for all $n \geq 1$. By definition, for every integer $N \geq 1$, one has $P(n) > N$ as soon as $n > b_N$. Therefore, $P(n)$ tends to infinity when n does so.

Theorem 2. *For every integer $r \geq 1$, there exists a constant $d_r \geq 0$ such that for every integer $n \geq 1$,*

$$D(C_n^r) \leq r(n-1) + 1 + d_r \left(\frac{n}{P(n)} - 1 \right).$$

Now, since $P(n)$ tends to infinity when n does so, Theorem 2 allows us to deduce that, for every integer $r \geq 1$, the gap between the Davenport constant and its conjectural value

$$\partial(C_n^r) = D(C_n^r) - (r(n-1) + 1)$$

is actually $o(n)$. This theorem will be obtained via the inductive method, which involves another key combinatorial invariant we now proceed to define.

By $\eta(G)$ we denote the smallest integer $t \geq 1$ such that every sequence S over G of length $|S| \geq t$ contains a non-empty zero-sum subsequence $S' \mid S$ with $|S'| \leq \exp(G)$. It is readily seen that $D(G) \leq \eta(G)$ for every finite Abelian group G .

A natural construction shows that, for all integers $n, r \geq 1$, one has

$$(3) \quad (2^r - 1)(n-1) + 1 \leq \eta(C_n^r).$$

Indeed, if (e_1, \dots, e_r) is a basis of C_n^r , it is easily checked that the sequence S consisting of $n-1$ copies of $\sum_{i \in I} e_i$ for each non-empty subset $I \subseteq \llbracket 1, r \rrbracket$ contains no non-empty zero-sum subsequence of length at most n .

The exact value of $\eta(C_n^r)$ is known to match the lower bound in (3) for all n when $r \leq 2$ [13, Theorem 5.8.3], and for all r when $n = 2^\alpha$, with $\alpha \geq 1$ [14, Satz 1]. Besides these two results, $\eta(C_n^r)$ is currently known only when $r = 3$ and $n = 3^\alpha 5^\beta$, with $\alpha, \beta \geq 0$ [9, Theorem 1.7], in which case $\eta(C_n^3) = 8n - 7$, or $n = 2^\alpha 3$, with $\alpha \geq 1$ [9, Theorem 1.8], in which case $\eta(C_n^3) = 7n - 6$. When $n = 3$, note that the problem of finding $\eta(C_3^r)$ is closely related to the well-known cap-set problem, and that for $r \geq 4$, the only known values so far are $\eta(C_3^4) = 39$ [20], $\eta(C_3^5) = 89$ [4] and $\eta(C_3^6) = 223$ [21]. For more details on this fascinating topic, see [3, 5] and the references contained therein.

In another direction, Alon and Dubiner showed [2] that when r is fixed, $\eta(C_n^r)$ grows linearly in the exponent n . More precisely, they proved that for every integer $r \geq 1$, there exists a constant $c_r > 0$ such that for every integer $n \geq 1$,

$$(4) \quad \eta(C_n^r) \leq c_r(n-1) + 1.$$

From now on, we will identify c_r with its smallest possible value in this theorem.

On the one hand, it follows from (3) that $c_r \geq 2^r - 1$, for all $r \geq 1$. Since, as already mentioned, $\eta(C_n) = n$ and $\eta(C_n^2) = 3n - 2$ for all $n \geq 1$, it is possible to choose $c_1 = 1$ and $c_2 = 3$, with equality in (4).

On the other hand, the method used in [2] yields $c_r \leq (cr \log r)^r$, where $c > 0$ is an absolute constant, and it is conjectured in [2] that there actually is an absolute constant $d > 0$ such that $c_r \leq d^r$ for all $r \geq 1$.

We can now state and prove our main technical result, which is the following.

Theorem 3. *For all integers $n, r \geq 1$,*

$$D(C_n^r) \leq r(n-1) + 1 + (c_r - r) \left(\frac{n}{P(n)} - 1 \right).$$

Proof of Theorem 3. We set $G = C_n^r$ and denote by $H = G_{P(n)}$ the largest Sylow subgroup of G . Since $H \simeq C_{P(n)}^r$ is a p -group, it follows from [18] that

$$D(H) = r(P(n) - 1) + 1.$$

In addition, since the quotient group $G/H \simeq C_{n/P(n)}^r$ has rank r and exponent $n/P(n)$, it follows from (4) that

$$\eta(G/H) \leq c_r \left(\frac{n}{P(n)} - 1 \right) + 1.$$

Now, from any sequence S over G such that

$$|S| \geq \exp(G/H) (D(H) - 1) + \eta(G/H),$$

one can sequentially extract at least $d = D(H)$ disjoint non-empty subsequences $S'_1, \dots, S'_d \mid S$ such that $\sigma(S'_i) \in H$ and $|S'_i| \leq \exp(G/H)$ for every $i \in \llbracket 1, d \rrbracket$. Since $T = \prod_{i=1}^d \sigma(S'_i)$ is a sequence over H of length $|T| = D(H)$, there exists a non-empty subset $I \subseteq \llbracket 1, d \rrbracket$ such that $T' = \prod_{i \in I} \sigma(S'_i)$ is a zero-sum subsequence of T . Then, $S' = \prod_{i \in I} S'_i$ is a non-empty zero-sum subsequence of S .

Therefore, we have

$$\begin{aligned} D(G) &\leq \exp(G/H) (D(H) - 1) + \eta(G/H) \\ &\leq \frac{n}{P(n)} (r(P(n) - 1)) + c_r \left(\frac{n}{P(n)} - 1 \right) + 1 \\ &= r(n-1) + 1 + (c_r - r) \left(\frac{n}{P(n)} - 1 \right), \end{aligned}$$

which completes the proof. \square

Note that Theorem 3 is sharp for all n when $r = 1$ and for all r when n is a prime power. Also, Theorems 1 and 2 are now direct corollaries of Theorem 3.

Proof of Theorem 2. The result follows from Theorem 3 by setting $d_r = c_r - r$. \square

Proof of Theorem 1. Since $P(n)$ tends to infinity when n does so, the desired result follows easily from (2) and Theorem 2. \square

To conclude this paper, we would like to offer a possibly useful extension of Theorem 3 to the following wider framework. Given any finite Abelian group L and any integer $r \geq 1$, we consider the groups defined by $L_n^r = L \oplus C_n^r$, where $n \geq 1$ is any integer such that $\exp(L) \mid n$. Note that the case $L = \{0\}$ brings one back to $L_n^r = C_n^r$ whose Davenport constant is already covered by Theorem 3.

Our aim in this more general context is to prove that, for every finite Abelian group L and every integer $r \geq 1$, $D(L_n^r)$ behaves asymptotically in the same way it already does, according to Theorem 3, when L is trivial. To do so, we establish the following theorem.

Theorem 4. *Let $L \simeq C_{n_1} \oplus \cdots \oplus C_{n_\ell}$, with $1 \leq n_1 \mid \cdots \mid n_\ell \in \mathbb{N}$, be a finite Abelian group. For every integer $n \geq 1$ such that $\exp(L) \mid n$ and every integer $r \geq 1$,*

$$D(L_n^r) \leq r(n-1) + 1 + (c_{\ell+r} - r) \left(\frac{n}{P(n)} - 1 \right) + \frac{n}{P(n)} \sum_{i=1}^{\ell} (\gcd(n_i, P(n)) - 1).$$

Proof of Theorem 4. We set $G = L_n^r$ and $H = G_{P(n)}$. On the one hand, since $H \simeq C_{n'_1} \oplus \cdots \oplus C_{n'_\ell} \oplus C_{P(n)}^r$, with $n'_i = \gcd(n_i, P(n)) \mid n_i$ for all $i \in \llbracket 1, \ell \rrbracket$ and $1 \leq n'_1 \mid \cdots \mid n'_\ell \mid P(n)$, is a p -group, it follows from [18] that

$$D(H) = \sum_{i=1}^{\ell} (n'_i - 1) + r(P(n) - 1) + 1.$$

On the other hand, since the quotient group G/H has exponent $n/P(n)$ and rank at most $\ell + r$, it follows from (4) that

$$\eta(G/H) \leq \eta \left(C_{\frac{n}{P(n)}^{\ell+r}} \right) \leq c_{\ell+r} \left(\frac{n}{P(n)} - 1 \right) + 1.$$

Therefore, just like in our proof of Theorem 3, we obtain

$$\begin{aligned} D(G) &\leq \exp(G/H) (D(H) - 1) + \eta(G/H) \\ &\leq \frac{n}{P(n)} \left(\sum_{i=1}^{\ell} (n'_i - 1) + r(P(n) - 1) \right) + c_{\ell+r} \left(\frac{n}{P(n)} - 1 \right) + 1 \\ &= r(n-1) + 1 + (c_{\ell+r} - r) \left(\frac{n}{P(n)} - 1 \right) + \frac{n}{P(n)} \sum_{i=1}^{\ell} (n'_i - 1), \end{aligned}$$

which is the desired upper bound. \square

Theorem 5. *For every finite Abelian group L and every integer $r \geq 1$,*

$$D(L_n^r) \underset{\substack{n \rightarrow +\infty \\ \exp(L) \mid n}}{\sim} rn.$$

Proof of Theorem 5. We write $L \simeq C_{n_1} \oplus \cdots \oplus C_{n_\ell}$, with $1 \leq n_1 \mid \cdots \mid n_\ell \in \mathbb{N}$. For every integer $n \geq 1$ such that $\exp(L) \mid n$, one has $\gcd(n_i, P(n)) \leq n_i$ for all $i \in \llbracket 1, \ell \rrbracket$. Since $P(n)$ tends to infinity when n does so, the result follows easily from (1) and Theorem 4. \square

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REFERENCES

- [1] W.R. ALFORD, A. GRANVILLE AND C. POMERANCE *There are infinitely many Carmichael numbers*, Annals of Math. **140** (3) (1994), 703-722.
- [2] N. ALON AND M. DUBINER *A lattice point problem and additive number theory*, Combinatorica **15** (1995), 301-309.
- [3] Y. EDEL, C. ELSHOLTZ, A. GEROLDINGER, S. KUBERTIN AND L. RACKHAM *Zero-sum problems in finite abelian groups and affine caps*, Q. J. Math. **58** (2007), 159-186.
- [4] Y. EDEL, S. FERRET, I. LANDJEV AND L. STORME *The classification of the largest caps in $AG(5,3)$* , J. Combin. Theory Ser. A **99** (2002), 95-110.
- [5] J. S. ELLENBERG AND D. GIJSWIJT *On large subsets of \mathbb{F}_q^n with no three-term arithmetic progression*, Annals of Math. **185** (1) (2017), 339-343.
- [6] P. VAN EMDE BOAS *A combinatorial problem on finite abelian groups II*, Reports ZW-1969-007, Math. Centre, Amsterdam (1969).
- [7] P. VAN EMDE BOAS AND D. KRUYSWIJK *A combinatorial problem on finite abelian groups III*, Reports ZW-1969-008, Math. Centre, Amsterdam (1969).
- [8] W. GAO AND A. GEROLDINGER *Zero-sum problems in finite abelian groups: a survey*, Expo. Math. **24** (2006), 337-369.
- [9] W. GAO, Q. H. HOU, W. A. SCHMID AND R. THANGADURAI *On short zero-sum subsequences II*, Integers **7** (2007), #A21.
- [10] A. GEROLDINGER *Additive group theory and non-unique factorizations*, In A. Geroldinger and I. Ruzsa, *Combinatorial Number Theory and Additive Group Theory*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser (2009), 1-86.
- [11] A. GEROLDINGER, M. LIEBMANN AND A. PHILIPP *On the Davenport constant and on the structure of extremal zero-sum free sequences*, Period. Math. Hungar. **64** (2) (2012), 213-225.
- [12] A. GEROLDINGER AND R. SCHNEIDER *On Davenport's constant*, J. Combin. Theory Ser. A **61** (1992), 147-152.
- [13] A. GEROLDINGER AND F. HALTER-KOCH *Non-unique factorizations. Algebraic, combinatorial and analytic theory*, Pure and Applied Mathematics **278**, Chapman & Hall/CRC (2006).
- [14] H. HARBORTH *Ein Extremalproblem für Gitterpunkte*, J. Reine Angew. Math. **262** (1973), 356-360.
- [15] M. MAZUR *A note on the growth of Davenport constant*, Manuscripta Math. **74** (1992), 229-235.
- [16] R. MESHULAM *An uncertainty inequality and zero subsums*, Discrete Math. **84** (1990), 197-200.
- [17] W. NARKIEWICZ *Elementary and analytic theory of algebraic numbers*, 3rd edition, Springer (2004).
- [18] J. E. OLSON *A combinatorial problem on finite abelian groups I*, J. Number Theory **1** (1969), 8-10.
- [19] J. E. OLSON *A combinatorial problem on finite abelian groups II*, J. Number Theory **1** (1969), 195-199.
- [20] G. PELLEGRINO *The maximal order of the spherical cap in $S_{4,3}$* , Matematiche **25** (1971), 149-157.
- [21] A. POTECHIN *Maximal caps in $AG(6,3)$* , Des. Codes Cryptogr. **46** (2008), 243-259.
- [22] K. ROGERS *A combinatorial problem in Abelian groups*, Proc. Camb. Philos. Soc. **59** (1963), 559-562.