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PDE for joint law of the pair of a continuous diffusion and its running maximum

Laure Coutin*, Monique Pontier†

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Abstract

Let X be a d -dimensional diffusion process and M the running supremum of the first component. In this paper, in case of dimension d , we first show that for any $t > 0$, the law of the pair (M_t, X_t) admits a density with respect to Lebesgue measure. In uni-dimensional case, we compute this one. This allows us to show that for any $t > 0$, the pair formed by the random variable X_t and the running supremum M_t of X at time t can be characterized as a solution of a weakly valued-measure partial differential equation.

Keywords: Partial differential equation, running supremum process, joint law.

A.M.S. Classification: 60J60, 60H07, 60H10.

In this paper one was interested in the joint law of the pair (a continuous diffusion process, its running maximum). In case of a Brownian motion the result is well known, see for instance [9]. For general Gaussian processes, the law of the maximum is studied in [1].

Concerning the maximum law, the main part of literature is devoted to maximum of martingales, their terminal value, their maximum at terminal time. For instance look at Rogers et al. [15, 7, 2]. Cox-Obloj [5] aim, given a price process S , is to exhibit an hedging strategy of the so-called “no touch option”, meaning that the payoff is the indicator of the set $\{\overline{S}_T < b; \underline{S}_T > a\}$. They are not concerned with the law of the pair (process, its running maximum). A lot of papers are mainly interested in the hedging of barrier option, for instance [2].

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The case of general Lévy processes is studied by Doney and Kyprianou [6]. In particular cases driven by a Brownian motion and a compound Poisson process, Roynette-Vallois-Volpi [16] provide the Laplace transform of undershot-overshot-hitting time law. In [11, 4] a weak partial integro differential equation for the pair (process-its running maximum) law density is done. Lagnoux-Mercier-Vallois [10] provide the law density of such a pair, but in case of reflected Brownian motion.

Concerning the diffusion processes, for instance the Ornstein Uhlenbeck process, the density of the running maximum law is given in [13]. Quote Yor et al. [9] for the one dimensional diffusion process: a PDE is obtained for the law density of the process stopped before hitting a moving barrier. In [8] a multi-dimensional diffusion (whose corresponding diffusion vector fields are commutative) joint distribution is studied at the time when a component attains its maximum on finite time interval; under regularity and ellipticity conditions the smoothness of this joint distribution is proved.

In [4] a Lévy process $(X_t, t \geq 0)$, starting from zero, right continuous left limited is considered: X is the sum of a drifted Brownian motion and a compound Poisson process, called a mixed diffusive-jump process, then the density function of the pair formed by the random variable X_t and its running supremum M_t is provided. Finally we quote [3] which proves that the hitting time law admits a density and we here use some of its basic ideas.

We here look for more general (but continuous) cases where this density exists. We have results in d -dimensional case, but without closed expression. In uni-dimensional case we get the existence and a closed expression for the joint law density.

The model is as following: on a filtered probability space $(\Omega, (\mathcal{F}_t = \sigma(W_u, u \leq t))_{t \geq 0}, \mathbb{P})$ where $W := (W_u, u \geq 0)$ is a d -dimensional Brownian motion. Let a diffusion process taking its values in \mathbb{R}^d , solution to

$$dX_t = B(X_t)dt + \sum_{i=1}^d A_i(X_t)dW_t, \quad X_0 = x \in \mathbb{R}^d, \quad t > 0,$$

where $B: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $A: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ satisfy

$$A \text{ and } B \in C_b^1. \tag{1}$$

Let $M_t := \sup_{s \leq t} X_s^1$. We first prove that the law of $V_t = (M_t, X_t)$ is absolutely continuous with respect to the Lebesgue measure in a general case with some standard assumptions on the coefficients A and B . Then in Section 2, we turn to the uni-dimensional case. Here the density of the pair (process, running maximum) is provided in a weak form. Section 3 is devoted to prove a PDE concerning this density. Finally, an Appendix gives some tools and intermediate results.

1 The law of V_t is absolutely continuous

Here it is proved that for any $t > 0$, the joint law of $V_t := (M_t, X_t)$ admits a density with respect to the Lebesgue measure. For this purpose, we use ‘‘Malliavin calculus’’ specifically Nualart’s results [12].

Proposition 1.1. *We assume that B and A satisfy Assumption (1) and there exists a constant $c > 0$ such that*

$$c\|v\|^2 \leq v'A(x)A(x)'v, \quad \forall v, x \in \mathbb{R}^d. \quad (2)$$

Then the joint law of $V_t := (M_t, X_t)$ admits a density with respect to the Lebesgue measure for all $t > 0$.

The next subsection recalls some useful definitions and results.

1.1 Short Malliavin calculus summary

The material of this subsection is taken in section 1.2 of [12]. Let $\mathbb{H} = L^2([0, T], \mathbb{R}^d)$ endowed with the usual scalar product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and the associated norm $\|\cdot\|_{\mathbb{H}}$.

For all $h, \tilde{h} \in \mathbb{H}$,

$$W(h) := \int_0^T h(t)dW_t$$

is a centered Gaussian variable with variance equal to $\|h\|_{\mathbb{H}}^2$. If $\langle h, \tilde{h} \rangle_{\mathbb{H}} = 0$ then the random variables $W(h)$ and $W(\tilde{h})$ are independent.

Let \mathcal{S} denote the class of smooth random variables F defined as following:

$$F = f(W(h_1), \dots, W(h_n))(W(h_1), \dots, W(h_n)) \quad (3)$$

where $n \in \mathbb{N}$, $h_1, \dots, h_n \in \mathbb{H}$ and f belongs to $C_b(\mathbb{R}^n)$.

Definition 1.2. *The derivative of a smooth variable F as (3) is the \mathbb{H} valued random variable given by*

$$DF = \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n))h_i.$$

Proposition 1.3. *The operator D is closable from $L^p(\Omega)$ into $L^p(\Omega, \mathbb{H})$ for any $p \geq 1$.*

For any $p \geq 1$, we denote the domain of the operator D in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$ meaning that $\mathbb{D}^{1,p}$ is the closure of the class of smooth random variables \mathcal{S} with respect to the norm

$$\|F\|_{1,p} = [\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{\mathbb{H}}^p]]^{1/p}.$$

Malliavin calculus is a powerful tool to prove the absolute continuity of random variables law. Namely Theorem 2.1.2 page 97 [12] states:

Theorem 1.4. *Let $F = (F^1, \dots, F^m)$ be a random vector satisfying the following conditions*

- (i) F^i belongs to $\mathbb{D}^{1,p}$ for $p > 1$ for all $i = 1, \dots, m$,
- (ii) the Malliavin matrix $\gamma_F = (\langle DF^i, DF^j \rangle_{\mathbb{H}})_{1 \leq i, j \leq m}$ is invertible.

Then the law of F is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^m .

According to this theorem, the proof of Proposition 1.1 will be a consequence of the following that we have to prove:

- X_t^i , $i = 1, \dots, d$ and M_t belongs to $\mathbb{D}^{1,p}$ $p > 1$, Lemma 1.5;
- the $(d+1) \times (d+1)$ matrix $\gamma_V(t) := (\langle DV_t^i, DV_t^j \rangle)_{1 \leq i, j \leq d+1}$ is almost surely invertible, Proposition 1.6.

1.2 Malliavin differentiability of the supremum

Lemma 1.5. *We assume that B and A satisfy Assumption (1) then X_t^i , $i = 1, \dots, d$ and M_t belongs to $\mathbb{D}^{1,p}$ $\forall p \geq 1$ for all $t > 0$.*

Proof. Using Theorem 2.2.1 [12], under Assumption (1),

- X_t^i , $i = 1, d$ belong to $\mathbb{D}^{1,\infty}$ for all $t > 0$,
- $\forall t \leq T$, $\forall p > 0$, $\forall i = 1, \dots, d$, there exists a constant C_T^p such that

$$\sup_{0 \leq r \leq t} \mathbb{E} \left(\sup_{r \leq s \leq T} |D_r X_s^i|^p \right) = C_t \leq C_T^p < \infty, \quad (4)$$

- the Malliavin derivative $D_r X_t$ satisfies $D_r X_t = 0$ for $r > t$ almost surely and for $r \leq t$ almost surely, using Einstein's convention:

$$D_r X_t^i = A^i(X_r) + \int_r^t \bar{A}_{k,\alpha}^i(s) D_r(X_s^k) dW_s^\alpha + \int_r^t \bar{B}_k^i(s) D_r(X_s^k) ds \quad (5)$$

where $\bar{A}_{k,\alpha}^i(s) := \partial_k A_\alpha^i(X_s)$ and $\bar{B}_k^i := \partial_k B^i(X_s)$ are in \mathbb{R}^d .

In order to prove that M_t belongs to $\mathbb{D}^{1,p}$ we follow the same lines as the proof of Nualart's Proposition 2.1.10 with index p instead of 2. Then, for any $i = 1, \dots, d$, we establish that the \mathbb{H} valued process $(D_t X_t^i, t \in [0, T])$ has a continuous modification and satisfies $\mathbb{E}(\|D_t X_t^i\|_H^p) < \infty$.

We now use Appendix (A.11) in Nualart [12], as a corollary of Kolmogorov's continuity criterion. Namely if there exist positive real numbers α, β, K such that

$$E[\|D_t X_{t+\tau}^i - D_t X_t^i\|_H^\alpha] \leq K \tau^{1+\beta}, \quad \forall t \geq 0, \tau \geq 0$$

then DX^i admits a continuous modification. Moreover $\mathbb{E}(\sup_{s \in [0, T]} \|D.X_s^i\|_{\mathbb{H}}^\alpha) < \infty$. Let $\tau > 0$, Equation (5) yields

$$\begin{aligned} \Delta_\tau D_r(X_t^i) &:= D_r(X_{t+\tau}^i) - D_r(X_t^i) \\ &= \int_{\max(r, t)}^{\max(t+\tau, r)} \overline{B}_k^i(s) D_r(X_s^k) ds + \int_{\max(r, t)}^{\max(t+\tau, r)} \overline{A}_{k, \alpha}^i(s) D_r(X_s^k) dW_s^\alpha. \end{aligned}$$

Using the definition of \mathbb{H}

$$\|\Delta_\tau D.(X_t^i)\|_{\mathbb{H}}^2 = \int_0^T \left| \int_{\max(r, t)}^{\max(t+\tau, r)} \overline{B}_k^i(s) D_r(X_s^k) ds + \int_{\max(r, t)}^{\max(t+\tau, r)} \overline{A}_{k, \alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^2 dr.$$

According to Jensen's inequality for $p \geq 2$

$$\|\Delta_\tau D.(X_t^i)\|_H^p \leq T^{\frac{p}{2}-1} \int_0^T \left| \int_{\max(r, t)}^{\max(t+\tau, r)} \overline{B}_k^i(s) D_r(X_s^k) ds + \int_{\max(r, t)}^{\max(t+\tau, r)} \overline{A}_{k, \alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^p dr.$$

Using $(a + b)^p \leq 2^{p-1}(a^p + b^p)$,

$$\|\Delta_\tau D.(X_t)\|_H^p \leq 2^{p-1} T^{\frac{p}{2}-1} \int_0^T \left[\left| \int_t^{t+\tau} \overline{B}_k^i(s) D_r(X_s^k) ds \right|^p + \left| \int_t^{t+\tau} \overline{A}_{k, \alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^p \right] dr.$$

The expectation of the first term is bounded using Jensen's inequality and (4) for any $r \in [0, T]$:

$$E \left[\left| \int_t^{t+\tau} \overline{B}_k^i(s) D_r(X_s^k) ds \right|^p \right] \leq \|\overline{B}\|_\infty^p \tau^{p-1} \sup_r E \left[\sup_{r \leq s \leq T} |D_r(X_s^k)|^p \tau \right] = \|\overline{B}\|_\infty^p \tau^p C_T^p.$$

Using once again (4), Burkholder-Davis Gundy' and Jensen's inequalities, the expectation of the second term satisfies for any $r \in [0, T]$:

$$\begin{aligned} E \left[\left| \int_t^{t+\tau} \overline{A}_{k, \alpha}^i(s) D_r(X_s^k) dW_s^\alpha \right|^p \right] &\leq C_p E \left[\left(\int_t^{t+\tau} |\overline{A}_{k, \alpha}^i(s) D_r(X_s^k)|^2 ds \right)^{p/2} \right] \\ &\leq C_p \|\overline{A}\|_\infty^p \tau^{p/2-1} \int_t^{t+\tau} E \left(\sup_{r \leq s \leq T} |D_r(X_s^i)|^p \right) ds \leq C_p \|\overline{A}\|_\infty^p \tau^{p/2-1} C_T^p \tau, \end{aligned}$$

thus for any $\tau \in [0, 1]$ there exists a constant $D = T^{p/2} 2^{p/2-1} C_T^p (\|\overline{B}\|_\infty^p \tau^{p/2} + C_p \|\overline{A}\|_\infty^p)$ such that for any $i = 1, \dots, d$,

$$E[\|D.(X_{t+\tau}^i) - D.(X_t^i)\|_H^p] \leq D \tau^{p/2}.$$

Kolmogorov's lemma applied to the process $\{D.(X_t), t \in [0, T]\}$, taking it values in the Hilbert space \mathbb{H} , proves the existence of a continuous version, meaning: there exist positive real numbers α, β, K such that

$$E[\|D.(X_{t+\tau}^i) - D.(X_t^i)\|_H^\alpha] \leq K \tau^{1+\beta}.$$

With $\alpha = p > 2$, $\beta = p/2 - 1$, $K = D$, we get the existence of a continuous version of the process $t \mapsto D.(X_t)$ from $[0, T]$ to the Hilbert space \mathbb{H} . Finally, we conclude as Nualart's Proposition 2.1.10 proof with index p instead of 2. •

1.3 Invertibility of the Malliavin matrix

Proposition 1.6. *Assume that B and A are in C_b^1 and there exists a constant $c > 0$ such that*

$$c\|v\|^2 \leq v'A(x)A(x)'v, \quad \forall v \in \mathbb{R}^d, \quad \forall x \in \mathbb{R}^d$$

then for all $t > 0$ the matrix $\gamma_V(t) := \langle \langle DV_t^i, DV_t^j \rangle_{\mathbb{H}} \rangle_{1 \leq i, j \leq d+1}$ is almost surely invertible.

Proof. The key is to introduce a new matrix which will be invertible:

$$\text{for all } (s, t), \quad 0 < s < t, \quad \gamma_G(s, t) := \langle \langle DG^i(s, t), DG^j(s, t) \rangle_{\mathbb{H}} \rangle_{1 \leq i, j \leq 2(d+1)} \quad (6)$$

where $G^i(s, t) := X_t^i$, $i = 1, \dots, d$ and $G^{i+d}(s, t) = X_s^i$, $i = 1, \dots, d$.

On another hand we will prove, $t > 0$ being fixed, $\mathbb{P}(X_t^1 = M_t) = 0$.

Step 1: We introduce

- $N_{1,t} := \{\omega, \exists s \in [0, t], \quad DX_s^1 \neq DM_t \quad \text{and} \quad X_s^1 = M_t\}$,
- $N_{2,t} := \{\omega, \exists s \in [0, t], \quad \det(\gamma_G(s, t)) = 0\}$,
- $N_{3,t} := \{\omega, X_t^1 = M_t\}$,
- $N_t = \{\omega, \det(\gamma_V(t)) = 0\}$.

Then,

$$N_t \subset (N_t \cap \cap_{i=1}^3 N_{i,t}^c) \cup \cup_{i=1}^3 N_{i,t}.$$

Proof. Note that $\mathbb{P}(N_t \cap \cap_{i=1}^3 N_{i,t}^c) = 0$. Indeed if $\omega \in N_t \cap \cap_{i=1}^3 N_{i,t}^c$, since X^1 admits a continuous modification there exists s_0 such that $X_{s_0}^1 = M_t$. The fact that $\omega \in N_{3,t}^c$ implies that $s_0 < t$, and $\gamma_V(t) = (\Gamma_G^{i,j}(s_0, t))_{(i,j) \in \{1, \dots, d+1\}^2}$ is a sub matrix of $\gamma_G(s_0, t)$. The fact that $\gamma_V(t)$ is not invertible contradicts the fact that $\gamma_G(s_0, t)$ is invertible. Then, it remains to prove that $\mathbb{P}(N_{i,t}) = 0$ for $i = 1, \dots, d+1$.

Step 2: Using the same lines as the proof of Proposition 2.1.11 [12], we prove that almost surely

$$\{s : X_s^1 = M_t\} \subset \{s : DM_t = DX_s^1\}$$

meaning $\mathbb{P}(N_{1,t}) = 0$. We skip the details for simplicity.

Step 3: For all $t > 0$, almost surely for all $s < t$, the $2d \times 2d$ matrix $\gamma_G(s, t)$ is invertible, meaning that $\forall t$, the event $N_{2,t}$ is negligible.

Proof. This matrix $\gamma_G(s, t)$ is symmetrical and using (2.59) and (2.60) in [12] yields:

$$\gamma_G(s, t) = \begin{pmatrix} Y(t)C(t)Y(t)' & Y(s)C(s)Y(t)' \\ Y(t)C(s)Y(s)' & Y(s)C(s)Y(s)' \end{pmatrix} \quad (7)$$

where, using Einstein's convention to avoid $\sum_k, \sum_{k'}, \sum_l \dots$

$$C^{i,j}(t) := \int_0^t Y^{-1}(u)_k^i A_l^k(X_u) Y^{-1}(u)_{k'}^j A_l^{k'}(X_u) du$$

$$Y_j^i(t) := \delta_{i,j} + \int_0^t \bar{A}_{k,l}^i(u) Y_j^k(u) dW_u^l + \int_0^t \bar{B}_k^i(u) Y^k(u) du, \quad i, j \in \{1, \dots, d\}$$

Let us denote

$$C^{i,j}(s, t) := C^{i,j}(t) - C^{i,j}(s).$$

According to (2.58) [12] there exists a process Z such that almost surely for all $h \in [0, T]$

$$Z(h)Y(h) = Id$$

thus for all t the matrices $Y(t)$ are invertible.

Actually for all i and j

$$Y_j^i(t) = Y_j^i(s) + \int_s^t \bar{A}_{k,l}^i(u) Y_j^k(u) dW_u^l + \int_s^t \bar{B}_k^i(u) Y_j^k(u) du,$$

and multiplying this equality by $Y(s)^{-1}$ one deduces:

$$Y(t)Y(s)^{-1} = Id + \int_s^t \bar{A}_{k,l}(u) Y(u)Y(s)^{-1} dW_s^l + \int_s^t \bar{B}(u) Y(u)Y(s)^{-1} ds$$

so the (d, d) matrix $Y(s, t) := Y(t)Y(s)^{-1}$ is invertible.

Then $\gamma_G(s, t)$ (7) can be rewritten as a matrix composed with four (d, d) blocks:

$$\gamma_G(s, t) := \begin{pmatrix} Y(s, t)Y(s)[C(s) + C(s, t)]Y(s)'Y(s, t)' & Y(s)C(s)Y(s)'Y(s, t)' \\ Y(s, t)Y(s)C(s)Y(s)' & Y(s)C(s)Y(s)' \end{pmatrix}$$

The second line of blocks multiplied by $Y(s, t)'$ and this one subtracted to the first line yield:

$$\det[\gamma_G(s, t)] = \begin{vmatrix} Y(s, t)Y(s)C(s, t)Y(s)'Y(s, t)' & 0 \\ Y(s, t)Y(s)C(s)Y(s)' & Y(s)C(s)Y(s)' \end{vmatrix}.$$

The properties of block trigonal matrix determinants prove that

$$\det[\gamma_G(s, t)] = |Y(s, t)Y(s)C(s, t)Y(s)'Y(s, t)'Y(s)C(s)Y(s)'|$$

The processes Z are Y are diffusion processes so each of them admits a continuous modification satisfying $Z(h)Y(h) = Id, \forall h \in [0, T]$. Thus, almost surely the continuous process Z is invertible so satisfies almost surely for all $0 \leq s \leq t \leq T$

$$\int_s^t \det(Z(h))^2 dh > 0.$$

Let $\sigma(x) = \sum_{l=1}^d A_l(x)A_l(x)'$. Formula (2.61) page 127 [12] shows

$$C(s) = \int_0^s Y^{-1}(h)\sigma(X_h)(Y(h)^{-1})' dh, \quad C(s, t) = \int_s^t Y^{-1}(h)\sigma(X_h)(Y(h)^{-1})' dh$$

We now follow the proof of Theorem 2.3.1 page 127 [12]: for $v \in \mathbb{R}^d$, using the uniform ellipticity Assumption (2)

$$v'\sigma(X_s)v \geq c|v|^2, \quad \forall s.$$

With $v = (Y(h)^{-1})'u$ we get

$$u'Y(h)^{-1}\sigma(X_h)(Y(h)^{-1})'u \geq cu'Y(h)^{-1}(Y(h)^{-1})'u$$

and

$$u'C(s)u = \int_0^s u'Y(h)^{-1}\sigma(X(h))(Y(h)^{-1})'udh \geq c \int_0^s u'Y(h)^{-1}(Y(h)^{-1})'udh = c|u|^2 \int_0^s \det(Z(h))^2 dh.$$

Similarly

$$u'C(s,t)u = \int_s^t u'Y(h)^{-1}\sigma(X(h))(Y(h)^{-1})'udh \geq c|u|^2 \int_s^t \det(Z(h))^2 dh.$$

Thus almost surely for all $s \in]0, t[$, $C(s)$ and $C(s, t)$ are invertible. As a consequence, the matrix $\gamma_G(s, t)$ is invertible.

The process $t \rightarrow D(X_t)$ taking its values in \mathbb{H} admits a continuous modification and the sets of invertible matrix is an open set then,

$$\mathbb{P}(\{\omega, \exists s \in [0, t[, \det(\gamma_G(s, t)) = 0\}) = \mathbb{P}(N_{2,t}) = 0.$$

Step 4: Under Assumptions (1) and (2), time t being fixed, almost surely $M_t > X_t^1$ meaning the event $N_{3,t}$ is negligible. •

Proof. For sake of completeness we prove this result, more or less included in Proposition 18 [8] but stronger assumptions are used there.

The set $\{M_t = X_t^1\}$ is detailed as follows:

$$\begin{aligned} & \{\omega, M_t(\omega) = X_t^1(\omega)\} \\ & = \{\omega, \exists s < t \mid \forall u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\} \cup \{\omega \mid \forall u < t, X_u^1(\omega) < X_t^1(\omega)\}. \end{aligned} \quad (8)$$

Using (1) and (2), $A^{-1}B$ is bounded, thus an equivalent change of equivalent probability measure can be operated using Girsanov Theorem: the probability measure \mathbb{P}_0 is defined as

$$\frac{d\mathbb{P}_0}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = L_t, \quad L_t := \exp \left(- \int_0^t (BA^{-1})^i(X_s) dW_s^i - \frac{1}{2} \int_0^t \|(BA^{-1}(X_s))\|^2 ds \right).$$

Then X^1 is a $(\mathcal{F}, \mathbb{P}_0)$ martingale:

$$X_t^1 = X_0^1 + \int_0^t \sum_j A^{1,j}(X_s) d\tilde{W}_s^j \quad (9)$$

where \tilde{W} is a $(\mathcal{F}, \mathbb{P}_0)$ d -dimensional Brownian motion. The bracket of X^1 , actually independent of the probability measure in continuous case, is

$$\langle X^1, X^1 \rangle_t = \int_0^t \sum_j (A^{1,j}(X_s))^2 ds.$$

Assumption (2) on A applied to $v = (1, 0, \dots, 0)$ implies that for any x , $\sum_j A^{1,j}(x)A^{1,j}(x) \geq c > 0$. This has two consequences:

- For all rational numbers $q < q'$ in $[0, T]$

$$\langle X^1, X^1 \rangle_{q'} - \langle X^1, X^1 \rangle_q > c(q' - q) > 0.$$

According to Proposition 1.13 page 119 [14], for all rational numbers $q < q'$ in $[0, T]$ X^1 is not constant on the interval $[q, q']$. But

$$\{\omega, \exists s > t \mid u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\} \subset \cup_{q < q' < T, q, q' \in \mathbb{Q}} \{\omega \mid u \in [q, q'], X_u^1(\omega) = X_q^1(\omega)\}$$

thus

$$\mathbb{P}_0(\{\omega \mid \exists s > t, \forall u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\}) = 0.$$

The probability measures \mathbb{P}_0 and \mathbb{P} are equivalent so

$$\mathbb{P}(\{\omega, \exists s > t \mid \forall u \in [s, t], X_u^1(\omega) = X_t^1(\omega)\}) = 0. \quad (10)$$

- Using Dambis-Dubins-Schwarz' Theorem (cf. Theorem 1.6 Chapter V [14]), and once again that $\sum_j (A^{1,j}(X_s))^2 \geq c$, then

$$\langle X^1, X^1 \rangle_\infty = \int_0^\infty \sum_j (A^{1,j}(X_s))^2 ds = +\infty.$$

So there exists a \mathbb{P}_0 Brownian motion B such that

$$X^1(t) = B_{\langle X^1, X^1 \rangle_t}, \quad \forall t > 0.$$

Here is followed step by step the proof of Theorem 2.7 Chapter I [14] (Lévy's modulus of continuity), but without absolute value: for all $N \in \mathbb{R}$

$$\mathbb{P}_0 \left(\limsup_{\varepsilon \rightarrow 0} \left\{ \sup_{\substack{0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < \varepsilon}} \frac{B_{t_1} - B_{t_2}}{h(\varepsilon)} = 1 \right\} \right) = 1$$

where $h(s) = \sqrt{2s \log(1/s)}$, $s \in [0, 1]$.

This is equivalent to

$$\mathbb{P}_0 \left(\liminf_{\varepsilon \rightarrow 0} \left\{ \sup_{\substack{0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < \varepsilon}} \frac{B_{t_1} - B_{t_2}}{h(\varepsilon)} \neq 1 \right\} \right) = 0.$$

We remark that

$$\begin{aligned} & \{\omega, \forall s \in [0, t], X^1(s) < X^1(t)\} = \{\omega, \forall s \in [0, t], B_{\langle X^1, X^1 \rangle_s} < B_{\langle X^1, X^1 \rangle_t}\} \\ & \subset \cup_N \cup_n \cap_{k \geq n} \left\{ \begin{array}{l} \sup \\ 0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < 1/k \end{array} \frac{B_{t_1} - B_{t_2}}{h(1/k)} \leq 0 \right\} \\ & \subset \cup_N \liminf_{1/n \rightarrow 0} \left\{ \begin{array}{l} \sup \\ 0 \leq t_1, t_2 \leq N \\ t_1 - t_2 < 1/n \end{array} \frac{B_{t_1} - B_{t_2}}{h(1/n)} \neq 1 \right\} \end{aligned}$$

thus $\mathbb{P}_0(\{\omega, \forall s \in [0, t], X_s^1(\omega) < X_t^1(\omega)\}) = 0$.

The probability measures \mathbb{P} and \mathbb{P}_0 are equivalent, so

$$\mathbb{P}(\{\omega, \forall s \in [0, t], X_s^1(\omega) < X_t^1(\omega)\}) = 0. \quad (11)$$

Finally (8), (10), (11) prove that

$$\mathbb{P}(\{M_t = X_t^1\}) = \mathbb{P}(N_{3,t}) = 0. \quad (12)$$

•

As a conclusion, Proposition 1.1 proof is done.

1.4 Monotonous image of the Brownian motion

Before the study of the one-dimensional case, we start with a first simple example which is completely solvable.

• Firstly one recalls the standard result concerning the Brownian motion. The density of the law of (W_t^*, W_t) is well known, cf. [9] Section 3.2, and is defined on \mathbb{R}^2 by

$$p_W(b, a; t) := 2 \frac{(2b - a)}{\sqrt{2\pi t^3}} e^{-\frac{(2b-a)^2}{2t}} \mathbf{1}_{\sup(0, a), \infty[}(b). \quad (13)$$

It could be checked that Theorem 3.1 provides for all $t > 0$, a partial differentiable equation and boundary conditions for the density of the law of $V_t := (W_t^*, W_t)$:

$$\begin{aligned} \partial_t p_W(b, a; t) &= \frac{1}{2} \partial_{22}^2 p_W(b, a; t), \quad \forall b > \max(0, a), \quad t > 0 \\ \partial_1 p_W(b, b; t) + 2\partial_2 p_W(b, b; t) &= 0, \quad \forall b > 0, \quad t > 0, \\ \partial_t p_{W^*}(b, t) + \frac{1}{2}(\partial_1 p_W(b, b, t) + \partial_2 p_W(b, b, t)) &= 0. \end{aligned} \quad (14)$$

• Another case is easily deduced: the particular case of a 1-dimensional monotonous image of the Brownian motion, as following:

$$dX_t = \sigma(X_t) dW_t + 1/2 \sigma'(X_t) \sigma(X_t) dt, \quad X_0 = x_0$$

with $\sigma \in C^1$ and $\inf_{x \in \mathbb{R}} \sigma(x) \geq c > 0$. Let φ be the solution to the differential equation

$$\frac{d}{dx}\varphi(x) = \sigma(\varphi(x)), \quad \varphi(0) = x_0$$

then $X = \varphi(W)$. Moreover since φ is continuous and increasing, $M = \varphi(W^*)$ where W^* is the running maximum of the Brownian motion.

Thus, using p_W defined in (13), the law of the pair $V_t = (M_t, X_t) = (\varphi(W_t^*), \varphi(W_t))$ admits the density on \mathbb{R}^2 : $p_V(m, x; t) := \frac{1}{\sigma(m)\sigma(x)}p_W(\varphi^{-1}(m); \varphi^{-1}(x); t)$ and similarly $p_M(m; t) := \frac{1}{\sigma(m)}p_{W^*}(\varphi^{-1}(m); t)$.

Standard change of variables, starting from the partial differentiable system (14), provides for all $t > 0$, a partial differentiable system for the density of the law of $V_t = (M_t, X_t)$:

$$\begin{aligned} \partial_t p_V(m, x; t) &= \frac{1}{2}[\sigma'(x)^2 + \sigma\sigma''(x)]p_V(m, x; t) + \frac{3}{2}\sigma\sigma'(x)\partial_2 p(m, x; t) + \frac{1}{2}\sigma(x)^2\partial_{22}^2 p(m, x; t), \\ 3\sigma'(m)p_V(m, m; t) + \sigma(m)[\partial_1 + 2\partial_2](p_V)(m, m; t) &= 0, \quad \forall m > 0, \end{aligned} \quad (15)$$

$$\partial_t p_M(m, t) + \sigma'(m)p_V(m, m; t) + \frac{1}{2}\sigma(m)(\partial_1 + \partial_2)(p_V)(m, m; t) = 0.$$

Remark 1.7. *This example is the ‘‘Example 4’’ in [8], but there only the law of M_t is studied instead of the pair as we do.*

Remark 1.8. *Note that if \mathcal{L} is the infinitesimal generator of X then its adjoint is*

$$\mathcal{L}^*F(x) = \frac{1}{2}[\sigma'(x)^2 + \sigma\sigma''(x)]F(x) + \frac{3}{2}\sigma\sigma'(x)F'(x) + \frac{1}{2}\sigma(x)^2F''(x).$$

The aim of the following section is to prove such results in case of one dimensional Brownian diffusions. In Theorem 2.1 we compute the expectation of the Itô formula as [4] Theorem 2.1 for Lévy processes. We derive the associated partial differential equations in Section 3.

We stress that, in Proposition 2.11, we succeed to show the continuity of V law density closed to the diagonal and obtain useful bounds on p_V only in the one dimensional case.

2 Brownian diffusion model, one dimension

Let the following stochastic differential equation

$$dX_t = B(X_t)dt + A(X_t)dW_t, \quad X_0 = x, \quad t \in [0, T], \quad (16)$$

where $B \in C_b^1(\mathbb{R})$ and $A \in C_b^2(\mathbb{R})$ satisfy (2), W is a Brownian motion. In particular, Assumption (2) means that there exists $c > 0$ such that

$$(*) \quad \forall x \in \mathbb{R}, \quad A^2(x) \geq c.$$

Moreover this assumption with (*) above yield $A^{-1}B$ is bounded.

We denote $V := (M, X)$ and we recall the notation

$$M_t = \sup_{s \leq t} X_s, \quad t \in \mathbb{R}.$$

From Proposition 1.1, the law of the pair $V_t = (M_t, X_t)$ admits a density. Below, a weak PDE concerning this density is provided. The first step is the following expectation of Itô formula.

Theorem 2.1. *Let $B \in C_b^1(\mathbb{R})$ and $A \in C_b^2(\mathbb{R})$ which satisfies Assumption (2). Let be $\Phi \in C_b^2(\mathbb{R}^2, \mathbb{R})$,*

$$\begin{aligned} E[\Phi(V_t)] - \Phi(V_0) &= E\left[\int_0^t \left[B(X_s)\partial_2\Phi(V_s) + \frac{1}{2}A^2(X_s)\partial_{22}^2\Phi(V_s) \right] ds \right. \\ &\quad \left. + \frac{1}{2} \int_0^t E[\partial_1\Phi(M_s, M_s)]A^2(M_s)\frac{p_V(M_s, M_s, s)}{p_M(M_s, s)}] ds \right] \end{aligned}$$

where $p_V(m, x, t)$ is the density of the law of the pair V_t and $p_M(\cdot, t)$ is the one of M_t .

Remark 2.2. *This result is to be compared to Theorem 2.1 and Proposition 4.4 in [4].*

2.1 Reduction to a Brownian plus drift

Since we are in an one dimensional setting, we use a Lamperti transformation in order to reduce the problem to the case of a diffusion with additive noise. A priori $dX_t = B(X_t)dt + A(X_t)dW_t$. We look for an increasing function $\varphi \in C_b^2$ such that the coefficient of dW would be 1. Itô formula yields

$$d\varphi(X_t) = \varphi'(X_t)B(X_t)dt + \frac{1}{2}\varphi''(X_t)A^2(X_t)dt + \varphi'(X_t)A(X_t)dW_t.$$

A sufficient condition is to choose φ such that $\varphi' = \frac{1}{A}$. As is A , φ' is bounded above and below uniformly. Then $Y = \varphi(X)$ satisfies

$$dY_t = \left[\frac{B}{A} \circ \varphi^{-1}(Y_t) - \frac{1}{2}A' \circ \varphi^{-1}(Y_t) \right] dt + dW_t. \quad (17)$$

Remark that $\tilde{B} = \frac{B}{A} \circ \varphi^{-1} - \frac{1}{2}A' \circ \varphi^{-1} \in C_b^1$ as a consequence of (2), $A \in C_b^2$ and $B \in C_b^1$. Moreover φ' being positive, φ is increasing and $Y_t^* = \varphi(X_t^*)$. From Proposition 1.1, the law of the pair (Y_t^*, Y_t) admits a density with respect to Lebesgue measure.

Lemma 2.3. *We assume that B is $C_b^1(\mathbb{R})$ and $A \in C_b^2(\mathbb{R})$ satisfies (2). Then the density law of (M_t, X_t) $p_V(\cdot, \cdot, t)$ satisfies*

$$p_V(b, a; t) = \frac{p_{Y^*, Y}(\varphi(b), \varphi(a); t)}{A(b)A(a)}$$

where φ is defined by $\varphi'(x) = \frac{1}{A(x)}$ and $p_{Y^*, Y}(\cdot, \cdot, t)$ is the pair (Y_t^*, Y_t) law density.

Proof. It is enough to identify the density law of the pair $V_t = (M_t, X_t)$ using, for any bounded measurable F , the following

$$E[F(M_t, X_t)] = E[F(\varphi^{-1}(Y_t^*), \varphi^{-1}(Y_t))] = \int F(\varphi^{-1}(\beta), \varphi^{-1}(\alpha))p_{Y^*, Y}(\beta, \alpha; t)d\beta d\alpha.$$

We operate the change of variables $b = \varphi^{-1}(\beta)$, $a = \varphi^{-1}(\alpha)$, so $d\beta = \varphi'(b)db = \frac{1}{A(b)}db$ and $d\alpha = \varphi'(a)da = \frac{1}{A(a)}da$ get the result. •

Proposition 2.4. *We assume that B is $C_b^1(\mathbb{R})$ and $A \in C_b^2(\mathbb{R})$ satisfies (2) and that Theorem 2.1 is proved in case $A = 1$, $\tilde{B} = \frac{B}{A} \circ \varphi^{-1} - \frac{1}{2}A' \circ \varphi^{-1}$, then Theorem 2.1 is proved: for any $\Phi \in C_b^2(\mathbb{R}^2, \mathbb{R})$:*

$$\begin{aligned} E[\Phi(V_t)] &= \Phi(V_0) + E\left[\int_0^t \left[B(X_s)\partial_2\Phi(V_s) + \frac{1}{2}A^2(X_s)\partial_{22}^2\Phi(V_s) \right] ds \right. \\ &\quad \left. + \frac{1}{2} \int_0^t E[\partial_1\Phi(M_s, M_s)A^2(M_s) \frac{p_V(M_s, M_s, s)}{p_M(M_s, s)}] ds \right] \end{aligned}$$

where $p_V(m, x, t)$ is the pair $V_t = (M_t, X_t)$ law density and $p_M(\cdot, t)$ is the one of M_t .

Proof. We apply Theorem 2.1 to (Y_t^*, Y_t) and $F(\beta, \alpha) = \Phi(\varphi^{-1}(\beta), \varphi^{-1}(\alpha))$; since $\varphi' > 0$, then φ is increasing, $\varphi(X_t^*) = Y_t^*$ and $\Phi(V_t) = F(Y_t^*, Y_t)$.

Note that

$$\begin{aligned} \partial_2 F(\beta, \alpha) &= \partial_2 \Phi(\varphi(\beta)^{-1}, \varphi^{-1}(\alpha)) \frac{1}{\varphi'(\varphi^{-1}(\alpha))} = \partial_2 \Phi(\varphi(\beta)^{-1}, \varphi^{-1}(\alpha)) A(\varphi^{-1}(\alpha)), \\ \partial_{22}^2 F(\beta, \alpha) &= \partial_{22}^2 \Phi(\varphi(\beta)^{-1}, \varphi^{-1}(\alpha)) A^2(\varphi^{-1}(\alpha)) + \partial_2 \Phi(\varphi(\beta)^{-1}, \varphi^{-1}(\alpha)) (AA')(\varphi^{-1}(\alpha)), \end{aligned}$$

so

$$\tilde{B}(\alpha)\partial_2 F(\beta, \alpha) + \frac{1}{2}\partial_{22}^2 F(\beta, \alpha) = B(\varphi^{-1}(\alpha))\partial_2 \Phi(\varphi(\beta)^{-1}, \varphi^{-1}(\alpha)) + \frac{1}{2}A^2(\varphi^{-1}(\alpha))\partial_{22}^2 \Phi(\varphi(\beta)^{-1}, \varphi^{-1}(\alpha)).$$

Then,

$$\int_0^t \mathbb{E} \left[\tilde{B}(Y_s)\partial_2 F(Y_s^*, Y_s) + \frac{1}{2}\partial_{22}^2 F(Y_s^*, Y_s) \right] ds = E \left[\int_0^t \left[B(X_s)\partial_2\Phi(V_s) + \frac{1}{2}A^2(X_s)\partial_{22}^2\Phi(V_s) \right] ds \right].$$

Note that

$$\partial_1 F(\beta, \alpha) = \partial_1 \Phi(\varphi(\beta)^{-1}, \varphi^{-1}(\alpha)) \frac{1}{\varphi'(\varphi^{-1}(\beta))} = \partial_1 \Phi(\varphi(\beta)^{-1}, \varphi^{-1}(\alpha)) A(\varphi^{-1}(\beta))$$

thus

$$\int_0^t E[\partial_1 F(Y_s^*, Y_s^*) \frac{p_{Y^*, Y}(Y_s^*, Y_s^*, s)}{p_{Y^*}(Y_s^*, s)}] ds = \int_0^t \int_{\mathbb{R}_+} \partial_1 \Phi(\varphi(\beta)^{-1}, \varphi^{-1}(\alpha)) A(\varphi^{-1}(\beta)) p_{Y^*, Y}(\beta, \beta, s) d\beta ds.$$

We perform the change of variable $b = \varphi^{-1}(\beta)$

$$\int_0^t E[\partial_1 F(Y_s^*, Y_s^*) \frac{p_{Y^*, Y}(Y_s^*, Y_s^*, s)}{p_{Y^*}(Y_s^*, s)}] ds = \int_0^t \int_{\mathbb{R}_+} \partial_1 \Phi(b, b) p_{Y^*, Y}(\varphi(b), \varphi(b), s) db ds$$

and we use Lemma 2.3 which tells us $p_{Y^*, Y}(\varphi(b), \varphi(b), s) = p_V(b, b, s)A^2(b)$:

$$\int_0^t E[\partial_1 F(Y_s^*, Y_s^*) A(Y_s^*) \frac{p_{Y^*, Y}(Y_s^*, Y_s^*, s)}{p_{Y^*}(Y_s^*, s)}] ds = \int_0^t \int_{\mathbb{R}_+} \partial_1 \Phi(b, b) A^2(b) p_V(b, b, s) db ds$$

and finally

$$\int_0^t E[\partial_1 F(Y_s^*, Y_s^*) A(Y_s^*) \frac{p_{Y^*, Y}(Y_s^*, Y_s^*, s)}{p_{Y^*}(Y_s^*, s)}] ds = \int_0^t E[\partial_1 \Phi(M_s, M_s) A^2(M_s) \frac{p_V(M_s, M_s, s)}{p_M(M_s, s)}] ds$$

•

2.2 Theorem 2.1 proof in case $A = 1$, first step

Let Φ be a C_b^2 function on \mathbb{R}^2 . Before applying Itô formula to $s \rightarrow \Phi(V_s)$, one remarks that the component M_t is a non decreasing continuous process, so it is a finite variation process thus the brackets $d\langle X, M \rangle_t = 0$, and $d\langle M \rangle_t = 0$. Itô's formula yields

$$\Phi(V_t) - \Phi(V_0) = \int_0^t \frac{\partial \Phi}{\partial x}(M_s, X_s) dW_s + \int_0^t \frac{\partial \Phi}{\partial m}(V_s) dM_s + \int_0^t \mathcal{L}\Phi(V_s) ds$$

where \mathcal{L} is the infinitesimal generator $B(x)\partial_x + \frac{1}{2}\partial_{xx}^2$.

The first term on the right hand side above is a martingale so its expectation is equal to 0. So

$$E[\Phi(V_t) - \Phi(V_0)] = E\left[\int_0^t \mathcal{L}\Phi(V_s) ds\right] + E\left[\int_0^t \frac{\partial \Phi}{\partial m}(V_s) dM_s\right].$$

Now the deal is to look at

$$E\left[\int_0^t \frac{\partial \Phi}{\partial m}(V_s) dM_s\right].$$

Namely, without losing in general, we look at $E[\int_0^t \Psi(V_s) dM_s]$ where $\Psi \in C_b^1$.

Proposition 2.5. *Let $A = 1$ and B be C_b^1 on \mathbb{R} , Ψ a Borel bounded function, then the application $t \rightarrow E[\int_0^t \Psi(V_s) dM_s]$ is absolutely continuous on all finite interval $[0, T]$.*

Proof. (i) Using Criterion 11.7 page 364 [17] since $\left|E[\int_a^b \Psi(V_s) dM_s]\right| \leq \|\Psi\|_\infty E[M_b - M_a]$, it is enough to prove the result in case of $\Psi = 1$, meaning the study of $t \rightarrow E[M_t]$.

(ii) We operate an equivalent change of probability measure, $Q = L_T\mathbb{P}$, with

$$dL_t = -L_t B(X_t) dW_t, \quad L_t = \exp\left[\int_0^t -B(X_s) dW_s - \frac{1}{2} \int_0^t B^2(X_s) ds\right]$$

so that under Q the process X is a Brownian motion and

$$E_{\mathbb{P}}[M_t] = E_Q[L_t^{-1} X_t^*].$$

Lemma 2.6. *Let $A = 1$ and B be C_b^1 on \mathbb{R} . For all $T > 0$ there exists a constant C_T such that for all a, b , $0 < a < b \leq T$,*

$$E_{\mathbb{P}}[M_b - M_a] \leq C_T \left[\int_a^b \mathbb{E}_Q [c^2 \cosh(cW_s) W_s^*] ds + \int_a^b \mathbb{E}_Q \left[\cosh(cW_s) \frac{p_W(W_s^+, W_s; s)}{g_W(W_s; s)} \right] ds \right] \quad (18)$$

where p_W is introduced in (13) and $g_W(\cdot, s)$ is the density of the law of W_s .

Proof. Abusing of notation, under Q , we here use W instead of X which is a Q -Brownian motion. Moreover we denote the process L^{-1} as Z .

Denoting \mathcal{B} such that $\mathcal{B}' = B$, and $\mathcal{B}(0) = 0$, so $\mathcal{B}(W_t) = \int_0^t B(W_s) dW_s + \frac{1}{2} B'(W_s) ds$,

$$Z_t = \exp\left[\mathcal{B}(W_t) - \frac{1}{2} \int_0^t B'(W_s) ds - \frac{1}{2} \int_0^t B^2(W_s) ds\right]. \quad (19)$$

Firstly, B and B' are bounded, so there exists a constant C such that for any T , $Z_t \leq \exp[\mathcal{B}(W_t) + CT]$, secondly $\mathcal{B} \in C_b^1$ is sub linear, there exists C such that $|\mathcal{B}(W_t)| \leq C|W_t|$ thus $Z_t \leq Ce^{CT}(\exp(cW_t) + \exp(-cW_t)) = Ce^{CT} \cosh(cW_t)$. So

$$\begin{aligned} E_{\mathbb{P}}[M_b - M_a] &= E_Q[Z_b(W_b^* - W_a^*)] \leq Ce^{CT} E_Q[\cosh(cW_b)(W_b^* - W_a^*)] \\ &\leq Ce^{CT} \{E_Q[\cosh(cW_b)W_b^* - \cosh(cW_a)W_a^*] + |E_Q[(\cosh(cW_b) - \cosh(cW_a))W_a^*]|\}. \end{aligned} \quad (20)$$

We introduce two sequences of C_b^2 positive functions, f_N and g_N such that $f_N(x) \uparrow \cosh(cx)$, $g_N(x) \uparrow x$, $f_N''(x) \uparrow c^2 \cosh(cx)$, $g_N'(x) \uparrow 1$, when N goes to infinity. According to Theorem 2.1 [4] in case of a null drift,

$$\mathbb{E}_Q(g_N(W_t^*)f_N(W_t)) = g_N(0)f_N(0) + \frac{1}{2} \int_0^t \mathbb{E}[f_N''(W_s)g_N(W_s^*)] ds + \frac{1}{2} \int_0^t \mathbb{E} \left[g_N'(W_s^*)f_N(W_s) \frac{p_W(W_s^+, W_s; s)}{g_W(W_s; s)} \right] ds. \quad (21)$$

Both hands are monotonous with respect to N , so using Lebesgue's monotonous convergence Theorem, we get

$$\mathbb{E}_Q(\cosh(cW_t)W_t^*) = \frac{1}{2} \int_0^t \mathbb{E}_Q[c^2 \cosh(cW_s)W_s^*] ds + \frac{1}{2} \int_0^t \mathbb{E}_Q \left[\cosh(cW_s) \frac{p_W(W_s^+, W_s; s)}{g_W(W_s; s)} \right] ds. \quad (22)$$

Using Itô formula and the fact that W^* is an increasing process

$$|E_Q[(\cosh(cW_b) - \cosh(cW_a))W_a^*]| = E_Q \left[\frac{c^2}{2} \int_a^b \cosh(cW_s) ds W_a^* \right] \leq E_Q \left[\frac{c^2}{2} \int_a^b \cosh(cW_s) W_s^* ds \right]. \quad (23)$$

Plugging estimations (22) and (23) into (20) yields (18). •

This lemma concludes the proof of Proposition 2.5 using Criterion 11.7 page 364 [17] since for all $0 < a < b \leq T$

$$E_{\mathbb{P}}[M_b - M_a] \leq \frac{C_T}{2} \int_a^b \mathbb{E}[c^2 \cosh(cW_s)W_s^*] ds + C_T \frac{1}{2} \int_a^b \mathbb{E} \left[\cosh(cW_s) \frac{p_W(W_s^+, W_s; s)}{g_W(W_s; s)} \right] dt. \quad \bullet$$

Corollary 2.7. *Since the application $F_{\Psi} : t \rightarrow E[\int_0^t \Psi(V_s) dM_s]$ is absolutely continuous, there exists f_{Ψ} such that $E[\int_0^t \Psi(V_s) dM_s] = \int_0^t f_{\Psi}(s) ds$.*

2.3 Computation of F_{Ψ} derivative, case additive noise, second step

Let now a drifted Brownian motion be defined as: $dX_t = B(X_t)dt + dW_t$, $X_0 = x$, $t \in [0, T]$ where $B \in C_b^1$. Let $\Psi \in C_b^1$ and recall the function $F_{\Psi} : t \mapsto E \left(\int_0^t \Psi(V_s) dM_s \right)$. Using Corollary 2.7, this function F_{Ψ} is absolutely continuous with respect to Lebesgue measure, so almost differentiable with derivative that we denote f_{Ψ} . Actually $f_{\Psi} = \lim_{h \rightarrow 0} T_{\Psi}(h, \cdot)$ where

$$T_{\Psi}(h, t) := \frac{1}{h} \mathbb{E} \left(\int_t^{t+h} \Psi(V_s) dM_s \right). \quad (24)$$

Theorem 2.8. Let the process X be defined as $dX_t = B(X_t)dt + dW_t$, with $B \in C_b^1$, $M_t = \sup_{0 \leq s \leq t} X_s$ and $\Psi \in C_b^1$. Then $E[\int_0^t \Psi(V_s) dM_s] = \int_0^t f_\Psi(s) ds$ where

$$f_\Psi(t) = \frac{1}{2} E_{\mathbb{P}} \left[\Psi(M_t, M_t) \frac{p_X(M_t, M_t, t)}{p_M(M_t, t)} \right].$$

The proof is a consequence of the three following propositions.

Proposition 2.9. Let $A = 1$, B be C_b^1 , Ψ be C_b^1 then for any t such that $(T_\Psi(h, t))_h$ converges when $h \rightarrow 0$,

$$f_\Psi(t) = \lim_{h \rightarrow 0} T_\Psi^1(h, t) \text{ where } T_\Psi^1(h, t) = 2\mathbb{E} \left(\Psi(M_t, M_t) \frac{1}{\sqrt{h}} H \left(\frac{M_t - X_t}{\sqrt{h}} \right) \right)$$

where

$$H(B) := \frac{1}{\sqrt{2\pi}} e^{-\frac{B^2}{2}} - B\Phi_G(-B)$$

and Φ_G is the Gaussian distribution function.

Proof. (i) Adding and subtracting $\Psi(V_t) \frac{M_{t+h} - M_t}{h}$ we get

$$T_\Psi(h, t) = \mathbb{E} \left(\Psi(V_t) \frac{M_{t+h} - M_t}{h} \right) + \frac{1}{h} \mathbb{E} \left(\int_t^{t+h} [\Psi(V_s) - \Psi(V_t)] dM_s \right).$$

Since the function $\Psi \in C_b^1$, Cauchy-Schwartz inequality yields

$$\left| \frac{1}{h} \mathbb{E} \left(\int_t^{t+h} [\Psi(V_s) - \Psi(V_t)] dM_s \right) \right| \leq K \sqrt{\mathbb{E} \left(\sup_{t \leq u \leq t+h} \|V_u - V_t\|^2 \right)} \frac{1}{h} \mathbb{E}[(M_{t+h} - M_t)^2].$$

Using Lemmas 4.1 and 4.2 with $p = 2$ and the fact that $M_u - M_s \leq \sup_{t \leq s \leq u} |X_s - X_t|$,

$$\lim_{h \rightarrow 0} \left| \frac{1}{h} \mathbb{E} \left(\int_t^{t+h} [\Psi(V_s) - \Psi(V_t)] dM_s \right) \right| = 0$$

meaning, t being fixed, that

$$T_\Psi(h, t) \sim \mathbb{E} \left(\Psi(V_t) \frac{M_{t+h} - M_t}{h} \right).$$

(ii) We introduce the processes:

$$X_{t,h} := B(X_t)h + A(X_t)[W_{t+h} - W_t], \quad X_{t,0} = 0; \quad M_{t,h} := \sup_{u \leq h} X_{t,u}, \quad M_{t,0} = 0.$$

Then, according to Lemma 4.4

$$\mathbb{E} [|M_{t+h} - M_t - (M_{t,h} - X_t + M_t)_+|] = o(h)$$

so

$$T_\Psi(h, t) \sim \mathbb{E} \left(\Psi(V_t) \frac{(M_{t,h} - X_t + M_t)_+}{h} \right).$$

(iii) Concerning the right hand above, we use the \mathcal{F}_t conditional expectation and Proposition 4.8 in case $A = 1$:

$$\lim_{h \rightarrow 0} \left| T_\psi(h, t) - 2\mathbb{E} \left(\Psi(M_t, M_t) \frac{1}{\sqrt{h}} H \left(\frac{M_t - X_t}{\sqrt{h}} \right) \right) \right| = 0.$$

We now provide a decomposition of p_V .

Proposition 2.10. *Assuming $A = 1$, $B \in C_b^1$ and defining \mathcal{B} such that: $\mathcal{B}(0) = 0$ and $\mathcal{B}' := B$, $C := -\frac{1}{2}\mathcal{B}' - \frac{1}{2}\mathcal{B}^2$, for all $t > 0$, the density of the pair (M_t, X_t) satisfies*

$$p_V(b, a, t) = \sum_{i=1}^3 p_V^i(b, a, t)$$

where

$$\begin{aligned} p_V^1(b, a, t) &= e^{\mathcal{B}(a)} p_W(b, a, t) \\ p_V^2(b, a, t) &= \int_0^t E_{\mathbb{P}} \left[C(X_s) e^{\mathcal{B}(a) - \mathcal{B}(X_s)} \mathbf{1}_{M_s < b} p_W(b - X_s, a - X_s, t - s) \right] ds \\ p_V^3(b, a, t) &= \mathbf{1}_{b > a} \int_0^t \int_{-\infty}^b C(x) e^{\mathcal{B}(a) - \mathcal{B}(x)} p_V(b, x, s) \frac{e^{-\frac{(a-x)^2}{2(t-s)}} - e^{-\frac{(2b-a-x)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} dx ds, \end{aligned}$$

and p_W (13) denotes the density of the law of the pair Brownian motion-its running maximum.

Proof. (i) Let F be a positive function, remark that

$$E_{\mathbb{P}}[F(M_t, X_t)] = E_Q[Z_t F(M_t, X_t)]$$

and that under Q , X is a Brownian motion. Recall that Z (19) could be expressed as

$$Z_t = \exp \left(\mathcal{B}(X_t) + \int_0^t C(X_s) ds \right).$$

So Z could be expressed as

$$Z_t = \exp \mathcal{B}(X_t) \left(1 + \int_0^t C(X_s) e^{\int_0^s C(X_u) du} ds \right).$$

Then

$$E_{\mathbb{P}}[F(M_t, X_t)] = E_Q \left[\exp \mathcal{B}(X_t) \left(1 + \int_0^t C(X_s) e^{\int_0^s C(X_u) du} ds \right) F(M_t, X_t) \right].$$

(ii) Using $X_t = X_s + X_t - X_s$ and $M_t = \sup(M_s, X_s + \sup_{0 \leq u \leq t-s} X_{u+s} - X_s)$ and the independence under Q of $X_{\cdot+s} - X_s$ and \mathcal{F}_s ,

$$E_{\mathbb{P}}[F(M_t, X_t)] = \int_{\mathbb{R}^2} F(b, a) \exp(\mathcal{B}(a)) p_W(b, a, t) db da \quad (25)$$

$$+ \int_0^t \int_{\mathbb{R}^2} E_Q[\exp(\mathcal{B}(X_s + \alpha))C(X_s)e^{\int_0^s C(X_u)du}F(\sup(M_s, X_s + \beta), X_s + \alpha)]p_W(\beta, \alpha, t - s)d\beta d\alpha ds.$$

Recall that $e^{\int_0^s C(X_u)du} = Z_s \exp(-\mathcal{B}(X_s))$, the integrand of the second term is

$$E_Q[\exp(\mathcal{B}(X_s + \alpha) - \mathcal{B}(X_s))C(X_s)Z_s F(\sup(M_s, X_s + \beta), X_s + \alpha)]p_W(\beta, \alpha, t - s).$$

Under the probability measure $\mathbb{P} = Z_s.Q$ this one is

$$E_{\mathbb{P}}[\exp(\mathcal{B}(X_s + \alpha) - \mathcal{B}(X_s))C(X_s)F(\sup(M_s, X_s + \beta), X_s + \alpha)]p_W(\beta, \alpha, t - s).$$

Thus the expectation $E_{\mathbb{P}}[F(M_t, X_t)]$ satisfies

$$E_{\mathbb{P}}[F(M_t, X_t)] = \int_{\mathbb{R}^2} F(b, a) \exp(\mathcal{B}(a))p_W(b, a, t)dbda +$$

$$\int_0^t \int_{\mathbb{R}^2} E_{\mathbb{P}}[e^{\mathcal{B}(X_s + \alpha) - \mathcal{B}(X_s)}C(X_s)F(\sup(M_s, X_s + \beta), X_s + \alpha)]p_W(\beta, \alpha, t - s)d\beta d\alpha ds.$$

(iii) We split the second term according to the subset $\{M_s \leq X_s + \beta\}$ and its complement:

$$I_1 = \int_0^t \int_{\mathbb{R}^2} E_{\mathbb{P}}[e^{\mathcal{B}(X_s + \alpha) - \mathcal{B}(X_s)}C(X_s)F(X_s + \beta, X_s + \alpha)\mathbf{1}_{\{M_s \leq X_s + \beta\}}]p_W(\beta, \alpha, t - s)d\beta d\alpha ds,$$

$$I_2 = \int_0^t \int_{\mathbb{R}^2} E_{\mathbb{P}}[e^{\mathcal{B}(X_s + \alpha) - \mathcal{B}(X_s)}C(X_s)F(M_s, X_s + \alpha)\mathbf{1}_{\{M_s > X_s + \beta\}}]p_W(\beta, \alpha, t - s)d\beta d\alpha ds.$$

In the first term I_1 , we perform the change of variable $b = X_s + \beta$, $a = X_s + \alpha$, so

$$I_1 = \int_0^t \int_{\mathbb{R}^2} E_{\mathbb{P}}[e^{\mathcal{B}(a) - \mathcal{B}(X_s)}C(X_s)F(b, a)\mathbf{1}_{\{M_s \leq b\}}]p_W(b - X_s, a - X_s, t - s)dbdad\alpha ds. \quad (26)$$

(iv) Concerning I_2 , we use the density of the law under \mathbb{P} of the pair (M_s, X_s) that we denote as $p_X(m, x, s)$, so

$$I_2 = \int_0^t \int_{\mathbb{R}^4} e^{\mathcal{B}(x + \alpha) - \mathcal{B}(x)}C(x)F(m, x + \alpha)\mathbf{1}_{\{m > x + \beta\}}p_V(m, x, s)p_W(\beta, \alpha, t - s)d\beta d\alpha dm dx ds.$$

We operate the change of variable $b = m$, $a = x + \alpha$, so

$$I_2 = \int_0^t \int_{\mathbb{R}^4} e^{\mathcal{B}(x + \alpha) - \mathcal{B}(x)}C(x)F(b, a)\mathbf{1}_{\{b > x + \beta\}}p_V(b, x, s)p_W(\beta, a - x, t - s)d\beta dbdad\alpha dx ds.$$

Using the expression of p_W , [9] Section 3.2 or (13),

$$p_W(\beta, a, t) = 2 \frac{2\beta - a}{\sqrt{2\pi t^3}} e^{-\frac{(2\beta - a)^2}{2t}} \mathbf{1}_{\{\beta > 0, \beta > a\}},$$

we integrate with respect to β between $\max(0, a - x)$ and $b - x$ and note that $(2 \max(0, a - x))^2 = a^2$ then

$$I_2 = \int_0^t \int_{\mathbb{R}^3} e^{\mathcal{B}(a) - \mathcal{B}(x)}C(x)F(b, a) \frac{e^{-\frac{(a-x)^2}{2(t-s)}} - e^{-\frac{(2b-a-x)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} \mathbf{1}_{\{b > a\}}p_V(b, x, s)dbdad\alpha dx ds. \quad (27)$$

Gathering Expressions (25), (26), (27) and using Tonelli Theorem, we get the result. •

The following proposition proves that the density p_V is continuous closed to the diagonal when X is a one dimensional diffusion with additive noise. We also provides a useful estimation for using Lebesgue's dominated convergence Theorem.

Proposition 2.11. *Let $A = 1$ and $B \in C_b^1$, for all t, u , in $\mathbb{R}^+ - \{0\}$, $h \in (0, 1]$, then almost surely in $b \in \mathbb{R}^+$*

$$\lim_{h \rightarrow 0, h > 0} p_V(b, b - u\sqrt{h}, t) = p_V(b, b, t).$$

and there exists a constant C independent of t such that:

$$p_V^1(b, b - \sqrt{h}u, t) \leq C e^{\|B\|_\infty[u+b]} \frac{1}{\sqrt{2\pi t^2}} e^{-\frac{b^2}{4t}} \quad (28)$$

$$p_V^2(b, b - \sqrt{h}u, t) \leq C e^{3/2t\|B\|_\infty^2 t + \|B\|_\infty u} \quad (29)$$

$$p_V^3(b, b - \sqrt{h}u, t) \leq 4\|C\|_\infty \int_0^t \mathbb{E}_{\mathbb{P}}[e^{\|B\|_\infty[b-X_s+u]} | M_s = b] \frac{p_M(b, s)}{\sqrt{2\pi(t-s)}} ds. \quad (30)$$

Proof. According to Proposition 2.10, we recall the decomposition of $p_V(b, a; t)$ in the sum of the three terms $p_V^i(b, a; t)$, $i = 1, 2, 3$:

$$p_V^1(b, a, t) = e^{\mathcal{B}(a)} p_W(b, a, t)$$

$$p_V^2(b, a, t) = \int_0^t E_{\mathbb{P}} \left[C(X_s) e^{\mathcal{B}(a) - \mathcal{B}(X_s)} 1_{M_s < b} p_W(b - X_s, a - X_s, t - s) \right] ds$$

$$p_V^3(b, a, t) = 1_{\{b > a\}} \int_0^t \int_{-\infty}^b C(x) e^{\mathcal{B}(a) - \mathcal{B}(x)} p_V(b, x, s) \left[\frac{e^{-\frac{(a-x)^2}{2(t-s)}} - e^{-\frac{(2b-a-x)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} \right] dx ds.$$

(i) Bound of p_V^1 : Let b and t be fixed. Since \mathcal{B} is $\|B\|_\infty$ Lipschitz continuous and $a \mapsto p_W(b, a; t)$ defined in (13) is continuous on $] -\infty, b]$ then

$$\lim_{a \rightarrow b, a < b} p_V^1(b, a; t) = p_V^1(b, b; t).$$

Moreover the Lipschitz property of \mathcal{B} and the definition of $a \mapsto p_W(b, a; t)$ (13), for $u > 0$, $h \in (0, 1]$, $b > 0$ so $0 < b < 2b - (b - u\sqrt{h}) = b + u\sqrt{h} < b + u$, yield

$$\begin{aligned} p_V^1(b, b - \sqrt{h}u, t) &\leq e^{\|B\|_\infty[u+b]} \frac{2(b + u\sqrt{h})}{\sqrt{2\pi t^3}} e^{-\frac{(b+u\sqrt{h})^2}{2t}} \\ &\leq e^{\|B\|_\infty[u+b]} \frac{2C_1}{\sqrt{2\pi t^2}} e^{-\frac{(b+u\sqrt{h})^2}{4t}} \leq e^{\|B\|_\infty[u+b]} \frac{2C_1}{\sqrt{2\pi t^2}} e^{-\frac{b^2}{4t}} \end{aligned} \quad (31)$$

using $C_1 = \sup_{x>0} x e^{-\frac{x^2}{4}}$.

(ii) Bound of p_V^2 : Let b, u in \mathbb{R}_+ and t be fixed. The integrand in $p_V^2(b, b - u\sqrt{h}; t)$ is $p_V^2(b, u, t, s, \omega, h)$ where

$$p_V^2(b, u, t, s, \omega, h) = C(X_s) e^{\mathcal{B}(b-u\sqrt{h}) - \mathcal{B}(X_s)} 1_{M_s < b} p_W(b - X_s, b - X_s - u\sqrt{h}, t - s).$$

Once again the Lipschitz property of \mathcal{B} is used and since $a \mapsto p_W(b, a; t)$ (13) is continuous on $] - \infty, b]$ then

$$\lim_{h \rightarrow 0, h > 0} p_V^2(b, u, t, s, \omega, h) = p_V^2(b, u, t, s, \omega, 0).$$

Recall that the function C is bounded, then the factor $C(X_s)e^{\mathcal{B}(b-u\sqrt{h})-\mathcal{B}(X_s)}$ is bounded by $\|C\|_\infty e^{\|B\|_\infty [b-X_s+u]}$ on the set $\{\omega, M_s(\omega) < b\} \subset \{\omega, X_s(\omega) < b\}$. Now using $2(b-X_s) - (b-X_s-u\sqrt{h}) = b-X_s+u\sqrt{h} > 0$ and $C_\alpha = \sup_x x^\alpha e^{-x^2/4}$, $\forall h \in (0, 1]$,

$$\begin{aligned} p_W(b-X_s, b-X_s-u\sqrt{h}, t-s) &= 2 \frac{b-X_s+u\sqrt{h}}{\sqrt{2\pi(t-s)^3}} e^{-\frac{(b-X_s+u\sqrt{h})^2}{2(t-s)}} \mathbf{1}_{X_s \leq b} \\ &\leq 2C_{1+\varepsilon} \frac{1}{\sqrt{2\pi(t-s)^{2-\varepsilon}} (b-X_s+u\sqrt{h})^\varepsilon} e^{-\frac{(b-X_s+u\sqrt{h})^2}{4(t-s)}} \\ &\leq 2C_{1+\varepsilon} \frac{1}{\sqrt{2\pi(t-s)^{2-\varepsilon}} (b-X_s)^\varepsilon} e^{-\frac{(b-X_s)^2}{4(t-s)}} \mathbf{1}_{X_s \leq b}. \end{aligned}$$

We gather these both bounds to get:

$$|p_V^2(b, u, t, s, \omega, h)| \leq 2\|C\|_\infty e^{\|B\|_\infty [b-X_s+u]} C_{1+\varepsilon} \frac{1}{\sqrt{2\pi(t-s)^{2-\varepsilon}} (b-X_s)^\varepsilon} e^{-\frac{(b-X_s)^2}{4(t-s)}} \mathbf{1}_{X_s \leq b}.$$

According to Lemma 4.10 for $0 < \varepsilon < 1/2$ and $\alpha = \|B\|_\infty$ there exists a constant $C(\varepsilon) > 0$ such that $\forall t > 0, 0 < s < t, b > 0$

$$\mathbb{E}_{\mathbb{P}} \left[e^{\|B\|_\infty [(b-X_s)]} \mathbf{1}_{\{M_s < b\}} \frac{1}{\sqrt{2\pi(t-s)^{2-\varepsilon}} (b-X_s)^\varepsilon} e^{-\frac{(b-X_s)^2}{4(t-s)}} \right] \leq C(\varepsilon) \frac{e^{\|B\|_\infty^2 (t-s+t/2)}}{s^{1/4} (t-s)^{3/4}}. \quad (32)$$

Then, the dominating function

$$(s, \omega) \mapsto 2\|C\|_\infty e^{\|B\|_\infty [(b-X_s)+u]} C_\varepsilon \frac{1}{\sqrt{2\pi(t-s)^{2-\varepsilon}} (b-X_s)^\varepsilon} e^{-\frac{(b-X_s)^2}{4(t-s)}} \mathbf{1}_{\{X_s \leq b\}}$$

is integrable with respect to $ds.d\mathbb{P}$ on $[0, t] \times \Omega$ and from Lebesgue's dominated convergence Theorem

$$\lim_{h \rightarrow 0, h > 0} p_V^2(b, b - \sqrt{hu}; t) = p_V^2(b, b; t).$$

Integrating the bound (32) with respect to s we obtain for all $t > 0, b, u > 0, h \in (0, 1]$ and $0 < \varepsilon < \frac{1}{2}$ and using $B(\frac{1}{4}, \frac{3}{4}) = \int_0^t (t-s)^{-3/4} s^{-1/4} ds$

$$p_V^2(b, b - \sqrt{hu}, t) \leq C(\varepsilon) e^{3t/2 \|B\|_\infty^2 + u \|B\|_\infty} B\left(\frac{1}{4}, \frac{3}{4}\right). \quad (33)$$

(iii) Bound of p_V^3 : Let b, u in \mathbb{R}_+ and t be fixed. The integrand in $p_V^3(b, b - u\sqrt{h}; t)$ is $p_V^3(b, u, t, s, x, h)$ where

$$p_V^3(b, u, t, s, x, h) = C(x) e^{\mathcal{B}(b-u\sqrt{h})-\mathcal{B}(x)} p_V(b, x, s) \left[\frac{e^{-\frac{(b-x-u\sqrt{h})^2}{2(t-s)}} - e^{-\frac{(b-x+u\sqrt{h})^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} \right].$$

Once again the Lipschitz property of \mathcal{B} is used firstly to get

$$\lim_{h \rightarrow 0, < h} p_V^3(b, u, t, s, x, h) = p_V^3(b, u, t, s, x, 0) = 0$$

and secondly, with $h \in (0, 1]$, $u > 0$, to dominate the function $p_V^3(b, u, t, s, x, h)$ by

$$(b, u, t, x, s) \mapsto 2\|C\|_\infty e^{\|B\|_\infty[b-x+u]} \frac{p_V(b, x; s)}{\sqrt{2\pi(t-s)}}. \quad (34)$$

Applying Lemma 4.11 with $\alpha = \|B\|_\infty$, for any $s \geq 0$,

$$\int_{\mathbb{R}^2} e^{\|B\|_\infty[b-x]} p_V(b, x; s) dx db = \mathbb{E}_{\mathbb{P}} [\exp \|B\|_\infty[M_s - X_s]] \leq 2 \exp 4s\|B\|_\infty^2,$$

so

$$\int_0^t \int_{\mathbb{R}^2} e^{\|B\|_\infty[b-x]} \frac{p_V(b, x; s)}{\sqrt{2\pi(t-s)}} dx db ds \leq 4\sqrt{\frac{t}{2\pi}} e^{4t\|B\|_\infty^2}.$$

Then, almost surely in b ,

$$(s, x) \mapsto \|C\|_\infty e^{\|B\|_\infty[b-x+u]} \frac{p_V(b, x; s)}{\sqrt{2\pi(t-s)}}$$

is integrable with respect to $ds d\mathbb{P}$ on $[0, t] \times \Omega$ and from Lebesgue's dominated convergence Theorem

$$\lim_{h \rightarrow 0, h > 0} p_V^3(b, b - \sqrt{h}u; t) = p_V^3(b, b; t).$$

Moreover, integrating the dominating function (34) with respect to x and s yields for all $t > 0, b > 0, u > 0$ and $h \in (0, 1]$

$$p_V^3(b, b - u\sqrt{h}, t) \leq 4\|C\|_\infty \int_0^t \mathbb{E}_{\mathbb{P}} [e^{\|B\|_\infty[b-X_s+u]} | M_s = b] \frac{p_M(b, s)}{\sqrt{2\pi(t-s)}} ds.$$

Gathering these three cases we get the result •

2.4 Proof of Theorem 2.8, last step

The key of this proof is the uniform (with respect to b) bounds of $p_V^i(b, b - u\sqrt{h}, t)$, $i = 2, 3$ which are obtained in Proposition 2.11. Thus, firstly we prove Theorem 2.8 for Ψ with compact support. This proof is split in three parts according to p_V^i , $i = 1, 2, 3$, Proposition 2.10. Then, using Lebesgue's dominated convergence Theorem, we derive the case of bounded Ψ .

(i) According to Proposition 2.9, for any t such that $(T_\psi(h, t))_{h \rightarrow 0}$ converges,

$$f_\Psi(t) = \lim_{h \rightarrow 0} T_\Psi^1(h, t) \text{ where } T_\Psi^1(h, t) = 2\mathbb{E} \left(\Psi(M_t, M_t) \frac{1}{\sqrt{h}} H \left(\frac{M_t - X_t}{\sqrt{h}} \right) \right).$$

Using the density of the law of the pair (M_t, X_t)

$$T_\Psi^1(h, t) = \int_{\mathbb{R}^2} \Psi(b, b) \frac{1}{\sqrt{h}} H \left(\frac{b-a}{\sqrt{h}} \right) p_V(b, a, t) db da$$

We perform the change of variable $u = \frac{b-a}{\sqrt{h}}$, then

$$T_{\Psi}^1(h, t) = 2 \int_{\mathbb{R}_+^2} \Psi(b, b) H(u) p_V(b, b - u\sqrt{h}, t) dbdu. \quad (35)$$

Using Proposition 2.11, if Ψ is with compact support, using $H(u) \leq \frac{e^{-u^2/2}}{\sqrt{2\pi}}$, we only need to uniformly bound $\forall h$

$$\Psi(b, b) \frac{e^{-u^2/2}}{\sqrt{2\pi}} p_V(b, b - u\sqrt{h}, t)$$

by a function in $L^1(\mathbb{R}_+^2)$.

Using the decomposition of p_V in Proposition 2.10, $T_{\Psi}^1(h, t)$ is the sum of three terms

$$T_{\Psi}^1(h, t) = \sum_{i=1}^3 T_{\Psi}^{1,i}(h, t)$$

where

$$T_{\Psi}^{1,i}(h, t) \leq \int_{\mathbb{R}^2} \Psi(b, b) \frac{e^{-u^2/2}}{\sqrt{2\pi}} p_V^i(b, b - u\sqrt{h}, t) dbdu.$$

Now using the bound of each terms p_V^i $i = 1, 2, 3$ (cf. Proposition 2.11), we bound each terms separately to use Lebesgue's dominated convergence Theorem.

(ii) a) Using the bound (28), the integrand in $T_{\Psi}^{1,1}(h, t)$ is bounded by

$$\|\Psi\|_{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} C e^{\|B\|_{\infty}[u+b]} \frac{1}{\sqrt{2\pi t^2}} e^{-\frac{b^2}{4t}}.$$

Thus this dominating function is, up to a multiplicative constant, the product of two integrable functions on \mathbb{R}^+ : $u \rightarrow e^{\|B\|_{\infty}u} e^{-\frac{u^2}{2}}$ and $b \rightarrow e^{\|B\|_{\infty}b} e^{-\frac{b^2}{4t}}$. This allow to apply Lebesgue's dominated convergence Theorem and to obtain

$$\lim_{h \downarrow 0} T_{\Psi}^{1,1}(h, t) = \int_{\mathbb{R}_+^2} \Psi(b, b) H(u) e^{\mathcal{B}(b)} p_W(b, b; t) dbdu.$$

(ii) b) Using the bound (29), the integrand in $T_{\Psi}^{1,2}(h, t)$ is bounded by

$$|\Psi(b, b)| C e^{3/2\|B\|_{\infty}^2 t + \|B\|_{\infty} u} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}.$$

Lebesgue's dominated Theorem yields for Ψ with compact support

$$\lim_{h \downarrow 0} T_{\Psi}^{1,2}(h, t) = \int_{\mathbb{R}_+^2} \Psi(b, b) p_v^2(b, b, t) H(u) dbdu.$$

(ii) c) Using the bound (30), the integrand in $T_{\Psi}^{1,3}(h, t)$ is bounded by

$$\|\Psi\|_\infty \|C\|_\infty \frac{e^{-u^2/2}}{\sqrt{2\pi}} \int_0^t \mathbb{E}_{\mathbb{P}}[e^{\|B\|_\infty [b - X_s + u]} | M_s = b] \frac{p_M(b, s)}{\sqrt{2\pi(t-s)}} ds.$$

Note that

$$\begin{aligned} & \int_{\mathbb{R}_+^2} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \int_0^t \mathbb{E}_{\mathbb{P}}[e^{\|B\|_\infty [b - X_s + u]} | M_s = b] \frac{p_M(b, s)}{\sqrt{2\pi(t-s)}} ds du db \\ &= \int_{\mathbb{R}_+} \frac{e^{-u^2/2 + \|B\|_\infty u}}{\sqrt{2\pi}} du \int_0^t \mathbb{E}_{\mathbb{P}}[e^{\|B\|_\infty [M_s - X_s]}] \frac{1}{\sqrt{2\pi(t-s)}} ds. \end{aligned}$$

According to Lemma 4.11 with $\alpha = \|B\|_\infty$

$$\mathbb{E}_{\mathbb{P}}[\exp\|B\|_\infty [M_s - X_s]] \leq \exp\frac{3}{2}\|B\|_\infty s,$$

so

$$\int_0^t \mathbb{E}_{\mathbb{P}}[e^{\|B\|_\infty [M_s - X_s]}] \frac{1}{\sqrt{(t-s)}} ds \leq \int_0^t \frac{1}{\sqrt{(t-s)}} \exp[\frac{3}{2}\|B\|_\infty s] ds = 2\sqrt{t} \exp[\frac{3}{2}\|B\|_\infty t].$$

Then, using that $u \rightarrow \frac{e^{-u^2/2}}{\sqrt{2\pi}} e^{\|B\|_\infty u}$ is integrable on \mathbb{R}^+ , the dominated function is integrable with respect u this allow to apply Lebesgue's dominated convergence Theorem for Ψ with compact support.

iii) We now can exchange limit and integral in (35) thus

$$f_\Psi(t) = 2 \int_{(\mathbb{R}^+)^2} \Psi(b, b) H(u) p_V(b, b, t) db du.$$

Multiplying and dividing by the density of M_t and using $\int_0^\infty H(u) du = \frac{1}{4}$,

$$f_\Psi(t) = \frac{1}{2} E_{\mathbb{P}}[\Psi(M_t, M_t) \frac{p_V(M_t, M_t, t)}{p_M(M_t, t)}].$$

(iv) Let $(\Psi_n)_n$ an increasing sequence of continuous function with compact support converging to 1 then $t \mapsto \mathbb{E}(\int_0^t \Psi_n(M_s) dM_s)$ is absolutely continuous with respect to the Lebesgue measure, its derivative, namely $t \mapsto \frac{1}{4} \int_{\mathbb{R}_+} \Psi_n(b) p_V(b, b, t) db$, is integrable

$$\mathbb{E}(\int_0^t \Psi_n(M_s) dM_s) = \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} \Psi_n(b) p_V(b, b, s) db ds.$$

Then letting n going to infinity with monotonous convergence theorem

$$\mathbb{E}(M_t) = \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} p_V(b, b, s) db ds.$$

That means $(s, b) \mapsto p_V(b, b, s)$ belongs to $L^1([0, t] \times \mathbb{R}_+)$.

Now let Ψ be a continuous bounded function then, $(m, x) \mapsto \Psi_n(b)\Psi_n(x)\Psi(b, x)$ is continuous with support compact and

$$\begin{aligned} \mathbb{E}\left(\int_0^t \Psi(V_s) dM_s\right) &= \lim_{n \rightarrow \infty} \mathbb{E}\left(\int_0^t \Psi(V_s) \Psi_n(X_s) \Psi_n(M_s) dM_s\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} (\Psi_n(b))^2 \Psi(b, b) p_V(b, b, s) db ds \\ &= \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} \Psi(b, b) p_V(b, b, s) db ds \end{aligned}$$

since $(s, b) \mapsto p_V(b, b, s) \in L^1([0, t] \times \mathbb{R}_+)$. This achieves the proof of Theorem 2.8. •

3 Integration by parts to go to a weak PDE

Let $\Phi \in C_b^2(\mathbb{R}^+ \times \mathbb{R})$; via Ito's formula, since (M_t, X_t) law admits a density which is denoted $p_V(., ., t)$ (Proposition 1.1), and using Theorem 2.8:

$$E[\Phi(M_t, X_t)] = \Phi(0, 0) + \int_0^t \int_{m \geq x^+} \mathcal{L}\Phi(m, x) p_V(m, x; s) dx dm ds + \frac{1}{2} \int_0^t E_{\mathbb{P}} \left[\partial_m \Phi(M_s, M_s) \frac{p_V(M_s, M_s, s)}{p_M(M_s, s)} ds \right] \quad (36)$$

where \mathcal{L} is the infinitesimal generator of the diffusion X

$$X_t = B(X_t) + W_t, \quad M_t = \sup_{s \leq t} X_s.$$

Theorem 3.1. *Let B be C_b^1 on \mathbb{R} and $A = 1$. On the domain $\{(m, x), x < m\}$, p_V satisfies the PDE*

$$\partial_t p_V(m, x, s) = -B'(x) p_V(m, x, s) - B(x) \partial_x p_V(m, x, s) + \frac{1}{2} \partial_{xx}^2 p_V(m, x, s).$$

On the diagonal $x = m$ there is two boundary conditions

$$\begin{aligned} (i) \quad & B(m) p_V(m, m; t) - \frac{1}{2} [\partial_m + 2\partial_x](p_V)(m, m; t) = 0. \\ (ii) \quad & \partial_t p_M(m, t) = -\frac{1}{2} (\partial_m p_V(m, m, t) + \partial_x p_V(m, m, t)), \end{aligned}$$

Proof. split in four parts:

a) Firstly we consider $\Phi \in C^2(\mathbb{R}^+ \times \mathbb{R})$ with compact support included in $\{(m, x), m \geq x^+\} \setminus \{(0, 0)\}$. We use integration by part with respect to x to develop the first integrand in the right hand side of (36)

$$\begin{aligned} \int_{m \geq x^+} \mathcal{L}\Phi(m, x) p_V(m, x, s) dx dm &= \int \Phi(x, m) \mathcal{L}^*(p_V)(m, x, s) dx dm \\ &+ \int_0^t \int_{\mathbb{R}^+} \left[\left(B(m) p_V(m, m; s) - \frac{1}{2} \partial_x p_V(m, m; s) \right) \Phi(m, m) + \frac{1}{2} p_V(m, m; s) \partial_x \Phi(m, m) \right] dm \end{aligned}$$

with $\mathcal{L}^* = -B'(x)p_V(m, x, s) - B(x)\partial_x p_V(m, x, s) + \frac{1}{2}\partial_{xx}^2 p_V(m, x, s)$. Then equation (36) becomes

$$E[(\Phi(M_t), X_t)] = \int_0^t \int_{m \geq x^+} \mathcal{L}^* p_V(m, x; s) \Phi(m, x) dx dm ds + \int_0^t \int_{\mathbb{R}^+} [\{B(m)p_V(m, m; s) - \frac{1}{2}\partial_x p_V(m, m; s)\} \Phi(m, m) + \frac{1}{2}p_V(m, m; s)[\partial_x \Phi(m, m) + \partial_m \Phi(m, m)]] dm ds$$

Using integration by part with respect to m in the right hand side last two terms

$$E[\Phi(M_t), X_t] = \int_0^t \int_{m \geq x^+} \mathcal{L}^* p_V(m, x; s) \Phi(m, x) dx dm ds \quad (37) + \int_0^t \int_{\mathbb{R}^+} [B(m)p_V(m, m; s) - \frac{1}{2}[\partial_m + 2\partial_x]p_V(m, m; s)] \Phi(m, m) dm ds.$$

b) Now taking Φ with compact support in $\{(m, x), m > x^+\}$, (37) yields that on this domain $x < m$, p_V satisfies the PDE

$$\partial_t p_V(m, x, s) = -B'(x)p_V(m, x, s) - B(x)\partial_x p_V(m, x, s) + \frac{1}{2}\partial_{xx}^2 p_V(m, x, s).$$

c) Coming back to $\Phi \in C^2(\mathbb{R}^+ \times \mathbb{R})$ but with compact support included in $\{(m, x), m \geq x^+\} \setminus \{(0, 0)\}$, since the diagonal has null Lebesgue measure, (37) becomes

$$0 = \int_0^t \int_{\mathbb{R}^+} [B(m)p_V(m, m; s) - \frac{1}{2}[\partial_m + 2\partial_x]p_V(m, m; s)] \Phi(m, m) dm ds.$$

which leads to boundary condition (i).

d) Finally concerning the boundary condition (ii) we consider a regular function Φ with compact support in $\mathbb{R}^+ \setminus \{0\}$ so from Theorem 2.8

$$\mathbb{E}(\Phi(M_t)) = \frac{1}{2} \int_0^t \mathbb{E} \left(\Phi'(M_s) \frac{p_V(M_s, M_s; s)}{p_M(M_s; s)} \right) ds.$$

Actually we perform integration by part on \mathbb{R}^+

$$E_{\mathbb{P}} \left[\Phi'(M_s) \frac{p_V(M_s, M_s, s)}{p_M(M_s, s)} \right] = \int_{\mathbb{R}^+} \Phi'(m) p_V(m, m, s) dm = - \int_{\mathbb{R}^+} \Phi(m) (\partial_m p_V(m, m, s) + \partial_x p_V(m, m, s)) dm.$$

So we get

$$E[(\Phi(M_t))] = -\frac{1}{2} \int_0^t \int_{\mathbb{R}^+} \Phi(m) (\partial_m p_V(m, m, t) + \partial_x p_V(m, m, t)) dm.$$

With $p_M(m, t) = \int_{x^+ \leq m} p_V(m, x, t) dx$ it yields a boundary condition on the diagonal

$$\partial_t p_M(m, t) = -\frac{1}{2}(\partial_m p_V(m, m, t) + \partial_x p_V(m, m, t)).$$

•

In case of $A \neq 1$, this proposition provides a PDE for the pair (Y^*, Y) defined in Section 2.1 and p_V is deduced from $p_{Y^*, Y}$ via Lemma 2.3.

4 Appendix

4.1 Control in first step Euler scheme approximation

Here are provided some estimations controlling the processes X and M .

Lemma 4.1. *Under assumption (1), $\forall h > 0, \forall t, \forall s, \forall p \geq 1$*

$$\mathbb{E} \left(\sup_{s \leq h} |X_{t+s} - X_t|^p \right) \leq 2^{p-1} (1 + C_p) [h^p + h^{p/2}] 2K^p$$

where $K := \max(\|A\|_\infty, \|B\|_\infty)$, and C_p the constant in Burkholder Davis Gundy inequality.

Proof. Using $(a + b)^p \leq 2^{p-1} [a^p + b^p]$:

$$\sup_{s \leq h} |X_{t+s} - X_t|^p \leq 2^{p-1} \left[\sup_{u \leq h} \left(\int_t^{t+u} B(X_s) ds \right)^p + \sup_{u \leq h} \left(\int_t^{t+u} \sum_\alpha A(X_s) dW_s \right)^p \right].$$

The expectation then Burkholder Davis Gundy inequality imply

$$\mathbb{E}(\sup_{s \leq h} |X_{t+s} - X_t|^p) \leq 2^{p-1} (1 + C_p) \mathbb{E} \left[\left(\int_t^{t+h} |B(X_s)| ds \right)^p + \left(\int_t^{t+h} |A(X_s)|^2 ds \right)^{p/2} \right].$$

Jensen's inequality yields

$$\mathbb{E}(\sup_{s \leq t} |X_{t+s} - X_t|^p) \leq 2^{p-1} (1 + C_p) [h^{p-1} + h^{p/2-1}] \mathbb{E} \left[\int_t^{t+h} |B(X_s)|^p ds + \int_t^{t+h} \|A(X_s)\|^p ds \right].$$

Assumption (1) with $K = \max(\|A\|_\infty, \|B\|_\infty)$ concludes the proof:

$$\mathbb{E}(\sup_{s \leq t} |X_{t+s} - X_t|^p) \leq 2^{p-1} (1 + C_p) [h^p + h^{p/2}] 2K^p$$

•

Lemma 4.2. *Under Assumption (1), the following bound is satisfied, $\forall t \leq T, \forall p \geq 1$:*

$$\begin{aligned}\mathbb{E}(|M_{t+h} - M_t|^p) &\leq 2^p \sqrt{1 + C_{2p}}(h^p + h^{p/2})K^p, \\ \mathbb{E}(|M_{t+h} - M_t|^p) &= o(h^{p/2})\end{aligned}\tag{38}$$

where $K = \max(\|A\|_\infty, \|B\|_\infty)$ and C_p the constant in Burkholder Davis Gundy inequality.

Proof. One remarks that

$$M_{t+h} - M_t = \left(X_t - M_t + \sup_{0 \leq u \leq h} (X_{t+u} - X_t) \right)_+$$

where $(x)_+ = \max(x, 0)$. For any $a > 0$, $(x - a)_+ \leq |x| \mathbf{1}_{x > a}$ thus

$$0 \leq M_{t+h} - M_t \leq \left| \sup_{0 \leq u \leq h} (X_{t+u} - X_t) \right| \mathbf{1}_{\{\sup_{0 \leq u \leq h} (X_{t+u} - X_t) > M_t - X_t\}}.$$

Cauchy-Schwartz inequality yields:

$$0 \leq \mathbb{E}([M_{t+h} - M_t]^p) \leq \sqrt{\mathbb{E} \left(\left| \sup_{0 \leq u \leq h} (X_{t+u} - X_t) \right|^{2p} \right) \mathbb{P}(\{ \sup_{0 \leq u \leq h} (X_{t+u} - X_t) > M_t - X_t \})}.$$

Lemma 4.1 in case $2p$ proves the bound (38) and that the almost sure limit $\lim_{h \rightarrow 0} \sup_{0 \leq u \leq h} (X_u - X_t) = 0$.

According to Proposition 1.1 the pair (M_t, X_t) law admits a density, and Step 4 of Proposition 1.6 proves (12), meaning $\mathbb{P}(\{0 = M_t - X_t\}) = 0$. So actually $\mathbb{E}([M_{t+h} - M_t]^p)$ is bounded by the product of $h^{p/2}$ and a factor going to zero, so it is an $o(h^{p/2})$. \bullet

For any t let the process $(X_{t,h}, h \in [t, T])$ and its running maximum defined as

$$X_{t,h} := B(X_t)h + A(X_t)[W_{t+h} - W_t], \quad M_{t,h} := \sup_{0 \leq u \leq h} X_{t,u}.\tag{39}$$

Lemma 4.3. *Under Assumption (1), the following bound is satisfied $\forall t \leq T$:*

$$\mathbb{E} \left(\sup_{s \leq h} |X_{s+t} - X_t - X_{t,s}|^p \right) \leq 2^{2p-1} (1 + C_p)^2 (h^p + h^{p/2})^2 K^{2p}$$

where $K = \max(\|A\|_\infty, \|B\|_\infty, \|A'\|_\infty, \|B'\|_\infty)$ and C_p is the constant in Burkholder Davis Gundy inequality.

Proof. By definition, with $\tilde{W}_u := W_{t+u} - W_t, u \geq 0$

$$X_{s+t} - X_t - X_{t,s} = \int_0^s [B(X_{u+t}) - B(X_t)] du + \int_0^s [A(X_{u+t}) - A(X_t)] d\tilde{W}_u.$$

Using $(a + b)^p \leq 2^{p-1}(a^p + b^p)$

$$\sup_{s \leq h} |X_{s+t} - X_t - X_{t,s}| \leq 2^{p-1} \left[\left(\int_0^h |B(X_{u+t}) - B(X_t)| du \right)^p + \sup_{s \leq h} \left| \int_0^h [A(X_{u+t}) - A(X_t)] d\tilde{W}_u \right|^p \right].$$

Expectation and Burkholder Davis Gundy inequality prove:

$$\mathbb{E} \left(\sup_{s \leq h} |X_{s+t} - X_t - X_{t,s}|^p \right) \leq 2^{p-1}(1 + C_p) \left[\mathbb{E} \left(\int_0^h |B(X_{u+t}) - B(X_t)| du \right)^p + \mathbb{E} \left| \int_0^h [|A(X_{u+t}) - A(X_t)]|^2 du \right|^{p/2} \right].$$

A and B Lipschitz property and Jensen's inequality imply

$$\mathbb{E} \left(\sup_{s \leq h} |X_{s+t} - X_t - X_{t,s}|^p \right) \leq 2^{p-1}(1 + C_p)(h^{p-1} + h^{p/2-1})K^p \left[\mathbb{E} \left(\int_0^h |X_{u+t} - X_t|^p du \right) \right].$$

Then Lemma 4.1 allows to bound the last factor so

$$\mathbb{E} \left(\sup_{s \leq h} |X_{s+t} - X_t - X_{t,s}|^p \right) \leq 2^{p-1}(1 + C_p)(h^{p-1} + h^{p/2-1})K^p 2^{p-1}(1 + C_p)[h^p + h^{p/2}]2K^p h = 2^{2p-1}(1 + C_p)^2(h^p + h^{p/2})^2 K^{2p}.$$

•

Lemma 4.4. *Under Assumption (1)*

- (i) $\sup_{0 \leq t \leq T; 0 \leq h \leq 1} h^{-1} \mathbb{E} (|M_{t+h} - M_t - (M_{t,h} - M_t + X_t)_+|) < \infty,$
- (ii) $\lim_{h \rightarrow 0} h^{-1} \mathbb{E} (|M_{t+h} - M_t - (M_{t,h} - M_t + X_t)_+|) = 0.$

Proof. Considering that

$$M_{t+h} - M_t = \left(\sup_{0 \leq u \leq h} (X_{u+t} - X_t) - M_t + X_t \right)_+$$

and that for all $a \in \mathbb{R}$

$$|(x - a)_+ - (y - a)_+| \leq |x - y| [\mathbf{1}_{\{x > a\}} + \mathbf{1}_{\{y > a\}}],$$

yields

$$\begin{aligned} & |M_{t+h} - M_t - (M_{t,h} - M_t + X_t)_+| \leq \\ & \left| \sup_{0 \leq u \leq h} (X_{u+t} - X_t) - M_{t,h} \right| \left[\mathbf{1}_{\{\sup_{0 \leq u \leq h} (X_{u+t} - X_t) > M_t - X_t\}} + \mathbf{1}_{\{M_{t,h} > M_t - X_t\}} \right]. \end{aligned}$$

If x and y are functions on $[0, T]$, $\forall s \in [0, T]$,

$$x(s) - \sup_{u \leq T} y(u) \leq x(s) - y(s) \leq |x(s) - y(s)| \leq \sup_{v \leq T} |x(v) - y(v)|,$$

so

$$\sup_{s \leq T} x(s) - \sup_{u \leq T} y(u) \leq \sup_{v \leq T} |x(v) - y(v)|.$$

Here the role of y and x are symmetrical so

$$\sup_{s \leq T} y(s) - \sup_{u \leq T} x(u) \leq \sup_{v \leq T} |x(v) - y(v)|,$$

meaning that $|\sup_{s \leq T} y(s) - \sup_{u \leq T} x(u)| \leq \sup_{v \leq T} |x(v) - y(v)|$.

This is applied to the functions $u \rightarrow X_{u+t} - X_t$ and $u \rightarrow X_{t,u}$:

$$\left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t)_+ \right| \leq \sup_{u \leq h} |X_{u+t} - X_t - X_{t,u}| \left[\mathbf{1}_{\{\sup_{0 \leq u \leq h} (X_{u+t} - X_t) > M_t - X_t\}} + \mathbf{1}_{\{M_{t,h} > M_t - X_t\}} \right]$$

Cauchy Schwartz inequality and $(a + b)^2 \leq 2(a^2 + b^2)$ get

$$\begin{aligned} & \mathbb{E} \left(\left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t)_+ \right| \right) \\ & \leq \sqrt{2 \mathbb{E} \left(\sup_{u \leq h} |X_{u+t} - X_t - X_{t,u}|^2 \right)} \left(\mathbb{P} \left(\left\{ \sup_{0 \leq u \leq h} (X_{u+t} - X_t) > M_t - X_t \right\} \right) + \mathbb{P} \left(\{M_{t,h} > M_t - X_t\} \right) \right). \end{aligned}$$

Lemma 4.3 with $p = 2$ insures that $h \mapsto h^{-1} \sqrt{2 \mathbb{E} \left(\sup_{u \leq h} |X_{u+t}^1 - X_t^1 - X_{t,u}^1|^2 \right)}$ is uniformly bounded in t .

Concerning the second factor,

- firstly the almost sure continuity with respect to h insures that the limits $\lim_{h \rightarrow 0} \sup_{0 \leq u \leq h} (X_{u+t} - X_t)$ and $\lim_{h \rightarrow 0} M_{t,h}$ are equal to 0;
- secondly the pair (M_t, X_t) law admits a density according to Proposition 1.1 and $\mathbb{P}\{0 = M_t - X_t\} = 0$. Thus the limit of the second factor is equal to 0.

This concludes the proof. •

4.2 Conditional expectation and approximation

Proposition 4.5. For A and $B \in C_b^1$ and (2), meaning $\|A\| \geq c > 0$,

$$\mathbb{E} [(M_{s,h} - (M_s - X_s))_+ | \mathcal{F}_s] = 2 \|A(X_s)\| \sqrt{h} H \left(\frac{M_s - X_s - B(X_s)h}{\|A(X_s)\| \sqrt{h}} \right) - 2hR(s, h)$$

where

$$\begin{aligned} H(B) & := \int_B^\infty \frac{1}{\sqrt{2\pi}} (y - B) e^{-\frac{y^2}{2}} dy = \frac{1}{\sqrt{2\pi}} e^{-\frac{B^2}{2}} - B \Phi_G(-B), \\ R(s, h) & := B(X_s)h \int_{\frac{M_s - X_s}{\|A(X_s)\| \sqrt{h}}}^\infty \left(z - \frac{M_s - X_s}{\|A(X_s)\| \sqrt{h}} \right) e^{\frac{2zB(X_s)\sqrt{h}}{\|A(X_s)\|}} \Phi_G \left(- \left(z + \frac{B(X_s)\sqrt{h}}{\|A(X_s)\|} \right) \right) dz, \end{aligned} \tag{40}$$

Φ_G being the Gaussian distribution function.

Remark 4.6. Let us remark that $H'(b) = -\Phi_G(-b) \leq 0$, H is decreasing, bounded by $\frac{1}{\sqrt{2\pi}}e^{-\frac{B^2}{2}}$ when $B \geq 0$, where $\Phi_G(u) = \int_{-\infty}^u \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2t}} dv$ is the Gaussian distribution function. Thus H is bounded on \mathbb{R}^+ : $\forall x \geq 0, H(x) \leq H(0)$.

Proof. (i) Denoting

$$Z(m)_t = mt + W_t, \quad Z(m)_t^* = \sup_{0 \leq s \leq t} Z(m)_s,$$

the density of the pair $(Z(m)_t^*, Z(m)_t)$ is

$$2 \frac{(2b-a)}{\sqrt{2\pi t^3}} e^{-\frac{(2b-a)^2}{2t} + ma - \frac{m^2 t}{2}} \mathbf{1}_{\{b > 0, b > a\}}.$$

cf. [9] p. 145 et seq. Corollary 3.2.1.2. By integration with respect to a we get the density of $Z(m)_t^*$, cf. [11] (3.2.8):

$$p_{Z^*}(b; m, t) = 2 \mathbf{1}_{\{b > 0\}} \left[\frac{1}{\sqrt{2\pi t}} e^{-\frac{(b-mt)^2}{2t}} - m e^{2mb} \Phi_G\left(-\frac{b+mt}{\sqrt{t}}\right) \right].$$

We now introduce

$$Y_t = mt + \sigma W_t = \sigma \left[\frac{m}{\sigma} t + W_t \right], \quad Y_t^* = \sup_{u \leq t} Y_u = \sigma Z\left(\frac{m}{\sigma}\right)_t^*.$$

We remark that conditionally to \mathcal{F}_s , $(M_{s,h} - (M_s - X_s))_+$ law is the one of the process Y with $m = B(X_s)$, $\sigma = A(X_s)$. In order to use the Markov property we now deal with $E[(Y_t^* - B)_+]$.

(ii) For any $B > 0$

$$\mathbb{E}((Y_t^* - B)_+) = \sigma \mathbb{E} \left(\left(Z\left(\frac{m}{\sigma}\right)_t^* - \frac{B}{\sigma} \right)_+ \right). \quad (41)$$

We now deal with

$$\begin{aligned} \mathbb{E}((Z(m)_t^* - B)_+) &= \int_B^\infty (z - B)_+ p_{Z^*}(z; m, t) dz \\ &= 2 \int_B^\infty (z - B) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(z-mt)^2}{2t}} dz - 2m \int_B^\infty (z - B) e^{2zm} \Phi_G\left(-\frac{z+mt}{\sqrt{t}}\right) dz = 2(I_1 + I_2). \end{aligned}$$

We operate the change of variable $y = \frac{z-mt}{\sqrt{t}}$ in the first integral:

$$I_1 = \sqrt{t} \int_{\frac{B}{\sqrt{t}} - m\sqrt{t}}^\infty \left(y + m\sqrt{t} - \frac{B}{\sqrt{t}} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy,$$

developed as following

$$I_1 = \frac{\sqrt{t}}{\sqrt{2\pi}} \exp\left(-\frac{(B-mt)^2}{2t}\right) + (mt - B) \Phi_G\left(-\frac{B}{\sqrt{t}} + m\sqrt{t}\right) = \sqrt{t} H\left(\frac{B}{\sqrt{t}} - m\sqrt{t}\right).$$

Finally, we operate the change of variable $y = \frac{z}{\sqrt{t}}$ in the second integral:

$$\mathbb{E}((Z(m)_t^* - B)_+) = 2\sqrt{t}H\left(\frac{B}{\sqrt{t}} - m\sqrt{t}\right) - 2mt \int_{\frac{B}{\sqrt{t}}}^{\infty} \left(y - \frac{B}{\sqrt{t}}\right)_+ e^{2ym\sqrt{t}} \Phi_G(-(y + m\sqrt{t})) dy.$$

(iii) Recalling (41)

$$\mathbb{E}((Y_t^* - B)_+) = \sigma \mathbb{E}\left(\left(Z\left(\frac{m}{\sigma}\right)_t^* - \frac{B}{\sigma}\right)_+\right) = 2\sigma\sqrt{t}H\left(\frac{B - mt}{\sigma\sqrt{t}}\right) - 2mt \int_{\frac{B}{\sigma\sqrt{t}}}^{\infty} \left(z - \frac{B}{\sigma\sqrt{t}}\right)_+ e^{\frac{2zm\sqrt{t}}{\sigma}} \Phi_G(-(z + \frac{m\sqrt{t}}{\sigma})) dz$$

that we apply to $t = h$, $\sigma = \|A(X_s)\|$, $m = B(X_s)$, $B = M_s - X_s$ and using the Markov property we get the result:

$$\begin{aligned} \mathbb{E}((M_{s,h} - (M_s - X_s))_+ | \mathcal{F}_s) &= 2\|A(X_s)\|\sqrt{h}H\left(\frac{M_s - X_s - B(X_s)h}{\|A(X_s)\|\sqrt{h}}\right) \\ &\quad - 2B(X_s)h \int_{\frac{M_s - X_s}{\|A(X_s)\|\sqrt{h}}}^{\infty} \left(z - \frac{M_s - X_s}{\|A(X_s)\|\sqrt{h}}\right)_+ e^{\frac{2zB(X_s)\sqrt{h}}{\|A(X_s)\|}} \Phi_G(-(z + \frac{B(X_s)\sqrt{h}}{\|A(X_s)\|})) dz. \end{aligned}$$

•

Proposition 4.7. *Let A and $B \in C_b^1$ and (2), and let us denote for a bounded function Ψ*

$$\tilde{T}_\Psi(h, t) = \mathbb{E}\left[\Psi(V_t) \frac{(M_{t,h} - M_t + X_t)_+}{h}\right]$$

then

$$\tilde{T}_\Psi(h, t) \sim 2\mathbb{E}\left(\Psi(V_t) \frac{1}{\sqrt{h}} |A(X_t)| H\left(\frac{M_t - X_t - B(X_t)h}{|A(X_t)|\sqrt{h}}\right)\right).$$

Proof. Using Proposition 4.5 actually

$$\tilde{T}_\Psi(h, s) - \mathbb{E}\left(\Psi(V_s) \frac{2}{\sqrt{h}} \|A(X_s)\| H\left(\frac{M_s - X_s - B(X_s)h}{\|A(X_s)\|\sqrt{h}}\right)\right) = -2\mathbb{E}[\Psi(V_s)R(s, h)].$$

Since Ψ is bounded, we only need to prove that

$$\lim_{h \rightarrow 0} \mathbb{E}(R(s, h)) = 0.$$

From equation (40) in Proposition 4.5, the integrand is the product of the bounded $\|B(X_s)\|$ and the factor (where necessarily $z \geq 0$):

$$r(s, h, z) := \left(z - \frac{M_s - X_s}{\|A(X_s)\|\sqrt{h}}\right)_+ e^{\frac{2zB(X_s)\sqrt{h}}{\|A(X_s)\|}} \Phi_G\left(-\left(z + \frac{B(X_s)\sqrt{h}}{\|A(X_s)\|}\right)\right).$$

Firstly, since $\mathbb{P}(\{M_s - X_s \leq 0\}) = 0$, almost surely $\lim_{h \rightarrow 0} r(s, h, z) = 0$.

Secondly two cases are now to be checked:

- On the set $\{B(X_s) \geq 0\}$, $\frac{z+B(X_s)\sqrt{h}}{\|A(X_s)\|} \geq \frac{z}{\|A\|_\infty}$. Since Φ_G is increasing then $\Phi_G(-\frac{z+B(X_s)\sqrt{h}}{\|A(X_s)\|}) \leq \Phi_G(-\frac{z}{\|A\|_\infty})$ and using the standard formula $x\Phi_G(-x) \leq \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for $x \geq 0$, since $z \geq 0$ and c being the constant in Assumption (2):

$$r(s, h, z) \leq ze^{2\frac{z\|B\|_\infty}{c}}\Phi_G(-\frac{z}{\|A\|_\infty}) \leq \frac{\|A\|_\infty}{\sqrt{2\pi}}e^{2\frac{z\|B\|_\infty}{c}-\frac{z^2}{2\|A\|_\infty^2}}.$$

- On the set $\{B(X_s) < 0\}$, $\frac{z+B(X_s)\sqrt{h}}{\|A(X_s)\|} \geq \frac{z-\|B\|_\infty}{\|A\|_\infty}$ and $\Phi_G(-\frac{z+B(X_s)\sqrt{h}}{\|A(X_s)\|}) \leq \Phi_G(-\frac{z-\|B\|_\infty}{\|A\|_\infty})$. Note that $e^{\frac{2zB(X_s)\sqrt{h}}{\|A(X_s)\|}} \leq 1$ and for $x > 0$

$$\Phi_G(-x) \leq e^{-\frac{x^2}{4}} \int_x^\infty \frac{e^{-\frac{y^2}{4}}}{\sqrt{2\pi}} dy \leq \sqrt{2}e^{-\frac{x^2}{4}},$$

and since $z \geq 0$

$$r(s, h, z) \leq z\Phi_G(-\frac{z-\|B\|_\infty}{\|A\|_\infty}) \leq \sqrt{2}ze^{-\frac{(z-\|B\|_\infty)^2}{4\|A\|_\infty^2}} + \|B\|_\infty \mathbf{1}_{z \leq \|B\|_\infty}.$$

Thus we can apply Lebesgue's dominated convergence Theorem:

$$\lim_{h \rightarrow 0} \mathbb{E} \left(\int_0^\infty |r(s, h, z)| dz \right) = 0.$$

•

Proposition 4.8. For A and $B \in C_b^1$ and (2), and Ψ Lipschitz continuous bounded

$$\tilde{T}_\Psi(h, t) \sim 2\mathbb{E} \left[\Psi(M_t, M_t) \frac{1}{\sqrt{h}} \|A(X_t)\| H \left(\frac{M_t - X_t}{\|A(X_t)\|\sqrt{h}} \right) \right].$$

Proof. (i) Since A is bounded from below by c , using Proposition 4.7, the deal is to cancel $B(X_t)$ inside the function H , meaning to study the difference

$\frac{1}{\sqrt{h}} H \left(\frac{M_t - X_t - B(X_t)h}{\|A(X_s)\|\sqrt{h}} \right) - \frac{1}{\sqrt{h}} H \left(\frac{M_t - X_t}{\|A(X_s)\|\sqrt{h}} \right)$ when $h \rightarrow 0$. Since the function H is differentiable with negative derivative $x \mapsto -\Phi_G(-x)$ where Φ_G is the Gaussian distribution function, we apply the finite increments formula:

$$\begin{aligned} D_h &:= \frac{1}{\sqrt{h}} \left| H \left(\frac{M_s - X_s - B(X_s)h}{\|A(X_s)\|\sqrt{h}} \right) - H \left(\frac{M_s - X_s}{\|A(X_s)\|\sqrt{h}} \right) \right| \\ &\leq \frac{\|B\|_\infty}{c} \max \left[\Phi_G \left(-\frac{M_s - X_s}{\|A(X_s)\|\sqrt{h}} \right), \Phi_G \left(-\frac{M_s - X_s - B(X_s)h}{\|A(X_s)\|\sqrt{h}} \right) \right] \leq \frac{\|B\|_\infty}{c} \end{aligned}$$

so D_h is uniformly bounded.

On the set $\{M_s - X_s > 0\}$ which satisfies $\mathbb{P}\{M_s - X_s > 0\} = 1$, by Proposition 1.1

$$\lim_{h \rightarrow 0} \frac{M_s - X_s - B(X_s)h}{\|A(X_s)\|\sqrt{h}} = \lim_{h \rightarrow 0} \frac{M_s - X_s}{\|A(X_s)\|\sqrt{h}} = +\infty;$$

then since $\lim_{x \rightarrow -\infty} \Phi_G(x) = 0$, the almost sure $\lim_{h \rightarrow 0} D_h = 0$.

Since D_h is bounded for any h by $\frac{\|B\|_\infty}{c}$ on both subsets, Lebesgue's dominated convergence Theorem yields:

$$\tilde{T}_\Psi(h, t) \sim \lim_{h \rightarrow 0} 2\mathbb{E} \left(\Phi(V_t) \frac{1}{\sqrt{h}} |A(X_t)| H \left(\frac{M_t - X_t}{|A(X_t)|\sqrt{h}} \right) \right).$$

(ii) The Lipschitz property of Ψ yields

$$|\Psi(V_s) - \Psi(M_s, M_s)| \leq \|\Psi\|_{lip} (M_s - X_s).$$

Since $H(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} - x\Phi_G(-x)$, for $x \geq 0$ there exists $C > 0$ such that $0 \leq xH(x) \leq x \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \leq C e^{-\frac{x^2}{4}}$. Thus

$$|\Psi(V_s) - \Psi(M_s, M_s)| \frac{1}{\sqrt{h}} \|A(X_s)\| H \left(\frac{M_s - X_s}{\|A(X_s)\|\sqrt{h}} \right) \leq \|\Psi\|_{lip} \|A(X_s)\|^2 C e^{-\frac{(M_s - X_s)^2}{4h\|A(X_s)\|^2}}$$

Once again for all s $M_s - X_s > 0$ \mathbb{P} almost surely and A is bounded, so almost surely

$$\lim_{h \rightarrow 0} |\Psi(V_s) - \Psi(M_s, M_s)| \frac{1}{\sqrt{h}} H \left(\frac{M_s - X_s}{\|A(X_s)\|\sqrt{h}} \right) = 0.$$

By hypotheses B , $\|A\|$ and Ψ are bounded, and Remark 4.6 gives $0 \leq \inf_{x>0} H(x) \leq \sup_{x>0} H(x) \leq H(0) < \infty$, so Lebesgue's dominated convergence Theorem yields:

$$\tilde{T}_\Psi(h, s) \sim 2\mathbb{E} \left(\Psi(M_s, M_s) \frac{1}{\sqrt{h}} \|A(X_s)\| H \left(\frac{M_s - X_s}{\|A(X_s)\|\sqrt{h}} \right) \right).$$

•

4.3 Some more tools for Proposition 2.11 proof

Lemma 4.9. *Let $0 < \varepsilon < 1$ and G be the standard Gaussian variable, there exists a constant $C(\varepsilon)$ such that for all $T > 0$, $b > 0$*

$$\mathbb{E}[(b - G)^{-\varepsilon} \exp -\frac{(b - G)^2}{2T} \mathbf{1}_{\{b > G\}}] \leq C(\varepsilon) \sqrt{T}^{1-\varepsilon}.$$

Proof. Using the density of G ,

$$J := \mathbb{E}[(b - G)^{-\varepsilon} \exp - \frac{(b - G)^2}{2T} \mathbf{1}_{\{b > G\}}] = \int_{\mathbb{R}} (b - g)^{-\varepsilon} \exp[-\frac{(b - g)^2}{2T} - \frac{g^2}{2}] \mathbf{1}_{\{b > g\}} \frac{dg}{\sqrt{2\pi}}.$$

We integrate by part

$$J = \int_{\mathbb{R}} \frac{(b - g)^{1-\varepsilon}}{1 - \varepsilon} [\frac{(b - g)}{T} - g] \exp[-\frac{(b - g)^2}{2T} - \frac{g^2}{2}] \mathbf{1}_{\{b > g\}} \frac{dg}{\sqrt{2\pi}}.$$

We share $J = J_1 + J_2$ where:

$$J_1 := \int_{\mathbb{R}} \frac{(b - g)^{2-\varepsilon}}{T(1 - \varepsilon)} \exp[-\frac{(b - g)^2}{2T} - \frac{g^2}{2}] \mathbf{1}_{\{b > g\}} \frac{dg}{\sqrt{2\pi}},$$

and

$$J_2 := - \int_{\mathbb{R}} \frac{(b - g)^{1-\varepsilon}}{1 - \varepsilon} g \exp[-\frac{(b - g)^2}{2T} - \frac{g^2}{2}] \mathbf{1}_{\{b > g\}} \frac{dg}{\sqrt{2\pi}}.$$

Using $C_\alpha = \sup_{x > 0} x^\alpha e^{-\frac{x^2}{4}}$

$$J_1 \leq \frac{(\sqrt{T})^{1-\varepsilon}}{1 - \varepsilon} C_{2-\varepsilon} \int_{\mathbb{R}} \exp[-\frac{(b - g)^2}{4T}] \frac{dg}{\sqrt{2\pi T}} = \frac{(\sqrt{T})^{1-\varepsilon}}{1 - \varepsilon} C_{2-\varepsilon} \sqrt{2}.$$

Similarly

$$|J_2| \leq \frac{(\sqrt{T})^{1-\varepsilon}}{1 - \varepsilon} C_{1-\varepsilon} C_1 \int_{\mathbb{R}} \exp[-\frac{g^2}{4}] \frac{dg}{\sqrt{2\pi}} \leq \frac{(\sqrt{T})^{1-\varepsilon}}{1 - \varepsilon} C_{1-\varepsilon} C_1 \sqrt{2}.$$

The sum yields the result with $C(\varepsilon) = \frac{C_{2-\varepsilon}\sqrt{2}}{1-\varepsilon} + \frac{C_{1-\varepsilon}C_1\sqrt{2}}{1-\varepsilon}$. •

Lemma 4.10. *Assume that $d = 1$, $A = 1$ and B is C_b^1 . Let $\varepsilon \in]0, \frac{1}{2}[$ there exists a constant $C(\varepsilon)$ such that for all $0 < s < t$, $\alpha \in \mathbb{R}$ and $b \in \mathbb{R}$ then*

$$\mathbb{E}_{\mathbb{P}} \left[\exp \alpha(b - X_s) \mathbf{1}_{b > X_s} \frac{1}{(b - X_s)^\varepsilon} e^{-\frac{(b - X_s)^2}{2(t-s)}} \right] \leq C(\varepsilon) \frac{(t - s)^{\frac{1-2\varepsilon}{4}}}{s^{1/4}} e^{\alpha^2(t-s) + \frac{t\|B\|_\infty^2}{2}}.$$

Proof. Under the equivalent probability measure $Q = L_s \mathbb{P}$, where $L_s = \exp \left[\int_0^s -B(W_u) dW_u + \frac{1}{2} \int_0^s B(W_u)^2 du \right]$, the process X is a Q -Brownian motion so, abusing the notation W instead of X :

$$K := \mathbb{E}_{\mathbb{P}} \left[\exp \alpha(b - X_s) \frac{1}{(b - X_s)^\varepsilon} \mathbf{1}_{b > X_s} e^{-\frac{(b - X_s)^2}{2(t-s)}} \right] = \mathbb{E}_Q \left[L_s^{-1} \exp \alpha(b - W_s) \frac{1}{(b - W_s)^\varepsilon} \mathbf{1}_{b > W_s} e^{-\frac{(b - W_s)^2}{2(t-s)}} \right].$$

Using Cauchy-Schwartz inequality

$$K \leq \sqrt{\mathbb{E}_Q [L_s^{-2}]} \sqrt{\mathbb{E}_Q \left[\exp 2\alpha(b - W_s) \frac{1}{(b - W_s)^{2\varepsilon}} \mathbf{1}_{b > W_s} e^{-\frac{(b - W_s)^2}{(t-s)}} \right]}.$$

Using the inequality $2xy = 2\sqrt{2(t-s)}x\frac{y}{\sqrt{2(t-s)}} \leq 2(t-s)x^2 + \frac{y^2}{2(t-s)}$ applied to $x = \alpha$ and $y = b - W_s$ we obtain

$$2|\alpha(b - W_s)| \leq 2(t-s)\alpha^2 + \frac{(b - W_s)^2}{2(t-s)}$$

so

$$K \leq e^{\alpha^2(t-s)} \sqrt{\mathbb{E}_Q[L_s^{-2}]} \sqrt{\mathbb{E}_Q \left[\frac{1}{(b - W_s)^{2\varepsilon}} \mathbf{1}_{b > W_s} e^{-\frac{(b - W_s)^2}{2(t-s)}} \right]}. \quad (42)$$

On the one hand

$$L_s^{-2} = \exp \left[\int_0^s 2B(W_u)dW_u - 2 \int_0^s B(W_u)^2 du \right] \exp \int_0^s B(W_u)^2 du$$

so for any $s \in [0, t]$

$$\sqrt{\mathbb{E}_Q[L_s^{-2}]} \leq e^{\frac{t\|B\|_\infty^2}{2}}. \quad (43)$$

On the other hand, G denoting a standard Gaussian random variable, the law of $\frac{1}{(b - W_s)^{2\varepsilon}} \mathbf{1}_{b > W_s} e^{-\frac{(b - W_s)^2}{2(t-s)}}$ is the one of

$$\frac{1}{\sqrt{s}^{2\varepsilon}} \frac{1}{(b/\sqrt{s} - G)^{2\varepsilon}} \mathbf{1}_{b/\sqrt{s} > G} \exp -\frac{s}{2(t-s)} [b/\sqrt{s} - G]^2.$$

Using Lemma 4.9 for 2ε and $T = \frac{t-s}{s}$ we obtain

$$\sqrt{\mathbb{E}_Q \left[\frac{1}{(b - W_s)^{2\varepsilon}} \mathbf{1}_{b > W_s} e^{-\frac{(b - W_s)^2}{2(t-s)}} \right]} \leq C(\varepsilon) \left(\frac{t-s}{s} \right)^{\frac{1-2\varepsilon}{4}} \frac{1}{\sqrt{s}^\varepsilon} = C(\varepsilon) \frac{(t-s)^{\frac{1-2\varepsilon}{4}}}{s^{\frac{1}{4}}}. \quad (44)$$

Plugging inequalities (43) and (44) into (42) achieves the proof of the lemma. •

Lemma 4.11. *Assume that $A = 1$ and B is C_b^1 . For all $\alpha \in \mathbb{R}$, $s > 0$*

$$\mathbb{E}_\mathbb{P} [\exp \alpha(M_s - X_s)] \leq \sqrt{2} e^{2s(\alpha^2 + \|B\|_\infty^2)}.$$

Proof. Using once again the equivalent probability measure $Q = L_s \mathbb{P}$ where $L_s = \exp \left[\int_0^s -B(W_u)dW_u + \frac{1}{2} \int_0^s B(W_u)^2 du \right]$, under which X is a Q -Brownian motion:

$$\mathbb{E}_\mathbb{P} [\exp \alpha(M_s - X_s)] = \mathbb{E}_Q [L_s^{-1} \exp \alpha(W_s^* - W_s)].$$

Using Cauchy Schwartz inequality,

$$\mathbb{E}_\mathbb{P} [\exp \alpha(M_s - X_s)] \leq \sqrt{\mathbb{E}_Q [L_s^{-2}] \mathbb{E}_Q [\exp 2\alpha(W_s^* - W_s)]}.$$

Firstly recall (43) $\sqrt{\mathbb{E}_Q[L_s^{-2}]} \leq e^{\frac{s\|B\|_\infty^2}{2}}$.

Secondly under Q , the law of $(W_s^* - W_s)$ is the one of $|W_s|$, then

$$\mathbb{E}_Q[\exp 2\alpha(W_s^* - W_s)] = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{2\alpha\sqrt{s}x} e^{-\frac{x^2}{2}} dx = e^{4\alpha^2 s} \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{(x-2\alpha\sqrt{s})^2}{2}} dx \leq 2 \exp(4\alpha^2 s).$$

Finally, multiplying both bounds

$$\mathbb{E}_\mathbb{P}[\exp \alpha(M_s - X_s)] \leq \sqrt{2} e^{\frac{s\|B\|_\infty^2}{2} + 2\alpha^2 s}$$

which concludes the proof. •

References

- [1] **J. M. Azaïs, M. Wschebor**, On the regularity of the distribution of the maximum of one-parameter Gaussian processes. *Probab. Th. Rel. Fields* 119 (2001), no. 1, 70-98.
- [2] **H. Brown, D. Hobson, L.C.G. Rogers**, Robust hedging of barrier options. *Math. Finance* 11, 285-314 (2001)
- [3] **L. Coutin and D. Dorobantu**, First passage time law for some Lévy processes with compound Poisson: existence of a density. *Bernoulli* 17 (2011), no. 4, 1127-1135.
- [4] **L. Coutin, W. Ngom, M. Pontier**, Joint distribution of a Lévy process and its running supremum, 2017, submitted.
- [5] **A.M.G. Cox, J. Obloj**, Robust pricing and hedging of double no-touch options. *Finance Stochast.* 15, 573-605 (2011).
- [6] **R.A. Doney, A.E. Kyprianou**, Overshoots and undershoots of Lévy processes. *Ann. Appl. Probab.* 16 (2006), no. 1, 91-106.
- [7] **M. Duembgen, L. C. G. Rogers**, The Joint Law of the Extrema, Final Value and Signature of a Stopped Random Walk, Chapter in 'Memoriam Marc Yor', Séminaire de Probabilités XLVII, L. N. in Mathematics Vol. 2137 pp 321-338.
- [8] **M. Hayashi and A. Kohatsu-Higa**, Smoothness of the distribution of the supremum of a multi-dimensional diffusion process, *Potential Analysis*, 2013, 38/1, pp. 57-77.
- [9] **M. Jeanblanc, M. Yor, M. Chesney**, *Mathematical Methods for Financial Markets*, Springer, 2009.

- [10] **A. Lagnoux, S. Mercier, P. Vallois**, Probability that the maximum of the reflected Brownian motion over a finite interval $[0, t]$ is achieved by its last zero before t . *Electron. Commun. Probab.* 20 (2015), no. 62, 9 pp.
- [11] **W. Ngom**, thesis: Contributions à l'étude de l'instant de défaut d'un processus de Lévy en observation complète et incomplète, IMT, 2016.
- [12] **D. Nualart**, *The Malliavin calculus and related topics* Second Edition, Springer-Verlag New-Yor, 2006.
- [13] **L. Alili, P. Patie, J.L. Pedersen**, Representations of the first hitting time density of an Ornstein-Uhlenbeck process. *Stoch. Models* 21 (2005), no. 4, 967-980.
- [14] **A. Revuz, M. Yor**, *Continuous Martingales and Brownian Motion*, Third edition, Springer Verlag, 2004.
- [15] **L. C. G. Rogers**, The Joint Law of the Maximum and Terminal Value of a Martingale, *Probability Theory and Related Fields* 95(4):451-466 · December 1993.
- [16] **B. ROYNETTE, P. VALLOIS, A. VOLPI**, *Asymptotic behavior of the passage time, overshoot and undershoot for some Lévy processes* ESAIM PS VOL. 12, 2008, pp. 58-93.
- [17] **E.C. Titchmarsh**, *The theory of functions*, second edition, Oxford University Press, 1939.