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PDE for joint law of the pair of a continuous diffusion and its running maximum

Laure Coutin*, Monique Pontier†

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Abstract

Let X be a d -dimensional diffusion process and M the running supremum of the first component. In this paper, we first show that for any $t > 0$, the pair (M_t, X_t) density law with respect to Lebesgue measure can be characterized as a weak solution of a partial differential equation on an open set. In the one-dimensional case, this density is the unique weak solution of a partial differential equation (PDE) on the set $\{(m, x) \in \mathbb{R}^2, m \geq x\}$.

Keywords: Partial differential equation, running supremum process, joint law.

A.M.S. Classification: 60J60, 60G70, 60H10.

In a previous work [9], using Malliavin calculus and specifically Nualart's seminal book [21], we prove that the joint law of a pair (X, M) , X being a d -dimensional diffusion, and M the running maximum of its first component, is absolutely continuous with respect to Lebesgue's measure. In the present work, we go on, proving that in the open set $\{(m, x), m > x^1\}$ the density is a weak solution to a PDE. In the one-dimensional case, we prove moreover that there is boundary conditions on the diagonal $\{(m, x), m = x^1\}$. This work follows the previous paper [9] and Waly Ngom's thesis [20].

When the process X is a Brownian motion such results are well known, see for instance [17]. In the literature there exist other studies on the law of this running maximum: For instance, multidimensional diffusion case in [16], random walk in [13], reflected Brownian motion [18], martingale case is studied by Rogers [23, 13]. In case of boundaries problem, some numerical studies are provided by Hermann and Tanré [15] or Sacerdote et al. [25]. For general Gaussian processes, the law of the maximum is studied in [2]. The cases where the process X is a Lévy process are deeply studied in the literature, see for instance [12]. In case of linear diffusion, we refer to [11]. Some specific diffusion processes as Ornstein-Uhlenbeck [1] and [4] are also investigated.

Such studies concerning this running maximum are useful in financial area which get out hitting times, for instance for the pricing of barrier option. It is known that the law of such stopping time is closely related to the one of the running maximum, see [6, 7, 26]. An application of our work is

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the case of a financial product with price process X , which produces a benefit $f(X_t)$ at time t but only when the running maximum belongs to an interval $[a, b]$; in such a case the joint law is useful to compute such a benefit (for instance a barrier option).

The model is as following: on a filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t = \sigma(W_u, u \leq t))_{t \geq 0}, \mathbb{P})$, $W := (W_u, u \geq 0)$ is a d -dimensional Brownian motion, where the filtration \mathbb{F} is completed and càd. Let a diffusion process taking its values in \mathbb{R}^d , solution to

$$(1) \quad dX_t = B(X_t)dt + A(X_t)dW_t, \quad \text{law of } X_0 = \mu_0 \text{ on } \mathbb{R}^d, \quad t > 0$$

where X_0 is independent of the Brownian motion and A and B are defined on \mathbb{R}^d to respectively $d \times d$ matrices, \mathbb{R}^d . Let us denote $C_b^i(\mathbb{R}^d)$ the set of the functions i times differentiable, bounded, with bounded derivatives. We specify the cases where $X_0 = x_0 \in \mathbb{R}^d$, $t > 0$ meaning that the initial law is Dirac one.

We assume that B and A satisfy both hypotheses:

$$(2) \quad B \in C_b^1(\mathbb{R}^d) \text{ and } A \in C_b^2(\mathbb{R}^d), \quad \|B\| = \sup_{x \in \mathbb{R}^d} |B(x)|,$$

and there exists a constant $c > 0$ such that

$$(3) \quad c\|v\|^2 \leq v^T A(x)A^T(x)v, \quad \forall v, x \in \mathbb{R}^d.$$

Let $M_t := \sup_{s \leq t} X_s^1$. In [9], it is proved that for all $t > 0$, the law of (M_t, X_t) has a density p_V with respect to the Lebesgue measure. The aim of this article is to show that this density is solution of a PDE, and it is the unique one in dimension 1.

Section 1 presents the main results: in d dimension, p_V is a weak solution of a PDE on the open subset of $\mathbb{R}^{d+1} \setminus \{(m, x), m > \max(x^1, x_0^1)\}$. In one-dimensional case the density is the unique weak solution of a PDE with boundary conditions. Section 1.1 gives the proofs with the expectation of the Itô formula as [8] Theorem 2.1 for Lévy processes. The key Proposition 1.5 concerning d -dimensional case is proved in Section 2. The proof in one-dimensional case (Section 3) needs more sophisticated tools. This section develops some crucial properties of the density p_V in one dimension and Section 3.4 provides useful tools to conclude the proof in one dimension case. Section 4 provides a uniqueness theorem for this PDE and as a corollary the uniqueness of the weak solution in case $d = 1$.

1 Main results

We start with some useful notations:

$$\Delta = \{(m, x) \in \mathbb{R}^{d+1}, m \geq x^1, x = (x^1, \dots, x^d)\}.$$

According to the model (1), let us denote the infinitesimal generator

$$(4) \quad \mathcal{L} = B^i \partial_{x_i} + \frac{1}{2} \partial_{x_i x_j}^2 (AA^T)^{ij}.$$

Abusing of notation, even if the $d + 1$ coordinate m appears, keep the notation \mathcal{L} .

Recall Assumptions: $A \in C_b^2(\mathbb{R}^d)$, $B \in C_b^1(\mathbb{R}^d)$; there exists a constant $c > 0$ such that

$$c\|v\|^2 \leq v^T A(x)A^T(x)v, \quad \forall v, x \in \mathbb{R}^d$$

and the notations $M_t = \sup_{0 \leq s \leq t} \{X_s^1\}$ and $V_t = (M_t, X_t) \in \mathbb{R}^{d+1}$.

Theorem 1.1 *Let B and A satisfy Assumptions (2) and (3). Let the initial condition V_0 be a random variable taking its values in $\{x \in \mathbb{R}^{d+1}, x^1 = x^2\}$. Let $\Phi \in C_b^2(\mathbb{R}^{d+1}, \mathbb{R})$, such that $\text{supp } \Phi$ is included in the interior of Δ . Then*

$$(5) \quad E[\Phi(V_t) - \Phi(V_0)] = E \left[\int_0^t \mathcal{L}(\Phi)(V_s) ds \right].$$

According to [9], for all $t > 0$, the process V_t law admits a density $p_V(\cdot, t)$ with respect to Lebesgue's measure.

Corollary 1.2 *Let B and A satisfy Assumptions (2) and (3). Then the density $p_V(\cdot, \cdot, t)$ is a weak solution on the interior of Δ of the PDE*

$$\partial_t p_V(\cdot, t) = \mathcal{L}^* p_V(\cdot, t)$$

where \mathcal{L}^* is the adjoint operator of \mathcal{L} .

The proof of Corollary 1.2 is a standard consequence of some integration by parts.

In one-dimensional case, we can deal with the boundary conditions.

Theorem 1.3 *Let $B \in C_b^1(\mathbb{R})$, let $A \in C_b^2(\mathbb{R})$ satisfy Assumption (3) and let $\Phi \in C_b^2(\mathbb{R}^2, \mathbb{R})$. Let $p_V(\cdot, \cdot, t)$ be the law density of the pair $V_t = (M_t, X_t)$ and $p_M(\cdot, t)$ be the one of M_t :*

$$(6) \quad \begin{aligned} E[\Phi(V_t)] - \Phi(V_0) &= E \left[\int_0^t [\mathcal{L}\Phi(V_s)] ds \right. \\ &\quad \left. + \frac{1}{2} \int_0^t E[\partial_1 \Phi(M_s, M_s)] A^2(M_s) \frac{p_V(M_s, M_s; s)}{p_M(M_s; s)} ds \right]. \end{aligned}$$

Remark 1.4 *This result is to be compared to Theorem 2.1 and Proposition 4.4 in [8].*

1.1 Proof of Theorem 1.1

Let $\Phi \in C_b^2(\mathbb{R}^{d+1})$. Before applying Itô formula to $t \rightarrow \Phi(V_t)$, one remarks that the component M_t is a non decreasing continuous process, so it is a finite variation process thus the brackets $d\langle X, M \rangle_t = 0$, and $d\langle M \rangle_t = 0$. Itô's formula yields

$$\Phi(V_t) - \Phi(V_0) = \int_0^t \frac{\partial \Phi}{\partial x}(M_s, X_s) A(X_s) dW_s + \int_0^t \frac{\partial \Phi}{\partial m}(V_s) dM_s + \int_0^t \mathcal{L}\Phi(V_s) ds$$

where \mathcal{L} is the infinitesimal generator of diffusion process X . The first term on the right hand side above is a martingale so its expectation is equal to 0. So

$$(7) \quad E[\Phi(V_t) - \Phi(V_0)] = E \left[\int_0^t \mathcal{L}\Phi(V_s) ds \right] + E \left[\int_0^t \frac{\partial \Phi}{\partial m}(V_s) dM_s \right].$$

Now the deal is to look at

$$E \left[\int_0^t \frac{\partial \Phi}{\partial m}(V_s) dM_s \right].$$

Namely, without lost of generality, we look at $E[\int_0^t \Psi(V_s) dM_s]$ where $\Psi \in C_b^1(\mathbb{R}^{d+1})$.

The key result is the following proposition, proved below, Section 2:

Proposition 1.5 *Let B and A satisfy Assumptions (2) and (3). Let $\Psi \in C_b^1(\mathbb{R}^{d+1}, \mathbb{R})$. Denote $\|A^1(x)\|^2 = \sum_{j=1}^d A_j^1(x)^2$ and $X_t^M := (M_t, X_t^2, \dots, X_t^d)$ then:*

(i) *for all $T > 0$ there exists a constant $C > 0$, (depending on $\|A\|_\infty, \|B\|_\infty, \|\nabla A\|_\infty, \|\Psi\|_\infty, \|\nabla \Psi\|_\infty$ and T) such that $\forall t \in [0, T], h \in [0, 1]$,*

$$(8) \quad \left| E\left[\int_t^{t+h} \Psi(V_s) dM_s\right] - 2\sqrt{h}E[\Psi(M_t, X_t^M)\|A^1(X_t)\|H\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)] \right| \leq C\|\Psi\|_{C_b^1}h,$$

(ii) *for all $t > 0, h \in [0, 1]$,*

$$\left| E\left[\int_t^{t+h} \Psi(V_s) dM_s\right] - 2\sqrt{h}E[\Psi(M_t, X_t^M)\|A^1(X_t)\|H\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right)] \right| = o(h),$$

where using the standard Gaussian distribution function Φ_G :

$$(9) \quad H(B) := \int_B^\infty \frac{1}{\sqrt{2\pi}}(y - B)e^{-\frac{y^2}{2}} dy = \frac{e^{-\frac{B^2}{2}}}{\sqrt{2\pi}} - B\Phi_G(-B).$$

To conclude Theorem 1.1 proof, let Φ be a function with support included in the interior of Δ , then using Proposition 1.5 with $\Psi = \partial_1 \Phi$, the second term of (8) left hand including $\Psi(M_t, M_t, X_t^2, \dots, X_t^d)$ is nul, so this left hand satisfies

$$|E[\int_t^{t+h} \partial_1 \Phi(V_s) dM_s]| \leq C\|\partial_1 \Phi\|_\infty h.$$

Then the application $t \rightarrow E[\int_0^t \partial_1 \Phi(V_s) dM_s]$ is Lipschitz and, using Proposition 1.5 (ii), its derivative is nul almost everywhere. So, since $\Psi = \partial_1 \Phi$ is nul on the diagonal of Δ , this ends the proof of Theorem 1.1. •

2 Proof of Proposition 1.5, dimension d

Recall the notations $X_t^M = (M_t, X_t^2, \dots, X_t^d)$, and $M_t = \sup_{0 \leq s \leq t} X_s^1$. The proof of Proposition 1.5 is developed along this section.

2.1 Control in first step Euler scheme approximation

Here are provided some estimations of processes X and M increments expectation. Abusing of notations, when there is no ambiguity, X denotes X^1 .

Depending on the assumptions, let K denote either $\max(\|A\|_\infty, \|B\|_\infty)$ or $\max(\|A\|_\infty, \|B\|_\infty, \|\nabla A\|_\infty)$, let C_p be the Burkholder Davis Gundy inequality constant.

Lemma 2.1 *Let A and B be bounded; $\forall h > 0, \forall t, \forall s, \forall p \geq 1$*

$$\mathbb{E} \left(\sup_{s \leq h} \|X_{t+s} - X_t\|^p \right) \leq 2^p(1 + C_p)[h^p + h^{p/2}]K^p.$$

Proof: Using $(a + b)^p \leq 2^{p-1} [a^p + b^p]$, $a, b \geq 0$:

$$\sup_{s \leq h} \|X_{t+s} - X_t\|^p \leq 2^{p-1} \left[\sup_{u \leq h} \left(\left\| \int_t^{t+u} B(X_s) ds \right\| \right)^p + \sup_{u \leq h} \left(\left\| \int_t^{t+u} \sum_{\alpha} A_{\alpha}(X_s) dW_s^{\alpha} \right\| \right)^p \right].$$

The expectation then Burkholder Davis Gundy inequality imply

$$\mathbb{E}(\sup_{s \leq h} \|X_{t+s} - X_t\|^p) \leq 2^{p-1}(1 + C_p) \mathbb{E} \left[\left(\int_t^{t+h} \|B(X_s)\|^2 ds \right)^{p/2} + \left(\int_t^{t+h} \|A(X_s)\|^2 ds \right)^{p/2} \right].$$

Boundness of B and A yields $\mathbb{E}(\sup_{s \leq t} |X_{t+s} - X_t|^p) \leq 2^{p-1}(1 + C_p)[h^p K^p + h^{p/2} K^p]$. •

Lemma 2.2 *Let B and A satisfy Assumptions (2) and (3); the following bound is satisfied, $\forall t \leq T$, $\forall p \geq 1$:*

$$(10) \quad \mathbb{E}(|M_{t+h} - M_t|^p) \leq 2^p \sqrt{1 + C_{2p}} (h^p + h^{p/2}) K^p ; \quad \mathbb{E}(|M_{t+h} - M_t|^p) = o(h^{p/2}).$$

Proof: One remarks that $M_{t+h} - M_t = (X_t^1 - M_t + \sup_{0 \leq u \leq h} (X_{t+u}^1 - X_t^1))_+$ where $(x)_+ = \max(x, 0)$. For any $a \geq 0$, $(x - a)_+ \leq |x| \mathbf{1}_{x > a}$ thus

$$0 \leq M_{t+h} - M_t \leq \left| \sup_{0 \leq u \leq h} (X_{t+u}^1 - X_t^1) \right| \mathbf{1}_{\{\sup_{0 \leq u \leq h} (X_{t+u}^1 - X_t^1) > M_t - X_t^1\}}.$$

Cauchy-Schwartz inequality yields:

$$0 \leq \mathbb{E}([M_{t+h} - M_t]^p) \leq \sqrt{\mathbb{E} \left(\left| \sup_{0 \leq u \leq h} (X_{t+u}^1 - X_t^1) \right|^{2p} \right) \mathbb{P}(\{\sup_{0 \leq u \leq h} (X_{t+u}^1 - X_t^1) > M_t - X_t^1\})}.$$

Lemma 2.1 in case $2p$ proves the bound (10) and that the almost sure limit $\lim_{h \rightarrow 0} \sup_{0 \leq u \leq h} (X_u^1 - X_t^1) = 0$. According to Theorem 1.1 [9] the pair (M_t, X_t) law admits a density, thus $\mathbb{P}(\{M_t - X_t^1 = 0\}) = 0$. So actually $\mathbb{E}([M_{t+h} - M_t]^p)$ is bounded by the product of $h^{p/2}$ and a factor going to zero and it is an $o(h^{p/2})$. •

For any t let the process $(X_{t,h}, h \in [t, T])$ and the running maximum of its first component be defined as:

$$(11) \quad X_{t,h} := \sum_{\alpha} A_{\alpha}(X_t) [W_{t+h}^{\alpha} - W_t^{\alpha}], \quad M_{t,h} := \sup_{0 \leq u \leq h} X_{t,u}^1.$$

Lemma 2.3 *Under Assumptions (2) and (3), the following bound is satisfied: there exists a constant D_p such that $\forall t \leq T$, $\forall h \in (0, 1]$:*

$$\mathbb{E} \left(\sup_{s \leq h} \|X_{s+t} - X_t - X_{t,s}\|^p \right) \leq 2^{2p-1} (1 + C_p)^2 D_p h^p.$$

Proof: By definition, with $\tilde{W}_u := W_{t+u} - W_t$, $u \geq 0$

$$X_{s+t} - X_t - X_{t,s} = \int_0^s B^1(X_{u+t}) du + \int_0^s [A^1(X_{u+t}) - A^1(X_t)] d\tilde{W}_u.$$

Using once again $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, $a, b \geq 0$,

$$\sup_{s \leq h} \|X_{s+t} - X_t - X_{t,s}\| \leq 2^{p-1} \left[\left(\int_0^h \|B^1(X_{u+t})\| du \right)^p + \sup_{s \leq h} \left\| \int_0^h [A^1(X_{u+t}) - A^1(X_t)] d\tilde{W}_u \right\|^p \right].$$

Expectation and Burkholder Davis Gundy inequality prove:

$$\mathbb{E} \left(\sup_{s \leq h} \|X_{s+t} - X_t - X_{t,s}\|^p \right) \leq 2^{p-1}(1 + C_p) \left[\mathbb{E} \left(\int_0^h \|B^1(X_{u+t})\|^p du \right) + \mathbb{E} \left| \int_0^h \|A^1(X_{u+t}) - A^1(X_t)\|^2 du \right|^{p/2} \right].$$

The first term above is bounded by $K^p h^p$ since B is bounded. Assumption $A \in C_b^1$ and Jensen's inequality imply the second term is bounded by

$K^p h^{p/2-1} \int_0^h \mathbb{E} \|X_{u+t} - X_t\|^p du$ thus

$$\mathbb{E} \left(\sup_{s \leq h} \|X_{s+t} - X_t - X_{t,s}\|^p \right) \leq 2^{p-1}(1 + C_p) K^p h^{p/2-1} \left[h^{p/2+1} + \int_0^h \mathbb{E} \|X_{u+t} - X_t\|^p du \right].$$

Lemma 2.1 allows to uniformly bound $\mathbb{E} \|X_{u+t} - X_t\|^p \leq 2^p(1 + C_p)[h^p + h^{p/2}]K^p$ so

$$\mathbb{E} \left(\sup_{s \leq h} \|X_{s+t} - X_t - X_{t,s}\|^p \right) \leq 2^{p-1}(1 + C_p) K^p h^p [1 + 2^p K^p(1 + C_p)[1 + h^{p/2}]].$$

Lemma 2.4 *Under Assumptions (2) and (3)*

- (i) $\sup_{0 \leq t \leq T; 0 \leq h \leq 1} h^{-1} \mathbb{E} \left(\left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+ \right| \right) < \infty,$
- (ii) $\lim_{h \rightarrow 0} h^{-1} \mathbb{E} \left(\left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+ \right| \right) = 0.$

Proof: Considering that $M_{t+h} - M_t = (\sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1) - M_t + X_t^1)_+$ and that for all $a \in \mathbb{R}$

$$(12) \quad |(x - a)_+ - (y - a)_+| \leq |x - y| [\mathbf{1}_{\{x > a\}} + \mathbf{1}_{\{y > a\}}],$$

yields

$$\left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+ \right| \leq \left| \sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1) - M_{t,h} \right| \left[\mathbf{1}_{\{\sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1) > M_t - X_t^1\}} + \mathbf{1}_{\{M_{t,h} > M_t - X_t^1\}} \right].$$

Let f and g be functions on $[0, T]$, $\forall s \in [0, T]$,

$$f(s) - \sup_{u \leq T} g(u) \leq f(s) - g(s) \leq |f(s) - g(s)| \leq \sup_{v \leq T} |f(v) - g(v)|,$$

so

$$\sup_{s \leq T} f(s) - \sup_{u \leq T} g(u) \leq \sup_{v \leq T} |f(v) - g(v)|.$$

Here the role of f and g are symmetrical so $\sup_{s \leq T} g(s) - \sup_{u \leq T} f(u) \leq \sup_{v \leq T} |f(v) - g(v)|$, and $|\sup_{s \leq T} g(s) - \sup_{u \leq T} f(u)| \leq \sup_{v \leq T} |f(v) - g(v)|$. This is applied to the functions $u \rightarrow X_{u+t}^1 - X_t^1$ and $u \rightarrow X_{t,u}^1$:

$$\begin{aligned} & \left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+ \right| \leq \\ & \sup_{u \leq h} |X_{u+t}^1 - X_t^1 - X_{t,u}^1| \left[\mathbf{1}_{\{\sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1) > M_t - X_t^1\}} + \mathbf{1}_{\{M_{t,h} > M_t - X_t^1\}} \right]. \end{aligned}$$

Cauchy Schwartz inequality and $(a + b)^2 \leq 2(a^2 + b^2)$ get

$$\begin{aligned} & \mathbb{E} \left(\left| M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+ \right| \right) \\ & \leq \sqrt{2\mathbb{E} \left(\sup_{u \leq h} |X_{u+t}^1 - X_t^1 - X_{t,u}^1|^2 \right) \left(\mathbb{P} \left\{ \sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1) > M_t - X_t^1 \right\} + \mathbb{P} \{ M_{t,h} > M_t - X_t^1 \} \right)}. \end{aligned}$$

Lemma 2.3 with $p = 2$ insures that $h \mapsto h^{-1} \sqrt{2\mathbb{E} \left(\sup_{u \leq h} |X_{u+t}^1 - X_t^1 - X_{t,u}^1|^2 \right)}$ is uniformly bounded in t . Concerning the second factor,

- firstly the almost sure continuity with respect to h insures that the limits $\lim_{h \rightarrow 0} \sup_{0 \leq u \leq h} (X_{u+t}^1 - X_t^1)$ and $\lim_{h \rightarrow 0} M_{t,h}$ are equal to 0;
 - secondly the pair (M_t, X_t) law admits a density according to Theorem 1.1 [9] $\mathbb{P}\{0 = M_t - X_t^1\} = 0$. Thus the limit of the second factor is equal to 0.
- This concludes the proof of the lemma. •

2.2 Conditional expectation and approximation

Here too X actually denotes the first component X^1 . Here, from $(\mathcal{F}_t := \sigma\{X_0, W_s, s \leq t\}, t \geq 0)$, we deduce the completed càd filtration \mathbb{F} .

Proposition 2.5 *Under Assumptions (2) and (3), with H defined in (9):*

$$\mathbb{E} \left[(M_{s,h} - M_s + X_s^1)_+ | \mathcal{F}_s \right] = 2 \|A^1(X_s)\| \sqrt{h} H \left(\frac{M_s - X_s^1}{\|A^1(X_s)\| \sqrt{h}} \right).$$

Proof: Recall Definition (11),

$$X_{t,h} = \sum_{j=1}^d A_j(X_t) [W_{t+h}^j - W_t^j], \quad M_{t,h} = \sup_{0 \leq u \leq t} X_{t,h}^1, \quad h \in [0, 1].$$

Conditionally to \mathcal{F}_t , the process $(X_{t,h}^1, h \in [0, 1])$ has the same law as $(\sqrt{h} \|A^1(X_t)\| \tilde{W}_u, u \in [0, 1])$ where \tilde{W} is a Brownian motion independent of \mathcal{F}_t . Conditionally to \mathcal{F}_t , the random variable $M_{t,h}$ has the same law as $\sqrt{h} \|A^1(X_t)\| \sup_{u \leq 1} \tilde{W}_u$.

Following [17] Section 3.1.3, $\sup_{u \leq 1} \tilde{W}_u$ has the same law as $|G|$ where G is a standard Gaussian variable (independent of \mathcal{F}_t), with law density $\Phi'_G(z) = \frac{2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \mathbf{1}_{[0, +\infty[}(z)$. Then,

$$\begin{aligned} \mathbb{E} [(M_{t,h} - (M_t - X_t^1))_+ | \mathcal{F}_t] &= \int_0^\infty \left(\|A^1(X_t)\| \sqrt{h} z - (M_t - X_t^1) \right)_+ \frac{2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= 2 \|A^1(X_t)\| \sqrt{h} H\left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|}\right). \end{aligned}$$

Proposition 2.6 *Under Assumptions (2) and (3), let $\Psi \in C_b^1(\mathbb{R}^{d+1})$, $X_t^M := (M_t, X_t^2, \dots, X_t^d)$. (i) For all $T > 0$ there exists a constant C such that for all $t \in [0, T]$, $h \in [0, 1]$.*

$$\left| \mathbb{E} \left([\Psi(V_t) - \Psi(M_t, X_t^M)] \|A^1(X_t)\| \sqrt{h} H\left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|}\right) \right) \right| \leq C \|\nabla \Psi\|_\infty h.$$

(ii) For all $t > 0$,

$$\left| \mathbb{E} \left([\Psi(V_t) - \Psi(M_t, X_t^M)] \|A^1(X_t)\| \sqrt{h} H\left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|}\right) \right) \right| = o(h).$$

Proof: Using the fact $\Psi \in C_b^1$

$$\left| \mathbb{E} \left([\Psi(V_t) - \Psi(M_t, X_t^M)] \|A^1(X_t)\| \sqrt{h} H\left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|}\right) \right) \right| \leq \|\nabla \Psi\|_\infty h \mathbb{E} \left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \|A^1(X_t)\|^2 H\left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|}\right) \right).$$

Point i) follows from the facts that A is bounded and the map $u \mapsto uH(u)$ is bounded by c on $[0, +\infty[$ according to Remark 3.1 :

$$\left| \mathbb{E} \left([\Psi(V_t) - \Psi(M_t, X_t^M)] \|A^1(X_t)\| \sqrt{h} H\left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|}\right) \right) \right| \leq \|\nabla \Psi\|_\infty \|A^1\|_\infty^2 ch.$$

Point ii) is a consequence of Lebesgue dominated Theorem. Indeed, $\lim_{u \rightarrow \infty} uH(u) = 0$. Moreover when $h \rightarrow 0$ $\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \rightarrow \infty$ on $\{M_t - X_t^1 \neq 0\}$ and $\mathbb{P}\{M_t - X_t^1 \neq 0\} = 1$ since $M_t - X_t^1$ law is absolutely continuous (see Theorem 1.1 [9]). Then $\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|} \|A^1(X_t)\|^2 H\left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|}\right)$ is uniformly bounded (as A and $uH(u)$ are) and almost surely goes to 0 when $h \rightarrow 0$. So dominated Lebesgue theorem concludes the proof of (ii). •

2.3 Conclusion of Proposition 1.5 proof

The key of this proof is to write

$$E \left[\int_t^{t+h} \Psi(V_s) dM_s \right] - 2\sqrt{h} E \left[\Psi(M_t, X_t^M) \|A^1(X_t)\| H\left(\frac{M_t - X_t^1}{\sqrt{h} \|A^1(X_t)\|}\right) \right]$$

as the sum of three terms

$$\begin{aligned} & E\left[\int_t^{t+h} (\Psi(V_s) - \Psi(V_t))dM_s\right] + \\ & E\left[\Psi(V_t)(M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+)\right] + \\ & E\left[(\Psi(V_t)(M_{t,h} - M_t + X_t^1)_+ - 2\sqrt{h}\Psi(M_t, X_t^M)\|A^1(X_t)\|H\left(\frac{M_t - X_t^1}{\sqrt{h}\|A^1(X_t)\|}\right))\right]. \end{aligned}$$

(a) Since $\nabla\Psi$ is bounded, and process M is increasing, the first term is bounded

$$E\left[\int_t^{t+h} (\Psi(V_s) - \Psi(V_t))dM_s\right] \leq \|\nabla\Psi\|_\infty E\left[\sup_{t \leq s \leq t+h} \|V_s - V_t\|(M_{t+h} - M_t)\right].$$

Using Cauchy-Schwarz inequality

$$E\left[\sup_{t \leq s \leq t+h} \|V_s - V_t\|(M_{t+h} - M_t)\right] \leq (E\left[\sup_{t \leq s \leq t+h} \|V_s - V_t\|^2\right])^{1/2} (E[(M_{t+h} - M_t)^2])^{1/2}.$$

Remark that $\|V_s - V_t\|^2 = (M_s - M_t)^2 + \|X_s - X_t\|^2$ and using $(a + b)^2 \leq 2(a^2 + b^2)$

$$\sqrt{\frac{1}{2}}E\left[\sup_{t \leq s \leq t+h} \|V_s - V_t\|(M_{t+h} - M_t)\right] \leq \sqrt{E[(M_{t+h} - M_t)^2] + E\left[\sup_{t \leq s \leq t+h} \|X_s - X_t\|^2\right]} \sqrt{E[(M_{t+h} - M_t)^2]}$$

Lemmas 2.1 and 2.2 yield this term is an $O(h)$ uniformly with respect to t and an $o(h)$ for all t .

(b) Using boundedness of Ψ and Lemma 2.4 there exists a constant C

$$E[\Psi(V_t)[M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+]] \leq Ch$$

and

$$\lim h^{-1}E[\Psi(V_t)[M_{t+h} - M_t - (M_{t,h} - M_t + X_t^1)_+]] = 0.$$

(c) Here we use Proposition 2.5 to replace the factor $(M_{t,h} - M_t + X_t^1)_+$ by its \mathcal{F}_t -conditional expectation, so in the third term we have to deal with

$$E\left[(\Psi(V_t) - \Psi(M_t, X_t^M))2\|A^1(X_s)\|\sqrt{h}H\left(\frac{M_s - X_s^1}{\|A^1(X_s)\|\sqrt{h}}\right)\right]$$

Proposition 2.6 (i) and the boundness of $\nabla\Psi$ yield there exists a constant C :

$$E\left[(\Psi(V_t) - \Psi(M_t, X_t^M))2\|A^1(X_s)\|\sqrt{h}H\left(\frac{M_s - X_s^1}{\|A^1(X_s)\|\sqrt{h}}\right)\right] \leq Ch.$$

Proposition 2.6 (ii) proves that for $t > 0$ this is an $o(h)$. •

3 Proofs of one-dimensional results

Here we first operate a Lamperti transformation to reduce our case to the one with $A = 1$. In this case, we show that, for $\Psi \in C_b^1$ the map $t \mapsto \mathbb{E}(\int_0^t \Psi(M_s, X_s)dM_s)$ is absolutely continuous with respect to the Lebesgue measure on $[0, T]$ with density denoted as f_Ψ . From Proposition 1.5,

$$f_\Psi(t) = \lim_{h \rightarrow 0} 2\mathbb{E}\left(\Psi(M_t, M_t)\frac{1}{\sqrt{h}}H\left(\frac{M_t - X_t}{\sqrt{h}}\right)\right).$$

Using the continuity of p_V close to the diagonal $\{(m, m), m > x_0\}$ (Proposition 3.8), Theorem 3.7 proves that

$$f_\Psi(t) = \frac{1}{2} \int_{x_0}^{+\infty} \Psi(m, m) p_V(m, m; t) dm.$$

Finally, we will conclude with the proof of Proposition 3.9 which is the key for the proof of Theorem 1.3.

The following remark is often used:

Remark 3.1 *Let us remark that the definition (9) proves the bound $0 \leq H(B) \leq \frac{e^{-\frac{B^2}{2}}}{\sqrt{2\pi}}$, $\int_0^\infty H(u) du = 1/4$ and $BH(B) = \frac{1}{\sqrt{2\pi}} B e^{-\frac{B^2}{2}} - B^2 \Phi_G(-B)$. Thus there exists $c > 0$ such that $BH(B) \leq c$ for any $B \in \mathbb{R}^+$ since $BH(B) \rightarrow 0$ when $B \rightarrow \infty$.*

Along this section Assumptions (2) and (3) are in force and the initial law μ_0 is the Dirac one at x_0 .

3.1 Reduction to a Brownian plus drift

A Lamperti transformation allows us to reduce the problem to the case of a diffusion process with additive noise. A priori $dX_t = B(X_t)dt + A(X_t)dW_t$. We look for an increasing function $\varphi \in C_b^2$ such that the coefficient of dW would be 1. Itô formula yields

$$d\varphi(X_t) = \varphi'(X_t)B(X_t)dt + \frac{1}{2}\varphi''(X_t)A^2(X_t)dt + \varphi'(X_t)A(X_t)dW_t, \quad \varphi(X_0) = \varphi(x_0).$$

A sufficient condition is to choose φ such that $\varphi' = \frac{1}{A}$. As is A , φ' is bounded above and below uniformly. Then $Y = \varphi(X)$ satisfies

$$(13) \quad dY_t = \left[\frac{B}{A} \circ \varphi^{-1}(Y_t) - \frac{1}{2} A' \circ \varphi^{-1}(Y_t) \right] dt + dW_t.$$

Remark that $\tilde{B} = \frac{B}{A} \circ \varphi^{-1} - \frac{1}{2} A' \circ \varphi^{-1} \in C_b^1$ as a consequence of $B \in C_b^1$, $A \in C_b^2$. Moreover φ' being positive, φ is increasing and $Y_t^* = \sup_{0 \leq s \leq t} Y_s = \varphi(M_t)$. From Theorem 1.1 in [9], the law of the pair (Y_t^*, Y_t) admits a density with respect to the Lebesgue measure.

Lemma 3.2 *The density law of (M_t, X_t) $p_V(\cdot, \cdot; t)$ satisfies*

$$p_V(b, a; t) = \frac{p_{Y^*, Y}(\varphi(b), \varphi(a); t)}{A(b)A(a)}$$

where φ is defined by $\varphi'(x) = \frac{1}{A(x)}$ and $p_{Y^*, Y}(\cdot, \cdot, t)$ is the pair (Y_t^*, Y_t) law density.

Proof: It is enough to identify the density law of the pair $V_t = (M_t, X_t)$ using, for any bounded measurable F , the following

$$E[F(M_t, X_t)] = E[F(\varphi^{-1}(Y_t^*), \varphi^{-1}(Y_t))] = \int F(\varphi^{-1}(\beta), \varphi^{-1}(\alpha)) p_{Y^*, Y}(\beta, \alpha; t) d\beta d\alpha.$$

We operate the change of variables $b = \varphi^{-1}(\beta)$, $a = \varphi^{-1}(\alpha)$, so $d\beta = \varphi'(b)db = \frac{1}{A(b)}db$ and $d\alpha = \varphi'(a)da = \frac{1}{A(a)}da$ get the result. •

Proposition 3.3 Assume that Theorem 1.3 is proved in case $A = 1$, $\tilde{B} = \frac{B}{A} \circ \varphi^{-1} - \frac{1}{2}A' \circ \varphi^{-1}$. Then Theorem 1.3 is proved: for any $\Phi \in C_b^2(\mathbb{R}^2, \mathbb{R})$:

$$\begin{aligned} E[\Phi(V_t)] &= \Phi(V_0) + E\left[\int_0^t \left[B(X_s)\partial_2\Phi(V_s) + \frac{1}{2}A^2(X_s)\partial_{22}^2\Phi(V_s) \right] ds \right. \\ &\quad \left. + \frac{1}{2} \int_0^t E[\partial_1\Phi(M_s, M_s)]A^2(M_s) \frac{p_V(M_s, M_s; s)}{p_M(M_s, s)} ds \right] \end{aligned}$$

where $p_V(m, x; t)$ is the pair $V_t = (M_t, X_t)$ law density and $p_M(\cdot, t)$ is the one of M_t .

Proof: We apply Theorem 1.3 to (Y_t^*, Y_t) and $F(\beta, \alpha) = \Phi(\varphi^{-1}(\beta), \varphi^{-1}(\alpha))$; since $\varphi' > 0$, then φ is increasing, $\varphi(X_t^*) = Y_t^*$ and $\Phi(V_t) = F(Y_t^*, Y_t)$. Note $\varphi' = A^{-1}$ so

$$\begin{aligned} \partial_2 F(\beta, \alpha) &= \partial_2 \Phi(\varphi^{-1}(\beta), \varphi^{-1}(\alpha)) \frac{1}{\varphi'(\varphi^{-1}(\alpha))} = \partial_2 \Phi(\varphi^{-1}(\beta), \varphi^{-1}(\alpha)) A(\varphi^{-1}(\alpha)), \\ \partial_{22}^2 F(\beta, \alpha) &= \partial_{22}^2 \Phi(\varphi^{-1}(\beta), \varphi^{-1}(\alpha)) A^2(\varphi^{-1}(\alpha)) + \partial_2 \Phi(\varphi^{-1}(\beta), \varphi^{-1}(\alpha)) (AA')(\varphi^{-1}(\alpha)), \end{aligned}$$

so using $\tilde{B} = \frac{B}{A}\varphi^{-1} - \frac{1}{2}A' \circ \varphi^{-1}$

$$\tilde{B}(\alpha)\partial_2 F(\beta, \alpha) + \frac{1}{2}\partial_{22}^2 F(\beta, \alpha) = B(\varphi^{-1}(\alpha))\partial_2 \Phi(\varphi^{-1}(\beta), \varphi^{-1}(\alpha)) + \frac{1}{2}A^2(\varphi^{-1}(\alpha))\partial_{22}^2 \Phi(\varphi^{-1}(\beta), \varphi^{-1}(\alpha)).$$

Then,

$$\int_0^t \mathbb{E} \left[\tilde{B}(Y_s)\partial_2 F(Y_s^*, Y_s) + \frac{1}{2}\partial_{22}^2 F(Y_s^*, Y_s) \right] ds = E\left[\int_0^t \left[B(X_s)\partial_2\Phi(V_s) + \frac{1}{2}A^2(X_s)\partial_{22}^2\Phi(V_s) \right] ds \right]$$

Note that

$$\partial_1 F(\beta, \alpha) = \partial_1 \Phi(\varphi^{-1}(\beta), \varphi^{-1}(\alpha)) \frac{1}{\varphi'(\varphi^{-1}(\beta))} = \partial_1 \Phi(\varphi^{-1}(\beta), \varphi^{-1}(\alpha)) A(\varphi^{-1}(\beta))$$

thus

$$\int_0^t E\left[\partial_1 F(Y_s^*, Y_s^*) \frac{p_{Y^*, Y}(Y_s^*, Y_s^*, s)}{p_{Y^*}(Y_s^*, s)}\right] ds = \int_0^t \int_{\mathbb{R}} \partial_1 \Phi(\varphi^{-1}(\beta), \varphi^{-1}(\beta)) A(\varphi^{-1}(\beta)) p_{Y^*, Y}(\beta, \beta, s) d\beta ds.$$

We perform the change of variable $b = \varphi^{-1}(\beta)$

$$\int_0^t E\left[\partial_1 F(Y_s^*, Y_s^*) \frac{p_{Y^*, Y}(Y_s^*, Y_s^*, s)}{p_{Y^*}(Y_s^*, s)}\right] ds = \int_0^t \int_{\mathbb{R}} \partial_1 \Phi(b, b) p_{Y^*, Y}(\varphi(b), \varphi(b), s) db ds.$$

We use Lemma 3.2 which tells us $p_{Y^*, Y}(\varphi(b), \varphi(b), s) = p_V(b, b; s)A^2(b)$:

$$\int_0^t E\left[\partial_1 F(Y_s^*, Y_s^*) A(Y_s^*) \frac{p_{Y^*, Y}(Y_s^*, Y_s^*, s)}{p_{Y^*}(Y_s^*, s)}\right] ds = \int_0^t \int_{\mathbb{R}} \partial_1 \Phi(b, b) A^2(b) p_V(b, b, s) db ds$$

and finally

$$\int_0^t E\left[\partial_1 F(Y_s^*, Y_s^*) A(Y_s^*) \frac{p_{Y^*, Y}(Y_s^*, Y_s^*, s)}{p_{Y^*}(Y_s^*, s)}\right] ds = \int_0^t E\left[\partial_1 \Phi(M_s, M_s) A^2(M_s) \frac{p_V(M_s, M_s; s)}{p_M(M_s, s)}\right] ds$$

3.2 Study of $F_\Psi : t \rightarrow E[\int_0^t \Psi(V_s)dM_s]$, case $A = 1$

From now on, without lost of generality, we look at $E[\int_0^t \Psi(V_s)dM_s]$ where $\Psi \in C_b^1$. In several places we operate an equivalent change of probability measure, $Q = L_T\mathbb{P}$ with

$$(14) \quad dL_t = -L_t B(X_t)dW_t, \quad L_t = \exp\left[\int_0^t -B(X_s)dW_s - \frac{1}{2} \int_0^t B^2(X_s)ds\right]$$

so using $dW_s = dX_s - B(X_s)ds$,

$$L_t = \exp\left[\int_0^t -B(X_s)(dX_s - B(X_s)ds) - \frac{1}{2} \int_0^t B^2(X_s)ds\right] = \exp\left[\int_0^t -B(X_s)dX_s + \frac{1}{2} \int_0^t B^2(X_s)ds\right]$$

and

$$Z_t = L_t^{-1} = \exp\left[\int_0^t B(X_s)dX_s - \frac{1}{2} \int_0^t B^2(X_s)ds\right].$$

Let the function $\mathcal{B} : x \rightarrow \int_0^x B(u)du$. Ito formula applied to the process $\mathcal{B}(X_t)$ yields $\mathcal{B}(X_t) = \mathcal{B}(x_0) + \int_0^t B(X_s)dX_s + \frac{1}{2} \int_0^t B'(X_s)ds$ so $\int_0^t B(X_s)dX_s = \mathcal{B}(X_t) - \mathcal{B}(x_0) - \frac{1}{2} \int_0^t B'(X_s)ds$ and

$$(15) \quad Z_t = \exp\left[\mathcal{B}(X_t) - \mathcal{B}(x_0) - \frac{1}{2} \int_0^t [B'(X_s) + B^2(X_s)]ds\right],$$

which could be written as

$$(16) \quad Z_t = \exp[\mathcal{B}(X_t) - \mathcal{B}(x_0)] \left(1 - \frac{1}{2} \int_0^t [B'(X_s) + B^2(X_s)] \exp\left[-\frac{1}{2} \int_0^s (B'(X_u) + B^2(X_u))du\right] ds\right)$$

Proposition 3.4 *Let $B \in C_b^1(\mathbb{R})$ and $A = 1$, let Ψ be a Borel bounded function, then the application $t \rightarrow E[\int_0^t \Psi(V_s)dM_s]$ is absolutely continuous on all finite interval in \mathbb{R}^+ .*

Proof:

(i) Using Criterion 11.7 p. 364 [27] since $\left|E[\int_a^b \Psi(V_s)dM_s]\right| \leq \|\Psi\|_\infty E[M_b - M_a]$, it is enough to prove the result in case of $\Psi = 1$, meaning the study of $t \rightarrow E[M_t]$.

We operate the equivalent change of probability measure, $Q = L_T\mathbb{P}$, defined in (14) so that under Q the process $X - x_0$ is a Brownian motion and (with an abuse of notation in the last term)

$$(17) \quad \mathbb{E}_\mathbb{P}[M_t] = E_Q[L_t^{-1}M_t] = \mathbb{E}_\mathbb{P}(Z_t(x_0 + W_t^*)) = x_0 + \mathbb{E}_\mathbb{P}(Z_t W_t^*).$$

Recall the density of the pair (W_t^*, W_t) in case of Brownian motion [17] Section 3.2:

$$(18) \quad p_W(b, a; t) = 2 \frac{2b - a}{\sqrt{2\pi t^3}} \exp\left[-\frac{(2b - a)^2}{2t}\right] \mathbf{1}_{b > \sup(a, 0)}.$$

Lemma 3.5 *Let $B \in C_b^1(\mathbb{R})$ and $A = 1$. For all $T > 0$ there exists a constant C_T such that for all $a, b, 0 < a < b \leq T, c > 0$,*

$$(19) \quad E_\mathbb{P}[M_b - M_a] \leq C_T \left[\int_a^b \mathbb{E}_\mathbb{P} [c^2 \cosh(cW_s) W_s^*] ds + \int_a^b \mathbb{E}_\mathbb{P} \left[\cosh(cW_s) \frac{p_W(W_s, W_s; s)}{g_W(W_s; s)} \right] ds \right]$$

where p_W is introduced in (18) and $g_W(\cdot, s)$ is the density of the law of W_s .

As a corollary, for any $t \in [0, T]$, $M_t \in L^1$.

Proof: We bound $L_t^{-1} = Z_t$ defined in (15): Firstly, B and B' are bounded, and by definition $|\mathcal{B}(x + x_0) - \mathcal{B}(x_0)| = |\int_{x_0}^{x+x_0} B(u)du| \leq \|B\||x|$, so for any T ,

$$(20) \quad Z_t \leq \exp[\|B\||X_t - x_0| + \frac{1}{2}\|B'\|T],$$

thus $Z_t \leq e^{\frac{1}{2}\|B'\|T}[\exp(\|B\|(X_t - x_0)) + \exp(-\|B\|(X_t - x_0))] = 2e^{\frac{1}{2}\|B'\|T} \cosh(\|B\|(X_t - x_0))$. So

$$(21) \quad \begin{aligned} E_{\mathbb{P}}[M_b - M_a] &= E_Q[Z_b(M_b^* - M_a^*)] \leq 2e^{\frac{1}{2}\|B'\|T} E_{\mathbb{P}}[\cosh(\|B\|W_b)(W_b^* - W_a^*)] \\ &\leq 2e^{\frac{1}{2}\|B'\|T} \{E_{\mathbb{P}}[\cosh(\|B\|W_b)W_b^* - \cosh(\|B\|W_a)W_a^*] + |E_{\mathbb{P}}[(\cosh(\|B\|W_a) - \cosh(\|B\|W_b))W_a^*]|\}. \end{aligned}$$

(i) We introduce two sequences of C_b^2 positive functions, f_N and g_N such that $f_N(x) \uparrow \cosh(\|B\|x)$, $g_N(x) \uparrow x$, $f_N''(x) \uparrow \|B\|^2 \cosh(\|B\|x)$, $g_N'(x) \uparrow 1$, when N goes to infinity. According to Theorem 2.1 [8] in case of a null drift,

$$(22) \quad \mathbb{E}_{\mathbb{P}}[g_N(W_t^*)f_N(W_t)] = g_N(0)f_N(0) + \frac{1}{2} \int_0^t \mathbb{E}[f_N''(W_s)g_N(W_s^*)] ds + \frac{1}{2} \int_0^t \mathbb{E} \left[g_N'(W_s^*)f_N(W_s) \frac{p_W(W_s, W_s; s)}{g_W(W_s; s)} \right] ds.$$

Both hands are monotonous with respect to N , so using Lebesgue's monotonous convergence Theorem, we get

$$(23) \quad \mathbb{E}_{\mathbb{P}}(\cosh(\|B\|W_t)W_t^*) = \frac{1}{2} \int_0^t \mathbb{E}_{\mathbb{P}}[\|B\|^2 \cosh(\|B\|W_s)W_s^*] ds + \frac{1}{2} \int_0^t \mathbb{E}_{\mathbb{P}} \left[\cosh(\|B\|W_s) \frac{p_W(W_s, W_s; s)}{g_W(W_s; s)} \right] ds.$$

(ii) Using Itô formula and the fact that W^* is an increasing process

$$(24) \quad \begin{aligned} |E_{\mathbb{P}}[(\cosh(\|B\|W_b) - \cosh(\|B\|W_a))W_a^*]| &= E_{\mathbb{P}}\left[\frac{\|B\|^2}{2} \int_a^b \cosh(\|B\|W_s) ds W_a^*\right] \\ &\leq \frac{\|B\|^2}{2} \int_a^b E_{\mathbb{P}}[\cosh(\|B\|W_s)W_s^*] ds. \end{aligned}$$

Adding bounds (23) and (24) yields (19). •

To conclude Proposition 3.4 proof, we introduce the absolutely continuous function

$$f : x \rightarrow \int_a^x \mathbb{E} \left[c^2 \cosh(cW_s)W_s^* + \cosh(cW_s) \frac{p_W(W_s, W_s; s)}{g_W(W_s; s)} \right] ds.$$

Using the boundedness of the above integrand on interval $[a, b]$, $|f(x+h) - f(x)| \leq \int_x^{x+h} |f'(u)|du$. Lemma 3.5 yielding $E_{\mathbb{P}}[M_{x+h} - M_x] \leq C_T \int_x^{x+h} |f'(u)|du$, Criterion 11.7 p. 364 [27] concludes the proof. •

Corollary 3.6 *Since the application $F_{\Psi} : t \rightarrow E[\int_0^t \Psi(V_s)dM_s]$ is absolutely continuous, there exists a function f_{Ψ} such that $E[\int_0^t \Psi(V_s)dM_s] = \int_0^t f_{\Psi}(s)ds$.*

3.3 Computation of F_Ψ derivative, case additive noise

Let now a drifted Brownian motion be defined as: $dX_t = B(X_t)dt + dW_t$, $t \in [0, T]$, $X_0 = x_0$, where $B \in C_b^1(\mathbb{R})$. Let $\Psi \in C_b^1(\mathbb{R}^2)$ and recall the function $F_\Psi : t \mapsto \mathbb{E} \left(\int_0^t \Psi(V_s) dM_s \right)$. Using Corollary 3.6, this function F_Ψ is absolutely continuous with respect to Lebesgue measure, so it is almost surely differentiable with derivative denoted as f_Ψ . Actually $f_\Psi = \lim_{h \rightarrow 0} T_\Psi(h, \cdot)$ where

$$(25) \quad T_\Psi(h, t) := \frac{1}{h} \mathbb{E} \left(\int_t^{t+h} \Psi(V_s) dM_s \right).$$

Theorem 3.7 *Let the process X be defined as $dX_t = B(X_t)dt + dW_t$, $X_0 = x_0$, with $B \in C_b^1(\mathbb{R})$, $M_t = \sup_{0 \leq s \leq t} X_s$ and $\Psi \in C_b^1(\mathbb{R})$. Then $E[\int_0^t \Psi(V_s) dM_s] = \int_0^t f_\Psi(s) ds$ where*

$$f_\Psi(t) = \frac{1}{2} E_{\mathbb{P}} \left[\Psi(M_t, M_t) \frac{p_V(M_t, M_t; t)}{p_M(M_t, t)} \right].$$

The proof is a consequence of the two following propositions. In case of dimension 1 and $A = 1$, the first one is a direct consequence of Proposition 1.5 (ii).

Proposition 3.8 *Let $A = 1$, $B \in C_b^1(\mathbb{R})$, $\Psi \in C_b^1(\mathbb{R})$, then for any t such that $(T_\Psi(h, t))_h$ converges when $h \rightarrow 0$,*

$$f_\Psi(t) = \lim_{h \rightarrow 0} T_\Psi^1(h, t) \text{ where } T_\Psi^1(h, t) = 2\mathbb{E} \left(\Psi(M_t, M_t) \frac{1}{\sqrt{h}} H \left(\frac{M_t - X_t}{\sqrt{h}} \right) \right)$$

where $H(B) := \frac{1}{\sqrt{2\pi}} e^{-\frac{B^2}{2}} - B\Phi_G(-B)$ and Φ_G is the standard Gaussian distribution function.

And the following is proved in Section 3.4:

Proposition 3.9 *We assume $A = 1$, $B \in C_b^1(\mathbb{R})$ then $\forall t > 0$*

(i) *for all $b > x_0$, $a \mapsto p(b, a; t) \in C^0([-\infty, b])$.*

(ii) *$M > x_1 > x_0$, $(b, s) \mapsto p_V(b, b; s) \in L^1([x_1, M] \times [0, t], dbds)$.*

(iii) *$(u, b) \mapsto H(u) \sup_{a \in [b-u, b]} p_V(b, a; t) \in L^1(\mathbb{R}^+ \times \mathbb{R}, dudb)$, $H(u) = \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} - u\Phi(u)$.*

We now can prove Theorem 3.7.

Proof of Theorem 3.7:

According to Proposition 3.8, for any t such that $(T_\Psi(h, t))_{h \rightarrow 0}$ converges,

$$f_\Psi(t) = \lim_{h \rightarrow 0} T_\Psi^1(h, t) \text{ where } T_\Psi^1(h, t) = 2\mathbb{E} \left(\Psi(M_t, M_t) \frac{1}{\sqrt{h}} H \left(\frac{M_t - X_t}{\sqrt{h}} \right) \right).$$

Using the density of the law of the pair (M_t, X_t)

$$T_\Psi^1(h, t) = 2 \int_{\mathbb{R}^2} \Psi(b, b) \frac{1}{\sqrt{h}} H \left(\frac{b-a}{\sqrt{h}} \right) p_V(b, a; t) dbda.$$

We perform the change of variable $u = \frac{b-a}{\sqrt{h}}$, then

$$(26) \quad T_\Psi^1(h, t) = 2 \int_{\mathbb{R}^2} \Psi(b, b) H(u) p_V(b, b - u\sqrt{h}; t) dudb.$$

Proposition 3.9 (i) (iii) allows to use Lebesgue dominated Theorem, let h go to 0:

$$f_{\Psi}(t) = 2 \int_{\mathbb{R}^+ \times \mathbb{R}} \Psi(b, b) H(u) p_V(b, b; t) du db.$$

From the definition of H (9)

$$\int_0^{\infty} 2H(u) du = 2 \int_0^{\infty} du \int_u^{\infty} (y - u) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy = \frac{1}{2};$$

and

$$f_{\Psi}(t) = \frac{1}{2} \int_{\mathbb{R}^+} \Psi(b, b) p_V(b, b; t) db = \frac{1}{2} \mathbb{E}_{\mathbb{P}} \left[\Psi(M_t, M_t) \frac{p_V(M_t, M_t; t)}{p_M(M_t, t)} \right].$$

As a conclusion, Theorem 1.3 proof is achieved with this proposition. •

3.4 Proposition 3.9 proof

According to Theorem 1.1 of [9], if $A = 1$ and B is C_b^1 for all $t > 0$, the joint law of the pair $V_t = (M_t, X_t)$ has a density p_V . The aim of this section is to specify the regularity of p_V closed to the set $\{(m, m), m > x_0\}$ and to prove Proposition 3.9. For sake of simplicity, we work in the case $A = 1$. We start with a global estimation on p_V .

Lemma 3.10 *Let $A = 1$ and $B \in C_b^1$ and p_W the density of (W_t^*, W_t) . For all $t > 0$ and all $(m, x) \in \mathbb{R}^2, m \neq x$*

$$(27) \quad p_V(m, x; t) \leq e^{[\mathcal{B}(x) - \mathcal{B}(x_0) + CT]} p_W(m - x_0, x - x_0; t) \leq e^{[\|B\| \|x - x_0\| + CT]} p_W(m - x_0, x - x_0; t).$$

where $C = \frac{1}{2} \sup_{u \in \mathbb{R}} |B'(u) + B^2(u)|$ and $\mathcal{B}(x) = \int_0^x B(u) du$.

Proof: Let Q be the probability measure defined in (14) by $\frac{dQ}{d\mathbb{P}}|_{\mathcal{F}_T} = L_T$. Then, under Q the law of $(X_t - x_0)_{t \geq 0}$ is the one of W_t under \mathbb{P} and using (15)

$$Z_t \leq e^{\mathcal{B}(X_t) - \mathcal{B}(x_0) + CT}.$$

Let F be a test function, namely non negative Borelian bounded. Then

$$\begin{aligned} \int F(m, x) p_V(m, x; t) dm dx &= \mathbb{E}_{\mathbb{P}}(F(M_t, X_t)) = \mathbb{E}_Q(Z_t F(M_t, X_t)) \\ &\leq E_{\mathbb{P}}[e^{[\mathcal{B}(x_0 + W_t) - \mathcal{B}(x_0) + CT]} F(x_0 + W_t^*, x_0 + W_t)] = \int F(x_0 + m, x_0 + x) e^{[\mathcal{B}(x_0 + x) - \mathcal{B}(x_0) + CT]} p_W(m, x; t) dm \end{aligned}$$

Then for all $t > 0$, for almost all (m, x) ,

$$p_V(m, x; t) \leq e^{[\mathcal{B}(x) - \mathcal{B}(x_0) + CT]} p_W(m - x_0, x - x_0; t).$$

Since p_W and p_V are continuous on the open set $\{(m, x), m \neq x\}$, the estimation (27) is true for all $t > 0, m \neq x$. •

The following proposition presents a decomposition of p_V more tractable for our goal : the regularity of p_V close to the diagonal $\{(m, m), m > x_0\}$.

Proposition 3.11 *We assume $A = 1$, $B \in C_b^1(\mathbb{R})$, recall $\mathcal{B}(x) = \int_0^x B(u)du$, $\mathcal{C} := -\frac{1}{2}B' - \frac{1}{2}B^2$; p_W (18) denotes the density of the law of the pair Brownian motion-its running maximum. For all $t > 0$, the density of the pair (M_t, X_t) satisfies*

$$p_V(b, a; t) = \sum_{i=1}^3 p_V^i(b, a, t),$$

where

$$\begin{aligned} p_V^1(b, a, t) &:= e^{\mathcal{B}(a) - \mathcal{B}(x_0)} p_W(b - x_0, a - x_0; t) \\ p_V^2(b, a, t) &:= \int_0^t E_{\mathbb{P}} [\mathcal{C}(X_s) e^{\mathcal{B}(a) - \mathcal{B}(X_s)} 1_{\{M_s \leq b\}} p_W(b - X_s, a - X_s; t - s)] ds \\ p_V^3(b, a, t) &:= 1_{\{b > a\}} \int_0^t \int_{-\infty}^b \mathcal{C}(x) e^{\mathcal{B}(a) - \mathcal{B}(x)} p_V(b, x; s) \int_{(a-x)^+}^{b-x} 2 \frac{2\beta - a + x}{\sqrt{2\pi(t-s)^3}} e^{-\frac{(2\beta - a + x)^2}{2(t-s)}} d\beta dx ds, \end{aligned}$$

or

$$1_{\{b > a\}} \int_0^t \int_{-\infty}^b \mathcal{C}(x) e^{\mathcal{B}(a) - \mathcal{B}(x)} p_V(b, x; s) \frac{e^{-\frac{(a-x)^2}{2(t-s)}} - e^{-\frac{(2b-a-x)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} dx ds.$$

Proof :

(i) Let F be a positive function, remark that

$$E_{\mathbb{P}}[F(M_t, X_t)] = E_Q[Z_t F(M_t, X_t)]$$

and that under Q , $X - x_0$ is a Brownian motion. Recall that Z (16) could be expressed as

$$(28) \quad Z_t = \exp(\mathcal{B}(X_t) - \mathcal{B}(x_0)) + \int_0^t \mathcal{C}(X_s) e^{\mathcal{B}(X_t) - \mathcal{B}(X_s)} Z_s ds$$

Then

$$E_{\mathbb{P}}[F(M_t, X_t)] = E_Q \left[\left(\exp(\mathcal{B}(X_t) - \mathcal{B}(x_0)) + \int_0^t \mathcal{C}(X_s) Z_s e^{\mathcal{B}(X_t) - \mathcal{B}(X_s)} ds \right) F(M_t, X_t) \right].$$

(ii) Using for $s \leq t$ $X_t = X_s + X_t - X_s$ and $M_t = \sup(M_s, X_s + \sup_{0 \leq u \leq t-s} X_{u+s} - X_s)$ and the independence under Q of $X_{\cdot+s} - X_s$ and \mathcal{F}_s ,

$$(29) \quad E_{\mathbb{P}}[F(M_t, X_t)] = \int_{\mathbb{R}^2} F(b, a) \exp(\mathcal{B}(a) - \mathcal{B}(x_0)) p_W(b - x_0, a - x_0; t) db da +$$

$$\int_0^t \int_{\mathbb{R}^2} E_Q[\exp(\mathcal{B}(X_s + \alpha)) \mathcal{C}(X_s)] Z_s e^{-\mathcal{B}(X_s)} F(\sup(M_s, X_s + \beta), X_s + \alpha) p_W(\beta, \alpha; t - s) d\beta da ds.$$

Under probability measure $\mathbb{P} = Z_s \cdot Q$ this last term is the integral on $[0, t] \times \mathbb{R}^2$ of

$$E_{\mathbb{P}}[\exp(\mathcal{B}(X_s + \alpha) - \mathcal{B}(X_s)) \mathcal{C}(X_s) F(\sup(M_s, X_s + \beta), X_s + \alpha)] p_W(\beta, \alpha; t - s).$$

Thus the expectation $E_{\mathbb{P}}[F(M_t, X_t)]$ satisfies

$$E_{\mathbb{P}}[F(M_t, X_t)] = \int_{\mathbb{R}^2} F(b, a) \exp(\mathcal{B}(a) - \mathcal{B}(x_0)) p_W(b - x_0, a - x_0; t) db da + \\ \int_0^t \int_{\mathbb{R}^2} E_{\mathbb{P}}[e^{\mathcal{B}(X_s + \alpha) - \mathcal{B}(X_s)} \mathcal{C}(X_s) F(\sup(M_s, X_s + \beta), X_s + \alpha)] p_W(\beta, \alpha; t - s) d\beta d\alpha ds.$$

(iii) We split the second term according to the subset $\{M_s \leq X_s + \beta\}$ and its complement:

$$I_1 = \int_0^t \int_{\mathbb{R}^2} E_{\mathbb{P}}[e^{\mathcal{B}(X_s + \alpha) - \mathcal{B}(X_s)} \mathcal{C}(X_s) F(X_s + \beta, X_s + \alpha) \mathbf{1}_{\{M_s \leq X_s + \beta\}}] p_W(\beta, \alpha; t - s) d\beta d\alpha ds, \\ I_2 = \int_0^t \int_{\mathbb{R}^2} E_{\mathbb{P}}[e^{\mathcal{B}(X_s + \alpha) - \mathcal{B}(X_s)} \mathcal{C}(X_s) F(M_s, X_s + \alpha) \mathbf{1}_{\{M_s > X_s + \beta\}}] p_W(\beta, \alpha; t - s) d\beta d\alpha ds.$$

In the first term I_1 , we perform the change of variable $b = X_s + \beta$, $a = X_s + \alpha$, so

$$(30) \quad I_1 = \int_0^t \int_{\mathbb{R}^2} E_{\mathbb{P}}[e^{\mathcal{B}(a) - \mathcal{B}(X_s)} \mathcal{C}(X_s) F(b, a) \mathbf{1}_{\{M_s \leq b\}}] p_W(b - X_s, a - X_s; t - s) db da ds.$$

Concerning I_2 , using $p_V(\cdot, \cdot; t)$ the density of the law under \mathbb{P} of the pair (M_s, X_s) ,

$$I_2 = \int_0^t \int_{\mathbb{R}^4} e^{\mathcal{B}(x + \alpha) - \mathcal{B}(x)} \mathcal{C}(x) F(m, x + \alpha) \mathbf{1}_{\{m > x + \beta\}} p_V(m, x; s) p_W(\beta, \alpha; t - s) d\beta d\alpha dm dx ds.$$

We operate the change of variable $b = m$, $a = x + \alpha$, so

$$I_2 = \int_0^t \int_{\mathbb{R}^4} e^{\mathcal{B}(a) - \mathcal{B}(x)} \mathcal{C}(x) F(b, a) \mathbf{1}_{\{b > x + \beta\}} p_V(b, x; s) p_W(\beta, a - x; t - s) d\beta db da dx ds.$$

Using the expression of p_W (18), and the constraints $\beta > 0$, $\beta > a - x$, $\beta < b - x$,

$$p_W(\beta, a - x; t - s) = 2 \frac{2\beta - a + x}{\sqrt{2\pi}(t - s)^3} e^{-\frac{(2\beta - a + x)^2}{2(t - s)}} \mathbf{1}_{\{\beta > 0, \beta > a\}},$$

$$(31) \quad I_2 = \int_0^t \int_{\mathbb{R}^3} e^{\mathcal{B}(a) - \mathcal{B}(x)} \mathcal{C}(x) F(b, a) \int_{(a-x)^+}^{b-x} 2 \frac{2\beta - a + x}{\sqrt{2\pi}(t - s)^3} e^{-\frac{(2\beta - a + x)^2}{2(t - s)}} d\beta p_V(b, x; s) db da dx ds.$$

Gathering (29), (30), (31) and using Tonelli Theorem, we get the result. •

To prove Proposition 3.9, we firstly prove some useful results for the Brownian motion.

Lemma 3.12 *Let $C_\theta := \sup_{x>0} (\frac{x}{\sqrt{t}})^\theta e^{-\frac{x^2}{4t}}$. For all $b > 0$, $t > 0$, $\varepsilon \geq 0$, $\alpha > 0$*

(i)

$$(32) \quad \sup_{a \in [-\infty, b]} p_W(b, a; t) \leq \frac{C_{1+\varepsilon}}{\sqrt{2\pi} t^{1-\varepsilon} b^\varepsilon} e^{-\frac{b^2}{4t}}.$$

(ii) $(b, t) \mapsto \sup_{a < b} e^{\alpha|a|} p_W(b, a; t) \in L^1(\mathbb{R}^+ \times [0, T], db dt)$.

(iii) For all $t \in [0, T]$ $(u, b) \mapsto H(u) \sup_{a \in [b-u, b]} e^{\alpha|a|} p_W(b, a; t) \in L^1(\mathbb{R}^+ \times \mathbb{R}^+, du db)$.

Proof: Recall $p_W(b, a; t) = 2 \frac{(2b-a)}{\sqrt{2\pi t^3}} e^{-\frac{(2b-a)^2}{2t}} \mathbf{1}_{\{b > \max(a, 0)\}}$.

(i) For any (b, a) such that $b > \max(a, 0)$, $\forall \varepsilon \geq 0$,

$$p_W(b, a; t) \leq \frac{C_1}{\sqrt{2\pi t}} e^{-\frac{(2b-a)^2}{4t}} \mathbf{1}_{\{b > \max(a, 0)\}}, \quad \text{and} \quad p_W(b, a; t) \leq \frac{C_{1+\varepsilon}}{\sqrt{2\pi t^{1-\varepsilon}}} (2b-a)^{-\varepsilon} e^{-\frac{(2b-a)^2}{4t}}$$

For $a \in]-\infty, b]$, $2b-a = b + b - a \geq b > 0$ and

$$\sup_{a \in]-\infty, b]} p_W(b, a; t) \leq \frac{C_1}{\sqrt{2\pi t}} e^{-\frac{b^2}{4t}}, \quad \sup_{a \in]-\infty, b]} p_W(b, a; t) \leq \frac{C_{1+\varepsilon}}{\sqrt{2\pi t^{1-\varepsilon} b^\varepsilon}} e^{-\frac{b^2}{4t}}$$

which is exactly (i).

(ii) Note that

$$(33) \quad \int_0^T \int_0^\infty e^{\alpha b} \frac{1}{\sqrt{2\pi t}} e^{-\frac{b^2}{4t}} db dt \leq \int_0^T \int_{-\infty}^\infty e^{\alpha b} \frac{1}{\sqrt{2\pi t}} e^{-\frac{b^2}{4t}} db dt = \int_0^T \frac{e^{4t\alpha^2}}{\sqrt{t}} dt < \infty.$$

Then, (33) and (32) with $\varepsilon = 0$, yield (ii).

(iii) Since for all $u > 0$, $0 \leq H(u) \leq \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}$ (Remark 3.1) (iii) is a consequence of (32) and (33). •

Proof of Proposition 3.9: Recall Lemma 3.10 with $c = \|B\|$:

$$p_V(b, a; t) \leq e^{[c|a-x_0|+CT]} p_W(b-x_0, a-x_0; t).$$

(ii) and (iii): Using Lemma 3.12 (ii) and (iii) with $\alpha = \|B\|$, p_V satisfies points (ii) and (iii). Moreover remark that, by definition, $p_V^3(b, b; t) = 0$.

(i) According to Proposition 3.11, we recall the decomposition of p_V in the sum of three terms $p_V^i(b, a; t)$, $i = 1, 2, 3$:

$$\begin{aligned} p_V^1(b, a, t) &= e^{\mathcal{B}(a)-\mathcal{B}(x_0)} p_W(b-x_0, a-x_0; t) \\ p_V^2(b, a, t) &= \int_0^t E_{\mathbb{P}} [\mathcal{C}(X_s) e^{\mathcal{B}(a)-\mathcal{B}(X_s)} \mathbf{1}_{\{M_s \leq b\}} p_W(b-X_s, a-X_s; t-s)] ds \\ p_V^3(b, a, t) &= \mathbf{1}_{\{b > a\}} \int_0^t \int_{-\infty}^b \mathcal{C}(x) e^{\mathcal{B}(a)-\mathcal{B}(x)} p_V(b, x; s) \left[\frac{e^{-\frac{(a-x)^2}{2(t-s)}} - e^{-\frac{(2b-a-x)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} \right] dx ds. \end{aligned}$$

The deal is now to prove Item (i) of Proposition 3.9 respectively for p_V^i $i = 1, 2, 3$ instead of p_V .

(1) Case of p_V^1 : Let b and t be fixed. Since \mathcal{B} is $\|B\|_\infty$ Lipschitz continuous and $a \mapsto p_W(b, a; t)$ defined in (18) is continuous on $] -\infty, b]$ then $\forall t > 0$, $b > x_0$, $a \rightarrow p_V^1(b, a, t)$ is continuous and

$$\lim_{a \rightarrow b, a < b} p_V^1(b, a, t) = p_V^1(b, b, t).$$

(2) Case of p_V^2 , using the change of probability it could be written as

$$p_V^2(b, a, t) = \int_0^t E_Q [Z_s \mathcal{C}(X_s) e^{\mathcal{B}(a)-\mathcal{B}(X_s)} \mathbf{1}_{\{M_s \leq b\}} p_W(b-X_s, a-X_s; t-s)] ds$$

where $X - x_0$ is now a Brownian motion. Since $a \mapsto p_W(\cdot, a; t)$ is continuous, then the integrand is also continuous with respect to a . It remains to bound this integrand.

Remark that using (15) and $\mathcal{C}(x) \leq \frac{1}{2}\|B'\|$,

$$Z_s e^{\mathcal{B}(a) - \mathcal{B}(X_s)} = \exp\left(\int_0^s \mathcal{C}(X_u) du + \mathcal{B}(a) - \mathcal{B}(x_0)\right) \leq e^{\|B\|\|a - x_0\| + \frac{1}{2}s\|B'\|}.$$

So the integrand above is bounded on $[0, t] \times \mathbb{R}$ (with respect to a multiplicative constant) as following

$$\exp[CT + \|B\|\|a - x_0\|] p_W(b - x, a - x; t - s) \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x - x_0)^2}{2s}}.$$

Using (32) Lemma 3.12 when $b - u \leq a < b$, this integrand is uniformly bounded by

$$\exp[CT + \|B\|(b + u + |x_0|)] \frac{C_{1+\varepsilon}}{\sqrt{2\pi}(t - s)^{1-\varepsilon}(b - x)^\varepsilon} e^{-\frac{(b-x)^2}{4(t-s)}} \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-x_0)^2}{2s}}.$$

With the Lebesgue dominated theorem $\lim_{a \rightarrow b, a < b} p_V^2(b, a, t) = p_V^2(b, b, t)$.

(3) Case of p_V^3 : since $a \rightarrow e^{\mathcal{B}(a) - \mathcal{B}(x_0)}$ is in $C^1(\mathbb{R})$, to prove item (i) we only need to study its factor in p_V^3 , which could be written as

$$1_{\{b > a\}} \left[\int_0^t \int_{-\infty}^b \mathcal{C}(x) e^{\mathcal{B}(x_0) - \mathcal{B}(x)} p_V(b, x; s) \left[\frac{e^{-\frac{(a-x)^2}{2(t-s)}} - e^{-\frac{(2b-a-x)^2}{2(t-s)}}}{\sqrt{2\pi}(t-s)} \right] dx ds \right].$$

Note that \mathcal{C} is bounded. We remark that

$$\frac{e^{-\frac{(a-x)^2}{2(t-s)}} - e^{-\frac{(2b-a-x)^2}{2(t-s)}}}{\sqrt{2\pi}(t-s)} = \int_{a-b}^{b-a} \frac{(b-x+u)e^{-\frac{(b-x+u)^2}{2(t-s)}}}{\sqrt{2\pi}(t-s)^3} du,$$

$$\text{so } p_V^3(b, a; t) = e^{\mathcal{B}(a) - \mathcal{B}(x_0)} \int_0^t ds \int_{-\infty}^b dx \int_{-(b-a)}^{b-a} P^3(b, u, x, s, t) du$$

where

$$P^3(b, u, x, s, t) := \mathcal{C}(x) e^{\mathcal{B}(x_0) - \mathcal{B}(x)} p_V(b, x; s) \frac{(b-x+u)e^{-\frac{(b-x+u)^2}{2(t-s)}}}{\sqrt{2\pi}(t-s)^3}.$$

We now prove that for any $b > x_0$

$$(34) \quad \int_0^t \int_{-\infty}^b \int_{-(b-a)}^{b-a} |P^3(b, u, x, s, t)| dx ds du < \infty,$$

indeed, $\forall b > x$, using (27) and (32)

$$(35) \quad p_V(b, x; s) e^{\mathcal{B}(x_0) - \mathcal{B}(x)} \leq e^{Ct} C_1 \frac{e^{-\frac{(b-x_0)^2}{4s}}}{\sqrt{2\pi s^2}} \leq e^{Ct} C_1^2 \frac{1}{\sqrt{s 2\pi (b-x_0)^2}},$$

and

$$\frac{(b-x+u)e^{-\frac{(b-x+u)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)^3}} \leq C_1 \frac{e^{-\frac{(b-x+u)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)^2}}$$

so the bound

$$|P^3(b, u, x, s, t)| \leq \text{const} \frac{1}{\sqrt{2\pi s(b-x_0)^2}} \frac{e^{-\frac{(b-x+u)^2}{4(t-s)}}}{\sqrt{2\pi(t-s)^2}}.$$

We operate the change of variable $y = b - x + u$ and integrate with respect to $y \in \mathbb{R}$

$$(36) \quad \int_0^t \int_{-\infty}^b |P^3(b, u, x, s, t)| dx ds \leq \text{const} \int_0^t \frac{1}{\sqrt{2\pi s(t-s)(b-x_0)^2}} ds = \text{const} \frac{1}{b-x_0} < \infty.$$

The right hand does not depend on u which belongs to any compact $[-M, M]$ $M > 0$, thus yields (34). So Fubini theorem yields

$$(37) \quad p_V^3(b, a; t) = e^{\mathcal{B}(a) - \mathcal{B}(x_0)} \int_{-(b-a)}^{b-a} du \int_0^t ds \int_{-\infty}^b P^3(b, u, x, s, t) dx.$$

This proves the continuity of $a \rightarrow p_V^3(b, a; t)$. •

4 Uniqueness Theorem

Theorem 4.1 *There exists a unique pair of functions (p, \tilde{p}) on $\mathbb{R}^2 \times \mathbb{R}^+$, respectively on $\mathbb{R} \times \mathbb{R}^+$, such that for all $T > 0$*

- (a) $p \in L^\infty([0, T]; L^1(\mathbb{R}^2))$,
- (b) $\tilde{p} \in L^1([0, T]; L^1(\mathbb{R}))$ and $\int_0^T (\int_{\mathbb{R}} \tilde{p}^2(m, s) dm)^{\frac{1}{2}} ds < \infty$.
- (c) there exists $\alpha > 0$, such that $\int_0^\infty \int_{\mathbb{R}} e^{-\alpha s} |\tilde{p}(m, s)| dm ds < \infty$,
- (d) there exists $f_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ such that for all $F \in C_b^2(\mathbb{R}^2, \mathbb{R})$,

$$(38) \quad \int_{\mathbb{R}^2} F(m, x) p(m, x; t) dm dx = \int_{\mathbb{R}} F(m, m) f_0(m) dm + \int_0^t \int_{\mathbb{R}^2} [B(x) \partial_2 F(m, x) + \frac{1}{2} \partial_{22}^2 F(m, x)] p(m, x; u) dm dx du + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \partial_1 F(m, m) \tilde{p}(m; u) dm du$$

where $B : \mathbb{R} \rightarrow \mathbb{R}$ is C_b^1 function.

As a corollary we prove in Section 4.4:

Corollary 4.2 *Let $V_0 = (X_0, X_0)$ and $f_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ be the law density of X_0 . The pair (p_V, \tilde{p}_V) , $\tilde{p}_V(m; t) = p_V(m, m; t)$ is the unique weak solution of the PDE (6) from Theorem 1.3 above.*

Proof of Theorem 4.1: Let two pairs (p_j, \tilde{p}_j) , $j = 1, 2$, satisfy $\tilde{p}_i(\cdot; 0) = f_0$ and the four properties of Theorem 4.1. The key of the proof is to show that these two pairs coincide, introducing $(p_1 - p_2, \tilde{p}_1 - \tilde{p}_2)$. So denote $q = p_1 - p_2$, $\tilde{q} = \tilde{p}_1 - \tilde{p}_2$. For any bounded function $H \in C^1(\mathbb{R})$ with compact support H' one study the function

$$(39) \quad q_H : (t, x) \rightarrow \int_x^\infty H(m)q(m, x; t)dm.$$

The proof is an application of Ball's Theorem [3] with $X = L^2(\mathbb{R})$, $A = \mathcal{L}^*$, $f(t) = (x \rightarrow G(t, x))$, where $G(t, x) = \frac{1}{2}H'(x)\tilde{p}(x; t)$ and the non bounded operator \mathcal{L}^* is defined on $H^{2,2}(\mathbb{R})$

$$\mathcal{L}^*(f) = -Bf' - B'f + \frac{1}{2}f'' ,$$

q_H will be the unique weak solution of the PDE:

$$(40) \quad q_H(x; t) = \int_0^t \left[\frac{1}{2}H'(x)q(x, x; s) + \mathcal{L}^*(q_H(\cdot; s))(x) \right] ds, \quad q_H(x; 0) = 0.$$

Below, Lemma 4.3 proves that \mathcal{L}^* is to be the associated to semi-group Q operator:

$$(41) \quad Q_t(f)(x) := \mathbb{E} \left(f(Y_t^x) \exp - \int_0^t B'(Y_u^x) du \right), \quad t \geq 0, \quad x \in \mathbb{R}, f \text{ bounded}$$

with density $d(\cdot, x; t)$, where the diffusion Y^x is solution of

$$(42) \quad dY_t^x = -B(Y_t^x)dt + dW_t, \quad Y_0^x = x.$$

Ball's Theorem needs to check the following:

- Operator \mathcal{L}^* is closed and the associated semi group Q is strongly continuous,
- $G \in L^1((0, T); L^2(\mathbb{R}))$,
- $q_H \in C((0, T); L^2(\mathbb{R}))$,
- $\forall v \in D(\mathcal{L}), t \rightarrow \langle q_H(t, \cdot), v \rangle$ absolutely continuous,
- $\frac{d}{dt} \langle q_H(t, \cdot), v \rangle = \langle q_H(t, \cdot), \mathcal{L}v \rangle + \langle G(t, \cdot), v \rangle$.

The last three items set that q_H is a "weak solution", cf. Ball's definition pages 1-2. So using Ball's Theorem, we could conclude that the unique weak solution of PDE (40) is $\int_0^t Q_{t-s}(G(\cdot, s))(x)ds$ meaning

$$q_H(x, t) = \frac{1}{2} \int_0^t \mathbb{E} \left(H'(Y_{t-s}^x) \tilde{p}(Y_{t-s}^x, s) \exp - \int_0^{t-s} B'(Y_u^x) du \right) ds, \quad t \geq 0, \quad x \in \mathbb{R}.$$

4.1 Definition and properties of the semi-group Q_t

Lemma 4.3 *Let $B \in C_b^1(\mathbb{R})$ and Y^x be solution of*

$$dY_t^x = -B(Y_t^x)dt + dW_t, \quad Y_0^x = x.$$

The operator $\mathcal{L}^(f) = -Bf' - B'f + \frac{1}{2}f''$ is closed and it is the generator of the semi-group Q :*

$$Q_t(f)(x) := \mathbb{E} \left(f(Y_t^x) \exp - \int_0^t B'(Y_u^x) du \right), \quad t \geq 0, \quad x \in \mathbb{R}, f \text{ bounded.}$$

Moreover Q is strongly continuous, and $Q_t(x; \cdot)$ is bounded and admits a density d : for any $f \in C_b(\mathbb{R})$,

$$Q_t f(x) = \int_{\mathbb{R}} f(y) d(x, y; t) dy$$

where d satisfies $\forall t > 0, \quad x, y \in \mathbb{R}$

$$e^{-\frac{3}{2}t\|B'\| - \frac{1}{2}\|B\|^2 t} e^{-\|B\|\|y-x\|} \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} \leq d(x, y; t) \leq e^{\frac{1}{2}\|B'\|t} e^{\|B\|\|y-x\|} \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}}.$$

Proof: According to Revuz-Yor (Proposition 3.10 du chapitre VIII page 343) [22], \mathcal{L}^* is the generator of the semi-groupe Q defined in the lemma.

Using Lunardi [19] the operator \mathcal{L}^* is "sectorial" (cf. page 34, definition page 71 and 73-74, and Theorem 3.1.3) thus the semi-group Q associated to operator \mathcal{L}^* on $L^2(\mathbb{R})$ is a strongly continuous semi-group of bounded linear operators.

Remark that the domain of the operator \mathcal{L}^* is $H^{2,2}(\mathbb{R})$, so it is dense in $L^2(\mathbb{R})$, \mathcal{L}^* is closed: $(\{h, \mathcal{L}^*(h)\}, h \in H^{2,2}(\mathbb{R}))$ is closed in $H^{2,2}(\mathbb{R}) \times L^2(\mathbb{R})$.

The existence of its density d is a standard consequence of the fact that Y^x is a Brownian diffusion with a compact support drift. We get these two bounds as above Lemma 3.10 using a change of probability measure such that under the new probability Y^x would be a Brownian motion starting from x , so Z satisfies (15) but with $-B$ instead of B , so Y^x instead of X :

$$(43) \quad Z_t = \exp[-\mathcal{B}(Y_t^x) + \mathcal{B}(x) + \frac{1}{2} \int_0^t B'(Y_s^x) ds - \frac{1}{2} \int_0^t B^2(Y_s^x) ds].$$

Let f be positive; $Q_t(f)(x) = \int_{\mathbb{R}} f(y) d(x, y; t) dy = \mathbb{E} \left(f(Y_t^x) \exp - \int_0^t B'(Y_u^x) du \right)$ admits the bounds

$$e^{-\|B'\|t} E[f(Y_t^x)] \leq \int_{\mathbb{R}} f(y) d(x, t; t) dy \leq e^{\|B'\|t} E[f(Y_t^x)].$$

On another hand, $E_{\mathbb{P}}[f(Y_t^x)] = E_Q[Z_t f(Y_t^x)]$ and one use the bounds for Z_t :

$$\exp[-\|B\|\|Y_t^x - x\| - \frac{1}{2}t\|B'\| - \frac{1}{2}\|B\|^2 t] \leq Z_t \leq \exp[\|B\|\|Y_t^x - x\| + \frac{1}{2}t\|B'\|].$$

So, for any positive f with a compact support, we get the following bounds for $\int_{\mathbb{R}} f(y) d(x, y; t) dy$, that ends the proof of the lemma:

$$\left[\int_{\mathbb{R}} \exp[-\|B\|\|u\| - \frac{3}{2}t\|B'\| - \frac{1}{2}\|B\|^2 t] \frac{f(x+u)}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}} du, \int_{\mathbb{R}} \exp[\|B\|\|u\| + \frac{1}{2}t\|B'\|] \frac{f(x+u)}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}} du \right].$$

4.2 Assumptions Ball's theorem verification, Items 2,3,4

Along this section q could be not the difference between p_1 and p_2 , but any p satisfying properties in Theorem 4.1.

4.2.1 Item 2

Lemma 4.4 *The application $(m, s) \rightarrow G(m; s) = \frac{1}{2}H'(m)\tilde{q}(m; s)$ belongs to $L^1((0, T); L^2(\mathbb{R}))$.*

Proof: Let us recall $G : (t, x) \rightarrow \frac{1}{2}H'(x)\tilde{q}(x; t)$.

(i) Since the support of H' is compact, for any t , the support of $G(t, \cdot)$ is too compact. Using Property (b) of Theorem 4.1, $G(t, \cdot)$ belongs to $L^2(\mathbb{R})$ almost surely on $(0, T]$.

(ii) Then one looks for the convergence of the following integral:

$$\int_0^T \left(\int (H'(x)\tilde{q}(x; t))^2 dx \right)^{\frac{1}{2}} dt.$$

Since H' support is compact, H' is bounded so the bound of the integrand with respect to time

$$t \rightarrow \|H'\|_\infty \left(\int \tilde{q}^2(x; t) dx \right)^{\frac{1}{2}}.$$

Using Assumption (b) in Theorem 4.1 $\int_0^T (\int_{\mathbb{R}} \tilde{q}^2(x, s) dx)^{\frac{1}{2}} ds < \infty$ meaning that the norm $\|G(\cdot, t)\|_2$ is integrable on $(0, T]$. •

4.2.2 q_H defined in (39) belongs to $C([0, T], L^2(\mathbb{R}))$

Property (a) of Theorem 4.1 for any $t \geq 0$ proves that $p_H(\cdot; t) \in L^1(\mathbb{R})$, so one can consider its Fourier transform.

Lemma 4.5 *For any $t \in (0, T]$, the Fourier transform of p_H ,*

$$\hat{p}_H(x, t) = \int_{\mathbb{R}} e^{ix\xi} p_H(\xi, t) d\xi \text{ satisfies}$$

$$\begin{aligned} \hat{p}_H(x, t) &= e^{-\frac{1}{2}x^2 t} \hat{p}_H(x, 0) \\ &+ \int_0^t e^{-\frac{1}{2}x^2(t-u)} \left[\int_{\mathbb{R}^2} i\xi B(x) e^{i\xi x} H(m) p(m, x; u) dm dx + \frac{1}{2} \int_{\mathbb{R}} H'(m) e^{i\xi x} H(m) \tilde{p}(m; u) dm \right] du. \end{aligned}$$

Proof: One applies the relation (38) to the real and imaginary parts of the function $F(m, \xi) = H(m)e^{ix\xi}$ what one integrates between 0 et t :

$$\hat{p}_H(x, t) - \hat{p}_H(x, 0) = \int_0^t \int_{\mathbb{R}^2} [ixB(\xi) - \frac{1}{2}x^2] e^{i\xi x} H(m) p(m, \xi; u) dm d\xi du + \frac{1}{2} \int_0^t \int_{\mathbb{R}} H'(m) e^{imx} \tilde{p}(m; u) dm du.$$

So for any real number x the application $t \rightarrow \hat{p}_H(x, t)$ is solution of the differential equation

$$y'(t) = -\frac{1}{2}x^2 y(t) + \int_{\mathbb{R}^2} ixB(\xi) e^{i\xi x} H(m) p(m, \xi; t) dm d\xi + \frac{1}{2} \int_{\mathbb{R}} H'(m) e^{imx} \tilde{p}(m; t) dm,$$

and using Duhamel's principle

$$(44) \quad \hat{p}_H(x, t) = e^{-\frac{1}{2}x^2 t} \hat{p}_H(x, 0) + \int_0^t e^{-\frac{1}{2}x^2(t-u)} \int_{\mathbb{R}^2} ixB(\xi)e^{i\xi x} H(m)p(m, \xi; u) dmd\xi du + \frac{1}{2} \int_0^t e^{-\frac{1}{2}x^2(t-u)} \int_{\mathbb{R}} H'(m)e^{imx} \tilde{p}(m; u) dmdu.$$

Lemma 4.6 *The application on $[0, T]$ $\phi_1(\cdot, \cdot)$:*

$$t \rightarrow \left(x \rightarrow \int_0^t e^{-\frac{1}{2}x^2(t-u)} \int_{\mathbb{R}^2} ixB(\xi)e^{i\xi x} H(m)p(m, \xi; t) dmd\xi du \right)$$

is continuous in $L^2(\mathbb{R})$.

Proof: Using H and B bounded and Property (a) of Theorem 4.1, there exists a constant C_p such that for any u , $\int_{\mathbb{R}^2} |p(m, x; u)| dmdx \leq C_p$ so the bound, x being fixed:

$$(45) \quad \left| \int_{\mathbb{R}^2} ixB(\xi)e^{i\xi x} H(m)p(m, \xi; t) dmd\xi \right| \leq C_p |x|.$$

The increment $\phi_1(x, t) - \phi_1(x, s)$ is the sum of two terms:

$$\begin{aligned} & \int_0^s [e^{-\frac{1}{2}x^2(t-u)} - e^{-\frac{1}{2}x^2(s-u)}] \int_{\mathbb{R}^2} ixB(\xi)e^{i\xi x} H(m)p(m, \xi; t) dmd\xi du \\ & + \int_s^t e^{-\frac{1}{2}x^2(t-u)} \int_{\mathbb{R}^2} ixB(\xi)e^{i\xi x} H(m)p(m, \xi; t) dmd\xi du. \end{aligned}$$

In both terms appears the factor (45) so:

$$|\phi_1(x, t) - \phi_1(x, s)| \leq C_p |x| \left(\int_0^s [e^{-\frac{1}{2}x^2(s-u)} - e^{-\frac{1}{2}x^2(t-u)}] du + \int_s^t e^{-\frac{1}{2}x^2(t-u)} du \right).$$

On the set $\{|x| \leq 1\}$, the bound is $C_p [T(1 - e^{-\frac{1}{2}x^2(t-s)}) + (t-s)] \leq C_p (t-s)(T/2 + 1)$.

On the set $\{|x| \geq 1\}$, we operate the two integrals with respect to du , so the bound:

$$\begin{aligned} & C_p |x| \frac{2}{x^2} [(1 - e^{-\frac{1}{2}x^2(t-s)})(1 - e^{-\frac{1}{2}x^2 s}) + (1 - e^{-\frac{1}{2}x^2(t-s)})] \\ & \leq \frac{4C_p}{|x|} (1 - e^{-\frac{1}{2}x^2(t-s)}) \leq \frac{4C_p}{|x|} (x^2(t-s))^\eta \end{aligned}$$

$\forall \eta > 0$ on $\{|x| \geq 1\}$, which belongs to $L^2(\mathbb{R})$ as soon as $4\eta - 2 < -1$, meaning $0 < \eta < 1/4$.

Lemma 4.7 *The application on $[0, T]$ $\phi_2(x, \cdot) : t \rightarrow \int_0^t e^{-\frac{1}{2}x^2(t-u)} \int_{\mathbb{R}} H'(m)e^{imx} \tilde{p}(m; u) dmdu$ is continuous in $L^2(\mathbb{R})$.*

Proof: The function H' is bounded and Assumption (b) for $j = 1, 2$,

$\int_0^T [\int_{\mathbb{R}} |\tilde{p}(m; u)|^j dm]^{1/j} du < \infty$, so almost surely for $u \in [0, T]$ the function $\psi(\cdot, u) : m \rightarrow H'(m)\tilde{p}(m; u)$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Its Fourier transform, $\hat{\psi}(\cdot, u) : x \rightarrow \int_K e^{ixm} H'(m)\tilde{p}(m; u) dm$ belongs to $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ and its $L^{j^*}(\mathbb{R})$ -norm ($j^* = \infty, 2$ is the conjugate exponent of j) is bounded by

$$(46) \quad \|\hat{\psi}(\cdot; u)\|_{L^{j^*}(\mathbb{R})} \leq \|H'\|_\infty \|\tilde{p}(\cdot; u)\|_{L^j(\mathbb{R})}, \quad j = 1, 2$$

One operates an integration by parts ($x \neq 0$) on the function $\hat{\psi}$ thus

$$\phi_2(x, t) = \left[e^{-\frac{1}{2}x(t-u)} \int_0^u \hat{\psi}(x; r) dr \right]_0^t + \frac{1}{2}x^2 \int_0^t e^{-\frac{1}{2}x^2(t-u)} \int_0^u \hat{\psi}(x; r) dr du.$$

Using Fubini Theorem, almost surely on $\mathbb{R} \times [0, T]$ one gets

$$(47) \quad \begin{aligned} \phi_2(x, t) &= \int_0^t \hat{\psi}(x; r) dr + \frac{1}{2}x^2 \int_0^t \hat{\psi}(x; r) \int_r^t e^{-\frac{1}{2}x^2(t-u)} du dr \\ &= \int_0^t \hat{\psi}(x; r) dr + \int_0^t \hat{\psi}(x; r) (1 - e^{-\frac{1}{2}x^2(t-r)}) dr, \end{aligned}$$

$$(48) \quad \text{or} \quad \phi_2(x, t) = 2 \int_0^t \hat{\psi}(x; r) dr - e^{\frac{1}{2}x^2 t} \int_0^t \hat{\psi}(x; r) e^{\frac{1}{2}x^2 r} dr.$$

The relation (48) proves that $t \rightarrow \phi_2(x, t)$ is continuous, x being fixed

Bounding $1 - e^{-a}$ by 1 when $a \geq 0$, the relation (47) gets

$$\left| \sup_{t \in [0, T]} \phi_2(x, t) \right| \leq 2 \int_0^T |\hat{\psi}(x; r)| dr.$$

The function $x \rightarrow \int_0^T |\hat{\psi}(x; r)| dr \in L^2(\mathbb{R})$ since

$$\int_{\mathbb{R}} \left[\int_0^T |\hat{\psi}(x; r)| dr \right]^2 dx = \int_{\mathbb{R}} \int_{[0, T]^2} |\hat{\psi}(x; r)| |\hat{\psi}(x; u)| dr du dx.$$

Using Fubini Theorem then Cauchy-Schwartz inequality, yields

$$\begin{aligned} &\int_{\mathbb{R}} \left[\int_0^T |\hat{\psi}(x; r)| dr \right]^2 dx \leq \int_{[0, T]^2} \int_{\mathbb{R}} |\hat{\psi}(x; r)| |\hat{\psi}(x; u)| dx dr du \\ &\leq \int_{[0, T]^2} \sqrt{\int_{\mathbb{R}} |\hat{\psi}(x; r)|^2 dx} \sqrt{\int_{\mathbb{R}} |\hat{\psi}(x; u)|^2 dx} dr du \\ &= \left(\int_{[0, T]} \|\hat{\psi}(\cdot; r)\|_{L^2(\mathbb{R})}^2 dr \right) \leq \left(\|H'\|_\infty \int_0^T \|\tilde{p}(\cdot; u)\|_{L^2(\mathbb{R})}^2 du \right)^2 \end{aligned}$$

using (46) to insure the last bound; then the continuity is a consequence of Lebesgue dominated convergence Theorem. •

Lemma 4.8 *The application on $[0, T]$ $t \rightarrow \hat{p}_H(x; t)$ is continuous in $L^2(\mathbb{R})$.*

Proof: Using the two lemmas above, one has to deal with the L^2 -norm of the first term: $e^{-\frac{1}{2}x^2t}\hat{p}_H(x; 0)$ in $\hat{p}_H(x; t)$, inside the expression given in Lemma 4.5, since the other terms are continuous for the $L^2(\mathbb{R})$ topology.

Firstly $m \rightarrow p_H(m, 0) = H(m)f_0(m) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ since $f_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and H is bounded. Thus $\hat{p}_H(x, 0) \in L^2(\mathbb{R})$ and $|e^{-\frac{1}{2}x^2t}\hat{p}_H(x, 0)| \leq |\hat{p}_H(x, 0)|$; x being fixed, $t \rightarrow e^{-\frac{1}{2}x^2t}\hat{p}_H(x, 0)$ is continuous almost surely for $x \in \mathbb{R}$. The continuity in $L^2(\mathbb{R})$ is a consequence of Lebesgue dominated convergence Theorem. \bullet

4.2.3 Absolute continuity and PDE whose solution is q_H

Lemma 4.9 *For any $v \in D(\mathcal{L})$, $t \rightarrow \langle q_H(t, \cdot), v \rangle$ is absolutely continuous and its derivative is*

$$t \rightarrow \langle \mathcal{L}(v), q_H(\cdot; t) \rangle_{L^2(\mathbb{R})} + \langle v, \frac{1}{2}H'\tilde{p}(\cdot; s) \rangle_{L^2(\mathbb{R})}$$

Proof: Let $v \in C_b^2(\mathbb{R})$ which is a dense subset of $D(\mathcal{L})$. The scalar product $\langle q_H(t, \cdot), v \rangle$ actually is the difference between two integrals

$$\int_{\mathbb{R}} v(x)q_H(x; t)dx = \int_{\mathbb{R}} v(x) \int_K H(m)q(m, x; t)dmdx,$$

where $q = p_1 - p_2$ satisfies Properties in Theorem 4.1 with $F(m, x) = H(m)v(x)$; $i = 1, 2$:

$$(49) \quad \int p_i(m, x; t)H(m)v(x)dmdx = \int H(m)v(m)f_0(m)dm + \\ + \int_0^t \int \left[B(x)v'(x) + \frac{1}{2}v''(x) \right] H(m)p_i(m, x; s)dmdxds + \frac{1}{2} \int_0^t \int_K H'(m)v(m)\tilde{p}_i(m; s)dmds.$$

Since $\Phi = Hv$ with a bounded H such that support of H' is compact and $v \in C_b^2$ Fubini Theorem is applied to left hand:

$$\langle q_H(t, \cdot), v \rangle = \int q(m, x; t)H(m)v(x)dmdx.$$

Right hand is absolutely continuous:

- there exists a constant c such that the integrand on $\mathbb{R} \times \mathbb{R}$ satisfies:

$$\left| [B(x)v'(x) + \frac{1}{2}v''(x)]H(m) \right| |q(m, x; s)| \leq c|q(m, x; s)|$$

and this bound has the good integrability properties (a) in Theorem 4.1,

- the integrand on K , the compact support of H' , is bounded by $\tilde{p}_i(m; s)$ which satisfies Property (c).

Let now v be in $H^{2,2}$, domain of the operator \mathcal{L}^* ; there exists a series $v_n \in C_b^2$ going to v in $H^{2,2}$; for any n , v_n satisfies the equality (49), so v satisfies the same, n going to infinity. Indeed, one observes that the following applications defined on $C^2(\mathbb{R})$ are continuous with respect to the norm $\|\cdot\|_{H^{2,2}}$, using that q_H belongs to $C([0, T], L^2(\mathbb{R}))$:

$$v \rightarrow \int_{\mathbb{R}} v(x)q_H(t, x)dx ; v \rightarrow \int_0^t \int_{\mathbb{R}} [B(x)v'(x) + \frac{1}{2}v''(x)]q_H(x; s)dxds ; v \rightarrow \int_0^t \int_{\mathbb{R}} v(m)H'(m)\tilde{q}(m, s)dmds.$$

Then using $t \mapsto q_H(\cdot; t)$ is in $C([0, T], L^2(\mathbb{R}))$ so bounded in $L^2(\mathbb{R})$:

$$\sup_{t \in [0, T]} \langle \mathcal{L}(v); q_H(x, t) \rangle_{L^2(\mathbb{R})} < \infty$$

$$\text{thus } t \rightarrow \int_0^t \int_{\mathbb{R}} [B(x)v'(x) + \frac{1}{2}v''(x)]q_H(x; s)dx ds$$

is absolutely continuous with derivative given by $s \rightarrow \int_{\mathbb{R}} [B(x)v'(x) + \frac{1}{2}v''(x)]q_H(x; s)dx$.
Moreover since $\forall s$

$$\left| \int_K v(m)H'(m)\tilde{q}(m, s)dm \right| \leq \|H'\|_{\infty} \|v\|_{L^2(\mathbb{R})} \|\tilde{q}(\cdot; s)\|_{L^2(\mathbb{R})}$$

(where K is H' support) using Assumption (b) $\int_0^T \|\tilde{q}(\cdot; s)\|_{L^2(\mathbb{R})} ds < \infty$:

$$t \rightarrow \int_0^t \int_K v(m)H'(m)\tilde{q}(m, s)dmds$$

is absolutely continuous with derivative given by $s \rightarrow \int_K v(m)H'(m)\tilde{q}(m, s)dm$. •

Finally using Equation (49) applied to $q = p_1 - p_2$, duality and Fubini theorem, one gets the PDE

$$(50) \quad q_H(x; t) = \int_0^t \left[\frac{1}{2}H'(x)q(x, x; s) - B'(x)q_H(x; s) - B(x)\partial_1 q_H(x; s) + \frac{1}{2}\partial_{11}^2 q_H(x; s) \right] ds, \quad q_H(x; 0) = 0$$

which admits q_H as unique weak solution.

As a conclusion, the following is proved:

Proposition 4.10 *Equation (40) admits a unique solution in $C([0, T], L^2(\mathbb{R}))$. Moreover this solution has the following mild representation*

$$q_H(x; t) = \int_0^t \int_{\mathbb{R}} \frac{1}{2}H'(y)\tilde{q}(y; s)d(y, x; t-s)dy ds$$

where d is the density of semi group Q defined in Lemma 4.3.

Lemmas 4.4 to 4.9 prove that q_H and G fulfill all assumptions of Ball's Theorem [3].

4.3 Proof of Uniqueness Theorem 4.1

Proof: The proof has five steps.

Step 1. Let x be fixed and $H \in C_b^1$ with a compact support, included in $(-\infty, x)$.
By definition, for any t , $q_H(x; t) = \int_{x < m} H(m)q(m, x; t)dm = 0$.

Step 2. One uses Proposition 4.10 :

$$q_H(x; t) = \int_0^t \int_{\mathbb{R}} \frac{1}{2}H'(y)\tilde{q}(y; s)d(y, x; t-s)dy ds$$

where y belongs to compact support of H . Using Assumption (b) in Theorem 4.1 yields:

$$\int_0^T \left(\int_{\mathbb{R}} \tilde{q}^2(y; t) dy \right)^{\frac{1}{2}} dt < \infty.$$

The integral of the product $q_H(x; t) \cdot e^{-\beta t}$ with respect to dt , with $\beta > \beta_0 := \sup(\alpha, 2\|B\|^2 + \frac{1}{2}\|B'\|)$ satisfies:

$$0 = \int_0^\infty e^{-\beta t} q_H(x; t) dt = \int_0^\infty e^{-\alpha t} \int_0^t \int_{\mathbb{R}} \frac{1}{2} H'(y) \tilde{q}(y; s) d(y, x; t-s) dy ds dt.$$

Let $u = t - s$ and $e^{-\beta t} = e^{\beta s} e^{-\beta(t-s)}$, then one applies Fubini Theorem using the bound of density d (Lemma 4.3) $d(y, x; u) \leq e^{\frac{1}{2}\|B'\|u} e^{\|B\||y-x|} \frac{e^{-(x-y)^2/2u}}{\sqrt{2\pi u}}$. The integrand is bounded by

$$1_{y \in K} e^{-\beta s} e^{-\beta u} \|H'\| |\tilde{q}(y; s)| e^{\frac{1}{2}\|B'\|u} e^{\|B\||y-x|} \frac{e^{-(x-y)^2/2u}}{\sqrt{2\pi u}}.$$

On one hand by hypothesis $\int e^{-\alpha s} \int |\tilde{q}(y; s)| dy ds < \infty$, so is $\int e^{-\beta s} \int |\tilde{q}(y; s)| dy ds < \infty$. On the other hand one has to bound for any $y \in K$,

$$1_{y \in K} e^{\|B\||y-x|} \|H'\| e^{-\beta u} e^{\frac{1}{2}\|B'\|u} \frac{e^{-(x-y)^2/2u}}{\sqrt{2\pi u}} \leq C e^{-\beta u} e^{\|B\||x|} e^{\frac{1}{2}\|B'\|u} \frac{1}{\sqrt{2\pi u}}.$$

This bound is integrable on $[0, s]$ since $\beta > \frac{1}{2}\|B'\|$. Using Fubini Theorem yields:

$$0 = \int_K H'(y) \int \int e^{-\beta s} e^{-\beta u} \tilde{q}(y; s) d(x, y; u) ds du dy.$$

So for any $(\beta > \beta_0, x)$ there exists a constant $C_{\beta, x}$ such that almost surely in y

$$(51) \quad C_{\beta, x} = \int \int e^{-\beta s} e^{-\beta u} \tilde{q}(y; s) d(x, y; u) ds du.$$

This equality could be integrated for $y \in [-N-1, -N]$:

$$C_{\beta, x} = \int_{-N-1}^{-N} \int \int e^{-\beta s} e^{-\beta u} \tilde{q}(y; s) d(x, y; u) ds du dy.$$

When $N \rightarrow \infty$, $1_{[-N-1, -N]} d(x, t; u)$ goes almost surely to 0, and since x is fixed, let $N \geq 2x$. With $|y| \leq N+1$, $x-y \geq N/2$, using the bound in Lemma 4.3 yields

$$d(x, y; u) \leq e^{\|B\||x|} e^{\frac{1}{2}\|B'\|u + (N+1)\|B\|} \frac{e^{-N^2/8u}}{\sqrt{2\pi u}}$$

so

$$d(x, y; u) \leq e^{\|B\||x|} e^{\frac{1}{2}\|B'\|u} \frac{e^{-(N-4\frac{N+1}{N}\|B\|u)^2/8u}}{\sqrt{2\pi u}} e^{2(\frac{N+1}{N})^2\|B\|^2 u}.$$

Since $\beta > \sup(\alpha, 2\|B\|^2 + \frac{1}{2}\|B'\|)$, with the bound of $e^{-\beta u} d(x, y; u)$ Dominated Convergence Theorem could be applied so for any (β, x) , $C_{\beta, x} = 0$.

Step 3. Let $\beta > 0$, in $C_{\beta, x}$ definition, we use Fubini Theorem and the change of variable $u = t - s$ so

$$C_{\beta, x} = 0 = \int_{\mathbb{R}^+} e^{-\beta t} \int_0^t \tilde{q}(y; s) d(y, x; t-s) ds dt, \quad \text{almost surely for } y \in \mathbb{R}.$$

Recall that $\tilde{q} = \tilde{p}_1 - \tilde{p}_2$ so almost surely for $y \in \mathbb{R}$

$$\int_{\mathbb{R}^+} e^{-\beta t} \int_0^t \tilde{p}_1(y; s) d(y, x; t-s) ds dt = \int_{\mathbb{R}^+} e^{-\beta t} \int_0^t \tilde{p}_2(y, y; s) d(y, x; t-s) ds dt.$$

Once again we use $e^{-\beta t} = e^{-\beta s} e^{-\beta(t-s)}$ and Tonelli Theorem on both hands:

$$\int_{\mathbb{R}^+} \int_0^t e^{-\beta s} \tilde{p}_1(y; s) e^{-\beta(t-s)} d(y, x; t-s) ds dt = \int_{\mathbb{R}^+} \int_0^t e^{-\beta s} \tilde{p}_2(y; s) e^{-\beta(t-s)} d(y, x; t-s) ds dt.$$

The change of variable $u = t - s$ yields

$$\begin{aligned} \int_{\mathbb{R}^+} e^{-\beta s} p_1(y, y; s) ds \int_0^t e^{-\beta u} d(y, x; u) du &= \int_{\mathbb{R}^+} e^{-\beta s} p_2(y, y; s) ds \int_0^t e^{-\beta u} d(y, x; u) du \\ \text{so } 0 &= \int_{\mathbb{R}^+} e^{-\beta s} \tilde{q}(y; s) ds \int_0^t e^{-\beta u} d(y, x; u) du. \end{aligned}$$

Using the low bound Lemma 4.3

$$e^{-\frac{3}{2}t\|B'\| - \frac{1}{2}\|B\|^2 t} e^{-\|B\||y-x|} \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} \leq d(x, y; t)$$

where y belongs to the compact support of $H \subset (-\infty, x)$, so $x - y > 0$. So this low bound is integrable on $[0, t]$ and $\int_0^t e^{-\beta u} d(y, x; u) du > 0$ so for any $\beta > 0$,

$$(52) \quad 0 = \int_{\mathbb{R}^+} e^{-\beta s} \tilde{q}(y; s) ds, \quad \text{almost surely for } y \in \mathbb{R}.$$

Step 4. Let $\alpha > 0$ from Property (c) and define the function g_y as

$$g_y : z \rightarrow \int_{\mathbb{R}^+} e^{zs} \tilde{q}(y; s) ds.$$

Almost surely for $y \in \mathbb{R}$, the function g_y is holomorphic on the complex domain $\Omega_\alpha := \{z : \Re(z) < -\alpha\}$: to prove this, one uses exercise 16 page 229 in [24] on the domain Ω_α with $\varphi(z, s) = e^{zs}$ and the measure $\mu(ds) = \tilde{q}(y; s) ds$.

Let (β_n) be a series of separate real numbers going to β in Ω_α and the associated set $A_n := \{y : \text{such that } g_y \text{ is holomorphic and } \int_0^\infty e^{-\beta_n s} \tilde{q}(y; s) ds = 0\}$. The complementary set $A_n^c = \{y : g_y \text{ is not holomorphic}\} \cup \{\int_0^\infty e^{-\beta_n s} \tilde{q}(y; s) ds \neq 0\}$ is negligible using both $g_y \in H(\Omega_\alpha)$ dy almost surely and (52) Step3. So $\cup_n A_n^c$ is negligible. Let $y \in \cap_n A_n$, the set $\{-\beta_n, n \geq 0\}$ is included in the set of g_y zeros, $\beta_n \rightarrow \beta$ so, $-\beta$ is limit point of g_y . So g_y is identically null using the corollary of Theorem 10.18 in Rudin [24].

As a particular case for any $a < -\alpha$ and $b \in \mathbb{R}$:

$$\int_{\mathbb{R}^+} e^{ibs} e^{-as} \tilde{q}(y; s) ds = 0.$$

This means that the Fourier transform of $s \rightarrow e^{-as} \tilde{q}(y; s)$ is null, so $\tilde{q}(y; s) = 0$ dy almost surely.

Step 5. Finally from $\tilde{q}(y; s) = 0$ and Duhamel principle stated in Proposition 4.10 one deduces for any t , any x and any compact support function H : $q_H(x; t) = 0$. By definition of q_H

$$0 = q_H(x; t) = \int H(m) q(m, x; t) dm,$$

thus $q(m, x; t) = 0$, and $p_1(m, x; t) = p_2(m, x; t)$ and the uniqueness is proved. •

4.4 Corollary 4.2 proof

We need to check that the pair (p_V, \tilde{p}_V) (where $\tilde{p}_V(m; t) = p_V(m, m; t)$) satisfies all Theorem 4.1 properties.

First of all let $(\mu(dm, dx; t))_{t \geq 0}$ be the family of probability measures such that $\mu(\cdot, \cdot; 0) = \delta_{m=x} f_0(x) dx$. So for any $t > 0$, $\mu(\cdot, \cdot; t)$ admits a density with respect to Lebesgue measure, denoted p_V , with support $\Delta = \{(m, x) \in \mathbb{R}^2 : x < m\}$ and which satisfies

$$p_V(m, x; t) = \int_{\mathbb{R}} p(m, x; t, x_0) \mu_0(dx_0).$$

Here $p(\cdot, \cdot; t, x_0)$ is the density law of the couple (M_t, X_t) when $X_0 = x_0$ and $\tilde{p}_V(m; t) = p_V(m, m; t)$. Below the majoration (27) Lemma 3.10 will be often useful:

$$(53) \quad p(m, x; t, x_0) \leq C e^{\|B\| |x-x_0|} p_W(m-x_0, x-x_0; t) = C e^{\|B\| |x-x_0|} \frac{2m-x-x_0}{\sqrt{2\pi t^3}} e^{-\frac{(2m-x-x_0)^2}{2t}}.$$

4.4.1 $p_V \in L^\infty([0, T]; L^1(\mathbb{R}^2))$

This property is a direct consequence from $p_V(\cdot, \cdot; t)$ being the law density of a probability measure, so it is positive, its integral on \mathbb{R}^2 is 1, bounded so integrable on $[0, T]$.

4.4.2 $\tilde{p}_V \in L^1([0, T]; L^i(\mathbb{R})), i = 1, 2$

(i) $i = 1$: For any t the function $\tilde{p}_V(\cdot; t)$ is positive; using (53) it is bounded by the integral

$$\int_{m \geq x_0} C e^{\|B\| |m-x_0|} \frac{m-x_0}{\sqrt{2\pi t^3}} e^{-\frac{(m-x_0)^2}{2t}} f_0(x_0) dx_0.$$

Any factor of the integrand is positive, one only needs to check the integrability on $[0, T] \times \mathbb{R}^2$, on \mathbb{R}^2 one integrates on $\{m \geq x_0\}$:

$$(54) \quad \int_0^T \int_{\{m \geq x_0\}} e^{\|B\| |m-x_0|} \frac{m-x_0}{\sqrt{2\pi s^3}} e^{-\frac{(m-x_0)^2}{2s}} f_0(x_0) dx_0 dm ds.$$

One operates the change of variable $u = \frac{m-x_0}{\sqrt{s}}$

$$\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}^+} e^{\|B\| u \sqrt{s}} \frac{u}{\sqrt{2\pi s}} e^{-\frac{u^2}{2}} f_0(x_0) dx_0 du ds = \int_{\mathbb{R}} f_0(x_0) dx_0 \int_0^T \frac{ds}{\sqrt{2\pi s}} \int_{\mathbb{R}^+} e^{\|B\| u \sqrt{s}} u e^{-\frac{u^2}{2}} du < \infty$$

using $e^{\|B\| u \sqrt{s}} \leq e^{\|B\| u \sqrt{T}}$ and the three integrals are finite.

(ii) $i = 2$: One has to look for the integrability on $[0, T]$ of

$$s \rightarrow \sqrt{\int_{\mathbb{R}} \tilde{p}_V^2(m; s) dm} \leq \sqrt{\int_{\mathbb{R}} \left(\int_{\{m \geq x_0\}} e^{\|B\| |m-x_0|} \frac{m-x_0}{\sqrt{2\pi s^3}} e^{-\frac{(m-x_0)^2}{2s}} f_0(x_0) dx_0 \right)^2 dm}.$$

Using f_0 is a density measure probability and $\|f_0\|_1 = 1$ and Cauchy-Schwartz inequality yields

$$\left(\int_{\{m \geq x_0\}} e^{\|B\| |m-x_0|} \frac{(m-x_0)}{\sqrt{2\pi s^3}} e^{-\frac{(m-x_0)^2}{2s}} f_0(x_0) dx_0 \right)^2 \leq \int_{\{m \geq x_0\}} e^{2\|B\| |m-x_0|} \frac{(m-x_0)^2}{2\pi s^3} e^{-\frac{(m-x_0)^2}{s}} f_0(x_0) dx_0.$$

Now one operates the integral in dm and the change of variable $\mu = \frac{m-x_0}{\sqrt{s}}$ so for any $s \in [0, T]$ one gets the bound

$$\int_{\mathbb{R}^+} e^{2\|B\|\mu\sqrt{s}} \frac{\mu^2 s \sqrt{s}}{2\pi s^3} e^{-\mu^2} d\mu \int_{\mathbb{R}} f_0(x_0) dx_0 \leq \frac{s\sqrt{s}}{2\pi s^3} \int_{\mathbb{R}^+} e^{2\|B\|\mu\sqrt{T}} \mu^2 e^{-\mu^2} d\mu.$$

With respect of a multiplicative coefficient, the square root of this bound is $s \rightarrow s^{-3/4}$, integrable on $[0, T]$.

4.4.3 Let $\alpha > \frac{1}{2}\|B\|$, so $\int_0^\infty \int_{\mathbb{R}} e^{-\alpha s} \tilde{p}(m, s) dm ds < \infty$.

One looks for this integral on $\mathbb{R}^+ \times \mathbb{R}$ and one uses the bound (54). Remark that all the factors in the integrand are positive so applying Tonelli Theorem one gets:

$$\int_{\mathbb{R}^+} e^{-\alpha s} \int_{\mathbb{R}} \tilde{p}_V(m; s) dm ds \leq \int_{\mathbb{R}^+} \int_{\{m \geq x_0\}} e^{-\alpha s} e^{\|B\|\|m-x_0\|} \frac{m-x_0}{\sqrt{2\pi s^3}} e^{-\frac{(m-x_0)^2}{2s}} f_0(x_0) dx_0 dm ds.$$

One operates the change of variable $\mu = \frac{m-x_0}{\sqrt{s}}$:

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-\alpha s} \tilde{p}_V(m; s) dm ds \leq \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} e^{-\alpha s + \|B\|\mu\sqrt{s}} \frac{\mu}{\sqrt{2\pi s}} e^{-\frac{\mu^2}{2}} d\mu ds \int_{\mathbb{R}} f_0(x_0) dx_0.$$

Remark that $\int_{\mathbb{R}} f_0(x_0) dx_0 = 1$ since f_0 is a law density. Once again Tonelli Theorem is applied:

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} e^{-\alpha s + \|B\|\mu\sqrt{s}} \frac{\mu}{\sqrt{2\pi s}} e^{-\frac{\mu^2}{2}} d\mu ds = \int_{\mathbb{R}^+} \mu e^{-\frac{\mu^2}{2}} \left(\int_{\mathbb{R}^+} e^{-\alpha s + \|B\|\mu\sqrt{s}} \frac{1}{\sqrt{2\pi s}} ds \right) d\mu.$$

One splits the time integral at $s = 1$:

$$\begin{aligned} \int_{\mathbb{R}^+} e^{-\alpha s + \|B\|\mu\sqrt{s}} \frac{1}{\sqrt{2\pi s}} ds &= \int_0^1 e^{-\alpha s + \|B\|\mu\sqrt{s}} \frac{1}{\sqrt{2\pi s}} ds + \int_1^\infty e^{-\alpha s + \|B\|\mu\sqrt{s}} \frac{1}{\sqrt{2\pi s}} ds \leq \\ &2 \frac{e^{\|B\|\mu}}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} \int_1^\infty e^{-\alpha s + \|B\|\mu\sqrt{s}} ds. \end{aligned}$$

One integrates this sum times the factor $\mu e^{-\frac{\mu^2}{2}}$ when $\mu \in \mathbb{R}^+$; the first integral is finite:

$$\int_{\mathbb{R}^+} \mu e^{-\frac{\mu^2}{2}} 2 \frac{e^{\|B\|\mu}}{\sqrt{2\pi}} d\mu < \infty.$$

One now deals with the second integral:

$$\int_{\mathbb{R}^+} \mu e^{-\frac{\mu^2}{2}} \frac{1}{\sqrt{2\pi}} \int_1^\infty e^{-\alpha s + \|B\|\mu\sqrt{s}} ds d\mu = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} \int_1^\infty \mu e^{-\frac{\mu^2}{2}} e^{-\alpha s + \|B\|\mu\sqrt{s}} ds d\mu.$$

Remark that $e^{-\frac{\mu^2}{2} - \alpha s + \|B\|\mu\sqrt{s}} = e^{-\frac{1}{2}(\mu - \|B\|\sqrt{s})^2} e^{-s(\alpha - \frac{1}{2}\|B\|)^2}$ so:

$$\begin{aligned} \int_1^\infty \int_{\mathbb{R}^+} \mu e^{-\frac{\mu^2}{2}} e^{-\alpha s + \|B\|\mu\sqrt{s}} d\mu ds &= \\ \int_1^\infty e^{-s(\alpha - \frac{1}{2}\|B\|)^2} \left(\int_{\mathbb{R}^+} (\mu - 2\|B\|\sqrt{s}) e^{-\frac{1}{2}(\mu - 2\|B\|\sqrt{s})^2} d\mu \right) ds &+ \int_1^\infty e^{-s(\alpha - \frac{1}{2}\|B\|)^2} 2\|B\|\sqrt{s} \int_{\mathbb{R}^+} e^{-\frac{1}{2}(\mu - 2\|B\|\sqrt{s})^2} d\mu \end{aligned}$$

The second term is bounded by $\int_1^\infty e^{-s(\alpha - \frac{1}{2}\|B\|)^2} 2\|B\|\sqrt{s} \sqrt{2\pi} ds$ which is finite since $\alpha > \frac{1}{2}\|B\|$. Concerning the first term, $\int_{\mathbb{R}^+} (\mu - 2\|B\|\sqrt{s}) e^{-\frac{1}{2}(\mu - 2\|B\|\sqrt{s})^2} d\mu \leq \int_{\mathbb{R}} |u| e^{-\frac{1}{2}u^2} du < \infty$, so Property (c) is proved. •

4.4.4 PDE, (d) Property

Recall that f_0 is the law density of X_0 , supposed to belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Using Theorem 1.3, one integrates the PDE which is satisfied by $p(\cdot, \cdot, \cdot; x_0)$ with respect to μ^0 : p_V is a weak solution of

$$(55) \quad \partial_t p(m, x; t) = -B(x)\partial_2 p(m, x; t) - B'(x)p(m, x; t) + \frac{1}{2}\partial_{2,2}^2 p(m, x; t), \quad m > x;$$

$$(56) \quad B(m)p(m, m; t) = \frac{1}{2}(\partial_1 + \partial_2)p(m, m; t) + \frac{1}{2}\partial_2 p(m, m; t)$$

One integrates the product of (55) with a compact support function $F \in C^2(\mathbb{R}^2, \mathbb{R})$ on $[0, t] \times \{(m, x), m \geq x\}$:

$$\int_0^t \int_{m \geq x} F(m, x) \partial_t p(m, x; s) dm dx ds =$$

$$\int_0^t \int_{m \geq x} F(m, x) (-B(x)\partial_2 p(m, x; s) - B'(x)p(m, x; s) + \frac{1}{2}\partial_{2,2}^2 p(m, x; s)) dm dx ds.$$

Using that F support is compact and p positive integrable, Fubini Theorem proves that the left hand is $\int_{m > x} F(m, x) p(m, x; s) dm dx$.

On the right hand, one operates some integrations by parts with respect to $dm dx$ and duality yields

$$(57) \quad \int_{\mathbb{R}^2} F(m, x) p(m, x; t) dm dx = \int_{\mathbb{R}} f(x, x) f_0(x_0) dx_0 + \int_0^t \int_{\mathbb{R}^2} (B(x)\partial_2 F(m, x)$$

$$+ \frac{1}{2}\partial_{2,2}^2 F(m, x)) ds p_V(m, x; s) dm dx + \frac{1}{2} \int_0^t \int_0^\infty \partial_1 F(m, m) \tilde{p}_V(m; s) dm ds,$$

Finally using that $C_k^2(\mathbb{R}^2)$ is dense in $C_b^2(\mathbb{R}^2)$ Property (d) is proved. •

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