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A Fuzzy Take on Graded Beliefs

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Abstract. Graded beliefs increase the expressiveness of any model representing beliefs, by allowing the introduction of modulation: it is then possible to model different levels of belief. The increased expressiveness raises issues regarding the interpretation of these levels as well as their combination. This paper addresses this issue proposing a fuzzy interpretation of graded beliefs, viewed as a fuzzy subset of the universe of well formed formulae, considering belief degrees as membership degrees to a belief set. It studies the consequences of this interpretation, for the definition of graded belief manipulation rules and of graded variants of the doxastic axioms KD45, leading to the proposition of conjunction, disjunction, negation, implication and introspection rules for graded beliefs.

Keywords: belief reasoning, doxastic logic, fuzzy set theory

1 Introduction

Reasoning about beliefs is an essential part of rational agent modelling. Indeed, believing is one of the essential ways that allow to evaluate observations and transmit, or acquire, information. The most common formal framework to model belief reasoning is doxastic logic [12], which is a variant of modal logic [6] offering axioms to manipulate a belief modality.

In order to increase its expressiveness, a belief can be modulated by a graduation, making it possible to model different levels of belief, for example making the difference between low, rather low, rather high and high beliefs. This notion of modulated belief is essential in belief representation, as for instance discussed in works in the domains of philosophy of mind and epistemology [11,21,20]. Indeed, binary beliefs offer a model that is too limited to be realistic. As an illustrative example, consider an astrophysicist who believes that the centre of the Milky Way contains a supermassive black hole and that life exists in another galaxy. As these two facts are not equally admitted in the scientific community, it appears likely that the astrophysicist has different levels of belief in them. Thus, it is relevant to allow the representation of modulated beliefs.

This paper addresses the task of graded belief modelling and manipulation, proposing a framework to reason about such graded beliefs, so as to allow an agent to infer from his own beliefs some other beliefs. The question is then to combine graded beliefs

Conjunction	$B_{\top(\alpha,\beta)}(\varphi \wedge \psi) \leftrightarrow (B_\alpha\varphi \wedge B_\beta\psi)$
Disjunction	$(B_\alpha\varphi \vee B_\beta\psi) \rightarrow B_{\perp(\alpha,\beta)}(\varphi \vee \psi)$
Negation (D)	$B_\alpha\neg\varphi \rightarrow \begin{cases} B_\beta\varphi & \text{if } \beta \leq 1-\alpha \\ \neg B_\beta\varphi & \text{otherwise} \end{cases}$
Implication (K)	$B_\alpha(\varphi \rightarrow \psi) \rightarrow (B_\beta\varphi \rightarrow B_{\top(\alpha,\beta)}\varphi)$
Introspection (4) and (5)	$B_\alpha\varphi \rightarrow B_1 B_\alpha\varphi$ $\neg B_\alpha\varphi \rightarrow B_1\neg B_\alpha\varphi$

Table 1. Proposed doxastic axioms for graded beliefs

and aggregate their respective degrees to derive new graded beliefs. It can be illustrated by the example: if the formula φ is α-believed and the formula ψ is β-believed, how much the formula $\varphi \wedge \psi$ is believed? The paper considers the case of all connectives, proposing rules for conjunction, disjunction, negation and implication and graded equivalents of the classic doxastic axioms, KD45.

The manipulation of graded beliefs as combination of their degrees depends on their interpretation and on the choice on an arithmetical structure to represent them. This paper discusses the meaning of belief degree proposing an interpretation based on fuzzy set theory [22]: graded beliefs are viewed as a fuzzy subset of the universe of well formed formulae, considering belief degrees as membership degrees to a belief set. The paper then studies the consequences of this fuzzy interpretation on all manipulation rules mentioned above.

The propositions of the paper are summarised in Table 1 and respectively discussed in the following sections, after a section briefly presenting related works about weighted modal logics and graded belief representation.

2 Related Works

The problem of representing graded beliefs refers to doxastic logic, which is the variant of modal logic [6] that allows to reason with beliefs [12] and it requires a weighted logic framework to manipulate the degrees associated with the beliefs. This section shortly presents a literature review concerning these two issues, the representation of graded notions in modal logic and the common representation of belief degrees, and then describes the motivation for the proposed approach.

2.1 Weighted Modal Logics

This section reviews the main existing approaches of weighted modal logics, discussing the interpretation of the weights they rely on. They are based on enriching the classical modality $\Box$ with a numerical coefficient α, representing a finite or infinite set of weighted modalities $\Box_\alpha$. Some of them preserve the classical Kripke semantics, modifying the counting function used to define the modality semantics [9,8,13,16], but they do not consider the formal meaning of the weights. Others, detailed below, consider enriched Kripke frame definitions, assigning weights interpreted in several formal frameworks, possibility theory, fuzzy subset theory and exceptionality degrees.

Possibilistic Interpretation The classic Kripke frames can be enriched with possibility degrees [23], either assigned to the possible worlds [3] or to the accessibility relation [5].

In the first case, worlds are considered to be more or less possible; the accessibility relation is derived from these possibility degrees as an order relation: a world is linked to all worlds that are at least as possible as it is.

The second case, in which the relation is weighted, allows to modulate the accessibility between two worlds, insofar as a weight is individually set for each couple of worlds. The weights then allow to express doubts about the very existence of this relation. The weighted modality $\Box_\alpha$ is defined from the corresponding α-weighted relation, instead of the classical unique accessibility relation. Thus, each value of weight induces a separate modality.

Fuzzy Interpretation Fuzzy degrees [22] can be assigned to the accessibility relation [2]. This formal framework allows to represent an imprecision about this relation, as opposed to the uncertainty of the previous possibilistic case: worlds are more or less accessible, where the relation strength can be modulated, without doubting its existence.

Weighted modalities are then defined from this semantics, one for each value of the accessibility degree. As in the possibilistic model, the modal formula interpretation requires to relate the $\Box_\alpha$ modality to the corresponding α-weighted relation.

As will be detailed in Section 3, the graduality property of fuzzy set theory implies that the induced modalities satisfy a specific implication property, establishing dependencies between them.

Exceptionality Interpretation Another possible interpretation of weights is based on the notion of world exceptionality [15]. An exceptionality degree represents the non-representativeness of the world amongst all possible worlds of the frame. These degrees can be extended to formulae, which are thus more or less represented by exceptional worlds, defining weighted modal formulae. The proposed modal formula interpretation thus considers these degrees instead of the accessibility relation, so that $\Box_\alpha$ is not locally defined any more: its semantics does not depend on accessible worlds but the global exceptionality degree assigned to the considered formula.

2.2 Frameworks for Graded Belief Representation

There exist numerous approaches to graded belief representation in a non-modal setting, that assign degrees to logical formulae in various theoretical frameworks, such as subjective probabilities [1], possibilities [7] or evidential theory [18].

They mainly propose tools for graded belief inference and are not essentially based on the modal framework. As a consequence, these models are more a belief extension of graded logics, whereas our proposition is to study the graded extension of a logical model for beliefs, namely doxastic logic.

The common property of subjective probabilities, possibilities and evidential approaches, is to view graded beliefs as related to uncertainty. They consider the belief degree as a measure of the certainty attached to the occurrence of the underlying fact. As a consequence, they do not allow to model a total belief in an uncertain fact, although such a case should be allowed: for instance if a superstitious person finds a four-leaf clover, he will believe in his own luck, whereas being lucky is a totally uncertain state and it is not related to the fact of finding a four-leaf clover. Thus, we propose to consider beliefs as independent of the underlying facts.

Subjective probabilities, or bayesian probabilities, are often used for such issue [10,19] but they are still related to the fact truthfulness whereas we propose to consider beliefs from a modal point of view, i.e. as a non-factual observation independent from its truth. Indeed, belief, and some more complex notions based on it, as trust for exemple [4], can be defined beyond subjective probabilities.

2.3 Motivation for the Proposed approach

Graded extensions of modal logics have been proposed in order to increase the expressiveness of the latter, but there is no variant specifically dedicated to the manipulation of graded beliefs. On the contrary, some of them are even explicitly non-adapted to the doxastic case [3]. Thus we propose to define a graded doxastic logic, only considering the KD45 axiomatic.

Moreover, as we choose to interpret beliefs in a purely modal way, it is possible for an agent to believe anything, the objects considered in this work are non-factual since they represent these beliefs. Therefore, beliefs are not necessarily related to a potential certainty: even if uncertainty is commonly used in the definition of belief, one can consider that an uncertain piece of information can be more believed than a definite one.

The aim of the following sections is to present and study the proposition of a formal interpretation for belief degrees and their manipulation. The definition of manipulation rules obviously depends on the considered interpretation of the latter, i.e. the meaning given to 'believing at degree α' and, therefore, on the formalism they are associated to and the arithmetical structure it induces. We propose not to consider uncertainty formalisms but to consider a fuzzy arithmetical structure.

3 Proposed Fuzzy Reading of Belief Degrees

This section defines a weighted variant of doxastic logic, i.e. a weighted modal logic, as described in Section 2.1, dedicated to belief reasoning.

After presenting the notations, this section describes the proposed interpretation of belief degrees in a fuzzy framework and discusses the induced fuzzy properties in a doxastic context. The following sections reciprocally discuss the doxastic axioms in a fuzzy reading.

3.1 Syntax

The graded doxastic logic presented in this paper is defined by the following language, where $\mathbb{P}$ denotes a set of propositional variables, $p \in \mathbb{P}$ and $\alpha \in [0, 1]$ a numerical coefficient

$$F := p \mid F \mid \neg F \mid F \wedge F \mid F \vee F \mid F \rightarrow F \mid B_\alpha F$$

We propose to use the notation B to represent the belief modality, so as to underline its specificity as opposed a general $\Box$ modality. A formula $B_\alpha F$ is called a graded belief formula and is read "F is believed at a degree α" or "the agent α-believes F".

3.2 Belief Degree as Fuzzy Membership

Considering as universe the set of well-formed formulae defined according to the previous language, denoted $\mathcal{P}$, and given any agent, we propose to interpret his belief set, i.e. the set of his believed formulae $\mathcal{B}$ as a fuzzy subset of $\mathcal{P}$.

As a consequence, each formula of $\mathcal{B}$ is associated to a membership degree which is interpreted as the belief degree assigned to the graded belief formula.

Formally, the membership function of $\mathcal{B}$ is noted $\mu_\mathcal{B} : \mathcal{P} \mapsto [0, 1]$. A graded belief formula is then defined:

$$B_\alpha \varphi \quad \text{iff} \quad \mu_\mathcal{B}(\varphi) = \alpha \tag{1}$$

3.3 Doxastic Reading of Fuzzy Properties

As this interpretation of graded beliefs is based on fuzzy subset theory, it induces some major properties, regarding α-cut nesting and boundary values, which are here interpreted from a doxastic point of view.

α-cuts and Graduality For any fuzzy subset A defined on a universe $\mathcal{X}$, the α-cuts of A satisfy the following graduality property: $\forall \alpha, \beta \in [0, 1]$, if $\alpha \geq \beta$, $A_\alpha \subseteq A_\beta$.

In the graded belief context, the α-cut of a belief set $\mathcal{B}$ corresponds to the crisp set of formulae that are at least α-believed. Due to the graduality property, it holds that if a formula φ is α-believed, it is β-believed for all $\beta \leq \alpha$: an α-believed formula allows to infer it is also β-believed for all lower degrees. This property seems consistent with an intuitive interpretation of belief sets.

Boundary Values The boundary values for belief degrees, 0 and 1, as extreme membership degrees, play a specific role in fuzzy subset theory, they also have a specific doxastic interpretation.

A formula with membership degree 1 totally belongs to the belief set. Intuitively, this maximal belief can be interpreted as a classic belief taken from the doxastic framework. It is possible to better exploit the flexibility allowed by the belief degrees and to set a lower threshold on belief degrees to define beliefs to be taken into account when determining a less expressive model of beliefs: this is equivalent to considering an α-cut with $\alpha < 1$.

On the other hand, formulae with membership degree 0 appear to be more difficult to interpret: they can be considered as uninformative about the agent's beliefs. Indeed, 0-believing represents the minimal value of belief degree and corresponds to a trivial value that always applies, due to the previous graduality property, or, equivalently to the fact that the 0-cut actually equals the considered universe. If, however, 0 is proved to be the maximal value with which the belief can be established, it then gets more informative. Nevertheless, it can be considered as showing a lack of information on the formula belief or as indicating that the formula is actually disbelieved. These two very different interpretations refer to classic discussion about the interpretation of a 0 membership degree.

4 Fuzzy Reading of Graded Belief Conjunction and Disjunction

The combination of graded beliefs can be expressed as rules transforming their associated degrees. This issue can for instance be exemplified by the question: if a formula φ is α-believed and a formula ψ is β-believed, how much the formula $\varphi \wedge \psi$ is believed?

This conjunctive factorisation of two graded beliefs allows to combine them to possibly infer new beliefs. For instance graded beliefs in the existence of unicorns and in the existence of pegasus can be brought together, leading to believe to a certain degree that winged unicorns exist, since the derived joint existence of unicorns and pegasus may lead them to have babies together.

Similarly for disjunction, the question is: if it independently holds that φ is α-believed or that ψ is β-believed, how much the disjunction $\varphi \vee \psi$ is believed?

Reciprocally, the reverse has to be analysed to determine how much φ and ψ are independently believed, if it globally holds that the formula $\varphi \wedge \psi$ (resp.$\varphi \vee \psi$) is α-believed.

Fuzzy Conjunction and Disjunction In a fuzzy logic framework, noting (φ, α) a formula φ with truth degree α, it holds that

$$(\varphi, \alpha) \wedge (\psi, \beta) \Rightarrow (\varphi \wedge \psi, \top(\alpha, \beta)) \quad (2)$$

$$(\varphi, \alpha) \vee (\psi, \beta) \Rightarrow (\varphi \vee \psi, \perp(\alpha, \beta)) \quad (3)$$

where $\top$ and $\perp$ are respectively a t-norm and a t-conorm. The first one is a function $\top : [0,1] \times [0,1] \mapsto [0,1]$ which is associative, commutative, non-decreasing in its two arguments and has for neutral element 1. A t-conorm has the same properties but has for neutral element 0 instead of 1 (see e.g. [14]).

In the following, we consider belief degrees as such truth degrees: they are related to the membership function of a fuzzy subset the same way.

Factorisation over Conjunction The conjunctive combination of formulae in fuzzy logic, as defined in Eq. (2) applied to the fuzzy reading of belief degrees leads to set:

$$(B_\alpha \varphi \wedge B_\beta \psi) \rightarrow B_{\top(\alpha,\beta)}(\varphi \wedge \psi) \quad (4)$$

The purpose is now to detail the properties of this aggregation operator regarding this doxastic interpretation.

Firstly, monotonicity is consistent with belief aggregation: for $\gamma = \top(\alpha, \beta)$, increasing the belief degree α should increase the value of γ. Indeed, if the agent increases his belief that pegasus exist from 0.5 to 0.8, it is expected that the belief in the simultaneous existence of pegasus and unicorns increases.

Similarly, it appears relevant to consider value 1 as a neutral element: as discussed in the previous section, 1-believing a formula means that it is completely believed. Thus, if the agent completely believes that unicorns exist, the simultaneous belief in the existence of unicorns and pegasus should only be informed by believing in pegasus.

Commutativity and associativity can be seen as raising more debate: in particular they imply that, when combining several beliefs, the order in which the latter are considered does not influence the final belief degree. This property seems relevant from a logical point of view, in which the conjunction connective also is commutative and associative.

Moreover, from a semantic point of view, in a context of modelling the beliefs of an agent, these properties can also be considered as consistent. Indeed, the $\wedge$ operator between two beliefs is meant to be as a logical conjunction. It thus cannot be considered with a sequential meaning as in natural language where "A and B" is often understood as expressing an order and interpreted as meaning "first A, and then B". However, we do not consider beliefs in a dynamic context. Thus, commutativity is a reasonable property. The same argument can be used for associativity.

Distribution over Conjunction The reciprocal of the rule given by Eq. (4) raises the question of separating a believed conjunction into two components. Now, if the formula $\varphi \wedge \psi$ is α-believed, there exists no certainty on the repartition of the value α among the belief degrees for φ and ψ independently: it is only possible to have the guarantee that α is distributed between the degrees assigned to φ and ψ separately, according the two rules below:

$$B_\alpha(\varphi \wedge \psi) \rightarrow B_\beta\varphi \qquad \text{with } \beta \leq \alpha \tag{5}$$

$$B_\alpha(\varphi \wedge \psi) \rightarrow B_\gamma\psi \qquad \text{with } \gamma \leq \alpha \tag{6}$$

However, it is possible to jointly relate β and γ to α in order to establish a distributive principle. Indeed, the classical so-called normal axiom states $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$; its graded doxastic extension is noted $B_\alpha(\varphi \wedge \psi) \leftrightarrow (B_\beta\varphi \wedge B_\gamma\psi)$. Thus, according to the existing equivalence between distribution and factorisation, a distributive rule can be proposed. A solution is to impose that β and γ are such that $\alpha = \top(\beta, \gamma)$ leading to the following distributive rule:

$$B_\alpha(\varphi \wedge \psi) \rightarrow (B_\beta\varphi \wedge B_\gamma\psi) \text{ with } \alpha = \top(\beta, \gamma) \tag{7}$$

In combination with the implication given in Eq. (4), it is thus established

$$B_{\top(\alpha,\beta)}(\varphi \wedge \psi) \leftrightarrow (B_\alpha\varphi \wedge B_\beta\psi) \tag{8}$$

as stated in the first row of Table 1.

Disjunctive Variant Using the same principles as for the conjunction, transposed to the fuzzy tools manipulating disjunction, i.e. t-conorms, the following rule can be established:

$$(B_\alpha\varphi \vee B_\beta\psi) \rightarrow B_{\perp(\alpha,\beta)}(\varphi \vee \psi) \tag{9}$$

Similarly to the conjunctive case, it can be discussed that the properties of t-conorms (monotonicity, commutativity, associativity and neutral element 0) are relevant for the disjunctive aggregation of belief degrees.

However, the disjunctive case of distributivity is not as simple: if $\varphi \vee \psi$ is α-believed, it is not possible to establish conditions on the belief degree assigned to φ and ψ separately. Indeed, the only information given by the formula is that $\varphi \vee \psi$ α-belongs to the belief set of the agent. The repartition of this membership degree between φ and ψ needs to consider each formula independently from the other. The disjunction gives no guarantee on the membership of these formulae to the belief set, as it is totally possible that only φ, for example, belongs to it.

This observation is also consistent with the classical case in modal logic: there exists no axiom that establishes distributivity of $\Box$ over disjunction. The normal axiom only allows its factorisation.

5 Fuzzy Reading of Graded Belief Negation (Axiom D)

Negation is a major issue in the formalisation of graded beliefs [17]: the aim of this section is to examine how fuzzy negation can be used for this issue, i.e. if it provides tools to determine the belief degree for a formula φ from the one associated to $\neg\varphi$ and reciprocally. In other terms, the issue here is to determine if a formula and its contrary can both belong to the belief set and how much they can.

Formally, this question aims at defining manipulation rules of the form

$$B_\alpha\neg\varphi \rightarrow B_\beta\varphi \qquad\qquad B_\alpha\neg\varphi \rightarrow \neg B_\beta\varphi$$

They correspond to two cases that can be distinguished for the graded belief in φ: either β is sufficiently low for φ to belong to the belief set even if $\neg\varphi$ does too, which allows to infer $B_\beta\varphi$, or it is too high and the membership of φ does not hold, which induces $\neg B_\beta\varphi$. The purpose is to determine the limit value of β for which φ switches from a membership to a non-membership to the belief set, i.e. between the two implications above.

In the fuzzy extension of classical sets, a complementation operator is defined, according to which, if an element α-belongs to a fuzzy set, its complement $(1-\alpha)$-belongs to it. In the previous case, φ and $\neg\varphi$ are related by a complement negation, since they are defined in propositional logic.

Therefore the fuzzy complementation property can be applied in the case of belief-set: if $\neg\varphi$ is α-believed then φ is at the most $(1-\alpha)$-believed:

$$B_\alpha\neg\varphi \rightarrow B_{1-\alpha}\varphi$$

It can be noted that the graduality property can be applied and allows to infer $B_\beta\varphi$ for all $\beta \leq 1-\alpha$. Therefore, the rules given at the beginning of this section can be rewritten:

$$B_\alpha \neg\varphi \rightarrow \begin{cases} B_\beta\varphi & \text{if } \beta \leq 1-\alpha \\ \neg B_\beta\varphi & \text{otherwise} \end{cases} \tag{10}$$

This definition of a negated graded belief formula can be compared to the classical modal axiom (D): $\Box\varphi \supset \neg\Box\neg\varphi$. Indeed, in doxastic logic, this axiom expresses that a formula and its negation cannot be believed together. The proposed graded version of this axiom allows to determine up to which belief value a formula and its negation can be believed together.

6 Fuzzy Reading of Graded Belief Implication (Axiom K)

This section turns to the manipulation of graded beliefs over the implication connective, which allows a rational agent to make inference in his belief set: if, for instance, the agent believes the implication rule "if the centre of the Milky way is a supermassive blackhole then stars around the center have an orbital movement", then he must be able to infer that he believes the last part if he believes that this supermassive blackhole exists.

Modal logics can be seen as answering this question through the normal axiom (K), which allows to distribute modality over implication: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$. This axiom can be rewritten in the form of a unique implication instead of an implication sequence: it then highlights the *modus ponens* that allows to deduce $\Box\psi$ from the modal rule $\Box\varphi \rightarrow \psi$ and its modal premise formula $\Box\varphi$: $(\Box(\varphi \rightarrow \psi) \wedge \Box\varphi) \rightarrow \Box\psi$. In doxastic logic, this version of the axiom expresses that if an implication and its premise are both believed, then its conclusion is also believed. This form of the axiom reflects that an inference can be made from a believed implication if its premise also belongs to the belief set.

The purpose is now to discuss a graded doxastic extension of this inference axiom. It can be generally written:

$$(B_\alpha(\varphi \rightarrow \psi) \wedge B_\beta\varphi) \rightarrow B_\gamma\psi \tag{11}$$

where the implication and its premise are believed at different levels α and β and the aim is to combine these values in order to determine conditions on the value of γ.

Rewriting the premise leads to

$$\begin{aligned} B_\alpha(\varphi \rightarrow \psi) \wedge B_\beta\varphi &\leftrightarrow B_{\top(\alpha,\beta)}((\varphi \rightarrow \psi) \wedge \varphi) \\ &\leftrightarrow B_{\top(\alpha,\beta)}(\varphi \wedge \psi) \end{aligned}$$

where the first step is obtained using Eq (4) established in a previous section, and the second uses propositional logic rewriting rules. Eq (11) then implies that

$$B_{\top(\alpha,\beta)}(\varphi \wedge \psi) \rightarrow B_\gamma\psi$$

which corresponds to the distribution rule over conjunctive operator, given in Eq. (5). Therefore, according to the equivalence rule between distribution and factorisation over conjunction, given by Eq (8), a condition on γ can be determined: $\gamma = \top(\alpha, \beta)$.

Note that the distribution rule applied to $B_{\top(\alpha,\beta)}(\varphi \wedge \psi)$ also leads to $B_{\top(\alpha,\beta)}\varphi$, which is compatible with the initial hypothesis $B_\beta\varphi$. Indeed, due to the property of t-norms, $\top(\alpha, \beta) \leq \min(\alpha, \beta) \leq \beta$ which implies, using the decreasing graduality property, that $B_\gamma\varphi$ allows to infer $B_{\top(\alpha,\beta)}\varphi$. The second part of the distributivity rule is thus not informative in this case.

Finally, the inference axiom proposed for graded belief reasoning is written

$$B_\alpha(\varphi \to \psi) \to (B_\beta\varphi \to B_\gamma\varphi) \qquad \text{with } \gamma = \top(\alpha, \beta) \tag{12}$$

7 Fuzzy Reading of the Introspection Axioms (4 and 5)

Doxastic logic includes the so-called positive and negative introspection axioms, that respectively state $\Box\varphi \to \Box\Box\varphi$ and $\neg\Box\varphi \to \Box\neg\Box\varphi$.

These axioms, usually considered as essential in rational agent modelling, express the fact that an agent is conscious both of his own belief and absence of belief: if an agent believes φ (resp does not believe φ), then he believes that he does.

This section studies the extension of these axioms to the case of graded beliefs: it first underlines the limit of their fuzzy interpretation and then discusses them from a semantic point of view.

Introspection Issue Syntactically rewriting the introspection axiom using weights leads to question relevant values α, β and γ such that: $B_\alpha\varphi \to B_\beta B_\gamma\varphi$ and $\neg B_\alpha\varphi \to B_\beta\neg B_\gamma\varphi$.

The interpretation of such formulae when belief degrees are considered as membership degrees to the belief set turns out to be an issue because of the formulae involving two modalities. Indeed, the fuzzy reading of a formulae $B_\beta B_\gamma\varphi$ implies that the formula $B_\gamma\varphi$ β-belongs to the belief set, i.e. $\mu_\mathcal{B}(B_\gamma\varphi) = \beta$.

The problem is that there is no fuzzy tool that makes it possible to combine this piece of information with the translation of the internal formula $B_\gamma\varphi$ which states that $\mu_\mathcal{B}(\varphi) = \gamma$. Nor is there a relation with the fact that $\mu_\mathcal{B}(\varphi) = \alpha$, which translates the premise of the positive introspection axiom. Indeed, fuzzy operators do not allow to manipulate such recursive objects.

Graded Positive Introspection We thus discuss semantic arguments related to the desired meaning of a graded variant of the positive introspection axiom $B_\alpha\varphi \to B_\beta B_\gamma\varphi$, leading to set $\beta = 1$ and $\gamma = \alpha$.

Indeed, we first propose to set $\gamma = \alpha$, to express the fact the introspection axiom gives information regarding the agent's own belief: this interpretation considers that the axiom provides means to infer new beliefs about already existing ones, and not weakened ones, as would be obtained if $\gamma < \alpha$, or enriched ones, corresponding to $\gamma < \alpha$. The notion of weakening is here to be interpreted in line with the graduality property established in the previous section, which allows to infer $B_\gamma\varphi$ from $B_\alpha\varphi$ if $\gamma < \alpha$.

Regarding the value of β, it can be argued that $\beta < 1$ leads to model a partially conscious agent, who does not stand by his own beliefs insofar as he does not fully believe them and may thus introduce some doubts about them. Illustrating this principle linguistically using adverbs to express belief degrees, $\beta < 1$ may lead to beliefs of the form ‘the agent strongly believes that he strongly believes that unicorns exist’ (for an agent who does strongly believe that unicorns exist), conveying some reluctance regarding his own beliefs.

This interpretation leads us to propose setting $\beta = 1$, requiring an agent to fully believe his own beliefs. It can be underlined that this choice may open the discussion regarding the limit between graded beliefs and knowledge, and more particularly the beliefs of degree 1 and knowledge. The model proposed by [2] takes this step, whereas the interpretation of belief degree as membership degrees to a belief set does not impose to do so.

Graded Negative Introspection The graded variant of the negative introspection axiom takes the form $\neg B_\alpha\varphi \rightarrow B_\beta\neg B_\gamma\varphi$ and raises the same questions as its positive counterpart. Therefore, for the same reasons, we suggest to consider $\gamma = \alpha$ and $\beta = 1$.

It must however be underlined that negative introspection additionally raises the question of the interpretation of the negation of the graded belied, $\neg B_\alpha\varphi$, which is a complex issue: it has been partially discussed in Section 4.3, in the case where negation is understood as applying to the formula; a transfer to the degree or to the modality itself may be considered as well and opens the way for a discussion out of scope of this paper, along the discussion in [17].

8 Conclusion and future work

Reasoning about graded beliefs relies on two principles: logical manipulation and degree combination. This paper considered these two issues at the cross-roads of two formal frameworks, doxastic logic and fuzzy set theory. It proposed to interpret belief degrees as membership degrees to a belief set, interpreted as a fuzzy subset of the universe of all well-formed formulae. It then examined the consequences of this interpretation both for graded belief manipulation rules and doxastic axioms: it considered conjunctive and disjunctive combinations and decompositions of graded beliefs as well as an interpretation of their negation. It also discussed all four axioms of doxastic logic, KD45, from the fuzzy interpretation point of view. Table 1 sums up the manipulation rules proposed for graded beliefs.

Ongoing works aim at studying additional properties of specific fuzzy aggregation operators, such as idempotence, reinforcement or compensation to name a few, to examine their relevance for the manipulation of graded beliefs and to further enrich the expressiveness of models for rational agent reasoning.

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