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# Convergence analysis of upwind type schemes for the aggregation equation with pointy potential

F. Delarue\*, F. Lagoutière†, N. Vauchelet‡

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## Abstract

A numerical analysis of upwind type schemes for the nonlinear nonlocal aggregation equation is provided. In this approach, the aggregation equation is interpreted as a conservative transport equation driven by a nonlocal nonlinear velocity field with low regularity. In particular, we allow the interacting potential to be pointy, in which case the velocity field may have discontinuities. Based on recent results of existence and uniqueness of a Filippov flow for this type of equations, we study an upwind finite volume numerical scheme and we prove that it is convergent at order 1/2 in Wasserstein distance. This result is illustrated by numerical simulations, which show the optimality of the order of convergence.

**Keywords:** Aggregation equation, upwind finite volume scheme, convergence order, measure-valued solution, Markov chains.

**2010 AMS subject classifications:** 35B40, 35D30, 35L60, 35Q92, 49K20.

## 1 Introduction

This paper is devoted to the numerical approximation of measure valued solutions to the so-called aggregation equation in space dimension  $d$ . This equation reads

$$\partial_t \rho = \operatorname{div} ((\nabla_x W * \rho) \rho), \quad t > 0, \quad x \in \mathbb{R}^d, \quad (1.1)$$

with the initial condition  $\rho(0, \cdot) = \rho^{ini}$ . Here,  $W$  plays the role of an interaction potential whose gradient  $\nabla_x W(x - y)$  measures the relative force exerted by a unit mass localized at a point  $y$  onto a unit mass located at a point  $x$ .

This system appears in many applications in physics and population dynamics. In the framework of granular media, equation (1.1) is used to describe the large time dynamics of inhomogeneous kinetic models, see [3, 17, 52]. Models of crowd motion with a nonlinear term of the form  $\nabla_x W * \rho$  are also addressed in [18, 20]. In population dynamics, (1.1) provides a biologically meaningful description of aggregative phenomena. For instance, the description of the collective migration

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of cells by swarming leads to such a kind of PDEs with non-local interaction, see e.g. [44, 45, 51]. Another example is the modelling of bacterial chemotaxis. In this framework, the quantity  $S = W * \rho$  is the chemoattractant concentration, which is a substance emitted by bacteria allowing them to interact with one another. The dynamics can be macroscopically modelled by the Patlak-Keller-Segel system [38, 46]. In the kinetic framework, the most frequently used model is the Othmer-Dunbar-Alt system, the hydrodynamic limit of which leads to the aggregation equation (1.1), see [25, 26, 35]. In many of these examples, the potential  $W$  is usually mildly singular, i.e.  $W$  has a weak singularity at the origin. Because of this low regularity, smooth solutions of such systems may blow-up in finite time, see e.g. [41, 8, 5, 15]. In the latter case, finite time concentration may be regarded as a very simple mathematical way to account for aggregation of individuals, as opposed to diffusion.

Since finite time blow-up of smooth solutions may occur and since equation (1.1) conserves mass, a natural framework to study the existence of global in time solutions is to work in the space of probability measures. In this regard, two strategies have been proposed in the literature. In [15], the aggregation equation is seen as a gradient flow taking values in the Wasserstein space and minimizing the interaction energy. In [35, 36, 16, 40], this system is considered as a conservative transport equation with velocity field  $\nabla_x W * \rho$ . Then a unique flow, say  $Z = (Z(t, \cdot))_{t \geq 0}$ , can be constructed, hence allowing to define the solution as a pushforward measure by the flow, namely  $\rho = (\rho(t) = Z(t, \cdot) \# \rho^{ini})_{t \geq 0}$ . When the singularity of the potential is stronger than the mild form described above, such a construction has been achieved in the radially symmetric case in [4], but uniqueness is then lacking. Actually, the assumptions on the potential  $W$  that are needed to ensure the well-posedness of the equation in the space of measure valued solutions require a certain convexity property of the potential that allows only for a mild singularity at the origin. More precisely, we assume that the interaction potential  $W : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies the following properties:

- (A0)**  $W(x) = W(-x)$  and  $W(0) = 0$ ;
- (A1)**  $W$  is  $\lambda$ -convex for some  $\lambda \in \mathbb{R}$ , i.e.  $W(x) - \frac{\lambda}{2}|x|^2$  is convex;
- (A2)**  $W \in C^1(\mathbb{R}^d \setminus \{0\})$ ;
- (A3)**  $W$  is Lipschitz-continuous.

Such a potential will be referred to as a *pointy* potential. Typical examples are fully attractive potentials  $W(x) = 1 - e^{-|x|}$ , which is  $-1$ -convex, and  $W(x) = |x|$ , which is  $0$ -convex. Notice that the Lipschitz-continuity of the potential allows to bound the velocity field: there exists a nonnegative constant  $w_\infty$  such that for all  $x \neq 0$ ,

$$|\nabla W(x)| \leq w_\infty. \tag{1.2}$$

Observe also that **(A3)** forces  $\lambda$  in **(A1)** to be non-positive, as otherwise  $W$  would be at least of quadratic growth, whilst **(A3)** forces it to be at most of linear growth. However, we shall sometimes discard **(A3)**, when the initial datum is compactly supported. In this case, as  $W - \lambda|x|^2/2$  is convex, it is locally Lipschitz-continuous, so that  $W$  is locally Lipschitz-continuous, what will be sufficient for compactly supported initial data. In that case it perfectly makes sense to assume  $\lambda > 0$  in **(A1)**. For numerical analysis, we will assume in this case that the potential is radial, that is to say that  $W$  is a function of the sole scalar  $|x|$ ,  $W(x) = \mathcal{W}(|x|)$ .

Although very accurate numerical schemes have been developed to study the blow-up profile for smooth solutions, see [33, 34], very few numerical schemes have been proposed to simulate the behavior of solutions to the aggregation equation after blow-up. The so-called sticky particles method was shown to be convergent in [15] and used to obtain qualitative properties of the solutions such as the time of total collapse. However, this method is not so practical to catch the behavior

of the solutions after blow-up in dimension  $d$  larger than one. In dimension  $d = 1$ , this question has been addressed in [35]. In higher dimension, particle methods have been recently proposed and studied in [19, 12], but only the convergence of smooth solutions, before the blowup time, has been proved. Finite volume schemes have also been developed. In [37], the authors propose a finite volume scheme to approximate the behavior of the solution to the aggregation equation (1.1) after blow-up and prove that it is convergent. A finite volume method for a large class of PDEs including in particular (1.1) has been also proposed in [14], but no convergence result has been given. Finally, a finite volume scheme of Lax-Friedrichs type for general measures as initial data has been introduced and investigated in [16]. Numerical simulations of solutions in dimension greater than one have been obtained, allowing to observe the behavior after blow-up. Moreover, convergence towards measure valued solutions has been proved. However, no estimate on the order of convergence has been established so far. In the current work, we provide a precise estimate of the order of convergence in Wasserstein distance for an upwind type scheme. This scheme is based on an idea introduced in [35] and used later on in [37, 16]. It consists in discretizing properly the macroscopic velocity so that its product with the measure solution  $\rho$  is well-defined. In this paper, we introduce an upwind scheme for which this product is treated accurately, and we prove its convergence at order  $1/2$  in Wasserstein distance (the definition of which is recalled below).

For a given velocity field, the study of the order of convergence for the finite volume upwind scheme for the transport equation has received a lot of attention. This scheme is known to be first order convergent in  $L^\infty$  norm for any smooth initial data in  $C^2(\mathbb{R}^d)$  and for well-suited meshes, provided a standard stability condition (Courant-Friedrichs-Lewy condition) holds, see [10]. However, this order of convergence falls down to  $1/2$  in  $L^p$  norm when considering non-smooth initial data or more general meshes. This result has been first proved in the Cartesian framework by Kuznetsov in [39]. In [23], a  $1/2$  order estimate in the  $L^\infty([0, T], L^2(\mathbb{R}^d))$  norm for  $H^2(\mathbb{R}^d)$  initial data has been established. Finally in [43, 21], a  $1/2$  order estimate in  $L^1$  has been proved for initial data in  $L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ , whilst, for Lipschitz-continuous initial data, an estimate of order  $1/2 - \varepsilon$  in  $L^\infty$  for any  $\varepsilon > 0$  has been obtained in [42, 21]. We emphasize that the techniques used in [42, 43] and [21] are totally different. In the former, the strategy of proof is based on entropy estimates, whereas in the latter, the proof relies on the construction and the analysis of stochastic characteristics for the numerical scheme. Finally, when the velocity field is only  $L^\infty$  and one-sided Lipschitz-continuous, solutions of the conservative transport equation are defined only in the sense of measures. In this regard, Poupaud and Rasle [47] have proved that solutions of the conservative transport equation could be defined as the pushforward of the initial condition by a flow of characteristics. A stability estimate for such solutions has been stated later in [6]. In dimension  $d = 1$ , these solutions, as introduced in [47], are equivalent to duality solutions, as defined in [11]. Numerical investigations may be found in [29]. In such a framework with a low regularity, numerical analysis requires to work with a sufficiently weak topology, which is precisely what has been done in [22]. Therein, the convergence at order  $1/2$  of a finite volume upwind scheme has been shown in Wasserstein distance by means of a stochastic characteristic method, as done in [21]. Observe also that, recently, such an approach has been successfully used in [50] for the numerical analysis of the upwind scheme for the transport equation with rough coefficients. In the current work, we are particularly interested in the approach used in [22] since, as mentioned above, measure valued solutions of (1.1) are constructed as a pushforward of the initial condition by a flow. Precisely, our aim is to extend the strategy developed in [22] to the aggregation equation when the velocity field depends on the solution in a nonlinear nonlocal way.

The outline of the paper is the following. In the next section, we introduce the notations and recall the theory for the existence of a measure solution to (1.1). Then we present the upwind scheme and state the main result: the scheme is convergent at order  $1/2$ . In case when the potential

$W$  is strictly convex and radially symmetric and the initial condition has a bounded support, the rate is claimed to be uniform in time. Section 3 is devoted to the properties of the scheme and its probabilistic interpretation. In particular, we show how the stochastic characteristics accommodate the gradient flow structure of the aggregation equation. The proof of the main result for a Cartesian grid mesh is presented in Section 4. In Section 5, we explain briefly how to extend our result to more general  $P_1$  meshes. Finally, numerical illustrations are given in Section 6. In particular, we show that the order of convergence is optimal and we provide several numerical simulations in which we recover the behavior of the solutions after blow-up time.

## 2 Notations and main results

### 2.1 Notations

Throughout the paper, we will make use of the following notations. We denote by  $C_0(\mathbb{R}^d)$  the space of continuous functions from  $\mathbb{R}^d$  to  $\mathbb{R}$  that tend to 0 at  $\infty$ . We denote by  $\mathcal{M}_b(\mathbb{R}^d)$  the space of Borel signed measures whose total variation is finite. For  $\rho \in \mathcal{M}_b(\mathbb{R}^d)$ , we call  $|\rho|(\mathbb{R}^d)$  its total variation. The space  $\mathcal{M}_b(\mathbb{R}^d)$  is equipped with the weak topology  $\sigma(\mathcal{M}_b(\mathbb{R}^d), C_0(\mathbb{R}^d))$ . For  $T > 0$ , we let  $\mathcal{S}_{\mathcal{M}} := C([0, T]; \mathcal{M}_b(\mathbb{R}^d) - \sigma(\mathcal{M}_b(\mathbb{R}^d), C_0(\mathbb{R}^d)))$ . For  $\rho$  a measure in  $\mathcal{M}_b(\mathbb{R}^d)$  and  $Z$  a measurable map, we denote  $Z_{\#}\rho$  the pushforward measure of  $\rho$  by  $Z$ ; it satisfies, for any continuous function  $\phi$ ,

$$\int_{\mathbb{R}^d} \phi(x) Z_{\#}\rho(dx) = \int_{\mathbb{R}^d} \phi(Z(x)) \rho(dx).$$

We call  $\mathcal{P}(\mathbb{R}^d)$  the subset of  $\mathcal{M}_b(\mathbb{R}^d)$  of probability measures. For any probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  and any integrable random variable  $X$  from  $(\Omega, \mathcal{A})$  into  $(\mathbb{P}, \mathcal{B}(\mathbb{R}))$ , we write  $\mathbb{E}(X)$  for the expectation of  $X$ . We define the space of probability measures with finite second order moment by

$$\mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d), \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty \right\}.$$

Here and in the following,  $|\cdot|^2$  stands for the square Euclidean norm, and  $\langle \cdot, \cdot \rangle$  for the Euclidean inner product. The space  $\mathcal{P}_2(\mathbb{R}^d)$  is equipped with the Wasserstein distance  $d_W$  defined by (see e.g. [2, 53, 54, 49])

$$d_W(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu, \nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^2 \gamma(dx, dy) \right\}^{1/2} \quad (2.3)$$

where  $\Gamma(\mu, \nu)$  is the set of measures on  $\mathbb{R}^d \times \mathbb{R}^d$  with marginals  $\mu$  and  $\nu$ , i.e.

$$\Gamma(\mu, \nu) = \left\{ \gamma \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d); \forall \xi \in C_0(\mathbb{R}^d), \int \xi(y_1) \gamma(dy_1, dy_2) = \int \xi(y_1) \mu(dy_1), \right. \\ \left. \int \xi(y_2) \gamma(dy_1, dy_2) = \int \xi(y_2) \nu(dy_2) \right\}.$$

By a minimization argument, we know that the infimum in the definition of  $d_W$  is actually a minimum. A measure that realizes the minimum in the definition (2.3) of  $d_W$  is called an *optimal plan*, the set of which is denoted by  $\Gamma_0(\mu, \nu)$ . Then, for all  $\gamma_0 \in \Gamma_0(\mu, \nu)$ , we have

$$d_W(\mu, \nu)^2 = \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^2 \gamma_0(dx, dy).$$

We will make use of the following properties of the Wasserstein distance. Given  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$  and two  $\mu$ -square integrable Borel measurable maps  $X, Y : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , we have the inequality

$$d_W(X_{\#}\mu, Y_{\#}\mu) \leq \|X - Y\|_{L^2(\mu)}.$$

It holds because  $\pi = (X, Y)_{\#}\mu \in \Gamma(X_{\#}\mu, Y_{\#}\mu)$  and  $\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy) = \|X - Y\|_{L^2(\mu)}^2$ .

More generally, for any probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  and any two random variables  $X, Y : \Omega \rightarrow \mathbb{R}^d$  such that  $\mathbb{E}[|X|^2]$  and  $\mathbb{E}[|Y|^2]$  are finite, it holds that

$$d_W(X_{\#}\mathbb{P}, Y_{\#}\mathbb{P}) \leq \mathbb{E}(|X - Y|^2)^{1/2}, \quad (2.4)$$

where  $X_{\#}\mathbb{P}$  and  $Y_{\#}\mathbb{P}$  denote the respective distributions of  $X$  and  $Y$  under  $\mathbb{P}$ . The proof follows from the same argument as above:  $\pi = (X, Y)_{\#}\mathbb{P} \in \Gamma(X_{\#}\mathbb{P}, Y_{\#}\mathbb{P})$  and  $\int_{\mathbb{R}^d} |x - y|^2 \pi(dx, dy) = \mathbb{E}[|X - Y|^2]$ .

## 2.2 Existence of a unique flow

In this section, we recall the existence and uniqueness result for the aggregation equation (1.1) obtained in [16] (and extend it a bit for non-globally Lipschitz-continuous potentials). For  $\rho \in C([0, T], \mathcal{P}_2(\mathbb{R}^d))$ , we define the velocity field  $\widehat{a}_\rho$  by

$$\widehat{a}_\rho(t, x) := - \int_{\mathbb{R}^d} \widehat{\nabla W}(x - y) \rho(t, dy), \quad (2.5)$$

where we have used the notation

$$\widehat{\nabla W}(x) := \begin{cases} \nabla W(x), & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

Due to the  $\lambda$ -convexity of  $W$ , see **(A2)**, we deduce that, for all  $x, y$  in  $\mathbb{R}^d \setminus \{0\}$ ,

$$\langle \nabla W(x) - \nabla W(y), x - y \rangle \geq \lambda |x - y|^2. \quad (2.6)$$

Moreover, since  $W$  is even,  $\nabla W$  is odd and by taking  $y = -x$  in (2.6), we deduce that inequality (2.6) still holds for  $\widehat{\nabla W}$ , even when  $x$  or  $y$  vanishes:

$$\forall x, y \in \mathbb{R}^d, \quad \langle \widehat{\nabla W}(x) - \widehat{\nabla W}(y), x - y \rangle \geq \lambda |x - y|^2. \quad (2.7)$$

This latter inequality provides a one-sided Lipschitz-continuity (OSL) estimate for the velocity field  $\widehat{a}_\rho$  defined in (2.5), i.e. we have

$$\forall x, y \in \mathbb{R}^d, t \geq 0, \quad \langle \widehat{a}_\rho(t, x) - \widehat{a}_\rho(t, y), x - y \rangle \leq -\lambda |x - y|^2.$$

We recall that, for a velocity field  $b \in L^\infty([0, +\infty); L^\infty(\mathbb{R}^d))^d$  satisfying an OSL estimate, i.e.

$$\forall x, y \in \mathbb{R}^d, t \geq 0, \quad \langle b(t, x) - b(t, y), x - y \rangle \leq \alpha(t) |x - y|^2,$$

for  $\alpha \in L^1_{loc}([0, +\infty))$ , it has been established in [27] that a Filippov characteristic flow could be defined. For  $s \geq 0$  and  $x \in \mathbb{R}^d$ , a Filippov characteristic starting from  $x$  at time  $s$  is defined as a continuous function  $Z(\cdot; s, x) \in C([s, +\infty); \mathbb{R}^d)$  such that  $\frac{\partial}{\partial t} Z(t; s, x)$  exists for a.e.  $t \in [s, +\infty)$  and satisfies  $Z(s; s, x) = x$  together with the differential inclusion

$$\frac{\partial}{\partial t} Z(t; s, x) \in \{\text{Convex}(\widehat{a}_\rho)\}(Z(t; s, x)), \quad \text{for a.e. } t \geq s.$$

In this definition,  $\text{Convess}(E)$  denotes the essential convex hull of the set  $E$ . We remind briefly the definition for the sake of completeness (see [27, 1] for more details). We denote by  $\text{Conv}(E)$  the classical convex hull of  $E$ , i.e., the smallest closed convex set containing  $E$ . Given the vector field  $\widehat{a}_\rho(t, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , its essential convex hull at point  $x$  is defined as

$$\{\text{Convess}(\widehat{a}_\rho(t, \cdot))\}(x) := \bigcap_{r>0} \bigcap_{N \in \mathcal{N}_0} \text{Conv}[\widehat{a}_\rho(t, B(x, r) \setminus N)],$$

where  $\mathcal{N}_0$  is the set of zero Lebesgue measure sets.

Moreover, we have the semi-group property: for any  $t, \tau, s \in [0, +\infty)$  such that  $t \geq \tau \geq s$  and  $x \in \mathbb{R}^d$ ,

$$Z(t; s, x) = Z(\tau; s, x) + \int_\tau^t \widehat{a}_\rho(\sigma, Z(\sigma; s, x)) d\sigma. \quad (2.8)$$

From now on, we will make use of the notation  $Z(t, x) = Z(t; 0, x)$ . Using this characteristic, it has been established in [47] that solutions to the conservative transport equation with a given bounded and one-sided Lipschitz-continuous velocity field could be defined as the pushforward of the initial condition by the Filippov characteristic flow. Based on this approach, existence and uniqueness of solutions to (1.1) defined by a Filippov flow has been established in [16]. More precisely the statement reads:

**Theorem 2.1** [16, Theorem 2.5 and 2.9] (i) *Let  $W$  satisfy assumptions **(A0)**–**(A3)** and let  $\rho^{ini}$  be given in  $\mathcal{P}_2(\mathbb{R}^d)$ . Then, there exists a unique solution  $\rho \in C([0, +\infty); \mathcal{P}_2(\mathbb{R}^d))$  satisfying, in the sense of distributions, the aggregation equation*

$$\partial_t \rho + \text{div}(\widehat{a}_\rho \rho) = 0, \quad \rho(0, \cdot) = \rho^{ini}, \quad (2.9)$$

where  $\widehat{a}_\rho$  is defined by (2.5).

*This solution may be represented as the family of pushforward measures  $(\rho(t) := Z_\rho(t, \cdot) \# \rho^{ini})_{t \geq 0}$  where  $(Z_\rho(t, \cdot))_{t \geq 0}$  is the unique Filippov characteristic flow associated to the velocity field  $\widehat{a}_\rho$ .*

*Moreover, the flow  $Z_\rho$  is Lipschitz-continuous and we have*

$$\sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|Z_\rho(t, x) - Z_\rho(t, y)|}{|x - y|} \leq e^{|\lambda|t}, \quad t \geq 0. \quad (2.10)$$

*At last, if  $\rho$  and  $\rho'$  are the respective solutions of (2.9) with  $\rho^{ini}$  and  $\rho^{ini, t}$  as initial conditions in  $\mathcal{P}_2(\mathbb{R}^d)$ , then*

$$d_W(\rho(t), \rho'(t)) \leq e^{|\lambda|t} d_W(\rho^{ini}, \rho^{ini, t}), \quad t \geq 0.$$

(ii) *Let  $W$  satisfy **(A0)**–**(A2)** and be radial,  $\lambda$  be (strictly) positive and let  $\rho^{ini}$  be given in  $\mathcal{P}_2(\mathbb{R}^d)$  with compact support included in  $B_\infty(M_1, R)$ , where  $M_1$  is the first moment of  $\rho^{ini}$  (i.e. its center of mass) and  $B_\infty(M_1, R)$  the closed ball for the infinite norm on  $\mathbb{R}^d$  centered at  $M_1$  with radius  $R$ . Then, there exists a unique solution  $\rho \in C([0, +\infty); \mathcal{P}_2(\mathbb{R}^d))$  with support included in  $B_\infty(M_1, R)$  satisfying, in the sense of distributions, the aggregation equation (2.9) where  $\widehat{a}_\rho$  is defined by (2.5).*

*Moreover, the flow  $Z_\rho$  is Lipschitz-continuous and we have*

$$\sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|Z_\rho(t, x) - Z_\rho(t, y)|}{|x - y|} \leq e^{-\lambda t}, \quad t \geq 0. \quad (2.11)$$

*At last, if  $\rho^{ini}$  and  $\rho^{ini, t}$  have a bounded support, then,*

$$d_W(\rho(t), \rho'(t)) \leq d_W(\rho^{ini}, \rho^{ini, t}), \quad t \geq 0.$$

The stability estimates that are present in this result are Dobrushin type estimates in the quadratic Wasserstein distance, in the case where the kernel is not Lipschitz-continuous but only one-sided Lipschitz-continuous. See [24] and [28].

We mention that the solution, which is here represented by the Filippov characteristic flow, may be also constructed as a gradient flow solution in the Wasserstein space  $\mathcal{P}_2(\mathbb{R}^d)$ , see [15] and Subsection 3.3. Here it is also important to remark that (2.11) is true under the sole assumptions **(A0)**–**(A2)** whenever  $\lambda > 0$  (which is a mere consequence of (2.13) and (2.14) below). In that case, it ensures that  $B_2(M_1, R)$  (the closed Euclidean ball) is preserved by the flow *without the assumption that  $W$  is radial*. As a result, it may be tempting to address the analysis below without requiring the potential to be radial. Nevertheless, the problem is that the numerical scheme does not satisfy a similar property. Indeed, the Euclidean ball  $B_2(M_1, R)$  is not convex from a numerical point of view, that is to say, if we regard the mesh underpinning the scheme, then the union of the square cells whose center is included in  $B_2(M_1, R)$  is not convex. Due to this drawback, the flow associated to the scheme does not preserve the ball  $B_2(M_1, R)$ . This is in contrast with Lemma 3.3 below, which shows that, in the radial setting, the ball  $B_\infty(M_1, R + \Delta x)$  is kept stable by the scheme, where  $\Delta x$  is the step of the spatial mesh. This latter fact is the reason why we here assume that the potential is radial.

**Proof.** For the first two statements of the Theorem, existence of a unique solution and Lipschitz-continuity of the flow, we refer to [16]. These statements remain true whenever the sole **(A0)**–**(A2)** hold true,  $W$  is radial,  $\lambda$  is (strictly) positive and the support of  $\rho^{ini}$  is bounded, provided that the notion of solution is limited to collections  $(\rho(t, \cdot))_{t \geq 0}$  that have a compact support, uniformly in  $t$  in compact subsets. Indeed, if we denote by  $M_1(t)$  the center of mass of the solution at time  $t$ , namely  $M_1(t) := \int_{\mathbb{R}^d} \rho(t, dx)$ , then this center of mass is known to be preserved:  $M_1(t) = M_1(0) = M_1$  (see [16] or Lemma 3.2 below for the discrete counterpart). Now, if  $\lambda \geq 0$  and if  $W$  is radial,  $\nabla W(x - y)$  is positively proportional to  $x - y$ , so that  $-\nabla W(x - y)$  is parallel to  $x - y$  and directed from  $x$  to  $y$ . Thus, if  $\rho(t)$  is zero outside the ball  $B_\infty(M_1, R)$ , then, for any  $x \in \partial B_\infty(M_1, R)$ , the velocity  $\hat{a}_\rho(t, x)$  is directed toward the interior of  $B_\infty(M_1, R)$ . This shows that  $B_\infty(M_1, R)$  is preserved by the flow and guarantees that  $\rho(t)$  has its support included in  $B_\infty(M_1, R)$  for any time  $t \geq 0$ , if it is the case for  $t = 0$ . Given the fact that the support of  $\rho(t)$  remains bounded in  $B_\infty(M_1, R)$ , everything works as if  $W$  was globally Lipschitz-continuous. Existence and uniqueness of a solution to the aggregation equation can thus be proved by a straightforward localization argument. Indeed, observe that from the very definition of the velocity  $a$ , the Lipschitz-continuity constant of  $W$  that is involved in the existence and uniqueness theory is the local one of  $W$  on the compact subset  $B_\infty(M_1, R)$ , provided that the support of  $\rho^{ini}$  is included in  $B_\infty(M_1, R)$ .

Now it only remains to prove the two inequalities regarding the Wasserstein distance between solutions starting from different data. Under assumptions **(A0)**–**(A3)** on the potential, it was proven in [16], but with a constant  $2|\lambda|$  instead of  $|\lambda|$  in the exponential (as in [24] and [28], in which the convolution operator is however replaced with a slightly more general integral operator), thus we here provide a proof of the present better estimate.

We consider the two Filippov flows  $(Z_\rho(t, \cdot))_{t \geq 0}$  and  $(Z_{\rho'}(t, \cdot))_{t \geq 0}$  as defined in the statement of Theorem 2.2. We recall that

$$Z_\rho(t, \cdot) \# \rho^{ini} = \rho(t, \cdot), \quad Z_{\rho'}(t, \cdot) \# \rho^{ini, j} = \rho'(t, \cdot), \quad t \geq 0. \quad (2.12)$$



To simplify, we just write  $Z(t, \cdot) = Z_\rho(t, \cdot)$  and  $Z'(t, \cdot) = Z_{\rho'}(t, \cdot)$ . Then, for any  $x, y \in \mathbb{R}^d$  and  $t \geq 0$ ,

$$\begin{aligned} & \frac{d}{dt} |Z(t, x) - Z'(t, y)|^2 \\ &= -2 \left\langle Z(t, x) - Z'(t, y), \right. \\ & \quad \left. \int_{\mathbb{R}^d} \widehat{\nabla W}(Z(t, x) - Z(t, x')) \rho^{ini}(dx') - \int_{\mathbb{R}^d} \widehat{\nabla W}(Z'(t, y) - Z'(t, y')) \rho^{ini,'}(dy') \right\rangle. \end{aligned}$$

Call  $\pi \in \Gamma_0(\rho^{ini}, \rho^{ini,'})$  an optimal plan between  $\rho^{ini}$  and  $\rho^{ini,}'$ . Then,

$$\begin{aligned} & \frac{d}{dt} |Z(t, x) - Z'(t, y)|^2 \\ &= -2 \left\langle Z(t, x) - Z'(t, y), \int_{\mathbb{R}^d} [\widehat{\nabla W}(Z(t, x) - Z(t, x')) - \widehat{\nabla W}(Z'(t, y) - Z'(t, y'))] \pi(dx', dy') \right\rangle. \end{aligned}$$

Integrating in  $(x, y)$  with respect to  $\pi$ , we get

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |Z(t, x) - Z'(t, y)|^2 \pi(dx, dy) \\ &= -2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\langle Z(t, x) - Z'(t, y), \right. \\ & \quad \left. [\widehat{\nabla W}(Z(t, x) - Z(t, x')) - \widehat{\nabla W}(Z'(t, y) - Z'(t, y'))] \right\rangle \pi(dx, dy) \pi(dx', dy'). \end{aligned}$$

Thanks to the fact that  $\widehat{\nabla W}$  is odd, see **(A0)**, we can write, by a symmetry argument,

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |Z(t, x) - Z'(t, y)|^2 \pi(dx, dy) \\ &= - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\langle Z(t, x) - Z'(t, y) - (Z(t, x') - Z'(t, y')), \right. \\ & \quad \left. [\widehat{\nabla W}(Z(t, x) - Z(t, x')) - \widehat{\nabla W}(Z'(t, y) - Z'(t, y'))] \right\rangle \pi(dx, dy) \pi(dx', dy'). \end{aligned}$$

Using (2.7), we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |Z(t, x) - Z'(t, y)|^2 \pi(dx, dy) \\ & \leq -\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |Z(t, x) - Z'(t, y) - (Z(t, x') - Z'(t, y'))|^2 \pi(dx, dy) \pi(dx', dy'). \end{aligned} \tag{2.13}$$

Observe that the above right-hand side is equal to

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |Z(t, x) - Z'(t, y) - (Z(t, x') - Z'(t, y'))|^2 \pi(dx, dy) \pi(dx', dy') \\ &= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |Z(t, x) - Z'(t, y)|^2 \pi(dx, dy) - 2 \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (Z(t, x) - Z'(t, y)) \pi(dx, dy) \right|^2. \end{aligned} \tag{2.14}$$

**1st case.** If  $\lambda \leq 0$ , we deduce from (2.13) and (2.14) that

$$\frac{d}{dt} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |Z(t, x) - Z'(t, y)|^2 \pi(dx, dy) \leq 2|\lambda| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |Z(t, x) - Z'(t, y)|^2 \pi(dx, dy),$$

which suffices to complete the proof of the first claim by noting that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |Z(0, x) - Z'(0, y)|^2 \pi(dx, dy) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^2 \pi(dx, dy) = d_W(\rho^{ini}, \rho^{ini'})^2,$$

and

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |Z(t, x) - Z'(t, y)|^2 \pi(dx, dy) \geq d_W(\rho(t), \rho(t'))^2,$$

see (2.12).

**2nd case.** If  $\lambda \geq 0$ , we just use the fact that the right-hand side in (2.13) is non-positive. Proceeding as above, this permits to complete the proof of the second claim.  $\square$

## 2.3 Main result

The aim of this paper is to prove the convergence at order 1/2 of an upwind type scheme in distance  $d_W$  for the aggregation equation. The numerical scheme is defined as follows. We denote by  $\Delta t$  the time step and consider a Cartesian grid with step  $\Delta x_i$  in the  $i$ th direction,  $i = 1, \dots, d$ ; we then let  $\Delta x := \max_i \Delta x_i$ . We also introduce the following notations. For a multi-index  $J = (J_1, \dots, J_d) \in \mathbb{Z}^d$ , we call  $C_J := [(J_1 - \frac{1}{2})\Delta x_1, (J_1 + \frac{1}{2})\Delta x_1) \times \dots \times [(J_d - \frac{1}{2})\Delta x_d, (J_d + \frac{1}{2})\Delta x_d)$  the corresponding elementary cell. The center of the cell is denoted by  $x_J := (J_1\Delta x_1, \dots, J_d\Delta x_d)$ . Also, we let  $e_i := (0, \dots, 1, \dots, 0)$  be the  $i$ th vector of the canonical basis, for  $i \in \{1, \dots, d\}$ , and we expand the velocity field in the canonical basis under the form  $a = (a_1, \dots, a_d)$ .

For a given nonnegative measure  $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$ , we put, for any  $J \in \mathbb{Z}^d$ ,

$$\rho_J^0 := \int_{C_J} \rho^{ini}(dx) \geq 0. \quad (2.15)$$

Since  $\rho^{ini}$  is a probability measure, the total mass of the system is  $\sum_{J \in \mathbb{Z}^d} \rho_J^0 = 1$ . We then construct iteratively the collection  $((\rho_J^n)_{J \in \mathbb{Z}^d})_{n \in \mathbb{N}}$ , each  $\rho_J^n$  being intended to provide an approximation of the value  $\rho(t^n, x_J)$ , for  $J \in \mathbb{Z}^d$ . Assuming that the approximating sequence  $(\rho_J^n)_{J \in \mathbb{Z}^d}$  is already given at time  $t^n := n\Delta t$ , we compute the approximation at time  $t^{n+1}$  by:

$$\rho_J^{n+1} := \rho_J^n - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} \left( (a_{iJ})^+ \rho_J^n - (a_{iJ+e_i})^- \rho_{J+e_i}^n - (a_{iJ-e_i})^+ \rho_{J-e_i}^n + (a_{iJ})^- \rho_J^n \right). \quad (2.16)$$

The notation  $(a)^+ = \max\{0, a\}$  stands for the positive part of the real  $a$  and respectively  $(a)^- = \max\{0, -a\}$  for the negative part. The macroscopic velocity is defined by

$$a_{iJ}^n := - \sum_{K \in \mathbb{Z}^d} \rho_K^n D_i W_J^K, \quad \text{where} \quad D_i W_J^K := \widehat{\partial_{x_i} W}(x_J - x_K). \quad (2.17)$$

Since  $W$  is even, we also have:

$$D_i W_J^K = -D_i W_K^J. \quad (2.18)$$

The main result of this paper is the proof of the convergence at order 1/2 of the above upwind scheme. More precisely the statement reads:

**Theorem 2.2** (i) *Assume that  $W$  satisfies hypotheses (A0)–(A3) and that the so-called CFL condition holds:*

$$w_\infty \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} \leq 1, \quad (2.19)$$

with  $w_\infty$  as in (1.2).

For  $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$ , let  $\rho = (\rho(t))_{t \geq 0}$  be the unique measure solution to the aggregation equation with initial data  $\rho^{ini}$ , as given by Theorem 2.1. Define  $((\rho_j^n)_{j \in \mathbb{Z}^d})_{n \in \mathbb{N}}$  as in (2.15)–(2.16)–(2.17) and let

$$\rho_{\Delta x}^n := \sum_{J \in \mathbb{Z}^d} \rho_J^n \delta_{x_J}, \quad n \in \mathbb{N}.$$

Then, there exists a nonnegative constant  $C$ , only depending on  $\lambda$ ,  $w_\infty$  and  $d$ , such that, for all  $n \in \mathbb{N}^*$ ,

$$d_W(\rho(t^n), \rho_{\Delta x}^n) \leq C e^{|\lambda|(1+\Delta t)t^n} (\sqrt{t^n \Delta x} + \Delta x). \quad (2.20)$$

(ii) Assume that  $W$  is radial and satisfies hypotheses **(A0)**–**(A2)** with  $\lambda$  (strictly) positive, that  $\rho^{ini}$  is compactly supported in  $B_\infty(M_1, R)$  where  $M_1$  is the center of mass of  $\rho^{ini}$ , and that the CFL condition (2.19) holds, with  $w_\infty$  defined as

$$w_\infty = \sup_{x \in B_\infty(0, 2R+2\Delta x) \setminus \{0\}} |\nabla W(x)|, \quad (2.21)$$

Assume also that  $\Delta t \leq 1/2$  and  $2\lambda\Delta t < 1$ . Then, there exists a nonnegative constant  $C$ , only depending on  $\lambda$ ,  $w_\infty$ ,  $d$  and  $R$  such that, for all  $n \in \mathbb{N}^*$ , (2.20) is valid, as well as

$$d_W(\rho(t^n), \rho_{\Delta x}^n) \leq C (\sqrt{\Delta x} + \Delta x), \quad (2.22)$$

which proves that the error can be uniformly controlled in time.

We stress the fact that, under the setting defined in (ii), (2.20) is valid. In small time, it provides a better estimate than (2.22). As indicated in the statement, the constant  $C$  in (2.22) may depend on the value of  $R$  in the assumption  $\text{Supp}(\rho^{ini}) \subset B_\infty(M_1, R)$ .

We also point out that, although the computations below are performed for the sole upwind scheme, the first part of the statement, which holds true under the full set of hypotheses **(A0)**–**(A3)**, can be straightforwardly adapted to other diffusive schemes, see for instance our previous article [22]. As for (ii), the statement remains true provided that the supports of the approximating measures  $(\rho^n)_{n \geq 0}$  remain bounded as  $n$  grows up. It must be stressed that there are some schemes for which the latter property fails (e.g. Lax-Friedrichs' scheme).

Moreover, as already mentioned in Introduction, the convergence rate is optimal; this latter fact will be illustrated by numerical examples in Section 6.

**Example 2.3** In one dimension, the scheme (2.16) reads

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x} \left( (a_i^n)^+ \rho_i^n - (a_{i+1}^n)^- \rho_{i+1}^n - (a_{i-1}^n)^+ \rho_{i-1}^n + (a_i^n)^- \rho_i^n \right),$$

where  $i$  is just taken in  $\mathbb{Z}$ . The scheme has then the following interpretation. Given  $\rho_{\Delta x}^n = \sum_{j \in \mathbb{Z}} \rho_j^n \delta_{x_j}$ , we construct the approximation at time  $t^{n+1}$  by implementing the following two steps:

- The Delta mass  $\rho_i^n$  located at position  $x_i$  moves with velocity  $a_i^n$  to the position  $x_i + a_i^n \Delta t$ . Under the CFL condition  $w_\infty \Delta t \leq \Delta x$ , the point  $x_i + a_i^n \Delta t$  belongs to the interval  $[x_i, x_{i+1}]$  if  $a_i^n \geq 0$ , and to the interval  $[x_{i-1}, x_i]$  if  $a_i^n \leq 0$ .
- Then the mass  $\rho_i^n$  is split into two parts; if  $a_i^n \geq 0$ , a fraction  $a_i^n \Delta t / \Delta x$  of it is transported to the cell  $i+1$ , while the remaining fraction is left in cell  $i$ ; if  $a_i^n \leq 0$ , the same fraction  $|a_i^n| \Delta t / \Delta x$  of the mass is not transported to the cell  $i+1$  but to the cell  $i-1$ . This procedure may be regarded as a linear interpolation of the mass  $\rho_i^n$  between the points  $x_i$  and  $x_{i+1}$  if  $a_i^n \geq 0$  and between the points  $x_i$  and  $x_{i-1}$  if  $a_i^n \leq 0$ .

Finally, we emphasize that this scheme differs from the standard finite volume upwind scheme in which the velocity is computed at the interface  $a_{i+1/2}^n$ . This subtlety is due to the particular structure of the equation, as the latter requires the product  $\widehat{a}_\rho \rho$  to be defined properly. A convenient way to make it proper is to compute, in the discretization, the velocity and the density at the same grid points. This fact has already been noticed in [37, 31] and is also illustrated numerically in Section 6.

### 3 Numerical approximation

#### 3.1 Properties of the scheme

The following lemma explains why we called CFL the condition on the ratios  $(\Delta t / \Delta x_i)_{i=1, \dots, d}$  that we formulated in the statement of Theorem 2.2.

**Lemma 3.1** *Assume that  $W$  satisfies hypotheses (A0)–(A3) and that the CFL condition (2.19) is in force. For  $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$ , define  $(\rho_J^0)_{J \in \mathbb{Z}^d}$  by (2.15). Then the sequences  $(\rho_J^n)_{n \in \mathbb{N}, J \in \mathbb{Z}^d}$  and  $(a_{iJ}^n)_{n \in \mathbb{N}, J \in \mathbb{Z}^d}$ ,  $i = 1, \dots, d$ , given by the scheme defined in (2.16)–(2.17), satisfy, for all  $J \in \mathbb{Z}^d$  and  $n \in \mathbb{N}$ ,*

$$\rho_J^n \geq 0, \quad |a_{iJ}^n| \leq w_\infty, \quad i = 1, \dots, d,$$

and, for all  $n \in \mathbb{N}$ ,

$$\sum_{J \in \mathbb{Z}^d} \rho_J^n = 1.$$

**Proof.** The total initial mass of the system is  $\sum_J \rho_J^0 = 1$ . By summing equation (2.16) over  $J$ , we can show that the mass is conservative, namely, for all  $n \in \mathbb{N}^*$ ,  $\sum_J \rho_J^n = \sum_J \rho_J^0 = 1$ .

Also, we can rewrite equation (2.16) as

$$\rho_J^{n+1} = \rho_J^n \left[ 1 - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} |a_{iJ}^n| \right] + \sum_{i=1}^d \rho_{J+e_i}^n \frac{\Delta t}{\Delta x_i} (a_{iJ+e_i}^n)^- + \sum_{i=1}^d \rho_{J-e_i}^n \frac{\Delta t}{\Delta x_i} (a_{iJ-e_i}^n)^+. \quad (3.23)$$

We prove by induction on  $n$  that  $\rho_J^n \geq 0$  for all  $J \in \mathbb{Z}^d$  and for all  $n \in \mathbb{N}$ . Indeed, if, for some  $n \in \mathbb{N}$ , it holds  $\rho_J^n \geq 0$  for all  $J \in \mathbb{Z}^d$ , then, by definition (2.17) and assumption (1.2), we clearly have

$$|a_{iJ}^n| \leq w_\infty \sum_{K \in \mathbb{Z}^d} \rho_K^n = w_\infty, \quad i = 1, \dots, d.$$

Then, assuming that the condition (2.19) holds, we deduce that, in the relationship (3.23), all the coefficients in front of  $\rho_J^n$ ,  $\rho_{J-e_i}^n$  and  $\rho_{J+e_i}^n$ ,  $i = 1, \dots, d$ , are nonnegative. Thus, using the induction assumption, we deduce that  $\rho_J^{n+1} \geq 0$  for all  $J \in \mathbb{Z}^d$ .  $\square$

In the following lemma, we collect two additional properties of the scheme: the conservation of the center of mass and the finiteness of the second order moment.

**Lemma 3.2** *Let  $W$  satisfy (A0)–(A3) and condition (2.19) be in force. For  $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$ , define  $(\rho_J^0)_{J \in \mathbb{Z}^d}$  by (2.15). Then, the sequence  $(\rho_J^n)_{J \in \mathbb{Z}^d}$  given by the numerical scheme (2.16)–(2.17) satisfies:*

(i) *Conservation of the center of mass. For all  $n \in \mathbb{N}^*$ ,*

$$\sum_{J \in \mathbb{Z}^d} x_J \rho_J^n = \sum_{J \in \mathbb{Z}^d} x_J \rho_J^0.$$

We will denote the right-hand side (and thus the left-hand side as well) by  $M_{1,\Delta x}$ .

(ii) Bound on the second moment. There exists a constant  $C > 0$ , independent of the parameters of the mesh, such that, for all  $n \in \mathbb{N}^*$ ,

$$M_{2,\Delta x}^n := \sum_{J \in \mathbb{Z}^d} |x_J|^2 \rho_J^n \leq e^{Ct^n} (M_{2,\Delta x}^0 + C),$$

where we recall that  $t^n = n\Delta t$ .

**Proof.** We recall from Lemma 3.1 that, for all  $n \in \mathbb{N}$ , the sequence  $(\rho_J^n)_{J \in \mathbb{Z}^d}$  is nonnegative and that its sum is equal to 1.

(i) Using (2.16) together with a discrete integration by parts, we have:

$$\sum_{J \in \mathbb{Z}^d} x_J \rho_J^{n+1} = \sum_{J \in \mathbb{Z}^d} x_J \rho_J^n - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} \sum_{J \in \mathbb{Z}^d} \left( (a_{iJ}^n)^+ \rho_J^n (x_J - x_{J+e_i}) - (a_{iJ}^n)^- \rho_J^n (x_{J-e_i} - x_J) \right).$$

By definition of  $x_J$ , we deduce

$$\sum_{J \in \mathbb{Z}^d} x_J \rho_J^{n+1} = \sum_{J \in \mathbb{Z}^d} x_J \rho_J^n + \Delta t \sum_{i=1}^d \sum_{J \in \mathbb{Z}^d} a_{iJ}^n \rho_J^n.$$

By definition of the macroscopic velocity (2.17) and by (2.18), we also have

$$\begin{aligned} \sum_{J \in \mathbb{Z}^d} a_{iJ}^n \rho_J^n &= - \sum_{J \in \mathbb{Z}^d} \sum_{K \in \mathbb{Z}^d} D_i W_J^K \rho_K^n \rho_J^n = \sum_{J \in \mathbb{Z}^d} \sum_{K \in \mathbb{Z}^d} D_i W_K^J \rho_K^n \rho_J^n \\ &= \sum_{J \in \mathbb{Z}^d} \sum_{K \in \mathbb{Z}^d} D_i W_J^K \rho_K^n \rho_J^n, \end{aligned}$$

where we exchanged the role of  $J$  and  $K$  in the latter sum. We deduce that it vanishes. Thus,

$$\sum_{J \in \mathbb{Z}^d} x_J \rho_J^{n+1} = \sum_{J \in \mathbb{Z}^d} x_J \rho_J^n.$$

(ii) For the second moment, still using (2.16) and a similar discrete integration by parts, we get

$$\begin{aligned} \sum_{J \in \mathbb{Z}^d} |x_J|^2 \rho_J^{n+1} &= \sum_{J \in \mathbb{Z}^d} |x_J|^2 \rho_J^n \\ &\quad - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} \sum_{J \in \mathbb{Z}^d} \left[ (a_{iJ}^n)^+ \rho_J^n (|x_J|^2 - |x_{J+e_i}|^2) - (a_{iJ}^n)^- \rho_J^n (|x_{J-e_i}|^2 - |x_J|^2) \right]. \end{aligned}$$

By definition of  $x_J$ ,  $|x_J|^2 - |x_{J+e_i}|^2 = -2J_i \Delta x_i^2 - \Delta x_i^2$  and  $|x_{J-e_i}|^2 - |x_J|^2 = -2J_i \Delta x_i^2 + \Delta x_i^2$ . Therefore, we get

$$\sum_{J \in \mathbb{Z}^d} |x_J|^2 \rho_J^{n+1} = \sum_{J \in \mathbb{Z}^d} |x_J|^2 \rho_J^n + 2\Delta t \sum_{i=1}^d \sum_{J \in \mathbb{Z}^d} J_i \Delta x_i a_{iJ}^n \rho_J^n + \Delta t \sum_{i=1}^d \Delta x_i \sum_{J \in \mathbb{Z}^d} \rho_J^n |a_{iJ}^n|.$$

As a consequence of Lemma 3.1, we have  $|a_{iJ}^n| \leq w_\infty$ . Using moreover the mass conservation, we deduce that the last term is bounded by  $w_\infty \Delta t \sum_{i=1}^d \Delta x_i$ . Moreover, applying Young's inequality and using the mass conservation again, we get

$$\left| \sum_{J \in \mathbb{Z}^d} a_{iJ}^n \rho_J^n J_i \Delta x_i \right| \leq \frac{1}{2} \left( w_\infty^2 + \sum_{J \in \mathbb{Z}^d} |J_i \Delta x_i|^2 \rho_J^n \right) \leq \frac{1}{2} \left( w_\infty^2 + \sum_{J \in \mathbb{Z}^d} \rho_J^n |x_J^n|^2 \right).$$

We deduce then that there exists a nonnegative constant  $C$  only depending on  $d$  and  $w_\infty$  such that

$$\sum_{J \in \mathbb{Z}^d} |x_J|^2 \rho_J^{n+1} \leq (1 + C\Delta t) \sum_{J \in \mathbb{Z}^d} |x_J|^2 \rho_J^n + C\Delta t \left( \sum_{i=1}^d \Delta x_i + 1 \right).$$

We conclude the proof using a discrete version of Gronwall's lemma.  $\square$

In case when  $W$  is radial and satisfies **(A0)**–**(A2)**,  $\lambda$  is (strictly) positive and  $\rho^{ini}$  has a bounded support, Lemmas 3.1 and 3.2 become:

**Lemma 3.3** *Assume that  $W$  is radial and satisfies **(A0)**–**(A2)**,  $\lambda$  is (strictly positive) and  $\rho^{ini}$  has a bounded support, then the conclusions of Lemmas 3.1 and 3.2 remain true provided that  $w_\infty$  is defined as in (2.21).*

Moreover, for any  $R \geq 0$  such that  $\text{Supp}(\rho^{ini}) \subset B_\infty(M_1, R)$ , it holds, for any  $n \in \mathbb{N}$ ,

$$\text{Supp}(\rho_{\Delta x}^n) \subset B_\infty(M_{1, \Delta x}, R + \Delta x),$$

that is

$$\forall J \in \mathbb{Z}^d, \quad x_J \notin B_\infty(M_{1, \Delta x}, R + \Delta x) \Rightarrow \rho_J^n = 0.$$

The meaning of Lemma 3.3 is pretty clear. For  $R$  as in the statement, the mass, as defined by the numerical scheme, cannot leave the ball  $B_\infty(M_{1, \Delta x}, R + \Delta x)$ . We here recover the same idea as in Theorem 2.1.

**Proof.** As long as we can prove that the mass, as defined by the numerical scheme, cannot leave the ball  $B_\infty(M_{1, \Delta x}, R + \Delta x)$ , the proof is similar to that of Lemmas 3.1 and 3.2. So, we focus on the second part of the statement.

We first recall that  $\rho_J^0 = \int_{C_J} \rho^{ini}(x) dx$ , for  $J \in \mathbb{Z}^d$ . Hence, if  $x_J \notin B_\infty(M_{1, \Delta x}, R + \Delta x)$ , we have  $x_J \notin B_\infty(M_1, R + \Delta x/2)$  and then  $C_J \cap B_\infty(M_1, R) = \emptyset$  and thus  $\rho_J^0 = 0$ . Below, we prove by induction that the same holds true for any  $n \in \mathbb{N}$ . To do so, we assume that there exists an integer  $n \in \mathbb{N}$  such that, for all  $J \in \mathbb{Z}^d$ ,  $\rho_J^n = 0$  if

$$x_J \notin B_\infty(M_{1, \Delta x}, R + \Delta x). \quad (3.24)$$

The goal is then to prove that, for any  $J$  satisfying (3.24),  $\rho_J^{n+1} = 0$ . By (3.23), it suffices to prove that, for any coordinate  $i \in \{1, \dots, d\}$  and any  $J$  as in (3.24),

$$\rho_{J+e_i}^n (a_{i, J+e_i}^n)^- = 0, \quad \text{and} \quad \rho_{J-e_i}^n (a_{i, J-e_i}^n)^+ = 0. \quad (3.25)$$

Without any loss of generality, we can assume that there exists a coordinate  $i_0 \in \{1, \dots, d\}$  such that  $(x_J)_{i_0} > R + \Delta x + (M_{1, \Delta x})_{i_0}$  (otherwise  $(x_J)_{i_0} < -R - \Delta x + (M_{1, \Delta x})_{i_0}$  and the argument below is the same). Hence,  $(x_{J+e_{i_0}})_{i_0} > R + \Delta x + (M_{1, \Delta x})_{i_0}$  and, by the induction hypothesis,  $\rho_{J+e_{i_0}}^n = 0$ , which proves the first equality in (3.25) when  $i = i_0$ . In order to prove the second equality when  $i = i_0$ , we notice from (2.17) that

$$\begin{aligned} a_{i_0, J-e_{i_0}}^n &= - \sum_{K \in \mathbb{Z}^d} \rho_K^n \widehat{\partial_{x_{i_0}} W}(x_{J-e_{i_0}} - x_K) = - \sum_{K \in \mathbb{Z}^d: (x_K)_{i_0} \leq R + \Delta x + (M_{1, \Delta x})_{i_0}} \rho_K^n \widehat{\partial_{x_{i_0}} W}(x_{J-e_{i_0}} - x_K) \\ &= - \sum_{K \in \mathbb{Z}^d: (x_K)_{i_0} < (x_J)_{i_0}} \rho_K^n \widehat{\partial_{x_{i_0}} W}(x_{J-e_{i_0}} - x_K) \\ &= - \sum_{K \in \mathbb{Z}^d: (x_K)_{i_0} \leq (x_{J-e_{i_0}})_{i_0}} \rho_K^n \widehat{\partial_{x_{i_0}} W}(x_{J-e_{i_0}} - x_K). \end{aligned}$$

As  $W$  is radial and  $\lambda > 0$ ,  $\nabla W(x-y)$  is positively proportional to  $x-y$ . Hence,  $\widehat{\partial_{x_{i_0}} W}(x_{J-e_{i_0}} - x_K) \geq 0$  when  $(x_K)_{i_0} \leq (x_{J-e_{i_0}})_{i_0}$ . Therefore,  $(a_{i_0}^n_{J-e_{i_0}})^+ = 0$ , which proves the second equality in (3.25).

It remains to prove (3.25) for  $i \neq i_0$ . Obviously,  $(x_{J-e_i}^n)_{i_0} = (x_{J+e_i}^n)_{i_0} = (x_i^n)_{i_0} > R + \Delta x + (M_{1,\Delta x})_{i_0}$ . By the induction hypothesis,  $\rho_{J-e_i}^n = \rho_{J+e_i}^n = 0$ , which completes the proof.  $\square$

**Remark 3.4** *Lemma 3.3 is the main rationale for requiring  $W$  to be radial. Indeed, the counter-example below shows that the growth of the support of  $\rho^{ini}$  can be hardly controlled whenever  $\lambda > 0$  and  $W$  is just assumed to satisfy **(A0)**–**(A2)**. Consider for instance the following potential in dimension  $d = 2$ :*

$$W(x_1, x_2) = \frac{1}{2}(x_1 - qx_2)^2 + \frac{q^2}{2}x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where  $q$  is a free integer whose value will be fixed later on. It is well checked that

$$\partial_{x_1} W(x_1, x_2) = x_1 - qx_2, \quad \partial_{x_2} W(x_1, x_2) = q(qx_2 - x_1) + q^2x_2.$$

Standard computations show that the smallest eigenvalue of the Hessian matrix (which is independent of  $(x_1, x_2)$ ) is

$$\frac{(1 + 2q^2) - 2q^2\sqrt{1 + 1/(4q^4)}}{2} \underset{q \rightarrow \infty}{\sim} \frac{1}{2},$$

so that  $W$  is  $\lambda$ -convex with  $\lambda$  converging to  $1/2$  as  $q$  tends to  $\infty$ .

Take now a centered probability measure  $\rho$  and compute the first coordinate of the velocity field  $\widehat{a}_\rho$ . By centering,

$$(\widehat{a}_\rho)_1(x_1, x_2) = qx_2 - x_1.$$

In particular, if  $x_2 = 1$ , then  $(\widehat{a}_\rho)_1(x_1, 1) = q - x_1$ , which is non-negative as long as  $x_1 < q$ . Therefore, if the numerical scheme is initialized with some centered  $\rho_{\Delta x}^0$  supported by the unit square  $[-1, 1]^2$ , it holds

$$(\widehat{a}_{\rho_{\Delta x}^0})_1(1, 1) > 0,$$

if  $q > 1$ . Hence, provided that the CFL condition holds true,  $\rho^1$  charges the point  $(1 + \Delta x, 1)$ . Since the numerical scheme preserves the centering, we also have

$$(\widehat{a}_{\rho_{\Delta x}^1})_1(1 + \Delta x, 1) > 0,$$

if  $q > 1 + \Delta x$ , and then  $\rho_{\Delta x}^2$  also charges the point  $(1 + 2\Delta x, 1)$ , and so on up until  $(\Delta x \lfloor q/\Delta x \rfloor, 1)$ . This says that there is no way to control the growth of the support of the numerical solution in terms of the sole lower bound of the Hessian matrix. Somehow, the growth of  $\nabla W$  plays a key role. This is in stark contrast with the support of the real solution, which may be bounded independently of  $q$ , as emphasized in the proof of Theorem 2.1.

A possible way to overcome the fact that the numerical scheme does not preserve any ball containing the initial support in the general case when  $W$  is not radial would be to truncate the scheme. We feel more reasonable not to address this question in this paper, as it would require to revisit in deep the arguments used to tackle the case  $\lambda \leq 0$ .

### 3.2 Probabilistic interpretation

In order to address the convergence of the numerical scheme (2.17)–(2.18), we adapt to our framework the probabilistic interpretation proposed in [21, 22]. It consists in associating random characteristics with the scheme when the CFL condition defined in the statement of Theorem 2.2 is in force (the definition of  $w_\infty$  depending on the sign of  $\lambda$  as we already explained). In [21, 22], these characteristics are constructed as the trajectories of a Markov chain admitting  $\mathbb{Z}^d$  as state space. Here the representation is pretty similar, except for one major fact. In order to account for the aggregation structure in (1.1), the transition matrix of the Markov chain involved in the representation of the scheme must indeed depend on the current marginal law of the chain. Accordingly, the chain cannot be a mere (or a standard) Markov chain; instead, it will be called a *nonlinear* Markov chain. The word *nonlinear* will be used to indicate that the local dynamics of the chain at time  $n$  heavily depend upon its marginal statistical distribution at the same time  $n$ . Equivalently, the chain could be also said to be of the *McKean-Vlasov* type. Alternatively, it could be regarded as a true Markov chain, but on the enlarged state space  $\mathbb{Z}^d \times \mathcal{P}(\mathbb{Z}^d)$ , where  $\mathcal{P}(\mathbb{Z}^d)$  denotes the set of probability measures on  $\mathbb{Z}^d$ .

Throughout the analysis, we will denote by  $\Omega = (\mathbb{Z}^d)^\mathbb{N}$  the canonical space for the nonlinear Markov chain. The canonical process is denoted by  $(K^n)_{n \in \mathbb{N}}$  (namely  $K^n$  maps  $\omega = (\omega^n) \in \Omega$  onto the  $n^{\text{th}}$  coordinate  $\omega^n$  of  $\omega$ ):  $K^n$  must be understood as the  $n^{\text{th}}$  site occupied by a random process taking values in  $\mathbb{Z}^d$ .

We equip  $\Omega$  with the Kolmogorov  $\sigma$ -field  $\mathcal{A}$  generated by sets of the form  $\prod_{n \in \mathbb{N}} A^n$ , with  $A^n \subset \mathbb{Z}^d$  for all  $n \in \mathbb{N}$  and, for some integer  $n_0$ ,  $A^n = \mathbb{Z}^d$  for  $n \geq n_0$  (in other words,  $\mathcal{A}$  is the smallest  $\sigma$ -field that renders each  $K^n$ ,  $n \in \mathbb{N}$ , measurable). Moreover, the canonical filtration generated by  $(K^n)_{n \in \mathbb{N}}$  is denoted by  $\mathbb{F} = (\mathcal{F}^n = \sigma(K^0, \dots, K^n))_{n \in \mathbb{N}}$ . In words,  $\mathcal{F}^n$  can be interpreted as the information that an observer would collect by observing the realizations of  $K^0, \dots, K^n$ . Given this setting, the chain can be constructed as follows.

The first step is to recover the framework used in [21, 22]. In order to do so, we shall disentangle the initial condition of the chain and the initial condition used to initialize the interaction in the aggregation structure. It will be in a second time only that we identify the two of them. So, for the time being, we fix an initial condition  $\varrho \in \mathcal{P}(\mathbb{Z}^d)$ , which can be seen as a non-negative valued sequence  $(\varrho_J)_{J \in \mathbb{Z}^d}$  with sum 1. Letting  $\rho^0 = \varrho$ , we then implement (2.16) to define the sequence  $(\rho^n)_{n \in \mathbb{N}}$ . This permits to compute the corresponding velocities  $((a_j^n)_{j \in \mathbb{Z}^d})_{n \in \mathbb{N}}$  by means of (2.17). Once the velocities are given, we can follow [21, 22]. Namely, we can endow the pair  $(\Omega, \mathcal{A})$  with a collection of probability measures  $(\mathbb{P}_{[\varrho], \mu})_{\mu \in \mathcal{P}(\mathbb{Z}^d)}$ , such that, for all  $\mu \in \mathcal{P}(\mathbb{Z}^d)$ ,  $(K^n)_{n \in \mathbb{N}}$  is a Markov chain under  $\mathbb{P}_{[\varrho], \mu}$  with initial distribution  $\mu$ , namely  $K^0 \# \mathbb{P}_{[\varrho], \mu} = \mu$ , and with transition matrix:

$$P_{J,L}[\rho^n] := \begin{cases} 1 - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} |a_{iJ}^n| & \text{when } L = J, \\ \frac{\Delta t}{\Delta x_i} (a_{iJ}^n)^+ & \text{when } L = J + e_i, \text{ for } i = 1, \dots, d, \\ \frac{\Delta t}{\Delta x_i} (a_{iJ}^n)^- & \text{when } L = J - e_i, \text{ for } i = 1, \dots, d, \\ 0 & \text{otherwise,} \end{cases} \quad (3.26)$$

where we used the definition of  $a_{iJ}^n$  in terms of  $\rho^n$ , see (2.17), under the prescription that  $(\rho^n)_{n \in \mathbb{N}}$  starts from  $\rho^0 = \varrho$ . We notice that  $K^0 \# \mathbb{P}_{[\varrho], \mu} = \mu$  means that for all  $J \in \mathbb{Z}^d$ ,  $\mathbb{P}_{[\varrho], \mu}(K^0 = J) = \mu(J)$ , that is, the initial starting cell is picked randomly according to the law  $\mu$  (sometimes it is denoted  $K^0 \sim \mu$ ). Of course, the rationale for the definition (3.26) comes from the following observation:



the scheme (2.16) may be rewritten as

$$\rho_J^{n+1} = \rho_J^n P_{J,J}[\rho^n] + \sum_{i=1}^d (\rho_{J+e_i}^n P_{J+e_i,J}[\rho^n] + \rho_{J-e_i}^n P_{J-e_i,J}[\rho^n]). \quad (3.27)$$

This must be compared with the following Kolmogorov equation satisfied by the marginal law of the chain:

$$\begin{aligned} \mathbb{P}_{[\varrho],\mu}(K^{n+1} = J) &= \mathbb{P}_{[\varrho],\mu}(K^n = J) P_{J,J}[\rho^n] \\ &+ \sum_{i=1}^d \left( \mathbb{P}_{[\varrho],\mu}(K^n = J + e_i) P_{J+e_i,J}[\rho^n] + \mathbb{P}_{[\varrho],\mu}(K^{n+1} = J - e_i) P_{J-e_i,J}[\rho^n] \right). \end{aligned} \quad (3.28)$$

For any  $\mu \in \mathcal{P}(\mathbb{Z}^d)$ , we write  $\mathbb{E}_{[\varrho],\mu}$  for the expectation under  $\mathbb{P}_{[\varrho],\mu}$ . Also, for any  $\mu \in \mathcal{P}(\mathbb{Z}^d)$  and any  $n \in \mathbb{N}$ , the conditional probability  $\mathbb{P}_{[\varrho],\mu}(\cdot | \mathcal{F}_n)$  is just denoted by  $\mathbb{P}_{[\varrho],\mu}^n(\cdot)$ ; similarly, the conditional expectation  $\mathbb{E}_{[\varrho],\mu}(\cdot | \mathcal{F}_n)$  is denoted by  $\mathbb{E}_{[\varrho],\mu}^n(\cdot)$ . Moreover, in statements that are true independently of the initial distribution  $\mu \in \mathcal{P}(\mathbb{Z}^d)$ , we often drop the index  $\mu$  in the symbols  $\mathbb{P}_{[\varrho],\mu}$  and/or  $\mathbb{E}_{[\varrho],\mu}$ . For instance, we may write:

$$\forall n \in \mathbb{N}, \forall L \in \mathbb{Z}^d, \quad \mathbb{P}_{[\varrho]}^n(K^{n+1} = L) = P_{K^n,L}[\rho^n].$$

Whenever  $\mu$  is the Dirac mass at some  $J \in \mathbb{Z}^d$ , namely  $\mu = \delta_J$ , we write  $\mathbb{P}_{[\varrho],J}$  instead of  $\mathbb{P}_{[\varrho],\mu}$  and similarly for  $\mathbb{E}_{[\varrho]}$ . Notice that, for any  $\mu \in \mathcal{P}(\mathbb{Z}^d)$ ,  $\mathbb{P}_{[\varrho],\mu}$  is entirely determined by  $\mu$  and the collection  $(\mathbb{P}_{[\varrho],J})_{J \in \mathbb{Z}^d}$ :

$$\mathbb{P}_{[\varrho],\mu}(\cdot) = \sum_{J \in \mathbb{Z}^d} \mu_J \mathbb{P}_{[\varrho],J}(\cdot).$$

Then, it generally suffices to restrict the analysis of the Markov chain to the cases when  $\mu = \delta_J$ , for  $J \in \mathbb{Z}^d$ .

So far, the Markov chain has been defined for a frozen value of the interaction kernel  $(\rho_K^n)_{K \in \mathbb{Z}^d}$  used to define the discrete velocity field  $a_i^n$  in (2.17). This is similar to the way we defined the flow  $Z_\rho$  in the statement of Theorem 2.1 for a given solution  $\rho$  of (1.1). Anyhow, now that we have constructed the Markov chain for a given initial condition  $\rho^0$  of the scheme, we may define the *nonlinear* Markov chain by letting

$$\mathbb{P}_\varrho = \mathbb{P}_{[\varrho],\varrho}, \quad \text{with } \varrho = \rho^0.$$

In other words, the nonlinear chain is obtained by identifying the initial distribution  $\mu$  (which is the initial distribution of the chain when the interaction kernel is given) and the initial distribution used to define the scheme. Of course, the crux is to notice, from (3.27) and (3.28), that

$$\forall n \in \mathbb{N}, \forall J \in \mathbb{Z}^d, \quad \mathbb{P}_\varrho(K^n = J) = \rho_J^n, \quad (3.29)$$

where  $(\rho^n)_{n \in \mathbb{N}}$  is obtained by implementing (2.16) with  $\rho^0 = \varrho$  as initial condition. As above, we sometimes write  $\mathbb{P}$  instead of  $\mathbb{P}_\varrho$  in statements where the role of  $\varrho$  is transparent.

We now have all the ingredients to define the corresponding random characteristics. The random characteristics consist of a sequence of random variables  $(X^n)_{n \in \mathbb{N}}$  from  $(\Omega, \mathcal{A})$  into  $\mathbb{R}^d$ :

$$\forall n \in \mathbb{N}, \forall \omega \in \Omega, \quad X^n(\omega) = x_{K^n(\omega)}, \quad (3.30)$$

where we recall that  $x_J = (J_1 \Delta x_1, \dots, J_d \Delta x_d)$  for  $J = (J_1, \dots, J_d) \in \mathbb{Z}^d$ .

The following proposition states that the sequence  $(\rho_{\Delta x}^n)_{n \in \mathbb{N}}$ , as given by the upwind scheme (2.16)-(2.17), is defined as a pushforward of the initial condition  $\rho_{\Delta x}^0$  by the random characteristics and that the mean velocity of the latter ones is a discretization of  $\widehat{a}_\rho$ , as defined in (2.5).

**Proposition 3.5** *For any initial distribution  $\rho^0 \in \mathcal{P}(\mathbb{Z}^d)$ , let  $(\rho_{\Delta x}^n)_{n \in \mathbb{N}}$  be defined as in Theorem 2.2 (with the appropriate definition of  $w_\infty$  in the CFL condition) and  $(X^n)_{n \in \mathbb{N}}$  be the sequence of random variables defined by (3.30) through the nonlinear Markov chain  $(K^n)_{n \in \mathbb{N}}$  admitting  $(P[\varrho])_{\varrho \in \mathcal{P}(\mathbb{Z}^d)}$  in (3.26) as transition matrices.*

(i) *For all  $\varrho \in \mathcal{P}(\mathbb{Z}^d)$  and  $J \in \mathbb{Z}^d$ , we have, with probability one under  $\mathbb{P}_{[\varrho], J}$ ,*

$$\mathbb{E}_{[\varrho], J}^n(X^{n+1} - X^n) = -\Delta t \int_{\mathbb{R}^d} \widehat{\nabla W}(X^n - y) \rho_{\Delta x}^n(dy), \quad (3.31)$$

when we use the initialization  $\rho^0 = \varrho$  to compute  $(\rho_{\Delta x}^n)_{n \in \mathbb{N}}$ . In particular, the same holds true under  $\mathbb{P}_\varrho$  with  $\mathbb{E}_\varrho^n$  in lieu of  $\mathbb{E}_{[\varrho], J}^n$ .

(ii) *We have, for all  $n \in \mathbb{N}$ ,  $\rho_{\Delta x}^n = X^n \# \mathbb{P}_{\rho^0}$ .*

**Proof.** (i) For  $\varrho \in \mathcal{P}(\mathbb{Z}^d)$  and  $J \in \mathbb{Z}^d$ , we compute the conditional expectation given the trajectory of the chain up until time  $t^n$ :

$$\mathbb{E}_{[\varrho], J}^n(X^{n+1} - X^n) = \sum_{i=1}^d \left( \Delta x_i e_i (a_{iK^n}^n)^+ \frac{\Delta t}{\Delta x_i} - \Delta x_i e_i (a_{iK^n}^n)^- \frac{\Delta t}{\Delta x_i} \right) = \Delta t \sum_{i=1}^d a_{iK^n}^n e_i.$$

Using definition (2.17), we have

$$\sum_{i=1}^d a_{iJ}^n e_i = - \sum_{K \in \mathbb{Z}^d} \widehat{\nabla W}(x_J - x_K) \rho_K^n = - \int_{\mathbb{R}^d} \widehat{\nabla W}(x_J - y) \rho_{\Delta x}^n(dy),$$

and we obtain (3.31).

(ii) For any  $\varrho \in \mathcal{P}(\mathbb{Z}^d)$ , we have, for all  $n \in \mathbb{N}$ ,

$$X^n \# \mathbb{P}_\varrho = \sum_{L \in \mathbb{Z}^d} \delta_{x_L} \mathbb{P}_\varrho(K^n = L).$$

We conclude by (3.29). □

### 3.3 The characteristics as a stochastic gradient flow

Formula (3.31) may be rewritten as follows.

$$X^{n+1} - X^n = -\Delta t \widetilde{\mathbb{E}}[\widehat{\nabla W}(X^n - \widetilde{X}^n)] + \Delta M_n, \quad (3.32)$$

where  $(\widetilde{X}^n)_{n \geq 0}$  is a sequence of random variables obtained by duplicating  $(X_n)_{n \in \mathbb{N}}$  on a copy  $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbb{P}})$  of the probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  (here we do not specify the initial condition of the chain  $(X_n)_{n \geq 0}$  as it may be arbitrarily chosen) and  $\widetilde{\mathbb{E}}$  denotes the expectation on  $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbb{P}})$ . In the above formulation,  $\Delta M_n$  is a martingale increment, namely it may be written as  $\Delta M_n := M_{n+1} - M_n$ , where  $(M_n)_{n \geq 0}$  is a martingale on  $(\Omega, \mathcal{A}, \mathbb{P})$  with 0 as initial condition. As a martingale, it satisfies:

$$\mathbb{E}^n[\Delta M_n] = 0,$$

or, equivalently,

$$X^{n+1} - X^n = \mathbb{E}^n[X^{n+1} - X^n] + \Delta M^n.$$

As explained in [21], the sequence  $(M_n)_{n \in \mathbb{N}}$  provides a probabilistic description of the numerical diffusion induced by the scheme. In this regard, it is worth computing its local variance. We claim:

$$\mathbb{E}^n[|\Delta M_n|^2] \leq \mathbb{E}^n[|X^{n+1} - X^n|^2] \leq \Delta t \sum_{i=1}^d \Delta x_i |a_{iK^n}^n|, \quad (3.33)$$

which shows that the numerical diffusion decays with the velocity.

In the specific framework of aggregation, equation (3.32) suggests the following interpretation. The sequence  $(X^n)_{n \in \mathbb{N}}$  forms a sort of stochastic gradient flow in  $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$ . Indeed, in case when  $W$  is a smooth functional, the function

$$L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni X \mapsto \frac{1}{2} \mathbb{E} \tilde{\mathbb{E}} [W(X - \tilde{X})],$$

where, as above, the symbol  $\sim$  is used to denote copies of the various random variables in hand on the space  $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ , is Fréchet differentiable and its Fréchet derivative at point  $X \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$  can be represented through Riesz' theorem by means of the random variable

$$\tilde{\mathbb{E}}[\nabla W(X - \tilde{X})].$$

We refer for instance to Chapter 5 in [13]. Of course, this observation is completely reminiscent of the fact that the original solution of the aggregation equation (1.1) can be interpreted as a gradient flow in the Wasserstein space, see [15]. Following Lions' approach to the differential calculus on the Wasserstein space (see also Chapter 5 in [13]), we here feel more convenient to regard the gradient flow structure in the space  $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$  above the Wasserstein space. This permits to distinguish more properly the influence of the numerical diffusion.

It is then worth recalling that, along a deterministic gradient flow, the potential driving the flow is non-increasing, namely for a solution  $\rho$  to (1.1), the function

$$[0, +\infty) \ni t \mapsto \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x - y) \rho(t, dx) \rho(t, dy)$$

is non-increasing, see [15] again. Because of the stochastic nature of the dynamics, or equivalently because of the numerical diffusion, we cannot prove here a similar result in full generality for the numerical scheme. Below, we just investigate several cases for which the question can be explicitly addressed, including a counter-example showing that the result cannot be true in all cases.

First, it is pretty obvious to check that the potential is non-increasing whenever  $W$  is concave on the whole space. Although this case is far from being the most interesting in practical applications, we give the proof of it, as it makes clear the influence of the numerical diffusion.

**Proposition 3.6** *Assume that  $W$  is even, concave and continuously differentiable on the whole space. Assume also that  $\rho^{ini}$  is compactly supported. Then, the potential*

$$\left( \frac{1}{2} \mathbb{E} \tilde{\mathbb{E}} [W(X^n - \tilde{X}^n)] = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x - y) \rho_{\Delta x}^n(dx) \rho_{\Delta x}^n(dy) \right)_{n \in \mathbb{N}},$$

*is non-increasing.*

Observe that the assumption on the support of  $\rho^{ini}$  is for convenience only. In short, it permits to work as if  $\nabla W$  was bounded, which is the assumption we used before for defining the scheme but which may be a bit restrictive here as  $W$  is also assumed to be concave, and thus to impose an appropriate CFL condition. Indeed, since the scheme propagates at a finite rate (which may depend on the discretization parameter), the random characteristics, when starting from a compact support, stay in a compact support in finite time. We let the reader adapt properly the form of the CFL condition.

**Proof.** It suffices to notice that, for any  $n \in \mathbb{N}$ ,

$$W(X^{n+1} - \tilde{X}^{n+1}) \leq W(X^n - \tilde{X}^n) + \langle X^{n+1} - X^n - (\tilde{X}^{n+1} - \tilde{X}^n), \nabla W(X^n - \tilde{X}^n) \rangle,$$

which follows from the concavity of  $W$ . Taking the expectation and using the fact that  $\nabla W$  is odd, we get:

$$\begin{aligned} \mathbb{E}\tilde{\mathbb{E}}[W(X^{n+1} - \tilde{X}^{n+1})] &\leq \mathbb{E}\tilde{\mathbb{E}}[W(X^n - \tilde{X}^n)] \\ &\quad + \mathbb{E}\left[\langle \mathbb{E}^n[X^{n+1} - X^n], \tilde{\mathbb{E}}[\nabla W(X^n - \tilde{X}^n)] \rangle\right] \\ &\quad + \tilde{\mathbb{E}}\left[\langle \tilde{\mathbb{E}}^n[\tilde{X}^{n+1} - \tilde{X}^n], \mathbb{E}[\nabla W(\tilde{X}^n - X^n)] \rangle\right], \end{aligned} \quad (3.34)$$

where  $\tilde{\mathbb{E}}^n$  denotes the conditional expectation given the  $\sigma$ -field  $\tilde{\mathcal{F}}^n$  obtained by duplicating  $\mathcal{F}^n$ .

Appealing to Proposition 3.5, we obtain:

$$\mathbb{E}\left[\langle \mathbb{E}^n[X^{n+1} - X^n], \tilde{\mathbb{E}}[\nabla W(X^n - \tilde{X}^n)] \rangle\right] = -\mathbb{E}\left[\left|\tilde{\mathbb{E}}[\nabla W(X^n - \tilde{X}^n)]\right|^2\right] \Delta t,$$

and similarly for the last term in (3.34). The result easily follows.  $\square$

Whenever  $W$  is convex (instead of concave), the previous argument completely fails. In that case, there is a competition between the gradient flow structure and the numerical diffusion, but, as shown by the result right below, the result may remain true.

**Proposition 3.7** *Take  $W(x) = |x|^2$  and assume that  $\rho^{ini}$  is compactly supported. Then, the potential*

$$\left(\frac{1}{2}\mathbb{E}\tilde{\mathbb{E}}[|X^n - \tilde{X}^n|^2] = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^2 \rho_{\Delta x}^n(dx) \rho_{\Delta x}^n(dy)\right)_{n \in \mathbb{N}},$$

*is non-increasing.*

It must be stressed that the left-hand side in the above equation may be rewritten as  $\mathbb{E}[|X^n - \mathbb{E}[X^n]|^2] = \mathbb{E}[|X^n|^2] - |\mathbb{E}[X^n]|^2$ .

**Proof.** For any  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \frac{1}{2}\mathbb{E}\tilde{\mathbb{E}}[|X^{n+1} - \tilde{X}^{n+1}|^2] &= \frac{1}{2}\mathbb{E}\tilde{\mathbb{E}}[|X^n - \tilde{X}^n|^2] + \mathbb{E}\tilde{\mathbb{E}}\left[\langle X^{n+1} - X^n - (\tilde{X}^{n+1} - \tilde{X}^n), X^n - \tilde{X}^n \rangle\right] \\ &\quad + \frac{1}{2}\mathbb{E}\tilde{\mathbb{E}}[|\Delta X^n - \Delta \tilde{X}^n|^2], \end{aligned} \quad (3.35)$$

where we have let

$$\Delta X^n := X^{n+1} - X^n.$$

The second term in the right-hand side of (3.35) can be treated as in the concave case. Namely,

$$\begin{aligned}
& \mathbb{E}\tilde{\mathbb{E}}\left[\langle X^{n+1} - X^n - (\tilde{X}^{n+1} - \tilde{X}^n), X^n - \tilde{X}^n \rangle\right] \\
&= \mathbb{E}\left[\langle \mathbb{E}^n(X^{n+1} - X^n), \tilde{\mathbb{E}}(X^n - \tilde{X}^n) \rangle\right] + \tilde{\mathbb{E}}\left[\langle \tilde{\mathbb{E}}^n(X^{n+1} - X^n), \mathbb{E}(\tilde{X}^n - X^n) \rangle\right] \\
&= -2\mathbb{E}\left[\tilde{\mathbb{E}}(X^n - \tilde{X}^n)^2\right]\Delta t - 2\tilde{\mathbb{E}}\left[\mathbb{E}(\tilde{X}^n - X^n)^2\right]\Delta t \\
&= -4\mathbb{E}\left[\tilde{\mathbb{E}}(X^n - \tilde{X}^n)^2\right]\Delta t = -4\mathbb{E}\left[|X^n - \mathbb{E}(X^n)|^2\right]\Delta t.
\end{aligned}$$

Now, the last term in (3.35) can be handled as follows.

$$\begin{aligned}
\mathbb{E}\tilde{\mathbb{E}}[|\Delta X^n - \Delta\tilde{X}^n|^2] &= 2\mathbb{E}\tilde{\mathbb{E}}[|\Delta X^n|^2] - 2\mathbb{E}\tilde{\mathbb{E}}[\langle \Delta X^n, \Delta\tilde{X}^n \rangle] \\
&= 2\mathbb{E}[|\Delta X^n|^2] - 2|\mathbb{E}[\Delta X^n]|^2.
\end{aligned}$$

As the last term right above is (together with the sign minus) non-positive (by the conservation of the center of mass, it is in fact null), it suffices to upper bound the first one. By (3.33), we know that

$$\mathbb{E}[|\Delta X^n|^2] \leq \Delta t \sum_{i=1}^d \Delta x_i \mathbb{E}[|a_{iK^n}^n|].$$

Now, keep in mind that  $a_{iK^n}^n = 2[(X^n)_i - \mathbb{E}[(X^n)_i]]$ . Therefore, the only thing we have to prove is

$$4\mathbb{E}\left[|X^n - \mathbb{E}[X^n]|^2\right] \geq 2 \sum_{i=1}^d \Delta x_i \mathbb{E}\left[|(X^n)_i - \mathbb{E}[(X^n)_i]|\right]. \quad (3.36)$$

The trick is notice (as already emphasized above) that

$$2\mathbb{E}\left[|X^n - \mathbb{E}[X^n]|^2\right] = 2\mathbb{E}[|X^n|^2] - 2|\mathbb{E}X^n|^2 = \mathbb{E}\tilde{\mathbb{E}}\left[|X^n - \tilde{X}^n|^2\right].$$

Hence,

$$2\mathbb{E}\left[|X^n - \mathbb{E}[X^n]|^2\right] = \sum_{i=1}^d \mathbb{E}\tilde{\mathbb{E}}\left[|(X^n)_i - (\tilde{X}^n)_i|^2\right] \geq \sum_{i=1}^d \Delta x_i \mathbb{E}\tilde{\mathbb{E}}\left[|(X^n)_i - (\tilde{X}^n)_i|\right],$$

where we used the fact that  $|(X^n)_i - (\tilde{X}^n)_i| \geq \Delta x_i$  if  $(X^n)_i \neq (\tilde{X}^n)_i$ . By entering the expectation  $\tilde{\mathbb{E}}$  inside the absolute value, we complete the proof of (3.36).  $\square$

We now provide a counter example, which shows that the potential may increase in the convex regime, as a consequence of the numerical diffusion.

**Proposition 3.8** *Choose  $d = 2$ ,  $W(x) = |x|$  and take  $\Delta x_1 = \Delta x_2 = 1$ . Let the initial condition of the scheme, which we just denote by  $\rho^0$ , charge the points  $0 = (0, 0)$ ,  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$  with  $1 - p$ ,  $p/2$  and  $p/2$  as respective weights, where  $p \in (0, 1)$ .*

*Then, denoting by  $\rho^1$  the distribution at time 1 obtained by implementing the upwind scheme, it holds that:*

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x - y| \rho^1(dx) \rho^1(dy) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x - y| \rho^0(dx) \rho^0(dy) + (\sqrt{2} - 1)p^2(2p - 1)\Delta t + O(\Delta t^2), \quad (3.37)$$

where the Landau symbol  $O(\cdot)$  may depend upon  $p$ .

Choosing  $p > 1/2$  in the above statement, we see that the potential may increase at the same rate as the time step.

**Proof.** We first compute the potential at time 0. To do so, we compute  $\int_{\mathbb{R}^2} |x - y| \rho^0(dy)$ , for  $x \in \{0, e_1, e_2\}$ :

$$\int_{\mathbb{R}^2} |y| \rho^0(dy) = p, \quad \int_{\mathbb{R}^2} |e_1 - y| \rho^0(dy) = \int_{\mathbb{R}^2} |e_2 - y| \rho^0(dy) = (1 - p) + \frac{p}{\sqrt{2}},$$

so that

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x - y| \rho^0(dx) \rho^0(dy) = 2(1 - p)p + \frac{p^2}{\sqrt{2}}.$$

In order to compute the potential at time 1, we compute the velocity at each of the above points. Observing that the velocity at point  $x$  is given by the formula:

$$a_{ix}^0 = \int_{\mathbb{R}^2} \frac{y_i - x_i}{|y - x|} \rho^0(dy), \quad i = 1, 2, \quad \text{with the convention } \frac{0}{0} = 0,$$

we get:

$$\begin{aligned} a_{1(0,0)}^0 &= \frac{p}{2}, & a_{2(0,0)}^0 &= \frac{p}{2}, \\ a_{1(1,0)}^0 &= -(1 - p) - \frac{p}{2\sqrt{2}}, & a_{2(1,0)}^0 &= \frac{p}{2\sqrt{2}}, \\ a_{1(0,1)}^0 &= \frac{p}{2\sqrt{2}}, & a_{2(0,1)}^0 &= -(1 - p) - \frac{p}{2\sqrt{2}}. \end{aligned}$$

We then compute the new masses at time 1. There is one additional point which is charged:  $e_1 + e_2 = (1, 1)$ . We have:

$$\begin{aligned} \rho^1(0) &= (1 - p) + \frac{p^2}{2\sqrt{2}} \Delta t, \\ \rho^1(e_1) &= \rho^1(e_2) = \frac{p}{2} - \frac{p^2}{2\sqrt{2}} \Delta t, \\ \rho^1(e_1 + e_2) &= \frac{p^2}{2\sqrt{2}} \Delta t. \end{aligned}$$

We now have all the required data to compute the potential at time 1.

$$\begin{aligned} \int_{\mathbb{R}^2} |y| \rho^1(dy) &= p - \frac{p^2}{\sqrt{2}} \Delta t + \frac{p^2}{2} \Delta t, \\ \int_{\mathbb{R}^2} |e_1 - y| \rho^1(dy) &= \int_{\mathbb{R}^2} |e_2 - y| \rho^1(dy) = (1 - p) + \frac{p}{\sqrt{2}} + \frac{p^2}{\sqrt{2}} \Delta t - \frac{p^2}{2} \Delta t, \\ \int_{\mathbb{R}^2} |e_1 + e_2 - y| \rho^1(dy) &= (1 - p)\sqrt{2} + p + \frac{p^2}{2} \Delta t - \frac{p^2}{\sqrt{2}} \Delta t. \end{aligned}$$

Finally, the potential at time 1 is given by:

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x - y| \rho^1(dx) \rho^1(dy) &= \left( (1 - p) + \frac{p^2}{2\sqrt{2}} \Delta t \right) \left( p - \frac{p^2}{\sqrt{2}} \Delta t + \frac{p^2}{2} \Delta t \right) \\ &\quad + \left( p - \frac{p^2}{\sqrt{2}} \Delta t \right) \left( (1 - p) + \frac{p}{\sqrt{2}} + \frac{p^2}{\sqrt{2}} \Delta t - \frac{p^2}{2} \Delta t \right) \\ &\quad + \frac{p^2}{2\sqrt{2}} \Delta t \left( (1 - p)\sqrt{2} + p + \frac{p^2}{2} \Delta t - \frac{p^2}{\sqrt{2}} \Delta t \right). \end{aligned}$$

We expand the above right-hand side in powers of  $\Delta t$ . The zero-order term is exactly equal to  $\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x - y| \rho^0(dx) \rho^0(dy)$ . So, we just compute the terms in  $\Delta t$ . It is equal to

$$(1 - \sqrt{2})(1 - p)p^2 + (\sqrt{2} - 1)p^3 = (\sqrt{2} - 1)p^2(2p - 1),$$

which completes the proof.  $\square$

### 3.4 Comparison with a potential non-increasing scheme

It must be stressed that the scheme could be defined differently in order to force the potential to be non-increasing. Basically, this requires the velocity  $a$  to be defined as a discrete derivative.

For simplicity, we provide the construction of the scheme in dimension 1 only. For a probability measure  $\varrho \in \mathcal{P}(\mathbb{Z})$  and a cell  $I \in \mathbb{Z}$ , we consider the following two discrete convolutions of finite differences:

$$\begin{aligned} & \frac{1}{\Delta x} \sum_{J \in \mathbb{Z}} \left[ \left( W(\Delta x(I + 1 - J)) - W(\Delta x(I - J)) \right) \varrho_J \right] \\ &= \left[ \int_{\mathbb{R}^d} \frac{W(x + \Delta x - y) - W(x - y)}{\Delta x} \varrho_{\Delta x}(dy) \right]_{|x=I\Delta x} \\ \text{and } & \frac{1}{\Delta x} \sum_{J \in \mathbb{Z}} \left[ \left( W(\Delta x(I - 1 - J)) - W(\Delta x(I - J)) \right) \varrho_J \right] \\ &= \left[ \int_{\mathbb{R}^d} \frac{W(x - \Delta x - y) - W(x - y)}{\Delta x} \varrho_{\Delta x}(dy) \right]_{|x=I\Delta x}, \end{aligned}$$

where, as before,  $\varrho_{\Delta x}$  is obtained by pushing forward  $\varrho$  by the mapping  $y \mapsto \Delta x y$ . The two terms above define velocities at the interfaces of the cell  $I$ . Namely, we call the first term  $-a_{I+\frac{1}{2}}$  and the second one  $a_{I-\frac{1}{2}}$ . Of course, the sign  $-$  in the former term guarantees the consistency of the notation, that is  $a_{(I+1)-\frac{1}{2}}$  is equal to  $a_{I+\frac{1}{2}}$ .

Following (2.16), the scheme is defined by:

$$\rho_J^{n+1} := \rho_J^n - \frac{\Delta t}{\Delta x} \left( (a_{J+\frac{1}{2}}^n)^+ \rho_J^n - (a_{J+\frac{1}{2}}^n)^- \rho_{J+1}^n + (a_{J-\frac{1}{2}}^n)^- \rho_J^n - (a_{J-\frac{1}{2}}^n)^+ \rho_{J-1}^n \right), \quad (3.38)$$

for  $n \in \mathbb{N}$  and  $J \in \mathbb{Z}$ . It is shown in [14] that the potential is non-increasing for the semi-discretized version of this scheme, which is to say that, up to a remainder of order 2 in  $\Delta t$  (the value of  $\Delta x$  being fixed), the potential of the fully discretized scheme does not increase from one step to another. The proof of the latter claim follows from a direct expansion of the quantity

$$\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x - y) \rho_{\Delta x}^{n+1}(dx) \rho_{\Delta x}^{n+1}(dy)$$

by using the updating rule for  $\rho_J^{n+1}$  in terms of  $\rho_J^n$ ,  $\rho_{J-1}^n$  and  $\rho_{J+1}^n$ .

Of course, Proposition 3.8 shows that the same property does not hold true for the scheme investigated in this paper, as the first order term in (3.37) is positive if  $p > 1/2$ .

However, the same example, but in dimension 1, shows that the scheme (3.38) may not be convergent for certain forms of potential for which Theorem 2.2 applies, see Subsection 6.3.

## 4 Order of convergence

This section is devoted to the proof of Theorem 2.2, which is divided into several steps. We start with two lemmas in which we establish crucial estimates on the stability of the Filippov flow and on the distance between the Filippov flow and the random characteristics.

### 4.1 Approximation by a random variable

Throughout this subsection, we use the same notations as in Subsections 3.2 and 3.3. In particular,  $(\Omega, \mathcal{A}, \mathbb{P})$  is defined as in Subsection 3.2 and, similar to Subsection 3.3, we call  $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$  a copy of  $(\Omega, \mathcal{A}, \mathbb{P})$ .

For a given random variable  $Y_0$  constructed on  $(\Omega, \mathcal{A}, \mathbb{P})$ , we let

$$Y_t(\omega) = Z_\rho(t, Y_0(\omega)), \quad t \geq 0,$$

where  $\rho$  denotes the solution of (2.9) when initialized with the law of  $Y_0$ . It satisfies

$$\frac{d}{dt} Y_t(\omega) = -\tilde{\mathbb{E}}[\widehat{\nabla W}(Y_t(\omega) - \tilde{Y}_t)], \quad t \geq 0, \quad (4.39)$$

where, as above, the notation  $\tilde{Y}_t$  is used to denote a copy of the random variable  $Y_t$  on  $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ . Observe in particular that, whenever  $Y_0 = X^0$ , with  $X^0$  as in (3.30), and  $\mathbb{P} = \mathbb{P}_{\rho^0}$ , with  $\rho^0$  a distribution on  $\mathbb{Z}^d$ , then

$$Y_t(\omega) = Z_\rho(t, X^0(\omega)),$$

where  $(\rho(t))_{t \geq 0}$  is the solution of (2.9) with  $\rho_{\Delta x}^0$  as initial condition, see Theorem 2.2 for the notations. Then, the law of  $Y_t$  under  $\mathbb{P}_{\rho^0}$  is equal to

$$Y_{t\#} \mathbb{P}_{\rho^0} = \rho(t), \quad t \geq 0.$$

Then we have the following estimate:

**Lemma 4.1** (i) *Let  $W$  satisfy assumptions **(A0)**–**(A3)**.*

*Call  $\rho_{\Delta x}^0 := \sum_{J \in \mathbb{Z}^d} \rho_J^0 \delta_{x_J}$ , for some  $\rho^0 \in \mathcal{P}(\mathbb{Z}^d)$ . Under the CFL condition (2.19), consider the random characteristics  $(X^n)_{n \in \mathbb{N}}$  defined in (3.30). Then, with probability 1 under  $\mathbb{P}_{\rho^0}$ , for all  $n \in \mathbb{N}$  and all  $\omega \in \Omega$ ,*

$$\begin{cases} X^{n+1}(\omega) = X^n(\omega) - \Delta t \tilde{\mathbb{E}}_{\rho^0}[\widehat{\nabla W}(X^n(\omega) - \tilde{X}^n)] + h^n(\omega), \\ X^0 \sim \rho^0, \end{cases} \quad (4.40)$$

where  $h^n$  is a  $\mathcal{F}^{n+1}$ -measurable  $\mathbb{R}^d$ -valued random variable that satisfies

$$\mathbb{E}_{\rho^0}^n(h^n) = 0, \quad |h^n(\omega)| \leq 2\Delta x, \quad \mathbb{E}_{\rho^0}^n[|h^n|^2] \leq C\Delta t \Delta x, \quad (4.41)$$

with a constant  $C$  only depending on  $d$  and  $w_\infty$ .

Moreover, if  $(Y_t)_{t \geq 0}$  is defined as in (4.39) with  $Y_0 = X^0$ , then

$$\forall n \in \mathbb{N}, \quad \mathbb{E}_{\rho^0}[|X^n - Y_{t^n}|^2]^{1/2} \leq C e^{|\lambda|t^n} \sqrt{t^n \Delta x},$$

for a constant  $C$  only depending on  $\lambda$ ,  $w_\infty$  and the dimension  $d$ .

(ii) *Let  $W$  be radial and satisfy **(A0)**–**(A2)** with  $\lambda$  (strictly) positive, and  $\rho_{\Delta x}^0$  be compactly supported.*



Then the result above remains true, provided that  $w_\infty$  is defined as in (2.21) with  $\rho^{ini} = \rho_{\Delta x}^0$ . Moreover,

$$\forall n \in \mathbb{N}, \quad \mathbb{E}_{\rho^0} [ |X^n - Y_{t^n}|^2 ]^{1/2} \leq C \sqrt{\Delta x},$$

provided that  $\Delta t \leq 1/2$  and  $2\lambda\Delta t < 1$ .

**Proof. 1st step.** The expansion (4.40), with  $h^n$  satisfying  $\mathbb{E}_{\rho^0}^n(h^n) = 0$  for each  $n \in \mathbb{N}$ , is a direct consequence of (i) and (ii) in Proposition 3.5. By construction, we can write  $h^n = X^{n+1} - X^n - \mathbb{E}_{\rho^0}^n(X^{n+1} - X^n)$ . Since  $|X^{n+1} - X^n| \leq \Delta x$  from the CFL condition, we deduce that  $|h^n| \leq 2\Delta x$ . Moreover,

$$\mathbb{E}_{\rho^0}^n [|h^n|^2] = \mathbb{E}_{\rho^0}^n [|X^{n+1} - X^n|^2] - |\mathbb{E}_{\rho^0}^n [X^{n+1} - X^n]|^2 \leq \mathbb{E}_{\rho^0}^n [|X^{n+1} - X^n|^2].$$

Then, the bound for  $\mathbb{E}_{\rho^0}^n [|h^n|^2]$  follows from the fact that:

$$\begin{aligned} \mathbb{E}_{\rho^0}^n [|X^{n+1} - X^n|^2] &= \mathbb{E}_{\rho^0}^n [|X^{n+1} - X^n|^2 \mathbf{1}_{\{K^{n+1} \neq K^n\}}] \leq \sum_{i=1}^d \Delta x_i^2 \mathbb{P}_{\rho^0}^n (K_i^{n+1} \neq K_i^n) \\ &\leq \sum_{i=1}^d \Delta x_i^2 (P_{K^n, K^n + e_i} + P_{K^n, K^n - e_i}) \leq w_\infty \Delta x \Delta t. \end{aligned}$$

**2nd step.** Starting from (4.39)–(4.40), we have after an expansion

$$\begin{aligned} |X^{n+1} - Y_{t^{n+1}}|^2 &\leq |X^n - Y_{t^n}|^2 + 2\langle X^n - Y_{t^n}, h^n \rangle + C(\Delta t \Delta x + |h^n|^2) \\ &\quad - 2 \int_{t^n}^{t^{n+1}} \tilde{\mathbb{E}}_{\rho^0} [\langle X^n - Y_{t^n}, \widehat{\nabla W}(X^n - \tilde{X}^n) - \widehat{\nabla W}(Y_s - \tilde{Y}_s) \rangle] ds, \end{aligned} \quad (4.42)$$

where we used the CFL condition (2.19) together with the fact that  $\nabla W$  is bounded, see (1.2). As for the last term, we write

$$\begin{aligned} &\int_{t^n}^{t^{n+1}} \tilde{\mathbb{E}}_{\rho^0} [\langle X^n - Y_{t^n}, \widehat{\nabla W}(X^n - \tilde{X}^n) - \widehat{\nabla W}(Y_s - \tilde{Y}_s) \rangle] ds \\ &= \int_{t^n}^{t^{n+1}} \tilde{\mathbb{E}}_{\rho^0} [\langle X^n - Y_s, \widehat{\nabla W}(X^n - \tilde{X}^n) - \widehat{\nabla W}(Y_s - \tilde{Y}_s) \rangle] ds \\ &\quad + \int_{t^n}^{t^{n+1}} \tilde{\mathbb{E}}_{\rho^0} [\langle Y_s - Y_{t^n}, \widehat{\nabla W}(X^n - \tilde{X}^n) - \widehat{\nabla W}(Y_s - \tilde{Y}_s) \rangle] ds. \end{aligned}$$

From the bound (1.2) on the gradient of  $W$ , we know that  $|Y_s - Y_{t^n}| \leq w_\infty |s - t^n|$ . Then, we deduce that the last term in the above right-hand side is bounded by  $w_\infty^2 \Delta t^2$ . Recalling the CFL condition, see (2.19), (4.42) yields

$$\begin{aligned} |X^{n+1} - Y_{t^{n+1}}|^2 &\leq |X^n - Y_{t^n}|^2 + 2\langle X^n - Y_{t^n}, h^n \rangle + C(\Delta t \Delta x + |h^n|^2) \\ &\quad - 2 \int_{t^n}^{t^{n+1}} \tilde{\mathbb{E}}_{\rho^0} [\langle X^n - Y_s, \widehat{\nabla W}(X^n - \tilde{X}^n) - \widehat{\nabla W}(Y_s - \tilde{Y}_s) \rangle] ds, \end{aligned} \quad (4.43)$$

where we just replaced  $Y_{t^n}$  by  $Y_s$  in the last term.

Since  $\nabla W$  is odd, see assumption **(A0)**, we deduce from a symmetry argument:

$$\begin{aligned}
& 2 \int_{t^n}^{t^{n+1}} \mathbb{E}_{\rho^0} \tilde{\mathbb{E}}_{\rho^0} [\langle X^n - Y_s, \widehat{\nabla W}(X^n - \tilde{X}^n) - \widehat{\nabla W}(Y_s - \tilde{Y}_s) \rangle] ds \\
&= \int_{t^n}^{t^{n+1}} \mathbb{E}_{\rho^0} \tilde{\mathbb{E}}_{\rho^0} [\langle X^n - Y_s - \tilde{X}^n + \tilde{Y}_s, \widehat{\nabla W}(X^n - \tilde{X}^n) - \widehat{\nabla W}(Y_s - \tilde{Y}_s) \rangle] ds \\
&\geq \lambda \int_{t^n}^{t^{n+1}} \mathbb{E}_{\rho^0} \tilde{\mathbb{E}}_{\rho^0} [|X^n - Y_s - \tilde{X}^n + \tilde{Y}_s|^2] ds,
\end{aligned} \tag{4.44}$$

where we used the  $\lambda$ -convexity (2.7) for the last inequality.

**3rd step.** Now, due to the symmetry assumption in **(A0)** of the potential  $W$ , we have

$$\mathbb{E}_{\rho^0} \tilde{\mathbb{E}}_{\rho^0} [\widehat{\nabla W}(X^n - \tilde{X}^n)] = 0, \quad \mathbb{E}_{\rho^0} \tilde{\mathbb{E}}_{\rho^0} [\widehat{\nabla W}(Y_t - \tilde{Y}_t)] = 0.$$

By (4.40) and by definition of  $(Y_t)_{t \geq 0}$ , we recover the fact that the center of mass is conserved, i.e.

$$\mathbb{E}_{\rho^0}[X^n] = \mathbb{E}_{\rho^0}[X^0] \quad \text{and} \quad \mathbb{E}_{\rho^0}[Y_t] = \mathbb{E}_{\rho^0}[Y_0].$$

Since the characteristics  $X^0$  and  $\tilde{Y}^0$  have the same initial distribution, we deduce that

$$\mathbb{E}_{\rho^0}[X^n] = \mathbb{E}_{\rho^0}[Y_{t^n}], \quad n \in \mathbb{N}. \tag{4.45}$$

Then, expending the term in the right hand side in (4.44), we deduce, using (4.45),

$$\begin{aligned}
& 2 \int_{t^n}^{t^{n+1}} \mathbb{E}_{\rho^0} \tilde{\mathbb{E}}_{\rho^0} [\langle X^n - Y_s, \widehat{\nabla W}(X^n - \tilde{X}^n) - \widehat{\nabla W}(Y_s - \tilde{Y}_s) \rangle] ds \\
&\geq 2\lambda \int_{t^n}^{t^{n+1}} \mathbb{E}_{\rho^0} [|X^n - Y_s|^2] ds = 2\lambda \int_{t^n}^{t^{n+1}} \mathbb{E}_{\rho^0} [|X^n - Y_s|^2] ds.
\end{aligned} \tag{4.46}$$

**4th step.** Taking the expectation in equation (4.43) and injecting (4.41) and (4.46), we deduce

$$\begin{aligned}
\mathbb{E}_{\rho^0} [|X^{n+1} - Y_{t^{n+1}}|^2] &\leq \mathbb{E}_{\rho^0} [|X^n - Y_{t^n}|^2] - 2\lambda \int_{t^n}^{t^{n+1}} \mathbb{E}_{\rho^0} [|X^n - Y_s|^2] ds \\
&\quad + 2\mathbb{E}_{\rho^0} [\langle X^n - Y_{t^n}, h^n \rangle] + C\Delta t \Delta x.
\end{aligned}$$

Moreover, from the tower property  $\mathbb{E}_{\rho^0}[\cdot] = \mathbb{E}_{\rho^0}[\mathbb{E}_{\rho^0}^n[\cdot]]$ , we also have

$$\begin{aligned}
\mathbb{E}_{\rho^0} [\langle X^n - Y_{t^n}, h^n \rangle] &= \mathbb{E}_{\rho^0} [\mathbb{E}_{\rho^0}^n \langle X^n - Y_{t^n}, h^n \rangle] \\
&= \mathbb{E}_{\rho^0} [\langle X^n - Y_{t^n}, \mathbb{E}_{\rho^0}^n(h^n) \rangle] = 0.
\end{aligned}$$

Then,

$$\mathbb{E}_{\rho^0} [|X^{n+1} - Y_{t^{n+1}}|^2] \leq \mathbb{E}_{\rho^0} [|X^n - Y_{t^n}|^2] - 2\lambda \int_{t^n}^{t^{n+1}} \mathbb{E}_{\rho^0} [|X^n - Y_s|^2] ds + C\Delta t \Delta x. \tag{4.47}$$

Now, we must distinguish between the two cases  $\lambda \leq 0$  and  $\lambda > 0$ . Start with the case when  $\lambda \leq 0$ . Using Young's inequality, one has, for all  $\varepsilon > 0$ ,

$$\begin{aligned}
\mathbb{E}_{\rho^0} [|X^n - Y_s|^2] &\leq (1 + \varepsilon) \mathbb{E}_{\rho^0} [|X^n - Y_{t^n}|^2] + (1 + \frac{1}{\varepsilon}) \mathbb{E}_{\rho^0} [|Y_s - Y_{t^n}|^2] \\
&\leq (1 + \varepsilon) \mathbb{E}_{\rho^0} [|X^n - Y_{t^n}|^2] + (1 + \frac{1}{\varepsilon}) w_\infty^2 |s - t^n|^2.
\end{aligned}$$

By (4.47) and by the CFL condition (2.19), we get

$$\mathbb{E}_{\rho^0} [|X^{n+1} - Y_{t^{n+1}}|^2] \leq (1 + 2(1 + \varepsilon)|\lambda|\Delta t)\mathbb{E}_{\rho^0} [|X^n - Y_{t^n}|^2] + C\Delta t\Delta x(1 + \Delta t(1 + \frac{1}{\varepsilon})).$$

Applying a discrete Gronwall lemma, we get, for any  $n \in \mathbb{N}$ ,

$$\mathbb{E}_{\rho^0} [|X^n - Y_{t^n}|^2] \leq e^{2(1+\varepsilon)|\lambda|n\Delta t} \left( \mathbb{E}_{\rho^0} [|X^0 - Y_0|^2] + Cn\Delta t\Delta x(1 + \Delta t(1 + \frac{1}{\varepsilon})) \right).$$

By choosing  $\varepsilon = \Delta t$ , we complete the proof of the first claim.

We now handle the case  $\lambda > 0$ . We write

$$\begin{aligned} \mathbb{E}_{\rho^0} [|X^n - Y_{t^n}|^2] &\leq (1 + \varepsilon)\mathbb{E}_{\rho^0} [|X^n - Y_s|^2] + (1 + \frac{1}{\varepsilon})\mathbb{E}_{\rho^0} [|Y_s - Y_{t^n}|^2] \\ &\leq (1 + \varepsilon)\mathbb{E}_{\rho^0} [|X^n - Y_s|^2] + (1 + \frac{1}{\varepsilon})w_\infty^2 |s - t^n|^2, \end{aligned}$$

which shows that

$$\begin{aligned} -\mathbb{E}_{\rho^0} [|X^n - Y_s|^2] &\leq -\frac{1}{1 + \varepsilon}\mathbb{E}_{\rho^0} [|X^n - Y_{t^n}|^2] + \frac{1}{\varepsilon}w_\infty^2 |s - t^n|^2 \\ &\leq -(1 - \varepsilon)\mathbb{E}_{\rho^0} [|X^n - Y_{t^n}|^2] + \frac{1}{\varepsilon}w_\infty^2 |s - t^n|^2, \end{aligned}$$

provided that  $\varepsilon \in (0, 1)$ . Hence, by (4.47) and by the CFL condition (2.19), we get

$$\mathbb{E}_{\rho^0} [|X^{n+1} - Y_{t^{n+1}}|^2] \leq (1 - 2(1 - \varepsilon)\lambda\Delta t)\mathbb{E}_{\rho^0} [|X^n - Y_{t^n}|^2] + C\Delta t\Delta x(1 + \frac{\Delta t}{\varepsilon}).$$

Hence, if  $2(1 - \varepsilon)\lambda\Delta t < 1$ , we get by induction,

$$\begin{aligned} \mathbb{E}_{\rho^0} [|X^n - Y_{t^n}|^2] &\leq (1 - 2(1 - \varepsilon)\lambda\Delta t)^n \mathbb{E}_{\rho^0} [|X^0 - Y_0|^2] \\ &\quad + C\Delta t\Delta x(1 + \frac{\Delta t}{\varepsilon}) \sum_{k=0}^{n-1} (1 - 2(1 - \varepsilon)\lambda\Delta t)^k \\ &\leq (1 - 2(1 - \varepsilon)\lambda\Delta t)^n \mathbb{E}_{\rho^0} [|X^0 - Y_0|^2] + \frac{C}{2(1 - \varepsilon)\lambda} \Delta x(1 + \frac{\Delta t}{\varepsilon}). \end{aligned}$$

Choosing  $\varepsilon = \Delta t$ , we complete the proof, □

## 4.2 Proof of Theorem 2.2

Since  $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$ , we first notice from Lemma 3.2 that, for all  $n \in \mathbb{N}$ ,  $\rho_{\Delta x}^n$  belongs to  $\mathcal{P}_2(\mathbb{R}^d)$ . The proof of Theorem 2.2 is then a consequence of Theorem 2.1 and Lemma 4.1. Consider indeed the collection  $(Y_t)_{t \geq 0}$  defined in Lemma 4.1 with  $Y_0 = X^0$ , which has distribution  $\rho_{\Delta x}^0$  under  $\mathbb{P}_{\rho^0}$ . As it has been emphasized in Subsection 4.1 and in the statement of Theorem 2.1, we have  $\rho'(t^n) = Z_{\rho'}(t^n, \cdot) \# \rho_{\Delta x}^0 = Y_{t^n} \# \mathbb{P}_{\rho^0}$ , where  $\rho'$  is the solution of (2.9) with  $\rho_{\Delta x}^0$  as initial condition.

We also have from Proposition 3.5  $\rho_{\Delta x}^n = X^n \# \mathbb{P}_{\rho^0}$ . By the triangle inequality, we have, as a consequence of the above discussion, for  $n \in \mathbb{N}^*$ ,

$$d_W(\rho_{\Delta x}^n, \rho(t^n)) \leq d_W(X^n \# \mathbb{P}_{\rho^0}, Y_{t^n} \# \mathbb{P}_{\rho^0}) + d_W(\rho'(t^n), \rho(t^n)), \quad (4.48)$$

where  $\rho = (\rho(t, \cdot))_{t \geq 0}$  is the solution of (2.9) with  $\rho^{ini}$  as initial condition.

We estimate each term of the right hand side separately.

**1st term.** Applying the inequality (2.4),

$$d_W(X^n \# \mathbb{P}_{\rho^0}, Y_{t^n} \# \mathbb{P}_{\rho^0}) \leq \mathbb{E}_{\rho_{\Delta x}^0} [ |X^n - Y_{t^n}|^2 ]^{1/2}.$$

Then, we apply Lemma 4.1 and obtain

$$d_W(X^n \# \mathbb{P}_{\rho^0}, Y_{t^n} \# \mathbb{P}_{\rho^0}) \leq C e^{|\lambda| t^n} \sqrt{t^n \Delta x}, \quad (4.49)$$

when  $\lambda \leq 0$  and

$$d_W(X^n \# \mathbb{P}_{\rho^0}, Y_{t^n} \# \mathbb{P}_{\rho^0}) \leq C \sqrt{\Delta x}, \quad (4.50)$$

when  $\lambda > 0$  (under the appropriate assumption on  $W$ , see the statement of Theorem 2.1).

**2nd term.** By Theorem 2.1 (we apply (i) if  $\lambda \leq 0$  and (ii) if  $\lambda > 0$ ), we have

$$d_W(\rho'(t^n), \rho(t^n)) \leq e^{(\lambda)^- t} d_W(\rho_{\Delta x}^0, \rho^{ini}), \quad (4.51)$$

where  $(\lambda)^- = \max(-\lambda, 0)$  is the negative part of  $\lambda$ .

Let us define  $\tau : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  by  $\tau(\sigma, x) = \sigma x_J + (1 - \sigma)x$ , for  $x \in C_J$ . We have that  $\tau(0, \cdot) = \text{id}$  and  $\tau(1, \cdot) \# \rho^{ini} = \rho_{\Delta x}^0$ . Then

$$\begin{aligned} d_W(\rho_{\Delta x}^0, \rho^{ini})^2 &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 [(\text{id} \times \tau(1, \cdot)) \# \rho^{ini}](dx, dy) \\ &\leq \sum_{J \in \mathbb{Z}^d} \rho_J^0 \int_{C_J} |x - x_J|^2 \rho^{ini}(dx). \end{aligned} \quad (4.52)$$

We deduce  $d_W(\rho_{\Delta x}^0, \rho^{ini}) \leq \Delta x$ . Injecting this latter inequality into (4.51), we obtain

$$d_W(\rho'(t^n), \rho(t^n)) \leq C e^{(\lambda)^- t} \Delta x. \quad (4.53)$$

We conclude the proof by injecting inequalities (4.49), (4.50) and (4.53) into estimate (4.48).

## 5 Unstructured mesh

Following [22], we can extend our convergence result to more general meshes. For the sake of simplicity of the notation, we present the case of a triangular mesh in two dimensions. This approach can be easily extended to  $P_1$  mesh in any dimension.

### 5.1 Numerical algorithm

Let us consider a triangular mesh  $\mathcal{T} = (T_k)_{k \in \mathbb{Z}}$  with nodes  $(x_i)_{i \in \mathbb{Z}}$ . We assume this mesh to be conformal: A summit cannot belong to an open edge of the grid. The triangles  $(T_k)_{k \in \mathbb{Z}}$  are assumed to satisfy  $\bigcup_{k \in \mathbb{Z}} T_k = \mathbb{R}^2$  and  $T_k \cap T_l = \emptyset$  if  $k \neq l$  (in particular, the cells are here not assumed to be closed nor open). For any triangle  $T$  with summits  $x, y, z$ , we will use also the notation  $(x, y, z) = T$ . We denote by  $\mathcal{V}(T) = \mathcal{V}(x, y, z)$  the area of this triangle, and  $h(T)$  its height (defined as the minimum of the three heights of the triangle  $T$ ). We make the assumption that the mesh satisfies  $\tilde{h} := \inf_{k \in \mathbb{Z}} h(T_k) > 0$ .

For any node  $x_i$ ,  $i \in \mathbb{Z}$ , we denote by  $K(i)$  the set of indices indexing triangles that have  $x_i$  as a summit, and we denote by  $\mathcal{T}_i$  the set of all triangles of  $\mathcal{T}$  that have  $x_i$  as a summit: thus  $\mathcal{T}_i = \{T_k; k \in K(i)\}$ .

For any triangle  $T_k$ ,  $k \in \mathbb{Z}$ , we denote by

$$I(k) = \{I_1(k), I_2(k), I_3(k)\} \quad (5.54)$$

the set of indices indexing the summits of  $T_k$  (for some arbitrary order, whose choice has no importance for the sequel).

We consider the following scheme, which is an extension to the triangular mesh of the scheme proposed above:

- For an initial distribution  $\rho^{ini}$  of the PDE (1.1), define the probability weights  $(\rho_i^0)_{i \in \mathbb{Z}}$  through the following procedure: Consider the one-to-one mapping  $\iota : \mathbb{Z} \ni k \mapsto \iota(k) \in \mathbb{Z}$  such that, for each  $k \in \mathbb{Z}$ ,  $x_{\iota(k)}$  is a node of the triangle  $T_k$ ;  $\iota$  is thus a way to associate a node with a cell; then, for all  $i \in \mathbb{Z}$ , let  $\rho_i^0 = \sum_{k: \iota(k)=i} \rho^{ini}(T_k)$ . Observe from (4.52) that  $\rho_{\Delta x}^0 = \sum_{j \in \mathbb{Z}} \rho_j^0 \delta_{x_j}$  is an approximation of  $\rho^{ini}$ .
- Assume that, for a given  $n \in \mathbb{N}$ , we already have probability weights  $(\rho_i^n)_{i \in \mathbb{Z}}$  such that  $\rho_{\Delta x}^n = \sum_{j \in \mathbb{Z}} \rho_j^n \delta_{x_j}$  is an approximation of  $\rho(t^n, \cdot)$ , where  $\rho$  is the solution to (1.1) with  $\rho^{ini}$  as initial condition. For  $i \in \mathbb{Z}$ , we let

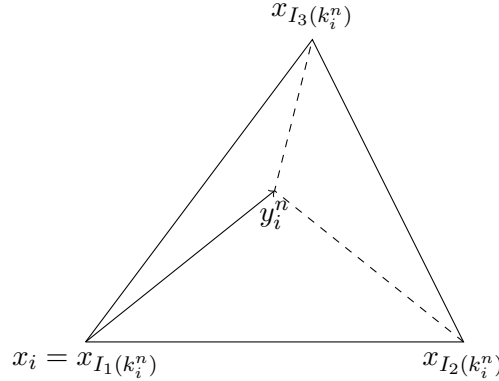
$$a_i^n := - \int_{\mathbb{R}^d} \widehat{\nabla W}(x_i - y) \rho_{\Delta x}^n(dy), \quad \text{and} \quad y_i^n := x_i + a_i^n \Delta t. \quad (5.55)$$

Under the CFL-like condition

$$w_\infty \Delta t \leq \hbar, \quad (5.56)$$

$y_i^n$  belongs to one (and only one) of the elements of  $\mathcal{T}_i$ . We denote by  $k_i^n$  the index of this triangle, namely  $y_i^n \in T_{k_i^n}$ .

- We use a linear splitting rule between the summits of the triangle  $T_{k_i^n}$ : the mass  $\rho_i^n$  is sent to the three points  $x_{I_1(k_i^n)}$ ,  $x_{I_2(k_i^n)}$ ,  $x_{I_3(k_i^n)}$  according to the *barycentric coordinates* of  $y_i^n$  in the triangle.



Let us make more precise the latter point. Let  $T = (x, y, z) \in \mathcal{T}$ , and  $\xi \in T$ . We define the barycentric coordinates of  $\xi$  with respect to  $x$ ,  $y$  and  $z$ ,  $\lambda_x^T$ ,  $\lambda_y^T$  and  $\lambda_z^T$ :

$$\lambda_x^T(\xi) = \frac{\mathcal{V}(\xi, y, z)}{\mathcal{V}(T)}, \quad \lambda_y^T(\xi) = \frac{\mathcal{V}(\xi, x, z)}{\mathcal{V}(T)}, \quad \lambda_z^T(\xi) = \frac{\mathcal{V}(\xi, x, y)}{\mathcal{V}(T)}, \quad (5.57)$$

and then have  $\xi = \lambda_x^T(\xi)x + \lambda_y^T(\xi)y + \lambda_z^T(\xi)z$ . Note also that  $\lambda_x^T(\xi) + \lambda_y^T(\xi) + \lambda_z^T(\xi) = 1$ . Therefore, we have the following fundamental property, which will be used in the sequel:

$$\lambda_x^T(\xi)(x - \zeta) + \lambda_y^T(\xi)(y - \zeta) + \lambda_z^T(\xi)(z - \zeta) = \xi - \zeta, \quad (5.58)$$

for any  $\zeta \in \mathbb{R}^2$ . For  $y_i^n \in T_{k_i^n}$  as above, we then use the barycentric coordinates of  $y_i^n$  with respect to the summits  $(x_j)_{j \in I(k_i^n)}$  of  $T_{k_i^n}$ . So, if, for notational convenience, we denote

$$\lambda_{i,j}^n := \lambda_{x_j}^T(y_i^n) \quad \text{when } T = T_{k_i^n},$$

then, the numerical scheme reads

$$\rho_j^{n+1} = \sum_{i \in \gamma(j)} \rho_i^n \lambda_{i,j}^n, \quad j \in \mathbb{Z}, \quad n \in \mathbb{N}, \quad (5.59)$$

where, for a given  $j \in \mathbb{Z}$ , we denote by  $\gamma(j)$  the set of all indices  $i \in \mathbb{Z}$  indexing nodes  $x_i$  such that  $x_i + a_i^n \Delta t$  belongs to a triangle that has  $x_j$  as a summit :

$$\gamma(j) = \{i \in \mathbb{Z} / \text{there exists } k \in K(j) \text{ such that } x_i + a_i^n \Delta t \in T_k\}.$$

## 5.2 Probabilistic interpretation

As in Section 3.2, we associate random characteristics with the sequence  $(\rho^n)_{n \in \mathbb{N}}$  defined in (5.59). Letting  $\Omega = \mathbb{Z}^{\mathbb{N}}$  and defining the canonical process  $(I^n)_{n \in \mathbb{N}}$  as we defined  $(K^n)_{n \in \mathbb{N}}$  above (the definition is the same but we use the letter  $I$  instead of  $K$ ), we equip  $\Omega$  with the Kolmogorov  $\sigma$ -field  $\mathcal{A}$  and with a collection of probability measures  $(\mathbb{P}_{[\varrho, \mu]})_{\varrho, \mu \in \mathcal{P}(\mathbb{Z})}$ , such that, for each  $\varrho, \mu \in \mathcal{P}(\mathbb{Z})$ ,  $(I^n)_{n \in \mathbb{N}}$  is a time-inhomogeneous Markov chain under  $\mathbb{P}_{[\varrho, \mu]}$ , with  $\mu$  as initial distribution and with transition matrix:

$$P_{i,j}^n[\rho^n] = \begin{cases} \lambda_{i,j}^n & \text{when } j \in I(k_i^n), \\ 0 & \text{otherwise,} \end{cases} \quad (5.60)$$

where  $(\rho^n)_{n \in \mathbb{N}}$  in (5.59) has  $\rho^0 = \varrho$  as initial condition and where we used the notation  $I(k_i^n)$  introduced in (5.54), that is to say, more precisely,

$$P_{i,j}^n[\rho^n] = \begin{cases} \lambda_{x_j}^{T_{k_i^n}}(y_i^n) & \text{when } j \in I(k_i^n), \\ 0 & \text{otherwise,} \end{cases}$$

with the same notation as in (5.57). Pay attention that the values of  $\rho^n$  in (5.60) are needed to compute the discrete velocity in (5.55). Observe also that the chain  $(I^n)_{n \in \mathbb{N}}$  here takes values in the set of indices indexing the nodes of the grid whilst the chain  $(K^n)_{n \in \mathbb{N}}$  used in the analysis of the upwind scheme (see Section 3.2) takes values in the set of indices indexing the cells of the grid. This is the rationale for using different letters.

We then define the *nonlinear* Markov chain by letting

$$\mathbb{P}_\varrho = \mathbb{P}_{[\varrho, \varrho]}, \quad \varrho \in \mathcal{P}(\mathbb{Z}).$$

Then, we let the random characteristics be the sequence of random variables  $(X^n)_{n \in \mathbb{N}}$  from  $(\Omega, \mathcal{A})$  into  $\mathbb{R}^2$  defined by

$$\forall n \in \mathbb{N}, \quad \forall \omega \in \Omega, \quad X^n(\omega) = x_{I^n(\omega)}. \quad (5.61)$$

We check that the properties in Proposition 3.5 still holds true with this definition of the random characteristics.

**Proposition 5.1** For any initial distribution  $\rho^0 \in \mathcal{P}(\mathbb{Z}^d)$ , let  $(\rho_{\Delta x}^n)_{n \in \mathbb{N}}$  be defined as in Theorem 2.2 and  $(X^n)_{n \in \mathbb{N}}$  be the sequence of random variables defined by (5.61) through the nonlinear Markov chain  $(I^n)_{n \in \mathbb{N}}$  admitting  $(P^n[\rho^n])_{n \in \mathbb{N}}$  in (5.60) as transition matrices.

(i) We have, for all  $n \in \mathbb{N}$ ,  $\rho_{\Delta x}^n = X^n \# \mathbb{P}_{\rho_{\Delta x}^0}$ .

(ii) For all  $\varrho \in \mathcal{P}(\mathbb{Z})$  and all  $j \in \mathbb{Z}$ , we have, with probability one under  $\mathbb{P}_{[\varrho],j} := \mathbb{P}_{[\varrho],\delta_j}$ ,

$$\mathbb{E}_{[\varrho],j}^n(X^{n+1} - X^n) = -\Delta t \int_{\mathbb{R}^d} \widehat{\nabla W}(X^n - y) \rho_{\Delta x}^n(dy),$$

when we use the initialization  $\rho^0 = \varrho$  to compute  $(\rho_{\Delta x}^n)_{n \in \mathbb{N}}$ . Above  $\mathbb{E}_{[\varrho],j}^n(\cdot)$  is the conditional expectation under  $\mathbb{P}_{[\varrho],j}$  given  $\sigma(I_0, \dots, I_n)$ .

**Proof.** (i) The proof of this result is similar to the proof of Proposition 3.5 (ii) with the transition matrix defined in (5.60).

(ii) From a direct computation, we have

$$\mathbb{E}_{[\varrho],j}^n(X^{n+1} - X^n) = \sum_{\ell \in I(k_{I^n}^n)} \lambda_{I^n, \ell}^n(x_\ell - x_{I^n}).$$

Thanks to (5.58), we get

$$\mathbb{E}_{[\varrho],j}^n(X^{n+1} - X^n) = y_{I^n}^n - x_{I^n} = a_{I^n}^n \Delta t,$$

which completes the proof. □

### 5.3 Convergence result

By the same token as in Section 4, we can use Proposition 5.1 and Theorem 2.1 and the analogue of Lemma 4.1 to prove that the numerical scheme (5.59) is of order 1/2:

**Theorem 5.2** Assume that  $W$  satisfies hypotheses **(A0)**–**(A3)**. For  $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$ , let  $(\rho(t))_{t \geq 0}$  be the unique measure solution to the aggregation equation with initial data  $\rho^{ini}$ , as given by Theorem 2.1. Let us also consider a triangular conformal mesh  $(T_k)_{k \in \mathbb{Z}}$  with nodes  $(x_j)_{j \in \mathbb{Z}}$  such that  $\hbar = \inf_{k \in \mathbb{Z}} h(T_k) > 0$  and the CFL condition (5.56) holds true. We denote by  $\Delta x$  the longest edge in the mesh.

Define  $(\rho_j^n)_{j \in \mathbb{Z}}_{n \in \mathbb{N}}$  as in (5.59) and let

$$\rho_{\Delta x}^n := \sum_{j \in \mathbb{Z}} \rho_j^n \delta_{x_j}, \quad n \in \mathbb{N}.$$

Then, there exists a nonnegative constant  $C$ , independent of the discretization parameters, such that, for all  $n \in \mathbb{N}^*$ ,

$$d_W(\rho(t^n), \rho_{\Delta x}^n) \leq C e^{|\lambda|(1+\Delta t)t^n} (\sqrt{t^n \Delta x} + \Delta x).$$

Importantly, we do not claim that (ii) in the statement of Theorem 2.2 remains true in the framework of Theorem 5.2. Indeed, it would require to prove that the support of the numerical solution remains included in a ball when the support of the initial condition is bounded. As made clear by the proof of Lemma 3.3, this latter fact depends on the geometry of the mesh.

## 6 Numerical illustrations

We now address several numerical examples. In Subsection 6.2, we show that the rate of convergence established in Theorem 2.2 is optimal in a one-dimensional example. This prompts us to start with a short reminder on the Wasserstein distance in dimension  $d = 1$ . In Subsection 6.3, we provide several numerical examples in dimension  $d = 1$  for the Newtonian potential, whilst examples in dimension  $d = 2$  are handled in Subsection 6.4.

### 6.1 Wasserstein distance in one dimension

The numerical computation of the Wasserstein distance between two probability measures in any dimension is generally quite difficult. However, in dimension  $d = 1$ , there is an explicit expression of the Wasserstein distance and this allows for direct computations, including numerical purposes, as shown in the pioneering work [30]. Indeed, any probability measure  $\mu$  on the real line  $\mathbb{R}$  can be described thanks to its cumulative distribution function  $F(x) = \mu((-\infty, x])$ , which is a right-continuous and non-decreasing function with  $F(-\infty) = 0$  and  $F(+\infty) = 1$ . Then we can define the generalized inverse  $Q_\mu$  of  $F$  (or monotone rearrangement of  $\mu$ ) by  $Q_\mu(z) = F^{-1}(z) := \inf\{x \in \mathbb{R} : F(x) > z\}$ ; it is a right-continuous and non-decreasing function, defined on  $[0, 1)$ . For every non-negative Borel-measurable map  $\xi : \mathbb{R} \rightarrow \mathbb{R}$ , we have

$$\int_{\mathbb{R}} \xi(x) \mu(dx) = \int_0^1 \xi(Q_\mu(z)) dz.$$

In particular,  $\mu \in \mathcal{P}_2(\mathbb{R})$  if and only if  $Q_\mu \in L^2((0, 1))$ . Moreover, in the one-dimensional setting, there exists a unique optimal transport plan realizing the minimum in (2.3). More precisely, if  $\mu$  and  $\nu$  belong to  $\mathcal{P}_p(\mathbb{R})$ , with monotone rearrangements  $Q_\mu$  and  $Q_\nu$ , then  $\Gamma_0(\mu, \nu) = \{(Q_\mu, Q_\nu)_\# \mathbb{L}_{(0,1)}\}$  where  $\mathbb{L}_{(0,1)}$  is the restriction to  $(0, 1)$  of the Lebesgue measure. Then we have the explicit expression of the Wasserstein distance (see [48, 54])

$$d_W(\mu, \nu) = \left( \int_0^1 |Q_\mu(z) - Q_\nu(z)|^2 dz \right)^{1/2}, \quad (6.62)$$

and the map  $\mu \mapsto Q_\mu$  is an isometry between  $\mathcal{P}_2(\mathbb{R})$  and the convex subset of (essentially) non-decreasing functions of  $L^2([0, 1))$ .

We will take advantage of this expression (6.62) of the Wasserstein distance in dimension 1 in our numerical simulations to estimate the numerical error of the upwind scheme (2.16). This scheme in dimension 1 on a Cartesian mesh reads, with time step  $\Delta t$  and cell size  $\Delta x$ :

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x} \left( (a_j^n)^+ \rho_j^n - (a_{j+1}^n)^- \rho_{j+1}^n - (a_{j-1}^n)^+ \rho_{j-1}^n + (a_j^n)^- \rho_j^n \right). \quad (6.63)$$

With this scheme, we define the probability measure  $\rho_{\Delta x}^n = \sum_{j \in \mathbb{Z}} \rho_j^n \delta_{x_j}$ . Then the generalized inverse of  $\rho_{\Delta x}^n$ , denoted by  $Q_{\Delta x}^n$ , is given by

$$Q_{\Delta x}^n(z) = x_{j+1}, \quad \text{for } z \in \left[ \sum_{k \leq j} \rho_k^n, \sum_{k \leq j+1} \rho_k^n \right). \quad (6.64)$$

### 6.2 Optimality of the order of convergence

Thanks to formula (6.62) in dimension  $d = 1$ , we can verify numerically the optimality of our result. Let us consider the potential  $W(x) = 2x^2$  for  $|x| \leq 1$  and  $W(x) = 4|x| - 2$  for  $|x| > 1$ ; such a potential



verifies our assumptions **(A0)**–**(A3)** with  $\lambda = 0$ . We choose the initial datum  $\rho^{ini} = \frac{1}{2}\delta_{-x_0} + \frac{1}{2}\delta_{x_0}$  with  $x_0 = 0.25$ . Then the solution to the aggregation equation (1.1) is given by

$$\rho(t) = \frac{1}{2}\delta_{-x_0(t)} + \frac{1}{2}\delta_{x_0(t)}, \quad x_0(t) = \frac{1}{4}e^{-4t}, \quad t \geq 0.$$

The generalized inverse  $Q_\rho(t, \cdot) = Q_{\rho(t)}$  of  $\rho(t)$  is given, for  $z \in [0, 1)$ , by  $Q_\rho(t, z) = -x_0(t)$  if  $z \in [0, 1/2)$ , and  $Q_\rho(t, z) = x_0(t)$  if  $z \in [1/2, 1)$ . Therefore, letting  $u_j^n := \sum_{k \leq j} \rho_k^n$  for  $j \in \mathbb{Z}$ , we can easily compute the error at time  $t^n = n\Delta t$  by means of the two formulas (6.62)–(6.64):

$$e_n := d_W(\rho(t^n), \rho_{\Delta x}^n) = \sum_{k \in \mathbb{Z}} \int_{u_{k-1}^n}^{u_k^n} |x_k - Q_\rho(t^n, z)| dz.$$

We then define the numerical error as  $e := \max_{n \leq T/\Delta t} e_n$ . We display in Figure 1 the numerical error with respect to the number of nodes in logarithmic scale, as computed with the above procedure (the time steps being chosen in a such a way that the ratio (2.19) in the CFL condition is kept constant). We observe that the computed numerical error is of order 1/2.

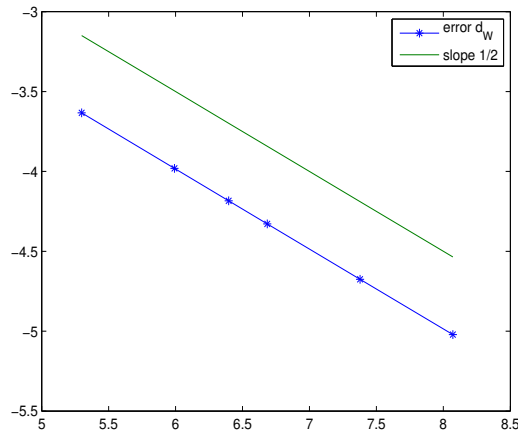


Figure 1: Numerical error with respect to the number of nodes in logarithmic scale for the upwind scheme in Wasserstein distance for the potential  $W$  defined by  $W(x) = 2x^2$  for  $|x| \leq 1$  and  $W(x) = 4|x| - 2$  for  $|x| > 1$ , and an initial data given by the sum of two Dirac deltas.

### 6.3 Newtonian potential in one dimension

An interesting and illustrative example is the Newtonian potential in dimension  $d = 1$ . Let us indeed consider the case  $W(x) = |x|$  and an initial datum given by the sum of two masses located at points  $x_{i_1}$  and  $x_{i_2}$  of the grid mesh, namely  $\rho^{ini} = \frac{1}{2}\delta_{x_{i_1}} + \frac{1}{2}\delta_{x_{i_2}}$ , with say  $x_{i_1} < x_{i_2}$ . The solution of the aggregation equation in Theorem 2.1 is given by  $\rho(t) = \frac{1}{2}\delta_{x_1(t)} + \frac{1}{2}\delta_{x_2(t)}$ , where

$$x_1(t) = x_{i_1} + \frac{t}{2}, \quad x_2(t) = x_{i_2} - \frac{t}{2}, \quad \text{for } t < x_{i_2} - x_{i_1}.$$

Indeed, recalling definition (2.5), we have, for  $t < x_{i_2} - x_{i_1}$ :

$$\widehat{a}_\rho(t, x) = \begin{cases} 1, & \text{if } x < x_1(t), \\ \frac{1}{2}, & \text{if } x = x_1(t), \\ 0, & \text{if } x_1(t) < x < x_2(t), \\ -\frac{1}{2}, & \text{if } x = x_2(t), \\ -1, & \text{if } x > x_2(t). \end{cases}$$

At  $t = x_{i_2} - x_{i_1}$ , the two particles collapse, then for  $t \geq x_{i_2} - x_{i_1}$ , we have  $\rho(t) = \delta_{\frac{1}{2}(x_{i_1} + x_{i_2})}$ .

**Standard finite volume upwind scheme.** This simple example explains why we have chosen the scheme (6.63) instead of the standard finite volume upwind scheme introduced in Subsection 3.4. In dimension  $d = 1$  and on a Cartesian grid, this latter one reads

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x} \left( (a_{i+1/2}^n)^+ \rho_i^n - (a_{i+1/2}^n)^- \rho_{i+1}^n - (a_{i-1/2}^n)^+ \rho_{i-1}^n + (a_{i-1/2}^n)^- \rho_i^n \right), \quad (6.65)$$

where  $a_{i+1/2}^n = -\sum_{k \in \mathbb{Z}} \rho_k^n \text{sign}(x_{i+1/2} - x_k)$ .

Assume indeed that, at time  $t^n$ , for some  $n \in \mathbb{N}$ , we have obtained the approximation  $\rho_i^n = 0$  for  $i \in \mathbb{Z} \setminus \{i_1, i_2\}$ , and  $\rho_{i_1}^n = \rho_{i_2}^n = 1/2$ . We then compute

$$a_{i+1/2}^n = \begin{cases} 1, & \text{for } i < i_1 \\ 0, & \text{for } i_1 \leq i < i_2 \\ -1, & \text{for } i \geq i_2. \end{cases}$$

So, when applying the upwind scheme for  $i \in \{i_1 - 1, i_1, i_1 + 1\}$ , we get

$$\begin{aligned} \rho_{i_1-1}^{n+1} &= \rho_{i_1-1}^n - \frac{\Delta t}{\Delta x} (\rho_{i_1-1}^n - \rho_{i_1-2}^n) = 0, \\ \rho_{i_1}^{n+1} &= \rho_{i_1}^n + \frac{\Delta t}{\Delta x} \rho_{i_1-1}^n = \rho_{i_1}^n, \\ \rho_{i_1+1}^{n+1} &= \rho_{i_1+1}^n = 0. \end{aligned}$$

Doing the same computation for  $i \in \{i_2 - 1, i_2, i_2 + 1\}$ , we deduce that  $\rho^{n+1} = \rho^n$ . Thus the above upwind scheme may be not able to capture the correct dynamics of Dirac deltas. The above computation is illustrated by the numerical results in Figure 2, where a comparison between the numerical results obtained with (6.65) (left) and with (6.63) (right) is displayed. We observe that the Dirac deltas are stationary when using the scheme (6.65), whereas the scheme (6.63) permits to catch the right dynamics. Another interesting numerical illustration of this phenomenon is provided by Figure 3. In this example, we choose the potential  $W(x) = 1 - e^{-2|x|}$ , which is  $-4$ -convex, and a smooth initial datum given by the sum of two Gaussian functions:  $\rho^{ini}(x) = \frac{1}{M}(e^{-20(x-0.5)^2} + e^{-20(x+0.5)^2})$ , where  $M = \|\rho^{ini}\|_{L^1}$  is a normalization coefficient. With this choice, we observe that the solution blows-up quickly. Dirac deltas appear in finite time and, as observed above, the scheme (6.65) (Fig. 3-left) does not allow to capture the dynamics after blow-up time, whilst the scheme (6.63) (Fig. 3-right) succeeds to do so. For these numerical simulations, the numerical spatial domain is  $[-1.25, 1.25]$ ; it is discretized with a uniform Cartesian grid of 800 nodes, and the ratio in the CFL condition is 0.5.

**Comparison with Burgers-Hopf equation.** Considering the potential  $W(x) = \frac{1}{2}|x|$ , it has been proved in [36] (see also [9]) that the following equivalence holds true:  $\rho$  is the solution in

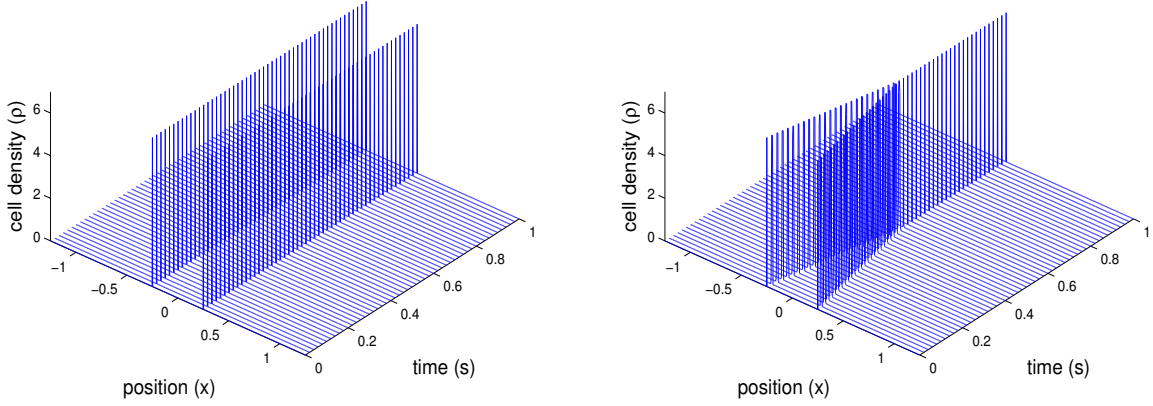


Figure 2: Numerical result for the one dimensional aggregation equation with  $W(x) = |x|$  and an initial datum given by two Dirac deltas. Left: Result obtained with the standard upwind scheme (6.65) with a velocity computed at the interfaces of the mesh. Right: Result with the scheme (6.63). As already emphasized in Example 2.3, this shows once again that a great care must be paid to the choice of the scheme in order to recover the correct dynamics of Dirac deltas.

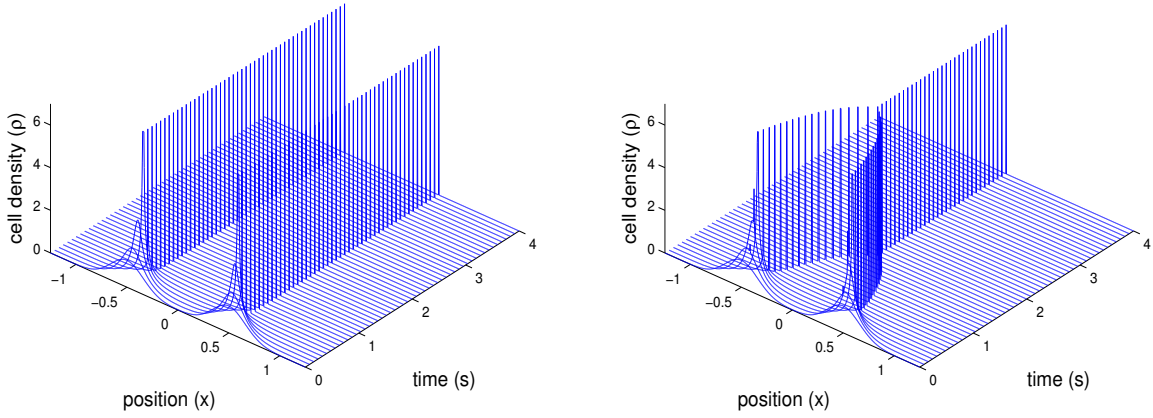


Figure 3: Numerical result for the one dimensional aggregation equation with  $W(x) = 1 - e^{-2|x|}$  and an initial datum given by the sum of two Gaussian functions. Left: Result obtained with the standard upwind scheme (6.65) with a velocity computed at the interfaces of the mesh. Right: Result with the scheme (6.63). As in Fig. 2, the upwind scheme (6.65) does not capture the right dynamics of the Dirac deltas after blow-up time.

Theorem 2.1 if and only if  $u = -W' * \rho$  is the entropy solution of the Burgers-Hopf equation  $\partial_t u + \frac{1}{2} \partial_x u^2 = 0$ .

Let  $(\rho_i^n)_{i \in \mathbb{Z}, n \in \mathbb{N}}$  be given by the scheme (2.16)–(2.17). By conservation of the total mass, see Lemma 3.2, we have  $\sum_{k \in \mathbb{Z}} \rho_k^n = 1$ . Introducing

$$u_i^n := \frac{1}{2} - \sum_{k \leq i} \rho_k^n, \quad i \in \mathbb{Z}, n \in \mathbb{N},$$

we deduce, by summing (2.16) and by using the fact that  $\rho_i^n = -(u_i^n - u_{i-1}^n)$ , that the family  $(u_i^n)_{i \in \mathbb{Z}, n \in \mathbb{N}}$  satisfies:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left( (a_i^n)^+ (u_i^n - u_{i-1}^n) - (a_{i+1}^n)^- (u_{i+1}^n - u_i^n) \right), \quad (6.66)$$

where, with (2.17), we have

$$a_i^n = -\frac{1}{2} \sum_{k \neq i} \rho_k^n \text{sign}(x_i - x_k).$$

Then

$$a_i^n = -\frac{1}{2} \left( \sum_{k < i} \rho_k^n - \sum_{k > i} \rho_k^n \right) = -\frac{1}{2} \left( \sum_{k < i} \rho_k^n - 1 + \sum_{k \leq i} \rho_k^n \right) = \frac{1}{2} (u_{i-1}^n + u_i^n).$$

Moreover, as  $\rho_i^n$  remains nonnegative under the CFL condition (see Lemma 3.1),  $u_i^n - u_{i-1}^n = -\rho_i^n \leq 0$ , so that

$$(a_i^n)^+ (u_i^n - u_{i-1}^n) = - (a_i^n (u_i^n - u_{i-1}^n))^- = -\frac{1}{2} \left( (u_i^n)^2 - (u_{i-1}^n)^2 \right)^-.$$

Similarly, we get

$$(a_{i+1}^n)^- (u_{i+1}^n - u_i^n) = - (a_{i+1}^n (u_{i+1}^n - u_i^n))^+ = -\frac{1}{2} \left( (u_{i+1}^n)^2 - (u_i^n)^2 \right)^+,$$

so that the scheme (6.66) for  $u$  finally rewrites

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x} \left( \left( (u_{i+1}^n)^2 - (u_i^n)^2 \right)^- - \left( (u_i^n)^2 - (u_{i-1}^n)^2 \right)^+ \right). \quad (6.67)$$

Then we may apply the main result of this paper and deduce the convergence at order 1/2 of the above scheme:

**Lemma 6.1** *Let  $u^{ini}$  be given in  $BV(\mathbb{R})$  such that  $\partial_x u^{ini} \leq 0$  and  $TV(u^{ini}) = 1$ . Define the family  $(u_i^n)_{i \in \mathbb{Z}, n \in \mathbb{N}}$  by means of (6.67), with the initial data  $u_i^0 := \frac{1}{2} + \partial_x u^{ini}(-\infty, x_{i+\frac{1}{2}})$ , and let  $u_{\Delta x}^n := \sum_{i \in \mathbb{Z}} u_i^n \mathbf{1}_{[x_i, x_{i+1})}$ . Let  $u$  be the entropy solution to the Burgers equation  $\partial_t u + \frac{1}{2} \partial_x u^2 = 0$  with  $u^{ini}$  as initial condition. Then, there exists  $C \geq 0$ , independent of the discretization parameters, such that if the CFL condition  $\Delta t \leq 2\Delta x$  is satisfied, one has*

$$\|u(t^n) - u_{\Delta x}^n\|_{L^1} \leq C(\sqrt{t^n \Delta x} + \Delta x).$$

**Remark 6.2** *We do not claim that the scheme converges for any initial datum for the Cauchy problem for the Burgers equation (and actually it does not). The convergence result above only applies to a non-increasing initial condition belonging to  $[-1/2, 1/2]$ .*

*Note that this scheme is not conservative, but, surprisingly (see [32]) this does not prevent it from converging toward the right solution.*

**Proof.** First remark that the CFL condition that is here required is  $w_\infty \Delta t \leq \Delta x$ , with  $w_\infty = 1/2$  as  $W(x) = \frac{1}{2}|x|$ .

The entropy solution  $u$  of the Burgers equation with a nonincreasing  $BV$  initial datum is a nonincreasing  $BV$  function. By Cauchy-Schwarz inequality, we have

$$\int_0^1 |Q_{\rho(t^n)}(z) - Q_{\rho_{\Delta x}^n}(z)| dz \leq \|Q_{\rho(t^n)} - Q_{\rho_{\Delta x}^n}\|_{L^2(0,1)} = d_W(\rho(t^n), \rho_{\Delta x}^n),$$

where  $(\rho(t))_{t \geq 0}$  is the solution of (1.1), with  $W(x) = \frac{1}{2}|x|$  as before and  $\rho^{ini} = -\partial_x u^{ini}$  as initial condition, and  $(\rho_{\Delta x}^n)_{n \geq 0}$  is the numerical solution obtained by Scheme (2.16) with  $d = 1$  together with initial condition (2.15) (numerical solution whose convergence at order 1/2 is stated in Theorem 2.2).

Observing that  $W$  is convex, we apply Theorem 2.2 with  $\lambda = 0$ . We obtain

$$\int_0^1 |Q_{\rho(t^n)}(z) - Q_{\rho_{\Delta x}^n}(z)| dz \leq d_W(\rho(t^n), \rho_{\Delta x}^n) \leq C(\sqrt{t^n \Delta x} + \Delta x).$$

The claim follows provided we prove that

$$\int_{\mathbb{R}} |u(t^n, x) - u_{\Delta x}^n(x)| dx = \int_0^1 |Q_{\rho(t^n)}(z) - Q_{\rho_{\Delta x}^n}(z)| dz. \quad (6.68)$$

In order to prove (6.68), we notice that, from a geometrical point of view, the left hand side of equality (6.68) corresponds to the area between the curves  $x \mapsto u(t^n, x)$  and  $x \mapsto u_{\Delta x}^n(x)$ . Also, the right hand side is a measure of the area between their generalized inverses. However, the graph of the pseudo-inverse of a function may be obtained by flipping the graph of the function with respect to the diagonal. Since this operation conserves the area, we deduce that both areas are equal, that is (6.68) holds.

Another way to prove the identity (6.68) is to observe that the solution  $u$  of the Burgers-Hopf equation reads:

$$u(t, x) = \frac{1}{2} [\rho(t, (x, +\infty)) - \rho(t, (-\infty, x))], \quad t \geq 0, \quad x \in \mathbb{R},$$

where  $\rho$  is the solution in Theorem 2.1. In fact, as the number of points  $x$  for which  $\rho(t, \{x\}) > 0$  is at most countable for any given  $t > 0$ , we have the almost everywhere equality:

$$u(t, x) = \rho(t, (x, +\infty)) - \frac{1}{2}.$$

Similarly,

$$\begin{aligned} u_{\Delta x}^n(t, x) &= \sum_{i \in \mathbb{Z}} u_i^n \mathbf{1}_{[x_i, x_{i+1})}(x) = \frac{1}{2} - \sum_{i \in \mathbb{Z}} \mathbf{1}_{[x_i, x_{i+1})}(x) \sum_{k \leq i} \rho_k^n \\ &= \frac{1}{2} - \sum_{i \in \mathbb{Z}} \mathbf{1}_{[x_i, x_{i+1})}(x) \rho_{\Delta x}^n(t, (-\infty, x_i]) = \frac{1}{2} - \rho_{\Delta x}^n(t, (-\infty, x]) = \rho_{\Delta x}^n(t, (x, +\infty)) - \frac{1}{2}. \end{aligned}$$

So, to complete the proof, it suffices to use the fact that, for any two probability measures  $\mu$  and  $\mu'$  on  $\mathbb{R}$ ,

$$\int_{\mathbb{R}} |\mu((x, +\infty)) - \mu'((x, +\infty))| dx = \int_0^1 |Q_{\mu}(z) - Q_{\mu'}(z)| dz,$$

see [7, Theorems 2.9 and 2.10], noticing that the function  $Q_{\mu}$  we use here is the right continuous version of the quantile function used in [7].

□

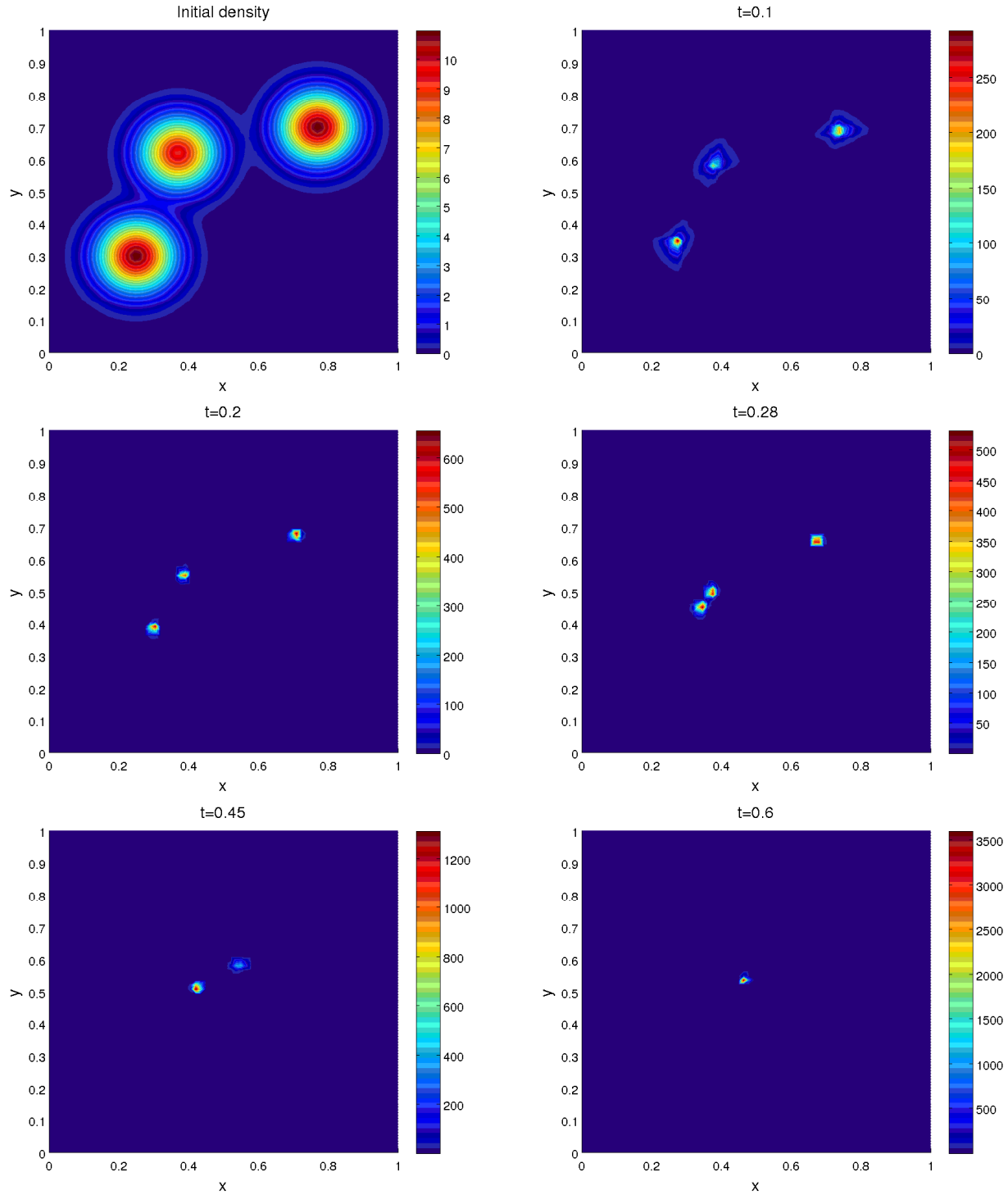


Figure 4: Time dynamics of the numerical solution of the aggregation equation (1.1) with  $W(x) = W_1(x) = 1 - e^{-5|x|}$  and an initial datum given by the sum of three bumps. Time increases from top left to bottom right.

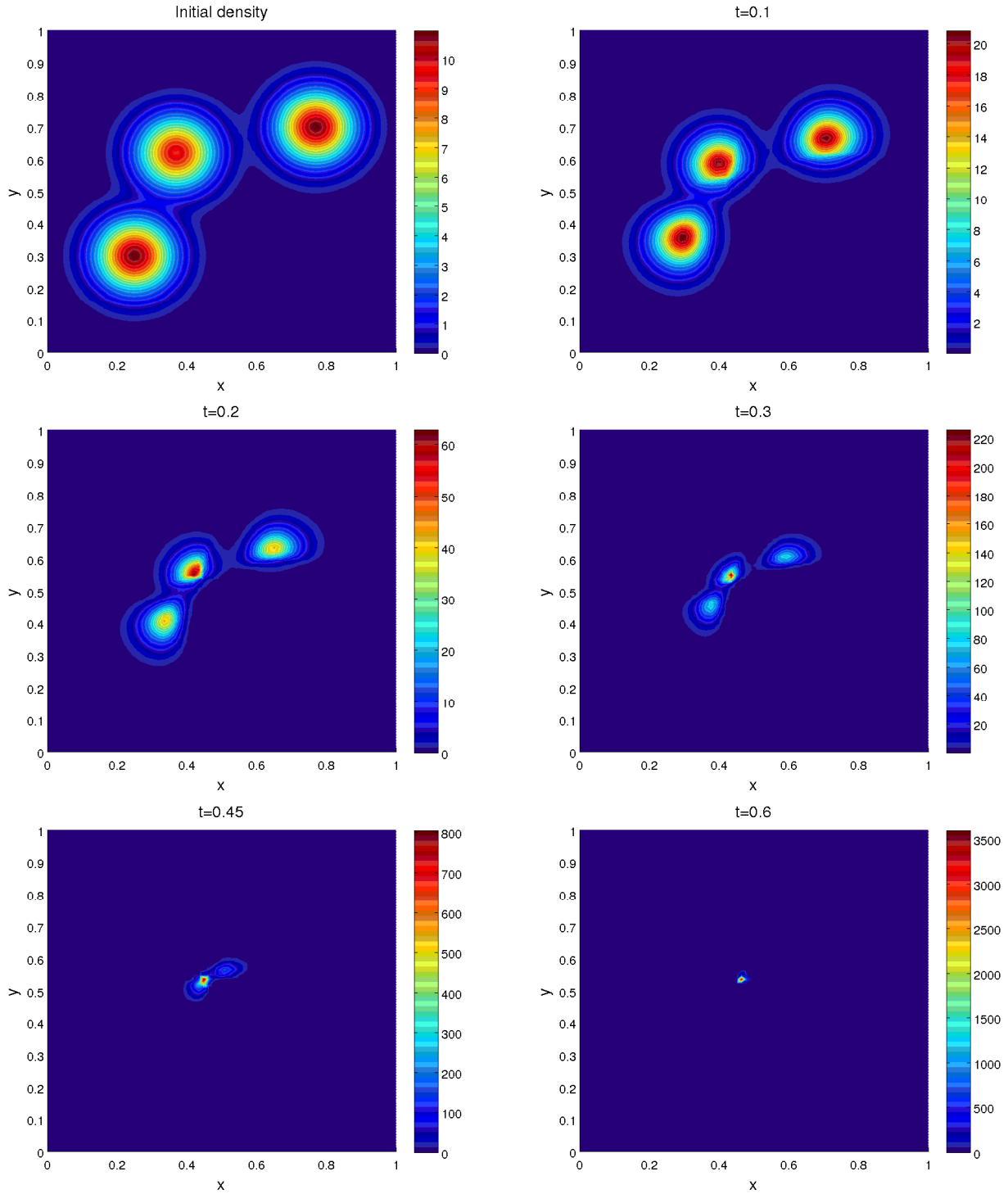


Figure 5: Time dynamics of the numerical solution of the aggregation equation (1.1) with  $W(x) = W_2(x) = 5|x|$  and an initial datum given by the sum of three bumps. Time increases from top left to bottom right.

#### 6.4 Numerical simulation in two dimensions

As an illustration, we propose now a numerical example in two dimensions. The spatial domain is the square  $[0, 1] \times [0, 1]$ ; it is discretized with  $N_x = 70$  nodes in the  $x$ -direction and  $N_y = 70$  nodes

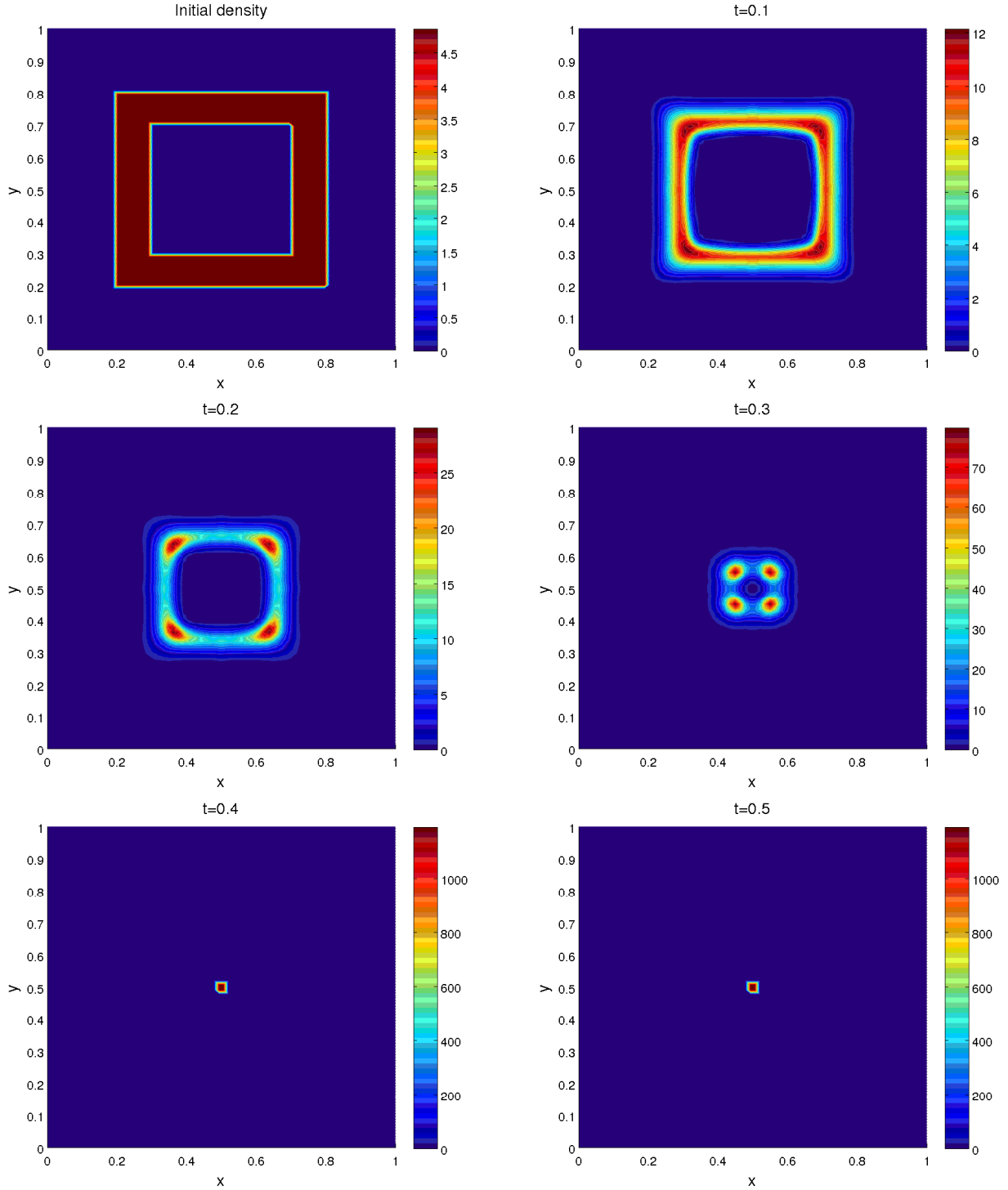


Figure 6: Time dynamics of the numerical solution of the aggregation equation (1.1) with  $W(x) = W_1(x) = 1 - e^{-5|x|}$  and an initial datum given by a square. Time increases from top left to bottom right.

in the  $y$ -direction; we take a time step  $\Delta t = 10^{-3}$ . We consider two different initial data: the sum



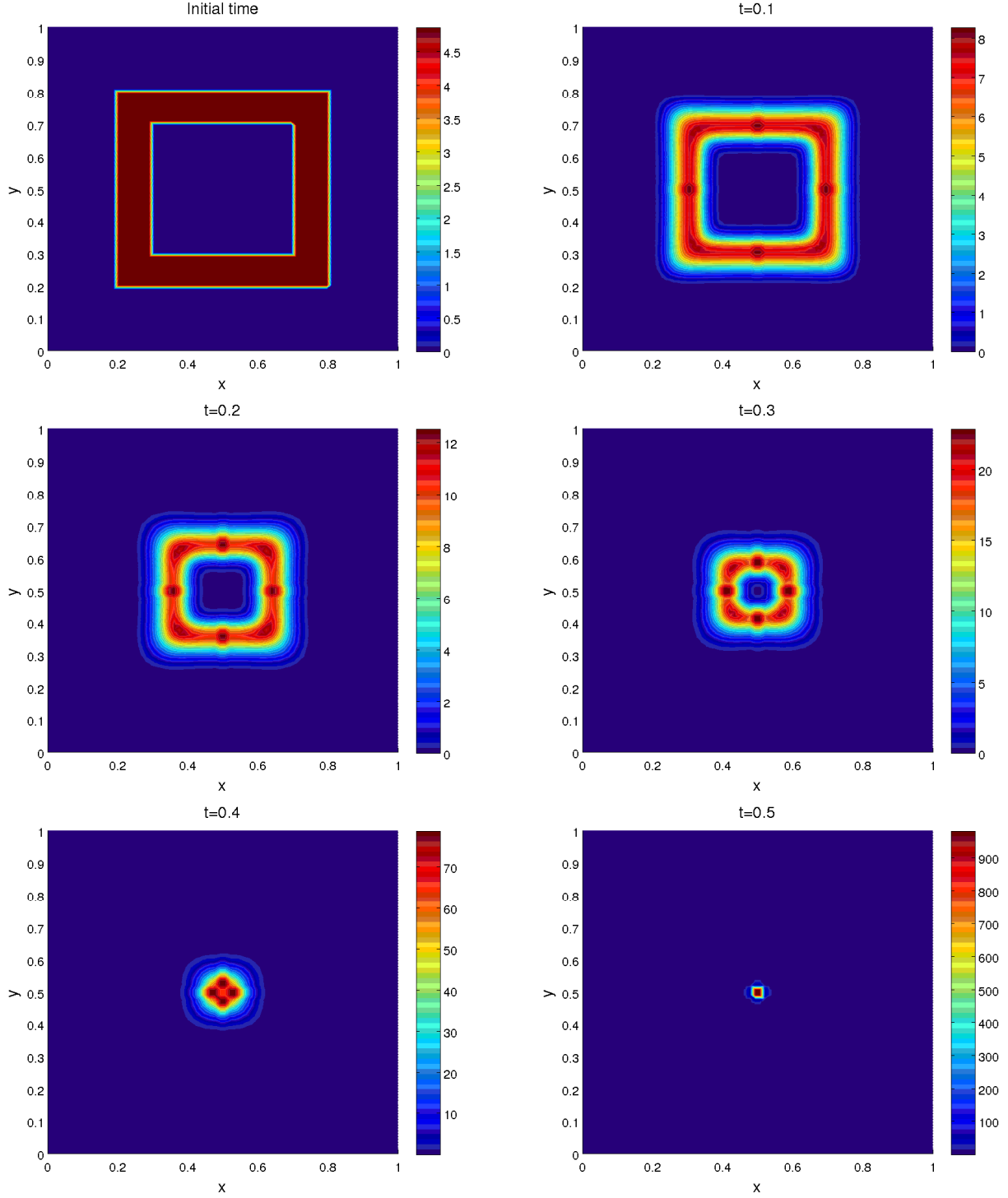


Figure 7: Time dynamics of the numerical solution of the aggregation equation (1.1) with  $W(x) = W_2(x) = 5|x|$  and an initial datum given by a square. Time increases from top left to bottom right.

of three bumps (as in [16])

$$\begin{aligned} & \rho^{ini}(t, x) \\ &= \frac{1}{M} \left( e^{-100((x_1-0.25)^2+(x_2-0.3)^2)} + e^{-100((x_1-0.77)^2+(x_2-0.7)^2)} + 0.9e^{-100((x_1-0.37)^2+(x_2-0.62)^2)} \right), \end{aligned}$$

where  $M$  is a normalization constant such that  $\|\rho^{ini}\|_{L^1} = 1$ ; and an initial density with a square shape

$$\rho^{ini}(t, x) = 5 \times \mathbf{1}_{[0.2, 0.8] \times [0.2, 0.8] \setminus [0.3, 0.7] \times [0.3, 0.7]}.$$

With these numerical data, we compare the numerical results between the two potentials  $W_1(x) = 1 - e^{-5|x|}$  and  $W_2(x) = 5|x|$ . For  $|x|$  close to 0, we have that  $\nabla W_1 \sim \nabla W_2$ . Then the short range interaction is similar between both potentials, but the long range interaction is different. The numerical results are displayed in Figures 4 and 6 for the potential  $W_1(x) = 1 - e^{-5|x|}$  and in Figures 5 and 7 for the potential  $W_2(x) = 5|x|$ .

In each case, we observe, as expected, the aggregation in finite time of  $\rho$  towards a Dirac delta. Indeed it has been proved in [15] that when the initial data is compactly supported, solutions converge towards a Dirac delta in finite time. We also observe that the time dynamics during this step of concentration is different between potentials  $W_1$  and  $W_2$ .

The case with an initial datum with three bumps has been implemented in [16] with a Lax-Friedrichs scheme. We obtain here similar results but we observe a smaller numerical diffusion. Then we can make similar comments for the comparison between the two potentials  $W_1$  and  $W_2$ . For the potential  $W_1$ , we observe that each bump coalesces into a Dirac delta, then the three remaining Dirac deltas merge into a single Dirac delta (see Fig 4). For the potential  $W_2$ , the solution seems to be more regular and Dirac deltas seems to appear for larger time (see Fig 5).

For the initial data with a square shape, the density  $\rho$  keeps, for both potentials, a shape similar to the initial square shape which tightens as time increases. However with the potential  $W_1$  (Fig 6), we notice a strong concentration at the corners of the square, whereas in the case of the potential  $W_2$  (Fig 7) the density is homogeneous along the edges of the square with a slight concentration in the middle of the edges.

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