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► **To cite this version:**

Guy Barles, Emmanuel Chasseigne. Corrigendum to ” (Almost) Everything You Always Wanted to Know About Deterministic Control Problems in Stratified Domains ”. 2017. hal-01591473v1

HAL Id: hal-01591473

<https://hal.science/hal-01591473v1>

Preprint submitted on 21 Sep 2017 (v1), last revised 19 Oct 2017 (v2)

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Corrigendum to “(Almost) Everything You Always Wanted to Know About Deterministic Control Problems in Stratified Domains”

G. Barles & E. Chasseigne ^{*†}

Abstract

The aim of this short note is: (i) to report an error in [1]; (ii) to explain why the comparison result of [1] lacks an hypothesis in the definition of subsolutions if we allow them to be discontinuous; (iii) to describe a simple counter-example; (iv) to show a simple way to correct this mistake, considering the classical Ishii’s definition of viscosity solutions; (v) finally, to give some elements of proof showing that this actually corrects the error.

Key-words: Optimal control, discontinuous dynamic, Bellman Equation, viscosity solutions.
AMS Class. No: 49L20, 49L25, 35F21.

In [1], we are considering deterministic control problems whose dynamic and cost (b, l) at any point $(x, t) \in \mathbb{R}^N \times [0, T]$ are chosen in a set $\mathbf{BL}(x, t)$. The classical Hamiltonian is

$$H(x, t, p) := \sup_{(b, l) \in \mathbf{BL}(x, t)} \{ -b \cdot p - l \}.$$

Having a stratified problem means that \mathbb{R}^N can be decomposed as

$$\mathbb{R}^N = \mathbf{M}^0 \cup \mathbf{M}^1 \cup \dots \cup \mathbf{M}^{N-1},$$

where, for any k , \mathbf{M}^k is a k dimensional submanifold of \mathbb{R}^N . The sets $\mathbf{M}^0, \mathbf{M}^1, \dots, \mathbf{M}^{N-1}$ are the locations where either $\mathbf{BL}(x, t)$ or H can have discontinuities. To take into account some particular control problems on \mathbf{M}^k , it is necessary to introduce the Hamiltonians which are defined on \mathbf{M}^k by

$$H^k(x, t, p) := \sup_{\substack{(b, l) \in \mathbf{BL}(x, t) \\ b \in T_x \mathbf{M}^k}} \{ -b \cdot p - l \},$$

where $T_x \mathbf{M}^k$ is the tangent space to \mathbf{M}^k at x .

The definition of super and subsolutions in [1] follows the ones of Bressan & Hong [2]: a lsc function v is a supersolution of the associated Hamilton-Jacobi-Bellman Equation if

$$v_t + H(x, t, Dv) \geq 0 \quad \text{on } \mathbb{R}^N \times (0, T],$$

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†This work was partially supported by the ANR HJnet ANR-12-BS01-0008-01 and by EU under the 7th Framework Programme Marie Curie Initial Training Network “FP7-PEOPLE-2010-ITN”, SADCO project, GA number 264735-SADCO.

while an usc function u is a subsolution if

$$u_t + H^k(x, t, Du) \leq 0 \quad \text{on } \mathbf{M}^k \times (0, T], \quad (1)$$

for any $k = 0, \dots, N$. And it is worth pointing out that these H^k - inequalities are really on $\mathbf{M}^k \times (0, T]$ -inequalities, *i.e.* they are obtained by considering maxima of $u - \phi$ on $\mathbf{M}^k \times (0, T]$ for any smooth test-function ϕ .

Unfortunately, this way to define subsolutions is not sufficient since it treats all the \mathbf{M}^k separately without linking them and this allows u to have artificial values on certain manifolds \mathbf{M}^k as the following example shows ⁽¹⁾: consider in \mathbb{R}^N the equation

$$|DU| + U = \min(|x|, 1) \quad \text{in } \mathbb{R}^N.$$

The "control" solution is given, in $\overline{B(0, 1)}$ by

$$U(x) = |x| + \exp(-|x|) - 1.$$

Now we can consider the stratification where $\mathbf{M}^{N-1} = S(0, 1) = \{x : |x| = 1\}$ and $\mathbf{M}^N = \mathbb{R}^N \setminus \mathbf{M}^{N-1}$. Then the above equation can be reformulated in terms of this stratified domain: for subsolutions, we have the same equation in \mathbf{M}^N while

$$|D_T u| + u \leq 1 \quad \text{on } \mathbf{M}^{N-1},$$

where $D_T u$ stands for the tangent derivative of u on the sphere $S(0, 1)$. For this stratified formulation,

$$u(x) := \begin{cases} 1 & \text{if } x \in S(0, 1), \\ U(x) & \text{if } x \in \mathbb{R}^N \setminus S(0, 1) \end{cases}$$

is obviously a subsolution which is upper semi-continuous since $U(x) = \exp(-1) < 1$ on $S(0, 1)$. But $v := U$ in all \mathbb{R}^N is also a supersolution and of course $u(x) \leq v(x)$ does not hold in \mathbb{R}^N because on $S(0, 1)$, $u(x) = 1 > v(x) = \exp(-1)$.

The key fact in this counter-example to comparison is that we can put artificial values on \mathbf{M}^{N-1} for the subsolution u since these values are unrelated with those of u on \mathbf{M}^N . A first way to correct our result is just to assume u to be continuous or slightly weaker (but difficult to check) requirement.

In order to solve this problem while keeping only usc subsolutions, we have to add a global Ishii type requirement for subsolutions, namely

$$u_t + H_*(x, t, Du) \leq 0 \quad \text{in } \mathbb{R}^N \times (0, T]. \quad (2)$$

Hence **the correct definition for a subsolution** consists in **the classical inequality due to Ishii's definition** (2), **reinforced by the \mathbf{M}^k inequalities** (1) in order to ensure that the control problem on \mathbf{M}^k (with perhaps low costs) is really seen in terms of HJB Equations.

The additional subsolution inequality (2) allows to prove, under the assumptions of [1] and, in particular the normal controllability that

- (i) For any $x \in \mathbf{M}^k$, $k < N - 1$, $u(x) = \limsup\{u(y), y \rightarrow x, y \notin \mathbf{M}^k\}$
- (ii) For any $x \in \mathbf{M}^{N-1}$, we also have $u(x) = \limsup\{u(y), y \rightarrow x, y \notin \mathbf{M}^{N-1}\}$ but this property is true when y stays in any of the (locally) two connected component of $\mathbb{R}^N \setminus \mathbf{M}^{N-1}$.

⁽¹⁾This difficulty does not appear in Bressan & Hong [2] since the subsolutions are assumed to be continuous.

Before giving an idea of the proof of (i) and (ii), we explain how to use it in the comparison proof. The first step is to regularize u by a tangent sup-convolution and then by a standard convolution. We claim in [1] that by doing so, we obtain a continuous u : the above counter-example shows that this statement was wrong. But this is correct assuming (i) and (ii), since (essentially) the values of u on each \mathbf{M}^k are limits of values of u outside \mathbf{M}^k .

To prove (i) we can assume that we are in the stationary case (for simplicity) and that \mathbf{M}^k is flat (this reduction is done in [1] using the regularity of \mathbf{M}^k). Consider the function

$$y \mapsto u(y) - \frac{|y-x|^2}{\varepsilon} - Ce \cdot (y-x) := u(y) - \phi(y),$$

where e is any unit vector normal to \mathbf{M}^k . If $u(x) > \limsup\{u(y), y \rightarrow x, y \notin \mathbf{M}^k\}$, the maximum of this function is necessarily achieved on \mathbf{M}^k at y_ε . Using the normal controllability in \mathbf{M}^N we deduce that H_* is coercive in any normal direction to \mathbf{M}^k so that if C is large enough we reach a contradiction in $H_*(y_\varepsilon, u(y_\varepsilon), D\phi(y_\varepsilon)) \leq 0$, which gives the desired property (i).

In the case of \mathbf{M}^{N-1} , the same proof shows that (i) also holds but this is not enough since, locally, $\mathbb{R}^N \setminus \mathbf{M}^{N-1}$ has two connected components and we have to show that the property is true separately for both connected components. Here, we assume again that \mathbf{M}^{N-1} is a hyperplane and we take the same test-function but with $e = +n$ or $-n$ where n is a normal vector to \mathbf{M}^{N-1} .

If $u(x) > \limsup\{u(y), y \rightarrow x, y \notin \mathbf{M}^N, n \cdot (y-x) > 0\}$, we consider

$$y \mapsto u(y) - \frac{|y-x|^2}{\varepsilon} + Cn \cdot (y-x).$$

The maximum cannot be achieved in the domain where $n \cdot (y-x) > 0$. But in the complementary of this set, the term $+Cn \cdot (y-x)$ has the right sign (*i.e.*, it is non-positive), allowing to show that a maximum is achieved and is converging to x as $\varepsilon \rightarrow 0$. Again, the normal controllability of H_* allows to get the contradiction.

In conclusion, the comparison result of [1] is correct if either we assume the subsolutions to satisfy the properties (i)-(ii) above (for example, continuous subsolutions) or if we assume them to satisfy (2) since this implies (i)-(ii). We also emphasize the fact that this change in the definition of subsolution does not affect the stability properties (for subsolutions) since Ishii's definition has nice stability properties.

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