Differential transcendence & algebraicity criteria for the series counting weighted quadrant walks

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DIFFERENTIAL TRANSCENDENCE & ALGEBRAICITY CRITERIA FOR THE SERIES COUNTING WEIGHTED QUADRANT WALKS

THOMAS DREYFUS AND KILIAN RASCHEL

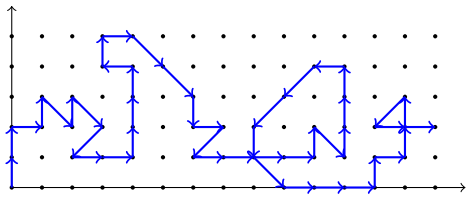
Abstract. We consider weighted small step walks in the positive quadrant, and provide algebraicity and differential transcendence results for the underlying generating functions: we prove that depending on the probabilities of allowed steps, certain of the generating series are algebraic over the field of rational functions, while some others do not satisfy any algebraic differential equation with rational functions coefficients. Our techniques involve differential Galois theory for difference equations as well as complex analysis (Weierstrass parameterization of elliptic curves). We also extend to the weighted case many key intermediate results, as a theorem of analytic continuation of the generating functions.

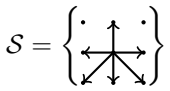
Introduction

Take a walk with small steps in the positive quadrant $\mathbb{Z}^2_{\geqslant 0}$, that is a succession of points

$$P_0, P_1, \ldots, P_k,$$

where each P_n lies in the quarter plane, where the moves (or steps) $P_{n+1}-P_n$ belong to a finite step set $\mathcal{S} \subset \{0,\pm 1\}^2$ which has been chosen a priori, and the probability to move in the direction $P_{n+1}-P_n=(i,j)$ is equal to some weight-parameter $d_{i,j}$, with $\sum_{(i,j)\in\mathcal{S}} d_{i,j}=1$. The following picture is an example of path with $\mathcal{S}=\{\leftarrow,\uparrow,\rightarrow,\searrow,\downarrow,\swarrow\}$ and starting point $P_0=(0,0)$:





Such objects are very natural both in combinatorics and probability theory: they are interesting for themselves and also because they are strongly related to other discrete structures, see [BMM10, DW15] and references therein.

Our main object of investigation is the probability

$$\mathbb{P}[P_0 \xrightarrow{k} (i,j)] \tag{1}$$

that the walk started at P_0 be at some generic position (i,j) after the kth step, with all intermediate points P_n remaining in the cone. More specifically we shall turn our attention to the generating function (or counting function)

$$Q(x,y;t)=\sum_{i,j,k\geqslant 0} \mathbb{P}[P_0 \xrightarrow{k} (i,j)]x^i y^j t^k. \tag{2}$$

Date: September 19, 2017.

2010 *Mathematics Subject Classification.* 05A15,30D05,39A06.

Key words and phrases. Random walks, Difference Galois theory, Elliptic functions, Transcendence.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under the Grant Agreement No 648132.

We are interesting in classifying the algebraic nature of the above series: to which of the following classes does the function in (2) belong to:

$$\text{(3)}\quad \{\text{rational}\} \subset \{\text{algebraic}\} \subset \{\text{holonomic}\} \subset \{\text{differentially algebraic}\}$$
$$\text{v.s.}\quad \{\text{differentially transcendent}\}?$$

Rational and algebraic functions are classical notions. By $Q(x,y;t)$ holonomic (resp. differentially algebraic) we mean that all of $x \mapsto Q(x,y;t)$, $y \mapsto Q(x,y;t)$ and $t \mapsto Q(x,y;t)$ satisfy a linear (resp. algebraic) differential equation with coefficients in $\mathbb{C}(x)$, $\mathbb{C}(y)$ and $\mathbb{C}(t)$, respectively. We say that $Q(x,y;t)$ is differentially transcendent if it is not differentially algebraic. Our main results give sufficient conditions on the weights $d_{i,j}$ to characterize the algebraic nature of the counting function.

Motivations to consider models of weighted walks. In this article we shall go beyond the classical hypothesis consisting in studying unweighted walks, that is walks with $d_{i,j} = 1/|\mathcal{S}|$ for all $(i,j) \in \mathcal{S}$. Indeed, motivations to consider weighted models are multiple: first, they offer a natural framework to generalize the numerous results established for unweighted lattice walks, see our bibliography for a non-exhaustive list of works concerning unweighted quadrant walks, especially [BMM10, BRS14, BK10, DHRS17a, FR10, KR12, KR15, Mis09, MM14, MR09]. Second, some models of unweighted walks in dimension 3 happen to be, after projection, equivalent to models of 2D weighted walks [BBMKM16]. Needless to mention that lattice walks in 3D represent a particularly challenging topic, see [BBMKM16, DHW16]. Third, these models with weights yield results in probability theory, where the hypothesis to have only uniform probabilities (case of unweighted lattice walks) is too restrictive. Fourth, since there exist infinitely many weighted models (compare with only 79 unweighted small step models!), case-by-case reasonings should be excluded, and in some sense only intrinsic arguments merge up, like in [DHRS17b].

Literature. There is a large literature on (mostly unweighted) walks in the quarter plane, focusing on various probabilistic and combinatorial aspects. Two main questions have attracted the attention of the mathematical community: first, finding a closed-form expression for the probability (1), or equivalently for the series (2); second, characterizing the algebraic nature of the series (2), according to the classes depicted in (3). The first question, combinatorial in nature, should not put the second one in the shade: knowing the nature of $Q(x,y;t)$ has consequences on the asymptotic behavior of the coefficients, and further allows to apprehend the complexity of these lattice paths problems (to illustrate this fact, let us remind that unconstrained walks are associated with rational generating functions, while walks confined to a half-plane admit algebraic counting functions [BMP00]). This is this second question that we shall consider in the present work.

To summarize the main results obtained so far in the literature, one can say that for unweighted quadrant models, the generating function (2) is holonomic (third class of functions in (3)) if and only if a certain group of transformations (simply related to the weights, see (9)) is finite; note that models having a finite group are models to which a variant of the well-known reflection principle applies. This is a very satisfactory result, as it connects combinatorial aspects to geometric features. Moreover, there are various tools for verifying whether given parameters lead to a finite group [BMM10, FR10, KY15].

Going back to the algebraic nature of the counting function, the pioneering result is [FIM99] (Chapter 4 of that book), which states that if the group is finite, the function (2) is holonomic, and even algebraic provided that some further condition be satisfied. Then Mishna [Mis09], Mishna and Rechnitzer [MR09], Mishna and Melczer [MM14] observed that there exist infinite group models such that the series is non-holonomic. Bousquet-Mélou and Mishna [BMM10], Bostan and Kauers [BK10] proved that for all unweighted quadrant models with finite group the

series is holonomic. In [KR12, BRS14] the converse statement is shown: for all infinite group models, the series is non-holonomic. The combination of all these works yields the aforementioned equivalence between finite group and holonomic generating function.

The question of differential algebraicity was approached more recently. Bernardi, Bousquet-Mélou and the second author of the present paper showed [BBMR17] that despite being non-holonomic, 9 unweighted quadrant models are differentially algebraic. In [DHRS17a, DHRS17b], the first author of this paper, Hardouin, Roques and Singer proved that all 47 remaining infinite group models are differentially transcendent. See also [FR10, KY15, DHW16] for related studies.

Main results. The above recap shows how actively the combinatorial community took possession of this question. It also illustrates that within a relatively small class of problems (only 79 unweighted different models!), there exists a remarkable variety of behaviors. This certainly explains the vivid interest in this model.

In this article we bring three main contributions, building on the recent works [FIM99, KR12, KR15, DHR15, DHRS17a, DHRS17b] and mixing techniques coming from complex analysis and Galois theory. The first contribution is about the techniques: along the way of proving our other contributions we generalize a certain number of results of [FIM99] (stationary probabilistic case) and [KR12] (unweighted combinatorial case). See below for more explanations.

The second one is a generalization of the differential Galois results of [DHRS17a], see our Theorems 34, 38 and 40, which provide differential transcendence sufficient conditions for weighted walks. These theorems are consequences of a more general result, Proposition 25, coming from [DHR15], which is a criteria (i.e., a necessary and sufficient condition) for differential transcendence. The latter is however not totally explicit in terms of the parameters $d_{i,j}$, contrary to our (easily verified) sufficient conditions. Note that the proofs here are similar to [DHRS17a].

Our third result is about models having a finite group. Theorem 41 and Corollary 42 show that the generating functions are then holonomic, and even algebraic if and only if a certain quantity vanishes (namely, the alternating sum of the monomial xy under the orbit of the group).

Structure of the paper.

- Section 1: statement of the kernel functional equation (5) satisfied by the generating function, study of the zero set defined by the kernel. Results in that section generalize results in [FIM99, KR12], at several places with new and minimal proofs.
- Section 2: elliptic parametrization of the zero set of the kernel and continuation of the generating functions. Results in this section are further and importantly used in Sections 3 and 4.
- Section 3: statement and proof of Theorems 34, 38 and 40, giving sufficient conditions for differential transcendence of the counting series.
- Section 4: Theorem 41 and Corollary 42 on algebraicity criteria for the generating function in the finite group case.

Acknowledgements. We would like to thank Charlotte Hardouin, Manuel Kauers, Julien Roques and Michael Singer for discussions and support of this work.

1. Kernel of the walk

1.1. **Functional equation.** Weighted walks with small steps in the quarter plane are sums of steps taken in a step set $\mathcal{S}$, itself being a subset of $\{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\}$, or alternatively

which will be identified with pairs $(i,j) \in \{0,\pm 1\}^2 \backslash \{(0,0)\}$. For $(i,j) \in \{0,\pm 1\}^2$, let $d_{i,j} \in [0,1]$ with $\sum d_{i,j} = 1$. We consider quadrant walks starting from $P_0 = (0,0) \in \mathbb{Z}^2_{\geqslant 0}$, which at each time move in the direction (i,j) (resp. stay at the same position) with probability $d_{i,j}$ (resp. $d_{0,0}$). A walk will be called unweighted if $d_{0,0} = 0$ and if in addition all non-zero $d_{i,j}$ take the same value.

As said in the introduction, we will mainly focus on the probability $\mathbb{P}[P_0 \xrightarrow{k} (i,j)]$ that the walk be at position (i,j) after k steps, starting from P_0 and with all intermediate points $P_1, \ldots, P_{k-1}$ in the quarter plane. The corresponding trivariate generating function $Q(x,y;t)$ is defined in (2). Being the generating functions of probabilities, $Q(x,y;t)$ converges for all $(x,y,t) \in \mathbb{C}^3$ such that $|x|, |y| < 1$ and $|t| \leqslant 1$. Notice that in several papers, as in particular in [BMM10], it is not assumed that $\sum d_{i,j} = 1$. However, after a rescaling of the t-variable, we may always reduce to this case.

The kernel of the walk is the polynomial defined by $K(x,y;t) := xy\{1 - tS(x,y)\}$, where

$$S(x,y) = \sum_{(i,j)\in\{0,\pm 1\}^2} d_{i,j} x^i y^j = A_{-1}(x)\frac{1}{y} + A_0(x) + A_1(x)y = B_{-1}(y)\frac{1}{x} + B_0(y) + B_1(y)x, \tag{4}$$

and $A_i(x) \in x^{-1}\mathbb{R}[x]$, $B_i(y) \in y^{-1}\mathbb{R}[y]$. Define further

$$F^1(x;t) := K(x,0;t)Q(x,0;t) \quad \text{and} \quad F^2(y;t) := K(0,y;t)Q(0,y;t).$$

The following is an adaptation of [BMM10, Lemma 4] to our context; the proof is omitted since it is exactly the same as in [BMM10].

Lemma 1. *The generating function $Q(x,y;t)$ introduced in (2) satisfies the functional equation*

$$K(x,y;t)Q(x,y;t) = xy - F^1(x;t) - F^2(y;t) + K(0,0;t)Q(0,0;t). \tag{5}$$

1.2. **Basic properties of the kernel.** From now, let us fix $t \in (0,1)$. The kernel curve $\overline{E}_t$ is defined as the zero set in $\mathbb{P}^1(\mathbb{C})^2$ of the homogeneous polynomial

$$\overline{K}(x_0,x_1,y_0,y_1;t) = x_0x_1y_0y_1 - t\sum_{i,j=0}^{2} d_{i-1,j-1}x_0^i x_1^{2-i} y_0^j y_1^{2-j}. \tag{6}$$

Working in $\mathbb{P}^1(\mathbb{C})$ rather than on $\mathbb{C}$ appears to be particularly convenient and allows avoiding tedious discussions (as in particular on the number of branch points, see [KR12, Section 2.1]).

To simplify the notations, for $x = [x_0 : x_1], y = [y_0 : y_1] \in \mathbb{P}^1(\mathbb{C})$, we shall alternatively write $\overline{K}(x,y;t)$, $\overline{K}(x,y_0,y_1;t)$ and $\overline{K}(x_0,x_1,y;t)$, instead of $\overline{K}(x_0,x_1,y_0,y_1;t)$.

We now need to discard some degenerate cases. Following [FIM99] we introduce the concept of singular model.

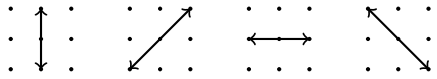
Definition 2. A walk is called singular if one of the following holds:

- $(x,y) \mapsto K(x,y;t)$ is reducible over $\mathbb{C}[X,Y]$;
- $(x,y) \mapsto K(x,y;t)$ has not bidegree $(2,2)$.

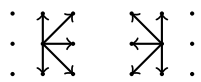
An analogue of the following result has been proved in [FIM99, Lemma 2.3.2] in the case $t = 1$. In what follows, $\mathbb{Q}(d_{i,j}) \subset \mathbb{R}$ denotes the field generated over $\mathbb{Q}$ by the $d_{i,j}$.

Lemma 3 ([DHRS17b], Proposition 1.2)**.** *Assume that t is transcendent over $\mathbb{Q}(d_{i,j})$. A walk is singular if and only if at least one of the following holds:*

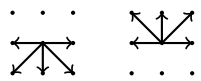
- *There exists $(i,j) \in \mathbb{Z}^2$, $|i|, |j| \leqslant 1$, such that only $d_{i,j}, d_{0,0}, d_{-i,-j}$ are different from zero. This corresponds to walks with steps supported in one of the following configurations:*



- *There exists* $i \in \{-1, 1\}$ *such that* $d_{i,-1} = d_{i,0} = d_{i,1} = 0$. *This corresponds to walks with steps supported in one of the following configurations:*



- *There exists* $j \in \{-1, 1\}$ *such that* $d_{-1,j} = d_{0,j} = d_{1,j} = 0$. *This corresponds to walks with steps supported in one of the following configurations:*



In what follows, we shall always assume that the walk is not singular. When t is transcendent over $\mathbb{Q}(d_{i,j})$, this only discards one-dimensional problems (resp. walks with support included in a half-plane), which are easier to study, as explained in [BMM10, Section 2.1].

Let us define the genus of the walk as the genus of the curve $\overline{E}_t$. As we will see, the genus may be equal to zero or one, but we will only discuss non-singular walks of genus one, the genus zero case being considered in [MR09, MM14, DHRS17b], and every weighted walk of genus zero has a generating series which is differentially transcendent, see [DHRS17b].

The following lemma, which gives a characterization of the genus of $\overline{E}_t$, has been proved in [DHRS17b] in the case t transcendent over $\mathbb{Q}(d_{i,j})$; the proof is exactly the same in our context. Let us remind that a singularity of $\overline{E}_t$ is a point of $\overline{E}_t$ at which the partial derivatives vanish. For instance, let $P = ([a:1], [b:1]) \in \overline{E}_t$ with $ab \neq 0$. Then, P is a singularity if and only if

$$b - t \sum_{i=1,j=0}^{2} i d_{i-1,j-1} a^{i-1} b^j = a - t \sum_{i=0,j=1}^{2} j d_{i-1,j-1} a^i b^{j-1} = 0.$$

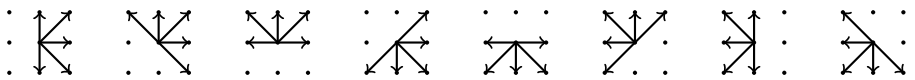
Lemma 4 ([DHRS17b], Lemma 1.4)**.** *The following facts are equivalent:*

(1) *The curve* $\overline{E}_t$ *is a genus zero curve;*
(2) *The curve* $\overline{E}_t$ *has exactly one singularity.*

Otherwise, the curve $\overline{E}_t$ *is a genus one curve with no singularity, i.e.,* $\overline{E}_t$ *is an elliptic curve.*

When t is transcendent over $\mathbb{Q}(d_{i,j})$ we have a very simple, geometric characterization of genus zero curves.

Lemma 5 ([DHRS17b], Lemma 1.5)**.** *Assume that* t *is transcendent over* $\mathbb{Q}(d_{i,j})$. *A walk whose curve* $\overline{E}_t$ *has genus zero is a walk whose steps are supported in one of the following configurations:*



1.3. **Unit circles on the kernel curve.** Due to the domain of convergence of the power series $Q(x, y; t)$, the domains where $|x| = 1$ and $|y| = 1$ are particularly interesting; some of their properties are studied in Lemmas 6 and 7. In what follows, we make the convention that the modulus of an element of $\mathbb{P}^1(\mathbb{C}) \setminus \{[1:0]\}$ is the modulus of the corresponding complex number. Furthermore, the point $[1:0]$ has modulus strictly bigger than that of any other element in $\mathbb{P}^1(\mathbb{C})$.

Lemma 6. *There are no* $x, y \in \mathbb{P}_1(\mathbb{C})$ *with* $|x| = |y| = 1$ *such that* $(x, y) \in \overline{E}_t$.

Proof. Let $x, y \in \mathbb{P}^1(\mathbb{C})$ with $|x| = |y| = 1$. Let $\overline{S}(x, y)$ be the continuation to $\mathbb{P}^1(\mathbb{C})^2$ of the function $S(x, y)$ defined in (4). The triangular inequality yields that $|\overline{S}(x, y)| \leqslant 1$ (recall that $\sum d_{i,j} = 1$) and finally $t|\overline{S}(x, y)| < 1$. Since $|xy| = 1$, we deduce the inequality $\overline{K}(x, y; t) \neq 0$. □

For $i \in \{-1,0,1\}$, let $\overline{A}_i(x)$ be the continuation of $A_i(x)$ to $\mathbb{P}^1(\mathbb{C})$. Let us see $y \mapsto \overline{K}(x,y;t)$ as a polynomial of degree two, and let

$$Y_\pm(x) = \frac{-t\overline{A}_0(x) + 1 \pm \sqrt{(t\overline{A}_0(x)-1)^2 - 4t^2\overline{A}_{-1}(x)\overline{A}_1(x)}}{2t\overline{A}_1(x)}$$

be the two roots, so that for all $x \in \mathbb{P}^1(\mathbb{C})$, we have $(x, Y_\pm(x)) \in \overline{E}_t$. Similarly, let the two roots of $x \mapsto \overline{K}(x,y;t)$ be denoted by

$$X_\pm(y) = \frac{-t\overline{B}_0(y) + 1 \pm \sqrt{(t\overline{B}_0(y)-1)^2 - 4t^2\overline{B}_{-1}(y)\overline{B}_1(y)}}{2t\overline{B}_1(y)}.$$

For all $y \in \mathbb{P}^1(\mathbb{C})$, we have $(X_\pm(y), y) \in \overline{E}_t$.

Introduce the sets

$$\Gamma_x = \overline{E}_t \cap \{|x| = 1\} \quad \text{and} \quad \Gamma_y = \overline{E}_t \cap \{|y| = 1\}.$$

The following result generalizes [FIM99, Lemma 2.3.4]. It is illustrated on Figure 1.

Lemma 7. *The set Γ_x is composed of two curves: Γ_x^- such that $(x,y) \in \Gamma_x^- \Rightarrow |y| < 1$, and Γ_x^+ such that $(x,y) \in \Gamma_x^+ \Rightarrow |y| > 1$. A symmetric statement holds for Γ_y.*

Proof. By definition,

$$\Gamma_x = \{(x, Y_-(x)) : |x| = 1\} \bigcup \{(x, Y_+(x)) : |x| = 1\}.$$

Consider the Taylor expansion at $t = 0$ of

$$Y_\pm(x) = \frac{-t\overline{A}_0(x) + 1 \pm \sqrt{(t\overline{A}_0(x)-1)^2 - 4t^2\overline{A}_{-1}(x)\overline{A}_1(x)}}{2t\overline{A}_1(x)}.$$

Note that $\overline{A}_1(x)$ is not identically zero since otherwise the walk would be singular. We have

$$Y_+(x) = \frac{1}{t\overline{A}_1(x)} + O(1), \qquad Y_-(x) = \frac{t}{4}\frac{4\overline{A}_{-1}(x)\overline{A}_1(x) - \overline{A}_0(x)^2}{\overline{A}_1(x)} + O(t^2),$$

proving that when $\overline{A}_1(x) \neq 0$, $Y_+(x)$ goes to infinity when t goes to 0 and $Y_-(x)$ goes to 0. Since the curves $\{(x, Y_\pm(x)) : |x| = 1\}$ can not intersect $\overline{E}_t$, see Lemma 6, we obtain that when t is close to zero, the curve $\{(x, Y_-(x)) : |x| = 1\}$ has y-coordinates with modulus strictly smaller than 1, while $\{(x, Y_+(x)) : |x| = 1\}$ has y-coordinates with modulus strictly bigger than 1. So the result stated in Lemma 7 is correct for t close to zero. Let us prove the result for an arbitrary t. To the contrary, assume that it is not the case. Since the two curves depend continuously upon $t \in (0,1)$, there must exist $t_0 \in (0,1)$ such that one of the two curves intersect $\overline{E}_{t_0} \cap \{|y| = 1\}$, thereby contradicting Lemma 6. □

1.4. **Discriminants.** For $[x_0 : x_1], [y_0 : y_1] \in \mathbb{P}^1(\mathbb{C})$, we denote by $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ the discriminants of the degree 2 homogeneous polynomials $y \mapsto \overline{K}(x_0, x_1, y; t)$ and $x \mapsto \overline{K}(x, y_0, y_1; t)$, respectively, i.e.,

$$\Delta^x_{[x_0:x_1]} = t^2\Big(\big(d_{-1,0}x_1^2 - \frac{1}{t}x_0x_1 + d_{0,0}x_0x_1 + d_{1,0}x_0^2\big)^2 - 4(d_{-1,1}x_1^2 + d_{0,1}x_0x_1 + d_{1,1}x_0^2)(d_{-1,-1}x_1^2 + d_{0,-1}x_0x_1 + d_{1,-1}x_0^2)\Big) \tag{7}$$

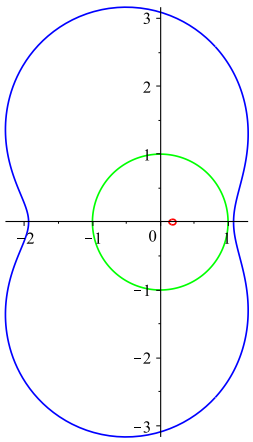


FIGURE 1. The unit circle (in green) and the curves $Y_+(\{|x|=1\})$ (blue) and $Y_-(\{|x|=1\})$ (red), for the model with jumps $d_{-1,1}=\frac{1}{2}$, $d_{-1,0}=d_{0,-1}=d_{1,1}=1/6$ (see on the left of Figure 5) and $t=0.96$

and

$$\Delta^y_{[y_0:y_1]} = t^2\Big(\big(d_{0,-1}y_1^2 - \frac{1}{t}y_0y_1 + d_{0,0}y_0y_1 + d_{0,1}y_0^2\big)^2 \\ - 4(d_{1,-1}y_1^2 + d_{1,0}y_0y_1 + d_{1,1}y_0^2)(d_{-1,-1}y_1^2 + d_{-1,0}y_0y_1 + d_{-1,1}y_0^2)\Big).$$

The polynomial $\Delta^x_{[x_0:x_1]}$ (resp. $\Delta^y_{[y_0:y_1]}$) is of degree four and so has four roots a_1,a_2,a_3,a_4 (resp. b_1,b_2,b_3,b_4). To go further we need additional informations about the kernel and the zeros of $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$.

Proposition 8 ([Dui10, §2.4.1, especially Proposition 2.4.3])**.** *The following facts are equivalent:*

1. *The curve $\overline{E}_t$ is a genus one curve with no singularities, i.e., $\overline{E}_t$ is an elliptic curve;*
2. *The discriminant $\Delta^x_{[x_0:x_1]}$ has simple roots in $\mathbb{P}_1(\mathbb{C})$;*
3. *The discriminant $\Delta^y_{[y_0:y_1]}$ has simple roots in $\mathbb{P}_1(\mathbb{C})$.*

Assumption 9. *From now we assume that $\overline{E}_t$ is an elliptic curve and the walk is non-singular.*

Lemma 10. *The four roots a_1,a_2,a_3,a_4 are real and distinct. Furthermore, two of them, namely a_1,a_2, satisfy $-1<a_1<a_2<1$ and the other two, namely a_3,a_4, satisfy $1<|a_3|,|a_4|$. The same holds for b_1,b_2,b_3,b_4.*

Remark 11. We choose to order the a_i in such a way that the cycle of $\mathbb{P}_1(\mathbb{R})$ starting from -1 and going to $+\infty$, and then from $-\infty$ to -1 crosses the a_i in the order a_1,a_2,a_3,a_4.

Proof of Lemma 10. Let us prove the statement for $\Delta^y_{[y_0:y_1]}$, the other case being clearly similar. We first claim that two roots have modulus bigger (resp. smaller) than 1. The polynomial $\Delta^y_{[y_0:y_1]}$ is homogeneous of degree four. More precisely, it is equal to

$$\begin{aligned}(8)\quad &\{d_{0,1}^2 - 4d_{1,1}d_{-1,1}\}t^2y_0^4 + \{2t^2d_{0,1}d_{0,0} - 2td_{0,1} - 4t^2(d_{1,0}d_{-1,1} + d_{1,1}d_{-1,0})\}y_0^3y_1 \\ &\quad + \{1 + t^2d_{0,0}^2 + 2t^2d_{0,-1}d_{0,1} - 4t^2(d_{1,-1}d_{-1,1} + d_{1,0}d_{-1,0} + d_{1,1}d_{-1,-1})\}y_0^2y_1^2 \\ &+ \{2t^2d_{0,-1}d_{0,0} - 2td_{0,-1} - 4t^2(d_{1,-1}d_{-1,0} + d_{1,0}d_{-1,-1})\}y_0y_1^3 + \{d_{0,-1}^2 - 4d_{1,-1}d_{-1,-1}\}t^2y_1^4.\end{aligned}$$

When $t \in \mathbb{R}$ is close to 0, the result is straightforward since $\Delta^y_{[y_0:y_1]}$ has two roots close to $[0:1]$ and two roots close to $[1:0]$. Since the roots depend continuously upon t, it is sufficient to prove that for any $t \in (0,1)$, the b_i's cannot have modulus 1. Assume to the contrary that $|b_i| = 1$. Then $x \mapsto \overline{K}(x, b_i; t)$ has a double root. But this contradicts Lemma 7, since one root should have a modulus strictly bigger than 1 and the other one a modulus strictly smaller than 1, proving they cannot be equal. This proves our claim.

With Proposition 8, the b_i's are two by two distinct. So it is now sufficient to prove that the four roots are real. Let us begin by proving the result when t is transcendent over $\mathbb{Q}(d_{i,j})$. Introduce the set of parameters

$$\mathcal{P} = \{d_{i,j} \in [0,1] : \sum d_{i,j} = 1\}$$

and consider an arbitrary element $p_0 \in \mathcal{P}$ such that the corresponding walk is non-singular of genus one. Let $\gamma : [0,1] \to \mathcal{P}$ be a continuous path such that

- $\gamma(0) = \{d_{1,-1} = d_{1,1} = d_{-1,1} = 1/3\}$;
- $\gamma(1/2) = p_0$;
- for all $\varepsilon \in (0,1)$, the walk corresponding to $\gamma(\varepsilon)$ is non-singular of genus one;
- $\gamma(1) = \{d_{1,-1} = d_{-1,-1} = d_{-1,1} = 1/3\}$.

The existence of such a path is justified by Lemma 5. As we may see in [DHRS17b, Lemma 1.7], the discriminant $\Delta^y_{[y_0:y_1]}$ corresponding to $\gamma(0)$ has a double root $[0:1]$ and two simple real roots. With the claim, this shows that the two simple real roots, namely b_3, b_4, have modulus strictly bigger than one. The roots depend continuously in the path γ. Let $b_3(\varepsilon), b_4(\varepsilon)$ be the roots on the path γ. As roots of a real polynomial, they are complex conjugate or real. But the two other roots have modulus strictly smaller than 1, as we have shown in the above claim, so for all $\varepsilon \in [0,1]$ we have two possibilities:

- $b_3(\varepsilon), b_4(\varepsilon)$ are real;
- $b_3(\varepsilon), b_4(\varepsilon)$ are complex conjugate.

Remind that $b_3(0), b_4(0)$ are real and distinct. But if $b_3(\varepsilon_0), b_4(\varepsilon_0)$ are complex conjugate for some $\varepsilon_0 \in (0,1]$, using the above dichotomy and the fact that they depend continuously upon ε, we conclude that they should be equal for some $\varepsilon_1 \leqslant \varepsilon_0$. With Proposition 8, we find that this implies that $\gamma(\varepsilon_1)$ has not genus one, and by construction $\varepsilon_1 = 1$. This shows that $b_3(\varepsilon), b_4(\varepsilon)$ are real for all $\varepsilon \in (0,1)$. Similarly, we prove that for all $\varepsilon \in (0,1)$, the two other roots $b_1(\varepsilon), b_2(\varepsilon)$ are real. We apply this result to $\varepsilon = 1/2$ in order to deduce the result when t transcendent over $\mathbb{Q}(d_{i,j})$.

Let us prove the general case. For t transcendent over $\mathbb{Q}(d_{i,j})$ such that the walk has genus one, the roots are real and distinct. Since they are solutions of a real polynomial equation, for $t \in (0,1)$ algebraic over $\mathbb{Q}(d_{i,j})$, they are complex conjugate or real. But they depend continuously upon t, and the set of t transcendent over $\mathbb{Q}(d_{i,j})$ is dense in $(0,1)$. This means that for t algebraic over $\mathbb{Q}(d_{i,j})$, the roots b_1, b_2, b_3, b_4 are real, and can possibly be equal. With Proposition 8, they are distinct when the walk has genus one. This shows the result. □

1.5. **Group of the walk.** Following [FIM99, Chapter 2], [BMM10, Section 3] or [KY15, Section 3], we attach to any model its group, which by definition is the group $\langle i_1, i_2 \rangle$ generated by the involutive birational transformations of $\mathbb{C}^2$ given by

$$\iota_1(x,y) = \left(x, \frac{A_{-1}(x)}{A_1(x)}\frac{1}{y}\right) \quad \text{and} \quad \iota_2(x,y) = \left(\frac{B_{-1}(y)}{B_1(y)}\frac{1}{x}, y\right). \tag{9}$$

We extend the definition of ι_1 and ι_2 to $\mathbb{P}^1(\mathbb{C})^2$.

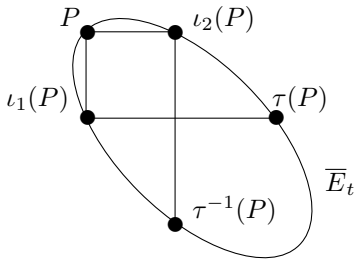


FIGURE 2. The maps ι_1, ι_2 restricted to the kernel curve $\overline{E}_t$

The kernel curve $\overline{E}_t$ is left invariant by the natural action of this group. For a fixed value of x, there are at most two possible values of y such that $(x, y) \in \overline{E}_t$. The involution ι_1 corresponds to interchanging these values, see Figure 2. A similar interpretation can be given for ι_2.

Let us finally define

$$\tau = \iota_2 \circ \iota_1.$$

Note that such a map is known as a QRT-map and has been widely studied, see [Dui10]. It is connected, for instance, with mathematical physics. As we will see later, the algebraic nature of the series (2), according to the classes depicted in (3), highly depend on the fact that τ has finite order or not.

2. Analytic continuation of the generating series

The generating series $Q(x, y; t)$ is analytic for $|x| < 1$, $|y| < 1$. The goal of this section is to prove that $F^1(x; t)$ and $F^2(y; t)$ defined in (5) may be continued into multivalued meromorphic functions on the elliptic curve $\overline{E}_t$. We are going to use a uniformization of $\overline{E}_t$ via the Weierstrass elliptic function in order to see the multivalued functions as univalued meromorphic functions on $\mathbb{C}$.

2.1. Uniformization of the elliptic curve $\overline{E}_t$.

Recall that Assumption 9 holds: the walk is non-singular and $\overline{E}_t$ is an elliptic curve. This implies (Proposition 8) that the discriminants $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ have four distinct zeros.

Since $\overline{E}_t$ is an elliptic curve, we can identify $\overline{E}_t$ with $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$, with $(\omega_1, \omega_2) \in \mathbb{C}^2$ basis of a lattice, via the $(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$-periodic map

$$\begin{array}{rccc} \Lambda: & \mathbb{C} & \to & \overline{E}_t \\ & \omega & \mapsto & (\mathfrak{q}_1(\omega), \mathfrak{q}_2(\omega)), \end{array}$$

where $\mathfrak{q}_1, \mathfrak{q}_2$ are rational functions of $\wp$ and its derivative $d\wp/d\omega$, and $\wp$ is the Weierstrass function associated with the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$:

$$\wp(\omega) := \frac{1}{\omega^2} + \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left\{ \frac{1}{(\omega + \ell_1\omega_1 + \ell_2\omega_2)^2} - \frac{1}{(\ell_1\omega_1 + \ell_2\omega_2)^2} \right\}.$$

Then, the field of meromorphic functions on $\overline{E}_t$ may be identified with the field of meromorphic functions on $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$, i.e., the field of functions meromorphic on $\mathbb{C}$ that are (ω_1, ω_2)-periodic. This latter field is equal to $\mathbb{C}(\wp, \wp')$.

The goal of this subsection is to give explicit expressions for $\mathfrak{q}_1(\omega), \mathfrak{q}_2(\omega), \omega_1, \omega_2$. Note that such computations have been performed in [FIM99, Section 3.3] when $t = 1$.

The maps ι_1, ι_2, τ may be lifted to the ω-plane. We will call them $\widetilde{\iota}_1, \widetilde{\iota}_2, \widetilde{\tau}$, respectively. More precisely, following [Dui10] (see in particular Proposition 2.5.2, Page 35 and Remark 2.3.8), there exist $\omega_3, \omega_4 \in \mathbb{C}$ such that

$$\widetilde{\iota}_1(\omega) = -\omega + \omega_4, \quad \widetilde{\iota}_2(\omega) = -\omega + \omega_3 \quad \text{and} \quad \widetilde{\tau}(\omega) = \omega - \omega_4 + \omega_3. \tag{10}$$

Up to a variable change of the form $\omega \mapsto \omega + \omega_4$, we may reduce to the case $\omega_4 = 0$. So let us assume that $\omega_4 = 0$ in the above formulas (10).

Let us write $D(x) := \Delta^x_{[x:1]} = \sum_{j=0}^4 \alpha_j x^j$ and $E(y) := \Delta^y_{[y:1]} = \sum_{j=0}^4 \beta_j y^j$. Using (7) we have

$$D(x) = B^2(x) - 4A(x)C(x),$$

with

$$\begin{cases} A(x) = t(d_{-1,1} + d_{0,1}x + d_{1,1}x^2), \\ B(x) = t(d_{-1,0} - \frac{1}{t}x + d_{0,0}x + d_{1,0}x^2), \\ C(x) = t(d_{-1,-1} + d_{0,-1}x + d_{1,-1}x^2). \end{cases}$$

Let $z = 2A(x)y + B(x)$.

The following proposition is the adaptation of [FIM99, Lemma 3.3.1] to our context.

Lemma 12. *The elliptic curve $\overline{K}(x, y; t) = 0$ admits a uniformization of the form:*

	$x(\omega)$	$z(\omega)$
$a_4 \neq [1:0]$	$\left[a_4 + \frac{D'(a_4)}{\wp(\omega) - \frac{1}{6}D''(a_4)} : 1\right]$	$\left[\frac{D'(a_4)\wp'(\omega)}{2(\wp(\omega) - \frac{1}{6}D''(a_4))^2} : 1\right]$
$a_4 = [1:0]$	$[\wp(\omega) - \alpha_2/3 : \alpha_3]$	$[-\wp'(\omega) : 2\alpha_3]$

Proof. The equality $\overline{K}(x, y; t) = 0$ can be reformulated as

$$z^2 = D(x). \tag{11}$$

The main idea is to reduce to the case $a_4 = [1:0]$ by performing a fractional linear transformation. Assume that $a_4 \neq [1:0]$, and introduce

$$u = \frac{D'(a_4)}{x - a_4} \quad \text{and} \quad v = \frac{2zD'(a_4)}{(x - a_4)^2}.$$

Note that $D'(a_4) \neq 0$ since otherwise a_4 would be a double zero of $D(x)$, which would contradict the fact that the discriminant has simple zeros. The Taylor formula $D(x) = \sum_{j=1}^4 \frac{(x-a_4)^j D^{(j)}(a_4)}{j!}$ allows us to express (11) as

$$v^2 = 4u^3 + 2D''(a_4)u^2 + \frac{2u}{3}D^{(3)}(a_4)D'(a_4) + \frac{D^{(4)}(a_4)D'(a_4)^2}{6}.$$

Letting finally $\mathbf{u} = u + \frac{D''(a_4)}{6}$, we obtain the Weierstrass canonical form

$$v^2 = 4\mathbf{u}^3 - g_2\mathbf{u} - g_3,$$

where

$$g_2 = 60 \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{\ell_1\omega_1 + \ell_2\omega_2}\right)^4, \quad \text{and} \quad g_3 = 140 \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{\ell_1\omega_1 + \ell_2\omega_2}\right)^6.$$

We just have to set $\mathbf{u} = \wp$ and $v = \wp'$ to obtain the result.

When $a_4 = [1:0]$, we have $\alpha_4 = 0$ and thus $\alpha_3 \neq 0$ (otherwise $[1:0]$ would be a double zero of the discriminant, contradicting our assumption). In this case we perform the changes of variable

$$x = \frac{\mathbf{u} - \alpha_2/3}{\alpha_3} \quad \text{and} \quad z = \frac{-v}{2\alpha_3}$$

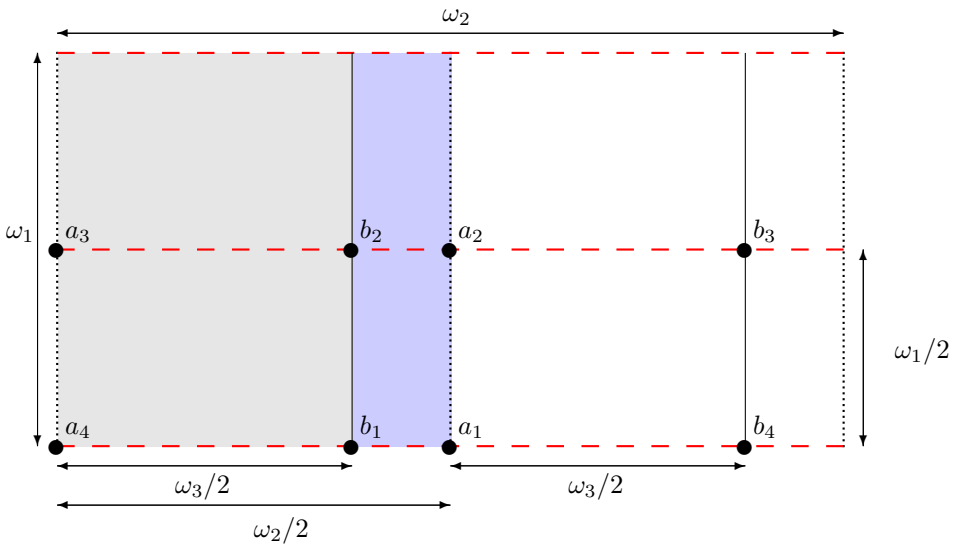


FIGURE 3. Real points of $x(\omega)$ and $y(\omega)$ on the fundamental parallelogram

in order to recover the Weierstrass canonical form. Again, we just have to set $\mathbf{u} = \wp$ and $v = \wp'$ to obtain the result announced in Lemma 12. □

Lemma 13. *One can choose the periods ω_1, ω_2 such that ω_1 is purely imaginary and ω_2 is real.*

Proof. As we see in the proof of Lemma 12, the invariants g_2, g_3 are real and the discriminant of $4\mathbf{u}^3 - g_2\mathbf{u} - g_3$ is strictly positive since we have three distinct real roots (this follows from the fact that the a_i's are real and distinct). Then the fact that we may choose ω_1, ω_2 real and purely imaginary may be deduced from [WW96, Section 20.32, Example 1]. □

Lemma 14. *The following holds:*

- $\omega \in \{\frac{\omega_1}{2}\mathbb{Z} + \omega_2\mathbb{R}\} \Longrightarrow x(\omega), y(\omega) \in \mathbb{P}_1(\mathbb{R})$ *(dashed line on Figure 3);*
- $\omega \in \{\omega_1\mathbb{R} + \frac{\omega_2}{2}\mathbb{Z}\} \Longrightarrow x(\omega) \in \mathbb{P}_1(\mathbb{R})$ *(dotted line on Figure 3);*
- $\omega \in \{\omega_1\mathbb{R} + \frac{\omega_2}{2}\mathbb{Z}\} \setminus \{\frac{\omega_1}{2}\mathbb{Z} + \frac{\omega_2}{2}\mathbb{Z}\} \Longrightarrow y(\omega) \notin \mathbb{P}_1(\mathbb{R})$.

Proof. The proof follows from the (well-known) location of the real points of $\wp, \wp'$ on the fundamental parallelogram, and the location of the purely imaginary points of $\wp'$ when one period is real and the other one purely imaginary.

For $\omega \in \mathbb{C}$, let $\overline{\omega}$ be the complex conjugate. Since by Lemma 13 the period ω_1 is purely imaginary and ω_2 is real, we have for all $\omega \in \mathbb{C}$

$$\overline{\wp(\omega)} = \wp(\overline{\omega}) \quad \text{and} \quad \overline{\wp'(\omega)} = \wp'(\overline{\omega}).$$

Moreover, we have

$$\wp(-\omega) = \wp(\omega) \quad \text{and} \quad \wp'(-\omega) = -\wp'(\omega). \tag{12}$$

Let $\omega \in \frac{\omega_1}{2}\mathbb{Z} + \omega_2\mathbb{R}$ (first item of Lemma 14). With (12) and the (ω_1, ω_2)-periodicity, we get that $\overline{\wp(\omega)} = \wp(\overline{\omega}) = \wp(\omega)$ and $\overline{\wp'(\omega)} = \wp'(\overline{\omega}) = \wp'(\omega)$. This shows that $\wp(\omega), \wp'(\omega) \in \mathbb{P}_1(\mathbb{R})$. Let now $\omega \in \omega_1\mathbb{R} + \frac{\omega_2}{2}\mathbb{Z}$ (second item of Lemma 14). Using again (12) and the (ω_1, ω_2)-periodicity, we obtain that $\overline{\wp(\omega)} = \wp(\overline{\omega}) = \wp(-\omega) = \wp(\omega)$ and $\overline{\wp'(\omega)} = \wp'(\overline{\omega}) = \wp'(-\omega) = -\wp'(\omega)$. This shows that $\wp(\omega) \in \mathbb{P}_1(\mathbb{R})$ and $\wp'(\omega) \in \mathbf{i}\mathbb{R} \cup [1:0]$.

Using Lemma 12, we deduce that $x(\omega) \in \mathbb{P}_1(\mathbb{R})$ for $\omega \in \{\omega_1\mathbb{R} + \frac{\omega_2}{2}\mathbb{Z}\} \bigcup \{\frac{\omega_1}{2}\mathbb{Z} + \omega_2\mathbb{R}\}$. Let us remind that the three zeros of $\wp'$ modulo $\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$ are $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ and that the unique triple pole is at 0. So $\wp'(\omega) \in \mathbf{i}\mathbb{R}^*$ for ω that belongs to $\{\omega_1\mathbb{R} + \frac{\omega_2}{2}\mathbb{Z}\} \setminus \{\frac{\omega_1}{2}\mathbb{Z} + \frac{\omega_2}{2}\mathbb{Z}\}$ (third item of Lemma 14). We then conclude with Lemma 12 that $\omega \in \{\frac{\omega_1}{2}\mathbb{Z}+\omega_2\mathbb{R}\}$ implies $z(\omega) \in \mathbb{P}_1(\mathbb{R})$, and $\omega \in \{\omega_1\mathbb{R} + \frac{\omega_2}{2}\mathbb{Z}\} \setminus \{\frac{\omega_1}{2}\mathbb{Z} + \frac{\omega_2}{2}\mathbb{Z}\}$ yields $z(\omega) \notin \mathbb{P}_1(\mathbb{R})$. The result on $y(\omega)$ follows by combining the results on $x(\omega)$ and $z(\omega)$. □

Recall that ω_3 is introduced in (10).

Lemma 15. *The following holds:*

- $(x(0), y(0)) = (a_4, Y_\pm(a_4))$;
- $(x(\frac{\omega_1}{2}), y(\frac{\omega_1}{2})) = (a_3, Y_\pm(a_3))$;
- $(x(\frac{\omega_1+\omega_2}{2}), y(\frac{\omega_1+\omega_2}{2})) = (a_2, Y_\pm(a_2))$;
- $(x(\frac{\omega_2}{2}), y(\frac{\omega_2}{2})) = (a_1, Y_\pm(a_1))$;
- $\Lambda\{\frac{\omega_3}{2}, \frac{\omega_1+\omega_3}{2}, \frac{\omega_2+\omega_3}{2}, \frac{\omega_1+\omega_2+\omega_3}{2}\} = \{(X_\pm(b_1), b_1), (X_\pm(b_2), b_2), (X_\pm(b_3), b_3), (X_\pm(b_4), b_4)\}$.

Proof. Remind that a_1, a_2, a_3, a_4 are the roots of the discriminant. By construction the roots of $\Delta^x_{[x_0:x_1]}$ correspond to points where $y \mapsto \overline{K}(a_j, y; t)$ has a double root. Therefore, the $(a_j, Y_\pm(a_j))$ are the fixed points by ι_1. Since a_1, a_2, a_3, a_4 are distinct and $\{0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\}$ are the four distinct fixed points modulo $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ of $\widetilde{\iota}_1(\omega) = -\omega$, we find the equality of sets

$$\Lambda\left\{0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\right\} = \{(a_1, Y_\pm(a_1)), (a_2, Y_\pm(a_2)), (a_3, Y_\pm(a_3)), (a_4, Y_\pm(a_4))\}.$$

Similarly, we prove

$$\Lambda\left\{\frac{\omega_3}{2}, \frac{\omega_1+\omega_3}{2}, \frac{\omega_2+\omega_3}{2}, \frac{\omega_1+\omega_2+\omega_3}{2}\right\} = \{(X_\pm(b_1), b_1), (X_\pm(b_2), b_2), (X_\pm(b_3), b_3), (X_\pm(b_4), b_4)\}.$$

By construction, see Lemma 12, we have $(x(0), y(0)) = (a_4, Y_\pm(a_4))$.

Consider a path in straight line from 0 to $\frac{\omega_1}{2}$. As we may see in Lemma 14, $x(\omega)$ is real and $y(\omega) \notin \mathbb{P}_1(\mathbb{R})$ except for $\omega = \{0, \frac{\omega_1}{2}\}$. Let us argue ad absurdum and assume that $x(\frac{\omega_1}{2})$ is not a_3. So $x(\frac{\omega_1}{2}) \in \{a_1, a_2\}$ and $|x(\frac{\omega_1}{2})| < 1 < |x(0)|$, see Lemma 10. Then, there is an ω_0 in the path such that $|x(\omega_0)| = 1$. But $y(\omega_0)$, which is not real due to Lemma 14, is one of the two roots of a real polynomial. The two roots have to be complex conjugate and thus have the same modulus. This contradicts Lemma 7 and proves that $(x(\frac{\omega_1}{2}), y(\frac{\omega_1}{2})) = (a_3, Y_\pm(a_3))$. The branch point $x(\frac{\omega_2}{2})$ should be the other neighbor of a_4 and we deduce that $(x(\frac{\omega_2}{2}), y(\frac{\omega_2}{2})) = (a_1, Y_\pm(a_1))$. □

Remark 16. The position of the four points b_1, b_2, b_3, b_4 is not specified yet. We may order them as in Figure 3, with

- $(x(\frac{\omega_3}{2}), y(\frac{\omega_3}{2})) = (X_\pm(b_1), b_1)$;
- $(x(\frac{\omega_1+\omega_3}{2}), y(\frac{\omega_1+\omega_3}{2})) = (X_\pm(b_2), b_2)$;
- $(x(\frac{\omega_1+\omega_2+\omega_3}{2}), y(\frac{\omega_1+\omega_2+\omega_3}{2})) = (X_\pm(b_3), b_3)$;
- $(x(\frac{\omega_2+\omega_3}{2}), y(\frac{\omega_2+\omega_3}{2})) = (X_\pm(b_4), b_4)$.

Proposition 17. *We may take the following formulas:*

$$\omega_1 = \mathbf{i}\int_{a_3}^{a_4} \frac{dx}{\sqrt{|D(x)|}} \in \mathbf{i}\mathbb{R}_{>0},$$
$$\omega_2 = \int_{a_4}^{a_1} \frac{dx}{\sqrt{D(x)}} \in \mathbb{R}_{>0},$$
$$\omega_3 = \int_{a_4}^{X_{\pm}(b_1)} \frac{dx}{\sqrt{D(x)}} \in (0,\omega_2).$$

Note a small misprint in [FIM99, Lemma 3.3.2], namely a (multiplicative) factor of 2 that should be 1.

Proof. Consider the inverse of the Weierstrass function. For all $\omega \in \mathbb{C}$, we have

$$\int_{\wp(\omega)}^{+\infty} \frac{d\mathbf{u}}{\sqrt{4\mathbf{u}^3 - g_2\mathbf{u} - g_3}} = \omega \mod \omega_1\mathbb{Z} + \omega_2\mathbb{Z}.$$

The modulo $\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$ comes from the fact that the Weierstrass elliptic function is not injective.

Following the proof of Lemma 12, let us perform the variable change

$$x = \begin{cases} a_4 + \dfrac{D'(a_4)}{\mathbf{u} - \frac{1}{6}D''(a_4)} & \text{when } a_4 \neq [1:0], \\ \dfrac{\mathbf{u} - \alpha_2/3}{\alpha_3} & \text{otherwise.} \end{cases}$$

Let us first assume that $a_4 \neq [1:0]$ and let us follow the notations of the proof of Lemma 12. Remind that $\sqrt{4\mathbf{u}^3 - g_2\mathbf{u} - g_3} = v = \frac{2zD'(a_4)}{(x-a_4)^2}$ and $z = \sqrt{D(x)}$. A straightforward computation yields $d\mathbf{u} = \frac{-D'(a_4)dx}{(x-a_4)^2}$. Therefore, we find

$$\int_{\wp(\omega)}^{+\infty} \frac{d\mathbf{u}}{\sqrt{4\mathbf{u}^3 - g_2\mathbf{u} - g_3}} = \int_{\wp(\omega)}^{+\infty} \frac{d\mathbf{u}}{v} = \int_{x(\omega)}^{a_4} \frac{-dx}{2z} = \frac{1}{2}\int_{a_4}^{x(\omega)} \frac{dx}{\sqrt{D(x)}}.$$

Similarly, when $a_4 = [1:0]$ we find $\sqrt{4\mathbf{u}^3 - g_2\mathbf{u} - g_3} = v = -2\alpha_3 z$, $z = \sqrt{D(x)}$, $d\mathbf{u} = \alpha_3 dx$, and

$$\int_{\wp(\omega)}^{+\infty} \frac{d\mathbf{u}}{\sqrt{4\mathbf{u}^3 - g_2\mathbf{u} - g_3}} = \int_{\wp(\omega)}^{+\infty} \frac{d\mathbf{u}}{v} = \int_{x(\omega)}^{a_4} \frac{-dx}{2z} = \frac{1}{2}\int_{a_4}^{x(\omega)} \frac{dx}{\sqrt{D(x)}}.$$

We thus have

$$\frac{1}{2}\int_{a_4}^{x(\omega)} \frac{dx}{\sqrt{D(x)}} = \omega \mod \omega_1\mathbb{Z} + \omega_2\mathbb{Z}.$$

Thus, using Lemma 15, we may take for the periods

$$\omega_1 = \int_{a_4}^{a_3} \frac{dx}{\sqrt{D(x)}}, \quad \omega_2 = \int_{a_4}^{a_1} \frac{dx}{\sqrt{D(x)}}.$$

Remind, see Remark 11, that we have ordered the a_i's in such a way that the cycle of $\mathbb{P}_1(\mathbb{R})$ starting from -1 to $+\infty$ and then from $-\infty$ to -1 cross the a_i's in the order a_1, a_2, a_3, a_4. As we may see in Lemma 14, for $\omega \in (0,\omega_2/2)$, both of $x(\omega)$ and $y(\omega)$ are real, proving that the discriminant $D(x)$ is positive in the expression $\omega_2 = \int_{a_4}^{a_1} \frac{dx}{\sqrt{D(x)}}$. This proves that $\omega_2 \in \mathbb{R}_{>0}$. Similarly, for $\omega \in (0,\omega_1/2)$, $x(\omega)$ is real but $y(\omega)$ is not real, proving that the discriminant $D(x)$ is negative. Then we obtain $\omega_1 = \int_{a_4}^{a_3} \frac{dx}{\sqrt{D(x)}} = \frac{1}{\mathbf{i}}\int_{a_4}^{a_3} \frac{dx}{\sqrt{|D(x)|}} = \mathbf{i}\int_{a_3}^{a_4} \frac{dx}{\sqrt{|D(x)|}} \in \mathbf{i}\mathbb{R}_{>0}$.

Let us now consider ω_3. Doing a similar reasoning we find

$$\omega_3 = \int_{a_4}^{X_\pm(b_1)} \frac{dx}{\sqrt{D(x)}}.$$

It remains to prove that we may take $\omega_3 \in (0, \omega_2)$.

Note that $y(\omega_3/2) = b_1$. Since the real polynomial $x \mapsto K(x, b_1; t)$ has a double root and $b_1 \in \mathbb{P}_1(\mathbb{R})$, we obtain that $X_\pm(b_1) \in \mathbb{P}_1(\mathbb{R})$. Consider the parallelogram $\mathcal{P}_\omega$ with vertices $0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$. With Lemmas 14 and 15, we deduce that the map that sends $\omega \in \mathcal{P}_\omega$ to $x(\omega)$ has a surjective image in $\mathbb{P}_1(\mathbb{R})$. Then, there exists $\omega_0 \in \mathcal{P}_\omega$ such that $x(\omega_0) = X_\pm(b_1)$. We have two possibilities: $(x(\omega_0), y(\omega_0)) = (X_\pm(b_1), b_1)$ or $(x(\omega_0), y(\omega_0)) = \iota_1(X_\pm(b_1), b_1)$. Since $\widetilde{\iota}_1(\omega_0) = -\omega_0$, we find that $\omega_3/2$ is one of the two complex numbers $\pm\omega_0$. With $\omega_0 \in \mathcal{P}_\omega$, this shows that $\omega_3/2 \in \{\omega_1\mathbb{R} + \frac{\omega_2}{2}\mathbb{Z}\} \bigcup \{\frac{\omega_1}{2}\mathbb{Z} + \omega_2\mathbb{R}\}$. Since $y(\omega_3/2)$ is also real, Lemma 14 implies that $\omega_3/2 \in \{\frac{\omega_1}{2}\mathbb{Z} + \omega_2\mathbb{R}\}$. So $\omega_3 \in \{\omega_1\mathbb{Z} + \omega_2\mathbb{R}\}$. Up to take a different lift of $\widetilde{\iota}_2$ we may modify ω_3 modulo $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ and assume that ω_3 is real with $0 \leqslant \omega_3 < \omega_2$. Note that $\omega_3 \neq 0$. since otherwise $\widetilde{\iota}_1 = \widetilde{\iota}_2$, which is not possible by definition of ι_1, ι_2. So $\omega_3 \in (0, \omega_2)$. □

2.2. **Analytic continuation.** The goal of this subsection is to prove that $F^1(x;t)$ and $F^2(y;t)$ admit a multivalued meromorphic continuation on $\overline{E}_t$ we will call r_x and r_y.

Let $\mathcal{D}_x := \overline{E}_t \cap \{|x| < 1\}$, $\mathcal{D}_y := \overline{E}_t \cap \{|y| < 1\}$ and $\mathcal{D}_{x,y} = \mathcal{D}_x \cap \mathcal{D}_y$. Remind that $Q(x, y; t)$ converges for $|x| < 1$ and $|y| < 1$. The same holds for $F^1(x;t)$ and $F^2(y;t)$. Due to the following lemma, the set $\mathcal{D}_{x,y}$ has a non-empty intersection with $\overline{E}_t$.

Lemma 18. *The following holds: $\mathcal{D}_{x,y} \neq \varnothing$.*

Proof. This is obvious from Lemma 7, as $\mathcal{D}_{x,y}$ is delimited by Γ_x^- and Γ_y^-. □

So we may define the three generating functions $Q(x, y; t), F^1(x; t), F^2(y; t)$ on $\mathcal{D}_{x,y}$. Let us further restrict the functional equation (5) on $\mathcal{D}_{x,y}$ (and more generally on $\overline{E}_t$) to obtain

$$0 = xy - F^1(x;t) - F^2(y;t) + K(0,0;t)Q(0,0;t). \tag{13}$$

Since $F^1(x;t)$ is analytic for $|x| < 1$, we may lift it to $\mathcal{D}_x$. As for $F^2(y;t)$, we lift it on $\mathcal{D}_x$ in the following way. For $(x, y) \in \mathcal{D}_x$, let us set

$$0 = xy - F^1(x;t) - F^2(y;t) + K(0,0;t)Q(0,0;t).$$

Similarly, we may lift $F^1(x;t)$ and $F^2(y;t)$ on $\mathcal{D}_y$.

Lemma 19. *The sets $\mathcal{D}_x$ and $\mathcal{D}_y$ are connected.*

Proof. Let us do the proof for $\mathcal{D}_x$, the other case being similar. By definition,

$$\mathcal{D}_x = \{(x, Y_-(x)) : |x| < 1\} \cup \{(x, Y_+(x)) : |x| < 1\}. \tag{14}$$

Since $\{(x, Y_-(x)) : |x| < 1\}$ and $\{(x, Y_+(x)) : |x| < 1\}$ are obviously connected, it suffices to prove that they have a non-empty intersection.

As we can see in Lemma 10, there exists $|a_1| < 1$ such that $y \mapsto \overline{K}(a_1, y; t)$ has a double root. This means that $Y_-(a_1) = Y_+(a_1)$, proving that

$$(a_1, Y_\pm(a_1)) \in \{(x, Y_-(x)) : |x| < 1\} \cap \{(x, Y_+(x)) : |x| < 1\}.$$

□

Since the union of two connected sets ($\mathcal{D}_x$ and $\mathcal{D}_y$) with non empty intersection (the set $\mathcal{D}_{x,y}$) is connected, we have proved that we may lift $F^1(x;t)$ and $F^2(y;t)$ as meromorphic functions on the connected domain

$$\mathcal{D} := \mathcal{D}_x \cup \mathcal{D}_y.$$

This domain is bounded by the curves Γ_x^+ and Γ_y^+.

The next step is to apply τ in order to lift $F^1(x;t)$ and $F^2(y;t)$ on a universal covering of $\overline{E}_t$. Let us identify $\mathcal{D}$ as $\widetilde{\mathcal{D}} \subset \mathbb{C}$, a connected set in the ω-plane, such that the map Λ restricted to $\widetilde{\mathcal{D}}$ is injective.

Lemma 20. *We have*

$$\bigcup_{\ell\in\mathbb{Z}} \widetilde{\tau}^{\ell}(\widetilde{\mathcal{D}}) = \mathbb{C}.$$

Proof. Let $\widetilde{\Gamma}_x^{\pm}, \widetilde{\Gamma}_y^{\pm} \subset \mathbb{C}$ be curves on the ω-plane such that $\Lambda(\widetilde{\Gamma}_x^{\pm}) = \Gamma_x^{\pm}$ and $\Lambda(\widetilde{\Gamma}_y^{\pm}) = \Gamma_y^{\pm}$. As we may deduce from the proof of Lemma 15, up to replace $\widetilde{\Gamma}_x^{\pm}$ by $\widetilde{\Gamma}_x^{\pm} + \ell_{\pm}\omega_2$, with $\ell_{\pm} \in \mathbb{Z}$, we may assume that one of the two curves $\widetilde{\Gamma}_x^-$, $\widetilde{\Gamma}_x^+$ should cross the parallelogram with vertices $\{0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\}$ in the open intervals $(0, \frac{\omega_2}{2})$ and $(\frac{\omega_1}{2}, \frac{\omega_1+\omega_2}{2})$. Since $\widetilde{\iota}_1(\widetilde{\Gamma}_x^-) = \widetilde{\Gamma}_x^+$ and $\widetilde{\iota}_1(\omega) = -\omega$, we deduce that we may assume that the same holds for the other curve but with the parallelogram with vertices $\{\frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}, \omega_2, \frac{\omega_1}{2} + \omega_2\}$. With $\widetilde{\Gamma}_x^- \cap \widetilde{\Gamma}_x^+ = \varnothing$, this proves that the domain bounded by $\widetilde{\Gamma}_x^-$ and $\widetilde{\Gamma}_x^+$ contains a vertical strip. Similarly, the domain bounded by $\widetilde{\Gamma}_y^-$ and $\widetilde{\Gamma}_y^+$ contains a vertical strip.

The domain $\mathcal{D}$ is delimited by Γ_x^+, Γ_y^+ and contains Γ_x^-, Γ_y^-. Then, we find that $\widetilde{\mathcal{D}}$ contains a vertical strip $\mathcal{S}_\omega$ in the ω-plane. Consequently the continuation of $F^1(x;t)$ in the ω-plane is defined on $\mathcal{S}_\omega$.

With $\iota_1(\Gamma_x^+) = \Gamma_x^-$ and $\iota_2(\Gamma_x^-) \in \mathcal{D}_y$ we deduce that $\tau(\Gamma_x^+) \subset \mathcal{D}$. In virtue of $\widetilde{\tau}(\omega) = \omega + \omega_3$, this shows that the width of the strip $\mathcal{S}_\omega$, which is bounded by the curves $\widetilde{\Gamma_x^+}$ and $\widetilde{\Gamma_y^+}$, is at least ω_3. The result follows from $\widetilde{\tau}(\omega) = \omega + \omega_3$. □

Theorem 21. *The functions $F^1(x;t)$ and $F^2(y;t)$ may be lifted to the universal cover of $\overline{E}_t$. We will call respectively r_x and r_y the continuations. Seen as functions of ω, they are meromorphic on $\mathbb{C}$ and satisfy*

$$r_y(\omega+\omega_3;t) = r_y(\omega;t) + x(\omega)\{y(-\omega) - y(\omega)\}, \tag{15}$$

$$r_y(\omega+\omega_1;t) = r_y(\omega;t),$$

$$r_x(\omega+\omega_3;t) = r_x(\omega;t) + y(-\omega)\{x(\omega+\omega_3) - x(\omega)\}, \tag{16}$$

$$r_x(\omega+\omega_1;t) = r_x(\omega;t). \tag{17}$$

Proof. Let us do the proof for r_x. The proof for r_y is the same. As we have seen in the proof of Lemma 20, the set $\widetilde{\mathcal{D}}$ contains a vertical strips of width bigger than ω_3. Since at this step of the continuation, F^1 is univalued as a function on $\mathcal{D}$, we find that the analytic continuation of r_x on $\mathcal{S}_\omega$ (and therefore on $\widetilde{\mathcal{D}}$) is ω_1-periodic, thereby proving (17).

To lift r_x on $\mathbb{C}$, we now apply successively $\widetilde{\tau}^{\pm}$ on r_x and use Lemma 20. The only point that remains to be proved is (15) and (16).

Consider $x(\omega)$ and $y(\omega)$ defined in Lemma 12. From (13), we deduce that for all $\omega \in \mathbb{C}$, we have

$$0 = x(\omega)y(\omega) - r_x(\omega;t) - r_y(\omega;t) + K(0,0;t)Q(0,0;t). \tag{18}$$

Let us apply $\widetilde{\iota}_1$ to the both sides of the equality. Using $r_x(\omega;t) = r_x(-\omega;t)$ (this follows from $\widetilde{\iota}_1(\omega) = -\omega$ and $\iota_1(F^1) = F^1$) we deduce that

$$0 = x(-\omega)y(-\omega) - r_x(\omega;t) - r_y(-\omega;t) + K(0,0;t)Q(0,0;t). \tag{19}$$

Let us apply $\widetilde{\iota}_2$ to the both sides of the equality. Similarly, using $r_y(-\omega;t) = r_y(\omega+\omega_3;t)$ (this follows from $\widetilde{\iota}_2(\omega) = -\omega + \omega_3$ and $\iota_2(F^2) = F^2$), we obtain

$$0 = x(\omega+\omega_3)y(\omega+\omega_3) - r_x(\omega+\omega_3;t) - r_y(-\omega;t) + K(0,0;t)Q(0,0;t). \tag{20}$$

Subtracting (18) to (19) (resp. (19) to (20)), we find

$$r_y(-\omega;t) - r_y(\omega;t) = x(-\omega)y(-\omega) - x(\omega)y(\omega),$$
$$r_x(\omega+\omega_3;t) - r_x(\omega;t) = x(\omega+\omega_3)y(\omega+\omega_3) - x(-\omega)y(-\omega).$$

To conclude the proof, we use $r_y(-\omega;t) = r_y(\omega+\omega_3;t)$, $x(-\omega) = x(\omega)$ and $y(-\omega) = y(\omega+\omega_3)$. □

Remark 22. So far we have considered walks that start at the point $(0,0)$. We could similarly handle models of walks starting at the point (i,j) with probability $p_{i,j}$, such that $\sum_{i,j} p_{i,j} = 1$. In this situation, the functional equation satisfied by the generating series is

$$K(x,y;t)Q(x,y;t) = \sum_{i,j\geqslant 0} p_{i,j}x^{i+1}y^{j+1} - F^1(x;t) - F^2(y;t) + K(0,0;t)Q(0,0;t).$$

Using exactly the same strategy, we may prove that the functions r_x and r_y may be continued to $\mathbb{C}$. They are ω_1-periodic and satisfy

$$r_y(\omega+\omega_3;t) - r_y(\omega;t) = \sum_{i,j\geqslant 0} p_{i,j}x^{i+1}(-\omega)y^{j+1}(-\omega) - \sum_{i,j\geqslant 0} p_{i,j}x^{i+1}(\omega)y^{j+1}(\omega),$$
$$r_x(\omega+\omega_3;t) - r_x(\omega;t) = \sum_{i,j\geqslant 0} p_{i,j}x^{i+1}(\omega+\omega_3)y^{j+1}(\omega+\omega_3) - \sum_{i,j\geqslant 0} p_{i,j}x^{i+1}(-\omega)y^{j+1}(-\omega).$$

3. Sufficient conditions for differential transcendence

Throughout this section, we assume that $\widetilde{\tau}$ has infinite order, which is equivalent to doing the hypothesis that the group is infinite. Our main results are to derive differential transcendence criteria for r_x and r_y. As we may see in Theorem 21, these functions satisfy "difference" equations of the form

$$\widetilde{\tau}(f) - f = b,$$

with $\widetilde{\tau}$ defined in (10). Galois theoretic methods to study the differential properties of such functions have been developed in [HS08, DHR15, DHRS17a], see also [Har16]. In this section we describe a consequence of the latter theory and show how it will be used to prove that in many cases $x \mapsto Q(x,0;t)$ and $y \mapsto Q(0,y;t)$ are differentially transcendent.

3.1. Background of difference Galois theory.

We remind that $\overline{E}_t$ is an elliptic curve. The field $\mathcal{M}er(\overline{E}_t)$ of meromorphic functions on the elliptic curve may be identified, via the Weierstrass elliptic function, to the field of meromorphic functions on $\mathbb{C}$ which are (ω_1,ω_2)-periodic. We have a natural derivation on this field given by the ω-derivative ∂_ω. Note that as Theorem 21 shows, the continuations of r_x and r_y belong to $\mathcal{M}er(\mathbb{C})$, the field of meromorphic functions on $\mathbb{C}$. The latter may be equipped with the derivation ∂_ω and the inclusion of differential fields holds $(\mathcal{M}er(\overline{E}_t),\partial_\omega) \subset (\mathcal{M}er(\mathbb{C}),\partial_\omega)$.

Definition 23. Let $(E,\partial_\omega) \subset (F,\partial_\omega)$ be differential fields. We say that $f \in F$ is differentially algebraic over E if it satisfies a non-trivial algebraic differential equation with coefficients in E, i.e., if for some m there exists a non-zero polynomial $P(y_0,\ldots,y_m) \in E[y_0,\ldots,y_m]$ such that

$$P(f,\partial_\omega(f),\ldots,\partial_\omega^m(f)) = 0.$$

We say that f is differentially transcendent over E if it is not differentially algebraic.

Our first remark is that the series $x \mapsto Q(x,0;t)$ is differentially algebraic over $\mathbb{C}(x)$ if and only if $\omega \mapsto r_x(\omega;t)$ is differentially algebraic over $\mathcal{M}er(\overline{E}_t)$. And symmetrically the same holds for $y \mapsto Q(0,y;t)$ and $\omega \mapsto r_y(\omega;t)$. The proof of this equivalence may be deduced from [DHRS17a, Lemmas 6.3 and 6.4]. We may therefore focus on r_x and r_y.

Definition 24. A $(\partial_\omega, \widetilde{\tau})$-field is a triple $(K, \partial_\omega, \widetilde{\tau})$, where K is a field, ∂_ω is a derivation on K, $\widetilde{\tau}$ is an automorphism of K, and where ∂_ω and $\widetilde{\tau}$ commute on K.

Since by (10) $\widetilde{\tau}(\omega) = \omega + \omega_3$, we deduce that on $\mathcal{M}er(\mathbb{C})$,

$$\widetilde{\tau} \circ \partial_\omega = \partial_\omega \circ \widetilde{\tau}.$$

Then, the triples $(\mathcal{M}er(\overline{E}_t), \partial_\omega, \widetilde{\tau})$ and $(\mathcal{M}er(\mathbb{C}), \partial_\omega, \widetilde{\tau})$ provide examples for Definition 24.

Proposition 2.6 of [DHR15] gives criteria for differential transcendence in the above general setting; we are now going to translate it in our context. The result of [DHR15] only requires the assumption to embed the solutions r_x and r_y into a $(\partial_\omega, \widetilde{\tau})$-field. This is done due to Theorem 21 and, remarkably, this is the only point were analytic tools are needed in this section.

Proposition 25 ([DHR15])**.** *Let $b \in \mathcal{M}er(\overline{E}_t)$, $f \in \mathcal{M}er(\mathbb{C})$, and assume that*

$$\widetilde{\tau}(f) - f = b.$$

If f is differentially algebraic over $\mathcal{M}er(\overline{E}_t)$, then there exist an integer $n \geqslant 0$, $c_0, \dots, c_{n-1} \in \mathbb{C}$ and $g \in \mathcal{M}er(\overline{E}_t)$, such that

$$\partial_\omega^n(b) + c_{n-1}\partial_\omega^{n-1}(b) + \cdots + c_1\partial_\omega(b) + c_0 b = \widetilde{\tau}(g) - g. \tag{21}$$

Let

$$b_1 = \iota_1(y)\{\tau(x) - x\} \quad \text{and} \quad b_2 = x\{\iota_1(y) - y\} \tag{22}$$

be the quantities that appear in the right-hand sides of Equations (15)–(16) of Theorem 21. As we may see in [DHRS17a, Corollary 3.9 and Proposition 3.10], Proposition 25 has the following consequence:

Corollary 26. *Assume that b_1 (resp. b_2) has a pole $P \in \overline{E}_t$ of order $m \geqslant 1$ such that none of the $\tau^k(P)$ with $k \in \mathbb{Z} \setminus \{0\}$ is a pole of order $\geqslant m$ of b_1 (resp. b_2). Then $x \mapsto Q(x, 0; t)$ and $y \mapsto Q(0, y; t)$ are differentially transcendent over $\mathbb{C}(x)$ and $\mathbb{C}(y)$ respectively.*

As suggested by Corollary 26, we are now interested in the poles of b_1 and b_2 in (22). Let $P_1, P_2 \in \overline{E}_t$ be the poles of x, let $Q_1, Q_2 \in \overline{E}_t$ be the poles of y. We are going to see these poles as elements of $\mathbb{P}_1(\mathbb{C})^2$.

Example 27. Consider the walk with $d_{-1,1} = d_{1,1} = d_{1,-1} = d_{0,-1} = 1/4$ and all other $d_{i,j} = 0$. The curve $\overline{E}_t$ is defined by

$$\overline{K}(x_0, x_1, y_0, y_1, t) = x_0 x_1 y_0 y_1 - \frac{t}{4}(x_1^2 y_0^2 + x_0^2 y_0^2 + x_0^2 y_1^2 + x_0 x_1 y_1^2).$$

Let us compute P_1 and P_2. To find the poles of x, we have to solve $\overline{K}(1, 0, y_0, y_1, t) = 0$, which gives the equation

$$\frac{t}{4}(y_0^2 + y_1^2) = 0.$$

The solutions of this equation in $\mathbb{P}^1(\mathbb{C})$ are $[\mathbf{i}:1]$ and $[-\mathbf{i}:1]$. Then

$$P_1, P_2 = ([1:0], [\pm\mathbf{i}:1]).$$

Similarly in order to compute Q_1, Q_2 we solve $\overline{K}(x_0, x_1, 1, 0, t) = 0$ and find

$$Q_1, Q_2 = ([\pm\mathbf{i}:1], [1:0]).$$

We now describe the poles of b_1 and b_2 in (22). Lemma 4.11 of [DHRS17a] gives in our context:

Lemma 28. *The set of poles of b_1 in $\overline{E}_t$ is contained in*

$$\mathcal{S}_1 = \{\iota_1(Q_1), \iota_1(Q_2), P_1, P_2, \tau^{-1}(P_1), \tau^{-1}(P_2)\}.$$

Similarly, the set of poles of b_2 in $\overline{E}_t$ is contained in

$$\mathcal{S}_2 = \{P_1, P_2, Q_1, Q_2, \iota_1(Q_1), \iota_1(Q_2)\} = \{P_1, P_2, Q_1, Q_2, \tau^{-1}(Q_1), \tau^{-1}(Q_2)\}.$$

Remark 29. As in the end of Section 2.2, we may consider walks starting at the point (i,j) with probability $p_{i,j}$, $\sum_{i,j} p_{i,j} = 1$. A similar criterion to Corollary 26 may be derived here but it will be less effective, since the second member of the equation appearing in Remark 22 may have many poles.

3.2. **Differential transcendence for genus one walks.** In this section we consider walks of genus one and derive criterias that ensure that the functions $x \mapsto Q(x,0;t)$ and $y \mapsto Q(0,y;t)$ are differentially transcendent over $\mathbb{C}(x)$ and $\mathbb{C}(y)$, respectively. These criteria are strong and tractable enough to show that these functions are differentially transcendent in many concrete weighted cases.

Theorem 21 allows us to generalize straightforwardly the results of [DHRS17a] in the weighted case, but the proofs will be made here in order to make our paper self-contained. We will only consider walks satisfying Assumption 9. Remind that $\widetilde{\tau}$ has infinite order; this implies that τ has infinite order as well and has the following consequence:

Lemma 30. *For all $P \in \overline{E}_t$ and all $k \in \mathbb{Z}$,*

$$\tau^k(P) = P \Longrightarrow k = 0. \tag{23}$$

Proof. Let us remind, see [Dui10, Proposition 2.5.2], that τ is the addition by a point of the elliptic curve: $P \mapsto P \oplus P_0$, for some $P_0 \in \overline{E}_t$. Since τ has infinite order, P_0 is a non-torsion point, and the result follows. □

We split the analysis in three cases: generic case (Theorem 34), double pole case (Theorem 38) and triple pole case (Theorem 40).

Generic case. Remind that $\mathbb{Q}(d_{i,j}) \subset \mathbb{R}$ denotes the field generated over $\mathbb{Q}$ by the $d_{i,j}$'s. Let $\mathbb{K} = \mathbb{Q}(d_{i,j})$ and consider $\mathbb{K}(t) \subset \mathbb{C}$. As we can see in Example 27, although the curve $\overline{E}_t$ has coefficients in $\mathbb{K}(t)$, the poles of b_2 may belong to an intermediate field extension $\mathbb{K}(t) \subset L \subset \mathbb{C}$. The goal of what follows is to prove that when the poles of b_2 are not defined over $\mathbb{K}(t)$, the generating series $x \mapsto Q(x,0;t)$ and $y \mapsto Q(0,y;t)$ are differentially transcendent over $\mathbb{C}(x)$ and $\mathbb{C}(y)$, respectively, see Theorem 34.

Let $\mathbb{K}(t) \subset L \subset \mathbb{C}$ be a field extension. For any L-point $P = ([x_0 : x_1], [y_0 : y_1])$ of $\overline{E}_t$, with $x_0, x_1, y_0, y_1 \in L$, and any $\sigma \in \mathrm{Aut}(L/\mathbb{K}(t))$, we set

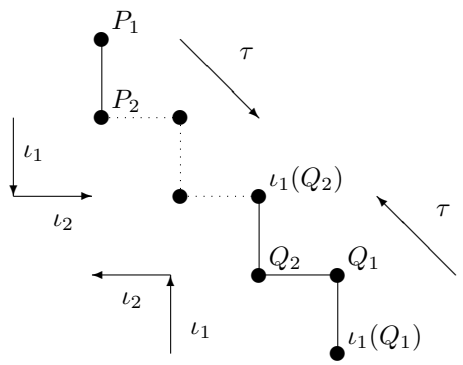
$$\sigma(P) = ([\sigma(x_0) : \sigma(x_1)], [\sigma(y_0) : \sigma(y_1)]).$$

Since $\overline{E}_t$ is defined over $\mathbb{K}(t)$, $\sigma(P)$ is an L-point of $\overline{E}_t$.

Next Proposition states the relation between $\mathrm{Aut}(L/\mathbb{K}(t))$ and the maps ι_1, ι_2, τ. It has been proved in [DHRS17a, Proposition 4.8] in the situation where t is transcendent and the $d_{i,j}$'s are rational, but the proof is exactly the same here.

Proposition 31. *Let $\mathbb{K}(t) \subset L \subset \mathbb{C}$ be a field extension and let $\sigma \in \mathrm{Aut}(L/\mathbb{K}(t))$. Let P be a L-point of $\overline{E}_t$. Then the following properties hold true:*

- *$\iota_1(P), \iota_2(P)$ and hence $\tau^n(P)$, for any $n \in \mathbb{Z}$, are L-points of $\overline{E}_t$;*
- *for any $k \in \{1,2\}$, $\iota_k \circ \sigma = \sigma \circ \iota_k$ on $\overline{E}_t(L)$ and hence $\tau \circ \sigma = \sigma \circ \tau$ on $\overline{E}_t(L)$.*



$$\begin{aligned} & \tau^n(P_1) = Q_1 \\ \Longrightarrow\ & \sigma\circ\tau^n(P_1) = \sigma(Q_1) \\ \Longrightarrow\ & \tau^n\circ\sigma(P_1) = \sigma(Q_1) \\ \Longrightarrow\ & \tau^n(P_2) = Q_2 \end{aligned} \qquad \begin{aligned} & \tau^n(P_1) = Q_1 \\ \Longrightarrow\ & \tau^{n-1}(Q_2) = P_2 \\ \Longrightarrow\ & \tau^{2n-1}(P_2) = P_2 \end{aligned}$$

FIGURE 4. Proof of $P_1 \not\sim Q_1$ in the case $\mathbb{K}(t) \subsetneq L_x, L_y$. See Lemma 33

Proof. We only prove the assertions concerning ι_1. The proofs for ι_2 are similar and the assertions concerning τ follow from those about ι_1 and ι_2 since $\tau = \iota_2 \circ \iota_1$.

Let us set $P = ([\alpha_0 : \alpha_1], [\beta_0 : \beta_1]) \in \overline{E}_t(L)$ with $\alpha_0, \alpha_1, \beta_0, \beta_1 \in L$, and consider the point $\iota_1(P) = ([\alpha_0 : \alpha_1], [\beta_0' : \beta_1'])$. The point $[\beta_0' : \beta_1']$ is characterized by the fact that $[\beta_0 : \beta_1]$ and $[\beta_0' : \beta_1']$ are the roots in $\mathbb{P}^1(\mathbb{C})$ of the homogeneous polynomial in y_0 and y_1 given by

$$A(\alpha_0, \alpha_1)y_0^2 + B(\alpha_0, \alpha_1)y_0y_1 + C(\alpha_0, \alpha_1)y_1^2 \tag{24}$$

where

$$\begin{cases} A(\alpha_0, \alpha_1) &= d_{-1,1}\alpha_1^2 + d_{0,1}\alpha_0\alpha_1 + d_{1,1}\alpha_0^2, \\ B(\alpha_0, \alpha_1) &= d_{-1,0}\alpha_1^2 - \frac{1}{t}\alpha_0\alpha_1 + d_{0,0}\alpha_0\alpha_1 + d_{1,0}\alpha_0^2, \\ C(\alpha_0, \alpha_1) &= d_{-1,-1}\alpha_1^2 + d_{0,-1}\alpha_0\alpha_1 + d_{1,-1}\alpha_0^2. \end{cases}$$

Since (24) has coefficients in L and $\beta_0, \beta_1 \in L$, we can assume that $\beta_0', \beta_1' \in L$ as well. Hence $[\beta_0' : \beta_1'] \in \mathbb{P}^1(L)$ and $\iota_1(P) \in \overline{E}_t(L)$, as desired.

Moreover, $[\sigma(\beta_0) : \sigma(\beta_1)]$ and $[\sigma(\beta_0') : \sigma(\beta_1')]$ are the roots in $\mathbb{P}^1(\mathbb{C})$ of

$$A(\sigma(\alpha_0), \sigma(\alpha_1))y_0^2 + B(\sigma(\alpha_0), \sigma(\alpha_1))y_0y_1 + C(\sigma(\alpha_0), \sigma(\alpha_1))y_1^2.$$

Therefore, $\iota_1(\sigma(P)) = ([\sigma(\alpha_0) : \sigma(\alpha_1)], [\sigma(\beta_0') : \sigma(\beta_1')]) = \sigma(\iota_1(P))$. □

Definition 32. We define an equivalence relation $\sim$ on $\overline{E}_t$ by

$$P \sim Q \quad \Longleftrightarrow \quad \exists n \in \mathbb{Z},\ \tau^n(P) = Q.$$

If $P \sim Q$ is not true, we shall write $P \not\sim Q$. An equivalence class for $\sim$ will be called a τ-orbit.

To use Corollary 26, we need to give criterias ensuring that $P_1 \not\sim P_2$, $Q_1 \not\sim Q_2$, and $P_i \not\sim Q_j$. Next lemma asserts this kind of results when one of the four points in not defined over $\mathbb{K}(t)$. It has been proved in [DHRS17a, Lemma 4.10] in the situation where t is transcendent and the $d_{i,j}$'s are rational, but the proof is exactly the same here.

Lemma 33. *Introduce*

$$L_x = \mathbb{K}(t)\left(\sqrt{\Delta^x_{[1:0]}}\right), \quad L_y = \mathbb{K}(t)\left(\sqrt{\Delta^y_{[1:0]}}\right), \quad L = \mathbb{K}(t)\left(\sqrt{\Delta^x_{[1:0]}}, \sqrt{\Delta^y_{[1:0]}}\right).$$

(L is the composite field of L_x and L_y.) The following properties hold true:

- *if $\mathbb{K}(t) \subsetneq L_x$ or $\mathbb{K}(t) \subsetneq L_y$ then, for all $i, j \in \{1, 2\}$, we have $P_i \not\sim Q_j$;*
- *if $\mathbb{K}(t) \subsetneq L_x$ (resp. $\mathbb{K}(t) \subsetneq L_y$) then $P_i \not\sim P_j$ (resp. $Q_i \not\sim Q_j$) for $i \neq j$.*

Proof. We recall that τ has infinite order. Let us prove the first assertion. Suppose to the contrary that, for instance, $P_1 \sim Q_1$ and that $\mathbb{K}(t) \subsetneq L_x$, the other cases being similar. So there exists $n \in \mathbb{Z}$ such that $\tau^n(P_1) = Q_1$. The fact that $P_1 = ([1:0], [\beta_0 : \beta_1])$ belongs to $\overline{E}_t$ means that

$$d_{1,-1}\beta_1^2 + d_{1,0}\beta_0\beta_1 + d_{1,1}\beta_0^2 = 0.$$

Since $\mathbb{K}(t) \subsetneq L_x$, we have that $\Delta^x_{[1:0]}/t^2 = d_{1,0}^2 - 4d_{1,-1}d_{1,1}$ is not a square in $\mathbb{K}(t)$. It follows that $P_1 \in \overline{E}_t(L_x) \setminus \overline{E}_t(\mathbb{K}(t))$. On the other hand, the fact that $Q_1 = ([\alpha_0 : \alpha_1], [1:0])$ belongs to $\overline{E}_t$ means that

$$d_{-1,1}\alpha_1^2 + d_{0,1}\alpha_0\alpha_1 + d_{1,1}\alpha_0^2 = 0.$$

So Q_1 belongs to $\overline{E}_t(L_y)$. Since $\tau^{-n}(Q_1) = P_1$, Proposition 31 ensures that $P_1 \in \overline{E}_t(L_y)$ as well. Therefore, $P_1 \in (\overline{E}_t(L_x) \setminus \overline{E}_t(\mathbb{K}(t))) \cap \overline{E}_t(L_y)$. In particular, $L_x \cap L_y$ is not reduced to $\mathbb{K}(t)$. Since L_x and L_y are field extensions of degree at most 2 of $\mathbb{K}(t)$, we get $L_x = L_y$. Let $\sigma \in \mathrm{Gal}(L_x/\mathbb{K}(t)) = \mathrm{Gal}(L_y/\mathbb{K}(t))$ be an element of order 2. We obviously have $\sigma(P_1) = P_2$ and $\sigma(Q_1) = Q_2$. Using Proposition 31, it follows that $\tau^n(P_2) = \tau^n(\sigma(P_1)) = \sigma(\tau^n(P_1)) = \sigma(Q_1) = Q_2$. On the other hand $\tau^n(P_1) = Q_1$ implies $\tau^n(\iota_1(P_2)) = \iota_2(Q_2)$. Since ι_2 is an involution, we get $\iota_2 \circ \tau^n \circ \iota_1(P_2) = Q_2$. We recommend the reader to refer to Figure 4 for a geometrical proof of the following statement. But, we have $\iota_2\tau^n\iota_1 = \tau^{-n+1}$ (because $\tau = \iota_2\iota_1$ and the ι_k are involutions). Then, we find $\tau^{-n+1}(P_2) = Q_2$. This gives $\tau^{-n+1}(P_2) = Q_2 = \tau^n(P_2)$. Therefore $\tau^{2n-1}(P_2) = P_2$, yielding a contradiction because of (23).

We shall now prove the second assertion. Assume that $\mathbb{K}(t) \subsetneq L_x$. In particular $\Delta^x_{[1:0]} \neq 0$ and hence $P_1 \neq \iota_1(P_1) = P_2$. Suppose to the contrary that $P_1 \sim P_2$, that is, that there exists $n \in \mathbb{Z}^*$ such that $\tau^n(P_1) = P_2$. Let $\sigma \in \mathrm{Gal}(L_x/\mathbb{K}(t))$ be an element of order 2. We obviously have $\sigma(P_1) = P_2$. Using Proposition 31, we get $\tau^n(P_2) = \tau^n(\sigma(P_1)) = \sigma(\tau^n(P_1)) = \sigma(P_2) = P_1$. Therefore, $\tau^{2n}(P_1) = P_1$. With (23), this implies that $n = 0$ and hence $P_1 = P_2$. This yields a contradiction. The proof of $\mathbb{K}(t) \subsetneq L_y \Rightarrow Q_1 \nsim Q_2$ is similar. □

Theorem 34. *If $\Delta^x_{[1:0]}/t^2 = d_{1,0}^2 - 4d_{1,-1}d_{1,1}$ is not a square in $\mathbb{Q}(d_{i,j}, t)$, then $x \mapsto Q(x, 0; t)$ and $y \mapsto Q(0, y; t)$ are differentially transcendent over $\mathbb{C}(x)$ and $\mathbb{C}(y)$, respectively.*

If $\Delta^y_{[1:0]}/t^2 = d_{0,1}^2 - 4d_{-1,1}d_{1,1}$ isn't a square in $\mathbb{Q}(d_{i,j}, t)$ then $x \mapsto Q(x, 0; t)$ and $y \mapsto Q(0, y; t)$ are differentially transcendent over $\mathbb{C}(x)$ and $\mathbb{C}(y)$, respectively.

Example 35. See Figure 5 for an example of a 3D model that projects to 2D, and to which Theorem 34 applies when t is transcendent. We have $\Delta^y_{[1:0]}/t^2 = d_{0,1}^2 - 4d_{-1,1}d_{1,1} = -1/3$, which obviously is not a square in $\mathbb{Q}(d_{i,j}, t)$ (remind that t is real).

Proof. Assume for instance that $\Delta^x_{[1:0]}/t^2$ is not a square in $\mathbb{K}(t)$, the other case being similar. Using Corollary 26, we see that it is sufficient to prove that P_1 is a pole of b_2 and that it is the only pole of b_2 of the form $\tau^n(P_1)$ with $n \in \mathbb{Z}$. The fact that P_1 is a pole of b_2 is clear (indeed, on the one hand, P_1 is a pole of x and, on the other hand, the y-coordinates of P_1 and $\iota_1(P_1)$ are distinct because $\Delta^x_{[1:0]} \neq 0$, and, hence, P_1 is not a zero of $\iota_1(y) - y$). Moreover, Lemma 33 implies that $P_1 \nsim P_2$ and $P_1 \nsim Q_i$ for $i = 1, 2$. The latter also implies that $P_1 \nsim \tau^{-1}(Q_i)$ for $i = 1, 2$. But Lemma 28 ensures that the set of poles of b_2 is included in $\{P_1, P_2, Q_1, Q_2, \tau^{-1}(Q_1), \tau^{-1}(Q_2)\}$. So P_1 is the only pole of b_2 of the form $\tau^n(P_1)$ with $n \in \mathbb{Z}$, as desired. □

Corollary 36. *Assume that the $d_{i,j} \in \mathbb{Q}$ and t is transcendent.*

If $d_{1,0}^2 - 4d_{1,-1}d_{1,1}$ is not a square in $\mathbb{Q}$, then $x \mapsto Q(x, 0; t)$ and $y \mapsto Q(0, y; t)$ are differentially transcendent over $\mathbb{C}(x)$ and $\mathbb{C}(y)$, respectively.

If $d_{0,1}^2 - 4d_{-1,1}d_{1,1}$ is not a square in $\mathbb{Q}$, then $x \mapsto Q(x, 0; t)$ and $y \mapsto Q(0, y; t)$ are differentially transcendent over $\mathbb{C}(x)$ and $\mathbb{C}(y)$, respectively.

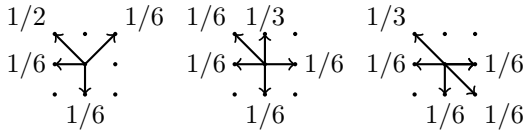


FIGURE 5. Three models, for which Theorem 34 (resp. Theorem 38, Theorem 40) applies

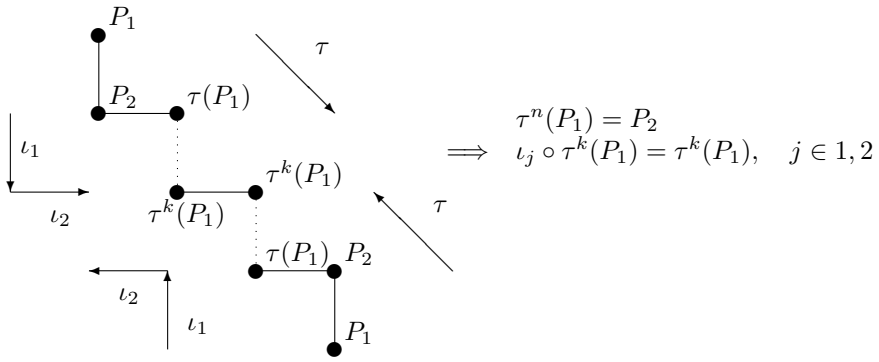


FIGURE 6. Proof of $P_1 \nsim P_2$ in Lemma 37

Let us consider unweighted quadrant walks. We then recover [DHRS17a, Proposition 5.1] that permits to conclude that when Assumption 9 is satisfied, 26 over the 51 unweighted quadrant models listed in [KR12] have a differentially transcendent generating series. In the weighted context, we find that almost every choice of the $d_{i,j}$'s leads to a differentially transcendent generating series.

Double pole case. Assume that $d_{1,1} = 0$ and $d_{1,0}d_{0,1} \neq 0$. As we may see in [DHRS17a, Section 5.2.1], the proof being similar in our situation, the function b_2 admits at least two double poles:

$$P_1 = ([1:0], [1:0]), \quad P_2 = ([1:0], [-d_{1,-1}:d_{1,0}]).$$

As we will see in the proof of Theorem 38, to deduce that $x \mapsto Q(x, 0; t)$ and $y \mapsto Q(0, y; t)$ are differentially transcendent over $\mathbb{C}(x)$ and $\mathbb{C}(y)$, respectively, it suffices to proves that $P_1 \not\sim P_2$. Let us give a criteria ensuring that $P_1 \not\sim P_2$.

Lemma 37. *Assume that there are no point of $\overline{E}_t(\mathbb{K}(t))$ that is fixed by ι_1 or ι_2. Then $P_1 \not\sim P_2$.*

Proof. To the contrary, assume that there exists $n \in \mathbb{Z}$ such that $\tau^n(P_1) = P_2$. It is easily seen that the equality $\tau^n(P_1) = P_2$, together with the fact that $\tau = \iota_2 \circ \iota_1$ is the composition of two involutions, implies if $n = 2k$ that $\iota_1(\tau^k(P_1)) = \tau^k(P_1)$, and if $n = 2k + 1$ that $\iota_2(\tau^k(\iota_2(P_2))) = \tau^k(\iota_2(P_2))$.

For $n = 2k$ we get that $\tau^k(P_1)$ is fixed by ι_1. With the second point of Proposition 31, we find that $\tau^k(P_1) \in \overline{E}_t(\mathbb{K}(t))$. But the assumption ensures that none of points of $\overline{E}_t(\mathbb{K}(t))$ is fixed by ι_1. This yields a contradiction.

For $n = 2k + 1$, we get that $\tau^k(\iota_2(P_2))$ is fixed by the involution ι_2. With the second point of Proposition 31, we find that $\tau^k(\iota_2(P_2)) \in \overline{E}_t(\mathbb{K}(t))$. But the assumption ensures that none of points of $\overline{E}_t(\mathbb{K}(t))$ is fixed by ι_2. This yields again a contradiction. □

Theorem 38. *Assume that $d_{1,1} = 0$ and $d_{1,0}d_{0,1} \neq 0$. Assume further that there are no point of $\overline{E}_t(\mathbb{K}(t))$ that is fixed by ι_1 or ι_2. Then $x \mapsto Q(x, 0; t)$ and $y \mapsto Q(0, y; t)$ are differentially transcendent over $\mathbb{C}(x)$ and $\mathbb{C}(y)$, respectively.*

Proof. As we may see in [DHRS17a, Section 5.2.1], the proof being similar in our situation, the function b_2 admits at least two double poles $([1:0],[1:0])$ and $([1:0],[-d_{1,-1}:d_{1,0}])$, that satisfy $([1:0],[1:0]) \not\sim ([1:0],[-d_{1,-1}:d_{1,0}])$, see Lemma 37. Since b_2 has at most six poles counted with multiplicities, it has at most three double poles. If b_2 has two double poles, we conclude with Corollary 26. If b_2 has three double poles, $([1:0],[1:0])$, $([1:0],[-d_{1,-1}:d_{1,0}])$ and P, then $P \not\sim ([1:0],[1:0])$ or $P \not\sim ([1:0],[-d_{1,-1}:d_{1,0}])$, due to $([1:0],[1:0]) \not\sim ([1:0],[-d_{1,-1}:d_{1,0}])$. We conclude again with Corollary 26. □

Example 39. Assume that t is transcendent. See Figure 5 for an example of model to which Theorem 38 applies. We have to check that there are no point of $\overline{E}_t(\mathbb{Q}(t))$ that is fixed by ι_1 or ι_2.

If we take the notations of Section 1.4, we find that the fixed points of ι_1 and ι_2 are of the form $(X_\pm(b_i), b_i)$ and $(a_i, Y_\pm(a_i))$. Furthermore, a_i's and b_i's are roots of the discriminant. Since t is transcendent over $\mathbb{Q}$ we may identify $\mathbb{Q}(t)$ with the field of rational functions in t and the discriminants become polynomials in coefficients in $\mathbb{Q}(t)$. We now have to prove that they have no roots in $\mathbb{Q}(t)$, i.e., that the two following polynomials have no roots in $\mathbb{Q}(t)$:

$$\big(x_1^2 - \frac{6}{t}x_0x_1 + x_0^2\big)^2 - 4x_0x_1(d_{-1,1}x_1^2 + 2x_0x_1) \quad \text{and} \quad \big(y_1^2 - \frac{6}{t}y_0y_1 + 2y_0^2\big)^2 - 4y_0y_1(y_0y_1 + y_0^2).$$

We begin by the first one. To the contrary, assume the existence of $x_0, x_1 \in \mathbb{Q}(t)$ with $(x_0, x_1) \neq (0,0)$ that cancel the discriminant. If $x_1 = 0$ then $x_0 = 0$ and so we may assume that $x_1 = 1$ in the projective coordinates. We have to consider

$$\big(1 - \frac{6}{t}x_0 + x_0^2\big)^2 - 4x_0(d_{-1,1} + 2x_0). \tag{25}$$

We find that x_0 has no zero and $t = 0$ may be its only pole. Furthermore, it is at most a simple pole. So $x_0 = a/t$ for some $a \in \mathbb{Q}$. Equating both sides of (25), we obtain $a = 6$ which is impossible since $x_0 = 6/t$ is not a root of (25). This proves that the first discriminant has no root in $\mathbb{Q}(t)$.

Let us now consider the second discriminant. As above let us assume that it has a root in $\mathbb{Q}(t)$, which leads to a root in $\mathbb{Q}(t)$ of

$$\big(1 - \frac{6}{t}y_0 + 2y_0^2\big)^2 - 4y_0(y_0 + y_0^2).$$

Again, y_0 has no zero and $t = 0$ is its only pole, of order one. So $y_0 = a/t$ and we find $a = 3$, which is impossible. We then conclude that Theorem 38 applies.

Let us consider unweighted quadrant walks. We then recover [DHRS17a, Theorem 5.3] that permits to conclude that when Assumption 9 is satisfied, 5 over the 51 unweighted quadrant models listed in [KR12] have a differentially transcendent generating series.

Triple pole case.

Theorem 40. *Assume that $d_{1,1} = d_{1,0} = 0$ and $d_{0,1} \neq 0$. Then $x \mapsto Q(x,0;t)$ and $y \mapsto Q(0,y;t)$ are differentially transcendent over $\mathbb{C}(x)$ and $\mathbb{C}(y)$, respectively.*

Similarly, assume that $d_{1,1} = d_{0,1} = 0$ and $d_{1,0} \neq 0$. Then $x \mapsto Q(x,0;t)$ and $y \mapsto Q(0,y;t)$ are differentially transcendent over $\mathbb{C}(x)$ and $\mathbb{C}(y)$, respectively.

Proof. Let us do the proof for the first case only. As we may see in [DHRS17a, Section 5.2.2], the proof being similar in our situation, the function b_2 admits a unique triple pole $([1:0],[1:0]) = P_1$ and no poles of order greater than 4. The result is now a consequence of Corollary 26. □

See Figure 5 for an example of model to which Theorem 40 applies when t is transcendent.

Let us consider unweighted quadrant walks. We then recover [DHRS17a, Theorem 5.4] that permits to conclude that when Assumption 9 is satisfied, 9 over the 51 unweighted quadrant models listed in [KR12] have a differentially transcendent generating series. Thus, the combination of the three above theorems allow us to conclude that 40 over the 51 unweighted quadrant models listed in [KR12] have a differentially transcendent generating series. Among the 11 remaining cases, 9 are differentially algebraic, see [BBMR17, DHRS17a], and 2 are differentially transcendent, see [DHRS17a].

4. A sufficient condition for algebraicity

Let $\langle \iota_1, \iota_2 \rangle$ be the group introduced in Section 1, see (9). In this section we restrict ourselves to the case of a finite group. Extending results of [BMM10, FR10, KR12, KR15] to the weighted case, we prove that the generating function $(x, y) \mapsto Q(x, y; t)$ is holonomic, and even algebraic if the orbit-sum

$$\sum_{\theta \in \langle \iota_1, \iota_2 \rangle} \operatorname{sign}(\theta) \cdot \theta(xy) \tag{26}$$

identically vanishes (above $\operatorname{sign}(\theta) = 1$ (resp. -1) if the number of elements ι_1, ι_2 used to write $\theta \in \langle \iota_1, \iota_2 \rangle$ is even (resp. odd)). More details are to come in Theorem 41 and Corollary 42. Following Theorem 21, we extend the notation (22) by

$$b_1(\omega) = y(\omega + \omega_3)\{x(\omega + \omega_3) - x(\omega)\} \quad \text{and} \quad b_2(\omega) = x(\omega)\{y(-\omega) - y(\omega)\}. \tag{27}$$

Remind that $\widetilde{\tau}(\omega) = \omega + \omega_3$ with $\omega_3 \in (0, \omega_2)$, see Proposition 17.

Theorem 41. *Let $0 < t < 1$ be such that the group $\langle \iota_1, \iota_2 \rangle$ restricted to the kernel curve $\overline{E}_t$ is finite (i.e., $\omega_3/\omega_2 = k/\ell \in \mathbb{Q}$, $\gcd(k, \ell) = 1$, see Proposition 17). Then the function $(x, y) \mapsto Q(x, y; t)$ is holonomic. Moreover, it is algebraic if and only if the orbit-sum $\mathcal{O}_1(\omega)$ (or equivalently $\mathcal{O}_2(\omega)$) defined by*

$$\mathcal{O}_1(\omega) = b_1(\omega) + \widetilde{\tau}(b_1(\omega)) + \cdots + \widetilde{\tau}^{\ell-1}(b_1(\omega)), \quad \mathcal{O}_2(\omega) = b_2(\omega) + \widetilde{\tau}(b_2(\omega)) + \cdots + \widetilde{\tau}^{\ell-1}(b_2(\omega)), \tag{28}$$

is identically zero.

Corollary 42. *If the group $\langle \iota_1, \iota_2 \rangle$ of birational transformation of $\mathbb{P}_1(\mathbb{C})^2$ is finite, then for all $0 < t < 1$, the function $(x, y) \mapsto Q(x, y; t)$ is holonomic. Moreover, if the orbit-sum (26) is zero for some $0 < t < 1$, then the above generating function is algebraic.*

Before proving these results, let us do a series of remarks.

• As an example, Corollary 42 applies to the three models of Figure 7, which admit a group of order 10, as shown in [KY15]. The orbit-sum (26) is 0, see again [KY15], from which it follows that these models are algebraic (as functions of x and y). This gives another proof of this fact, after the recent proof given in [BBMR17] (note, [BBMR17] proves the algebraicity of $Q(x, y; t)$ in the three variables).

• Theorem 41 and Corollary 42 apply exactly the same, for walks starting at point (i, j) with probability $p_{i,j}$, $\sum_{i,j} p_{i,j} = 1$. In the orbit sum (26), xy should be replaced by $\sum_{i,j} p_{i,j} x^{i+1} y^{j+1}$, and b_1 and b_2 in (27) become as in Remark 22.

• For all 23 finite group models of unweighted walks listed in [BMM10], Corollary 42 is proved in [BMM10] for 22 out of the 23 models, while the article [BK10] concludes the proof for the last model (Gessel's walk).

Moreover, for certain families of weighted walks, Corollary 42 is shown in [KY15]. Notice that these families in [KY15] completely describe the models having a group of order 4, 6 and 8.

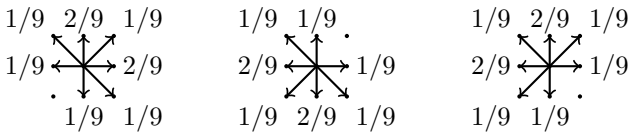


FIGURE 7. Three models with a group of order 10. These models, proposed in [KY15] and conjectured to be algebraic, were very recently solved in [BBMR17]

This is actually a refined version of Corollary 42 which is proved in [BMM10, BK10, KY15], as the holonomy and algebraicity of the generating functions are proved in the three variables x, y, t, whereas we show these properties only in the variables x and y.

• The above results hold when a certain group is finite, so it is natural to ask how often this property happens to be satisfied. Note a first subtlety: in Theorem 41 the group is defined on $\overline{E}_t$, while in Corollary 42 it is considered as acting on $\mathbb{P}_1(\mathbb{C})^2$. As obviously $\overline{E}_t$ is strictly included in $\mathbb{P}_1(\mathbb{C})^2$, the second group has an order larger than or equal to that of the first one. A precise comparison of the two groups may be found in [FR10, Section 2].

When the group $\langle \iota_1, \iota_2 \rangle$ acts on $\mathbb{P}_1(\mathbb{C})^2$ (Corollary 42), its only known possible orders are 4, 6, 8, 10 and ∞, see [KY15]. It is believed that the cardinality of finite groups in the weighted case may be bounded. On the other hand, restricted to $\overline{E}_t$ (as in Theorem 41), the group can take any order (even and equal to or larger than 4), see [FR11].

• From a methodological viewpoint, the proof of Theorem 41 is largely inspired by [KR15, Section 9]. However, it seems important to us to state this result with the most general hypotheses, namely, arbitrary starting point and weighted steps.

Let us begin by the proof of a lemma.

Lemma 43. *Let $f(\omega)$ be a $(\omega_1, k\omega_2)$-elliptic function. Then $f(\omega)$ is algebraic in $x(\omega)$.*

Proof. The functions $f(\omega)$ and $\wp(\omega)$ are $(\omega_1, k\omega_2)$-elliptic. Using a well-known property of elliptic functions, there must exist a non-zero polynomial P such that $P(f, \wp) = 0$, see [WW96, 20.54]. As a consequence $f(\omega)$ is algebraic in $\wp(\omega)$, and hence also in $x(\omega)$ due to Lemma 12. □

Proof of Theorem 41. For $0 \leqslant n \leqslant \ell - 1$, apply $\widetilde{\tau}^n$ to Equations (15) and (16) and sum these ℓ identities. We easily obtain that

$$\widetilde{\tau}^\ell(r_x(\omega;t)) - r_x(\omega;t) = \mathcal{O}_1(\omega) \quad \text{and} \quad \widetilde{\tau}^\ell(r_y(\omega;t)) - r_y(\omega;t) = \mathcal{O}_2(\omega). \tag{29}$$

With $\omega_3/\omega_2 = k/\ell$, we deduce that for every (ω_1, ω_2)-elliptic function $\widetilde{\tau}^\ell f(\omega) = f(\omega + \ell\omega_3) = f(\omega + k\omega_2) = f(\omega)$ showing that $\widetilde{\tau}^\ell$ restricted to the the field of (ω_1, ω_2)-elliptic functions equal to the identity. Consequently $\widetilde{\tau}^\ell(x(\omega)y(\omega)) = x(\omega)y(\omega)$. As we may see in the proof of Theorem 21, we have $b_1(\omega) + b_2(\omega) = \widetilde{\tau}(x(\omega)y(\omega)) - x(\omega)y(\omega)$. We conclude from (28) that $\mathcal{O}_1(\omega) = -\mathcal{O}_2(\omega)$.

Let us first assume that the orbit-sum $\mathcal{O}_1(\omega)$ identically vanishes. Let us rewrite the first identity of (29) as

$$r_x(\omega + \ell\omega_3; t) - r_x(\omega; t) = 0,$$

which reads that r_x is $\ell\omega_3$-periodic. Being in addition ω_1-periodic by (17), we deduce that r_x is $(\omega_1, \ell\omega_3)$-elliptic, and therefore $(\omega_1, k\omega_2)$-elliptic since $\omega_3/\omega_2 = k/\ell$. Using Lemma 43, we obtain that $r_x(\omega; t)$ is an algebraic function of $x(\omega)$. For the exact same reasons, $r_y(\omega; t)$ is an algebraic function of $y(\omega)$. This shows that $x \mapsto Q(x, 0; t)$ is algebraic (resp. $y \mapsto Q(0, y; t)$ is algebraic). We conclude with (5) that $(x, y) \mapsto Q(x, y; t)$ is algebraic.

Conversely, assume that the function $(x, y) \mapsto Q(x, y; t)$ is algebraic. Then $r_x(\omega; t)$ (resp. $r_y(\omega; t)$) is algebraic in $x(\omega)$ (resp. $y(\omega)$) and $r_x(\omega; t)$ is algebraic in $\wp(\omega)$, due to Lemma 12. As we may deduce from the proof of [DR15, Proposition 6], there exist $\ell_1, \ell_2 \in \mathbb{N}^*$ such that

$r_x(\omega;t)$ is $(\ell_1\omega_1, \ell_2\omega_2)$-elliptic. Note that due to Theorem 21, we may take $\ell_1 = 1$. We have $r_x(\omega + \ell_2\omega_3; t) - r_x(\omega; t) = 0$, and therefore $r_x(\omega + \ell\ell_2\omega_3; t) - r_x(\omega; t) = 0$. For $0 \leqslant n \leqslant \ell\ell_2 - 1$, apply $\widetilde{\tau}^n$ to Equations (15) and (16) and make the sum of these $\ell\ell_2$ identities. We easily obtain that

$$b_1(\omega) + \widetilde{\tau}(b_1(\omega)) + \cdots + \widetilde{\tau}^{\ell\ell_2-1}(b_1(\omega)) = 0.$$

Using $\widetilde{\tau}^{\ell}(b_1(\omega)) = b_1(\omega)$, we deduce that

$$0 = b_1(\omega) + \widetilde{\tau}(b_1(\omega)) + \cdots + \widetilde{\tau}^{\ell\ell_2-1}(b_1(\omega)) = \ell_2\big(b_1(\omega) + \widetilde{\tau}(b_1(\omega)) + \cdots + \widetilde{\tau}^{\ell-1}(b_1(\omega))\big) = \ell_2\mathcal{O}_1(\omega).$$

Then, $0 = \mathcal{O}_1(\omega)$. We conclude with $\mathcal{O}_1(\omega) = -\mathcal{O}_2(\omega)$ that $\mathcal{O}_2(\omega) = 0$.

We now assume that the orbit-sum $\mathcal{O}_1(\omega)$ is non-zero and want to prove the holonomy. Set $\Omega_1 = \omega_1$, $\Omega_2 = k\omega_2$ and consider $\zeta(\omega)$ that is meromorphic on the complex plane and is the (opposite of the) antiderivative of the Weierstrass $\wp$-function: $\zeta' = -\wp$, coupled with the condition $\lim_{\omega\to 0}\zeta(\omega) - \frac{1}{\omega} = 0$, see [WW96, 20.4]:

$$\zeta(\omega) := \frac{1}{\omega} + \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left\{ \frac{1}{\omega + \ell_1\Omega_1 + \ell_2\Omega_2} - \frac{1}{\ell_1\Omega_1 + \ell_2\Omega_2} + \frac{\omega}{(\ell_1\Omega_1 + \ell_2\Omega_2)^2} \right\}.$$

As in [FIM99, Equation (4.3.7)], the key idea is to introduce

$$\phi(\omega) = \frac{\Omega_1}{2i\pi}\zeta(\omega) - \frac{\omega}{i\pi}\zeta(\Omega_1/2) \tag{30}$$

and to prove that

$$\phi(\omega + \Omega_1) = \phi(\omega) \quad \text{and} \quad \phi(\omega + \Omega_2) = \phi(\omega) + 1. \tag{31}$$

Using the quasi-periodicity property of ζ, that is for $i \in \{1, 2\}$,

$$\zeta(\omega + \Omega_i) - \zeta(\omega) = 2\zeta(\Omega_i/2),$$

see [WW96, 20.41], we obtain that ϕ is Ω_1-periodic (first identity in (31)). Using further the relation

$$\zeta(\Omega_1/2)\Omega_2 - \zeta(\Omega_2/2)\Omega_1 = -\pi i,$$

see [WW96, 20.411] (the minus sign in front of πi differs from [WW96], as the role of Ω_1 and Ω_2 is reversed here), we deduce that the function ϕ satisfies $\phi(\omega + k\omega_2) = \phi(\omega) + 1$ (second identity in (31)).

With the help of the property (31) satisfied by ϕ and $\mathcal{O}_1$, we may rewrite the first identity in (29) as the fact that the function

$$\psi(\omega) = r_x(\omega) - \mathcal{O}_1(\omega)\phi(\omega) \tag{32}$$

is $k\omega_2$-periodic:

$$\psi(\omega + k\omega_2) = \psi(\omega).$$

With the ω_1 periodicity, we find using Lemma 43 that ψ is algebraic in the variable $x(\omega)$.

Writing (32) under the form $r_x(\omega) = \psi(\omega) + \mathcal{O}_1(\omega)\phi(\omega)$, one sees that $r_x(\omega)$ is the sum of an algebraic function in $x(\omega)$ and the function $\mathcal{O}_1(\omega)\phi(\omega)$. We claim that $\mathcal{O}_1(\omega)\phi(\omega)$ is holonomic in the variable $x(\omega)$. First, with $\widetilde{\tau}^{\ell}(x(\omega)y(\omega)) = x(\omega)y(\omega)$, we deduce that $\mathcal{O}_1(\omega)$ is Ω_2-periodic. With the Ω_1-periodicity, we obtain using Lemma 43 that $\mathcal{O}_1(\omega)$ is algebraic in $x(\omega)$. Second, $\phi'(\omega)$ is a rational function of $x(\omega)$, as $\zeta'(\omega) = -\wp(\omega)$. So $\mathcal{O}_1(\omega)\phi(\omega)$ is holonomic in the variable $x(\omega)$, proving the claim. We conclude using closure properties of holonomic functions □

Proof of Corollary 42. If the group $\langle \iota_1, \iota_2\rangle$ of birational transformation of $\mathbb{P}_1(\mathbb{C})^2$ is finite, then for any $0 < t < 1$, the group $\langle \iota_1, \iota_2\rangle$ restricted to the kernel curve $\overline{E}_t$ is finite too, see [FR10, Section 2]. The proof of Corollary 42 is then a straightforward consequence of Theorem 41. □

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