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► **To cite this version:**

Ying Hu, Jianhui Huang, Tianyang Nie. Linear-Quadratic-Gaussian Mixed Mean-field Games with Heterogenous Input Constraints. 2017. hal-01590971v1

**HAL Id: hal-01590971**

**<https://hal.science/hal-01590971v1>**

Preprint submitted on 20 Sep 2017 (v1), last revised 10 Oct 2017 (v2)

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# Linear-Quadratic-Gaussian Mixed Mean-field Games with Heterogenous Input Constraints

Ying Hu <sup>a</sup>, Jianhui Huang <sup>b</sup>, Tianyang Nie <sup>c,\*</sup>

<sup>a</sup>IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France  
and School of Mathematical Sciences, Fudan University, Shanghai 200433, China

<sup>b</sup>Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China

<sup>c</sup>School of Mathematics, Shandong University, Jinan, Shandong 250100, China

September 20, 2017

## Abstract

We consider a class of linear-quadratic-Gaussian mean-field games with a major agent and considerable heterogeneous minor agents with mean-field interactions. The individual admissible controls are constrained in closed convex subsets  $\Gamma_k$  of full space  $\mathbb{R}^m$ . The decentralized strategies for individual agents and consistency condition system are represented in an unified manner via a class of mean-field forward-backward stochastic differential equations involving projection operators on  $\Gamma_k$ . The well-posedness of consistency system is established in the local and global cases both by the contraction mapping and discounting method respectively. The related  $\varepsilon$ -Nash equilibrium property is also verified.

**Key words:**  $\varepsilon$ -Nash equilibrium, Forward-backward stochastic differential equation, Input constraint, Projection operator, Linear-quadratic mixed mean-field games.

**AMS Subject Classification:** 60H10, 60H30, 91A10, 91A23, 91A25, 93E20

## 1 Introduction

Mean-field games (MFGs) for stochastic large-population system have been well-studied in recent years due to their wide applications in various fields such as economics, engineering, social science and operational research, etc. The large-population system has the following significant features: on the one hand, there exist a large number of agents (or players) whose individual influence on the overall population is negligible. On the other hand, the impact of their statistical behaviors cannot be ignored in population scale. Mathematically, all agents are weakly-coupled in their dynamics and/or cost functionals via the state-average (in linear state case) or the general empirical measure (in nonlinear state case) which characterize the statistical impact generated by the population in macroscopic perspective. Due to above features, when the number

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\*Corresponding author: Tianyang NIE (nietianyang@sdu.edu.cn).

<sup>1</sup>The work of Ying Hu is partially supported by Lebesgue center of mathematics “Investissements d’avenir” program - ANR-11-LABX-0020-01, by ANR-15-CE05-0024 and by ANR-16-CE40-0015-01; The work of James Jianhui Huang is supported by G-YL04, RGC Grant 502412, 15300514; The work of Tianyang Nie is supported by the National Natural Sciences Foundations of China (No. 11601285, 11571205, 61573217), the Natural Science Foundation of Shandong Province (No. ZR2016AQ13) and the Fundamental Research Funds of Shandong University (No.2015HW023).

*E-mail addresses:* Ying.Hu@univ-rennes1.fr (Ying Hu); james.huang@polyu.edu.hk (James Jianhui Huang); nietianyang@sdu.edu.cn (Tianyang NIE).

of the agents are large, highly complicated coupling feature arises and it is unrealistic for a given agent to obtain all other agents' information. Consequently, it is thereby also intractable for him to design the centralized strategies based on the information of all its peers in large-population system. Alternatively, one reasonable and practical direction is to transfer such high-dimensional weakly-coupled problem to a low-dimensional and decoupled one hence reduce the complexity in both analysis and computation. One way is to investigate the related decentralized strategies based only on local information: that is, the related strategies are designed upon the individual state of given agent only, together with some mass-effect quantities but computed off-line.

Along this research direction, motivated by the theory of propagation of chaos, Lasry and Lions [30, 31, 32] proposed the distributed closed-loop strategy for each agent through solving the limiting problem which are formulated as a coupled nonlinear forward-backward system consisting of Hamilton-Jacobi-Bellman (HJB) equation and Fokker-Planck equation. Moreover, the limiting problem enables to find an approximate Nash equilibrium strategies. Independently, Caines, Huang and Malhamé [28] developed a similar program which they called Nash Certainty Equivalence (NCE) principle motivated by the analysis of large communications networks. In principle, the procedure of MFGs consists of the following four main steps (see [5, 13, 25, 27, 32], etc): Step 1 is to introduce a limiting mass-effect term which comes from the asymptotic behavior of mass-effect when the agent number  $N$  tends to infinity. This limiting term should be treated as some exogenous "frozen" undetermined term at this moment; In Step 2, by replacing the mass-effect by the frozen limiting term, we can introduce the related limiting optimization problem. The initial highly-coupled problem is thus decoupled and only parameterized by this generic frozen limit. Then using standard control techniques (see [44]), some HJB equation due to dynamic programming principle (DPP) or Hamiltonian system due to stochastic maximum principle (SMP) will be obtained to characterize the decentralized optimal strategies; Step 3 aims to establish the consistency condition to guarantee that the set of decentralized optimal strategies collectively replicates the mass-effect. Finally, Step 4 is to show that the derived decentralized strategies are  $\varepsilon$ -Nash equilibrium which justifies the above scheme for finding approximate Nash equilibrium.

For further analysis details of MFGs, the interested readers are referred to [1, 5, 13, 19, 25, 42, 43]. It is remarkable that there exist substantial literatures studying linear-quadratic-Gaussian (LQG) mean-field games, for example, [26] studied the LQG MFGs by using the approach of common Riccati equation, while [8] adopted stochastic maximum principle with help of adjoint equations; [2] and [33] studied ergodic LQG MFGs; [27] studied the LQG MFGs with nonuniform agents through the state-aggregation by empirical distribution. For further research, the readers are referred to [3, 7, 41] and the references therein.

All above mentioned works are standard MFGs, it (except [28]) requires that all the agents are statistically identical and the individual influence on the overall population of a single agent is negligible as the number of the agents tends to infinity. However, in the real world, there are some models in which there is a major agent who has a significant influence on other agents (called minor agents) no matter how large the number of the minor agents are. Such interaction exists in many socio-economic problems, see e.g. [29, 45]. This kind of games involving agents with different power are usually called mixed type games. Comparing with the MFGs with only minor agents, a distinctive feature of the mixed type MFGs is that the mean field behavior of the minor agents is impacted by the major agent and thus is a random process and the influence of the major agent to minor agents is not negligible in the limiting problem. To deal with such new features, conditional distribution with respect to the major's information flow will be introduced, see [37, 14]. Let us now recall some works on MFGs with a major agent and minor agents related

to our paper. To our best knowledge, the LQG MFGs with a major agent and minor agents was first studied by [24], in which the minor agents are from a total  $K$  classes. Then in [38], the authors studied the mean-field LQG mixed games with continuum-parameterized minor agents. [37] investigated nonlinear stochastic dynamic systems with major and minor agents. Recently, [11] studied nonlinear stochastic differential games involving a major agent and a large number of collectively acting minor agents as two-person zero-sum stochastic differential games of the type feedback control against feedback control, the limit behavior of the saddle point controls are also studied. For further research, the readers are referred to [4, 14] and the reference therein.

In this paper, we investigate a class of LQG MFGs with major agents and minor agents in the presence of control constraint. In all above mentioned papers about linear quadratic (LQ) control problems, the control is unconstrained, then the (feedback) control constructed from DPP or SMP is automatically admissible, while if we put constraints for the admissible control, the whole LQ approach fails to apply, see e.g. [15, 23]. We emphasize that the LQ control problems with control constraints have wide applications in finance and economics, for example, the mean-variance problem with prohibiting short-selling can be transferred to LQ control problems with positive control, see e.g. [6, 23]; the optimal investment problems where the agents have relative performance, i.e. their portfolio constraints are different, can also be tackled by the approach of LQ control problems with input constraint, see e.g. [20, 17]. Remark 3.1 of the current paper gives several other constraint sets  $\Gamma \subset \mathbb{R}^m$  as well as their applications. For the investigation of LQ problems with positive control or more general with the control constrained in a given convex cone, the readers are referred to [9] for deterministic case and [15, 23, 34] for stochastic case.

To our best knowledge, the current paper is the first one to study the constrained LQG MFGs with major agents and a large number of minor agents. Besides the control constraint is fully new, our paper have also the following novel points comparing with other related works: in [24, 38], the diffusion term takes simply a constant, while in the current paper, we will consider the mean-field LQG mixed games in which the diffusion term depends on the major agent's state, the minor agent's state as well as the individual control strategy. This will arise additional difficulties, especially when applying the general stochastic maximum principle; different to [37, 11] which studied nonlinear stochastic differential games, here we put ourselves in a linear quadratic mean-field framework with individual control constrained in a closed convex set, thus we can give explicitly the optimal strategies through projection operator; moreover, different to [37] which using DPP and verification theorem to characterize the optimal strategies, we use SMP to obtain the optimal strategies through Hamiltonian systems which are fully coupled forward-backward stochastic differential equations (FBSDEs). Here, we connect the consistency condition to a new type of conditional mean field forward-backward stochastic differential equations (MF-FBSDEs) involving projection operators. We establish its well-posedness under suitable conditions by using fixed point theorem both in local case and global case. We mention that, different to our previous paper [21], the current paper focus on the mixed game which is more realistic and more difficult. In fact, in this situation the consistency condition is a conditional MF-FBSDE which does not satisfy the usual monotonicity condition of [22]. Moreover, we need additional subtle analysis to deal with the major agent's influence to establish the approximate Nash equilibrium. Finally, motivated by [17], we believe that our results can be applied to solve the optimal investment problems with major agent and  $N$  minor agents.

The remainder of this paper is structured as follows: In Section 2, we give some notations. Section 3 formulates the LQG MFGs with control constraint involving major agent and minor agents. The decentralized strategies are derived through a FBSDE with projection operators.

The consistency condition is also established by some fully coupled FBSDEs which comes from the SMP. In Section 4, we prove the well-posedness of fully coupled conditional MF-FBSDEs which characterize the consistency condition in local time horizon case. Section 5 gets the wellposedness in global time case. Section 6 aims to verify the  $\varepsilon$ -Nash equilibrium of the decentralized strategies. Finally, as an appendix, in Section 7 we establish the global wellposedness of general fully coupled conditional MF-FBSDEs.

The main contributions of this paper can be summarized as follows:

- To introduce and analyze a new class of linear-quadratic-Gaussian mixed mean-field games using stochastic maximum principle. In our setting, both the major agent and minor agents are constrained in their control inputs.
- The diffusion terms of major and minor agents are both dependent on their states and control variables.
- The consistency condition system or Nash certainty equivalence (NCE) is represented via a new type of conditional mean-field type FBSDE with projection operators.
- The existence and uniqueness of such NCE system are established in local case (i.e., small time horizon) using contraction mapping method; and in global case (i.e., arbitrary time horizon) using discounting method.

## 2 Notations and terminology

Consider a finite time horizon  $[0, T]$  for fixed  $T > 0$ . We assume  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$  is a complete, filtered probability space satisfying usual conditions and  $\{W_i(t), 0 \leq i \leq N\}_{0 \leq t \leq T}$  is a  $(N+1)$ -dimensional Brownian motion on this space. Let  $\mathcal{F}_t$  be the natural filtration generated by  $\{W_i(s), 0 \leq i \leq N, 0 \leq s \leq t\}$  and augmented by  $\mathcal{N}_{\mathbb{P}}$  (the class of  $\mathbb{P}$ -null sets of  $\mathcal{F}$ ). Let  $\mathcal{F}_t^{W_0}, \mathcal{F}_t^{W_i}, \mathcal{F}_t^i$  be respectively the augmentation of  $\sigma\{W_0(s), 0 \leq s \leq t\}$ ,  $\sigma\{W_i(s), 0 \leq s \leq t\}$ ,  $\sigma\{W_0(s), W_i(s), 0 \leq s \leq t\}$  by  $\mathcal{N}_{\mathbb{P}}$ . Here,  $\{\mathcal{F}_t^{W_0}\}_{0 \leq t \leq T}$  stands for the information of the major agent, while  $\{\mathcal{F}_t^{W_i}\}_{0 \leq t \leq T}$  represents the individual information of  $i$ -th minor agent.

Throughout the paper,  $x'$  denotes the transpose of a vector or a matrix  $x$ ,  $\mathcal{S}^n$  denotes the set of symmetric  $n \times n$  matrices with real elements. For a matrix  $M \in \mathbb{R}^{n \times d}$ , we define the norm  $|M| := \sqrt{\text{Tr}(M'M)}$ . If  $M \in \mathcal{S}^n$  is positive (semi) definite, we write  $M > (\geq) 0$ . Let  $\mathcal{H}$  be a given Hilbert space, the set of  $\mathcal{H}$ -valued continuous functions is denoted by  $C(0, T; \mathcal{H})$ . If  $N(\cdot) \in C(0, T; \mathcal{S}^n)$  and  $N(t) > (\geq) 0$  for every  $t \in [0, T]$ , we say that  $N(\cdot)$  is positive (semi) definite, which is denoted by  $N(\cdot) > (\geq) 0$ . Now, for a given Hilbert space  $\mathcal{H}$  and a filtration  $\{\mathcal{G}_t\}_{0 \leq t \leq T}$ , we also introduce the following spaces which will be used in this paper:

$$L_{\mathcal{G}_T}^2(\Omega; \mathcal{H}) := \{\xi : \Omega \rightarrow \mathcal{H} \mid \xi \text{ is } \mathcal{G}_T\text{-measurable such that } \mathbb{E}|\xi|^2 < \infty\},$$

$$L_{\mathcal{G}}^2(0, T; \mathcal{H}) := \left\{x(\cdot) : [0, T] \times \Omega \rightarrow \mathcal{H} \mid x(\cdot) \text{ is } \mathcal{G}_t\text{-progressively measurable process such that}$$

$$\mathbb{E} \int_0^T |x(t)|^2 dt < \infty\right\},$$

$$L_{\mathcal{G}}^2(\Omega; C(0, T; \mathcal{H})) := \left\{x(\cdot) : [0, T] \times \Omega \rightarrow \mathcal{H} \mid x(\cdot) \text{ is } \mathcal{G}\text{-adapted continuous process such that}$$

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |x(t)|^2 \right] < \infty\right\}.$$

### 3 LQG mixed games with control constraint

We consider a linear-quadratic-Gaussian mixed mean-field game involving a major agent  $\mathcal{A}_0$  and a heterogeneous large-population with  $N$  individual minor agents  $\{\mathcal{A}_i : 1 \leq i \leq N\}$ . Unlike other works of LQG mixed games, our control domain is constrained in a closed convex set of the full Euclidean space. The states  $x_0$  for major agent  $\mathcal{A}_0$  and  $x_i$  for each minor agent  $\mathcal{A}_i$  are modeled by the following controlled linear stochastic differential equations (SDEs) with mean-field coupling:

$$\begin{aligned} dx_0(t) = & [A_0(t)x_0(t) + B_0(t)u_0(t) + F_0^1(t)x^{(N)}(t) + b_0(t)]dt \\ & + [C_0(t)x_0(t) + D_0(t)u_0(t) + F_0^2(t)x^{(N)}(t) + \sigma_0(t)]dW_0(t), \quad x_0(0) = x_0 \in \mathbb{R}^n, \end{aligned} \quad (1)$$

and

$$\begin{aligned} dx_i(t) = & [A_{\theta_i}(t)x_i(t) + B(t)u_i(t) + F_1(t)x^{(N)}(t) + b(t)]dt \\ & + [C(t)x_i(t) + D_{\theta_i}(t)u_i(t) + F_2(t)x^{(N)}(t) + Hx_0(t) + \sigma(t)]dW_i(t), \quad x_i(0) = x \in \mathbb{R}^n, \end{aligned} \quad (2)$$

where  $x^{(N)}(\cdot) = \frac{1}{N} \sum_{i=1}^N x_i(\cdot)$  is the state-average of minor agents. Recalling that  $\mathcal{F}_t^i$  is the individual decentralized information while  $\mathcal{F}_t$  is the centralized information driven by all Brownian motion components. We point out that the heterogeneous noise  $W_i$  is specific for individual agent  $\mathcal{A}_i$  whereas  $x_i(t)$  is adapted to  $\mathcal{F}_t$  instead of  $\mathcal{F}_t^i$  due to the coupling state-average  $x^{(N)}$ . The coefficients of (1) and (2) are deterministic matrix-valued functions with appropriate dimensions. The number  $\theta_i$  is a parameter of agent  $\mathcal{A}_i$  to model a heterogeneous population of minor agents, for more explanations, see [24]. For sake of notations, in (2), we only set the coefficients  $A(\cdot)$  and  $D(\cdot)$  (see also  $R(\cdot)$  in (4)) to be dependent on  $\theta_i$ . Similar analysis can be proceeded in case that all other coefficients depend also on  $\theta_i$ . In this paper, we assume that  $\theta_i$  takes values from a finite set  $\Theta := \{1, 2, \dots, K\}$  which means that totally  $K$  types of minor agents are considered. We call  $\mathcal{A}_i$  a  $k$ -type minor agent if  $\theta_i = k \in \Theta$ .

In this paper, we are interested in the asymptotic behavior as  $N$  tends to infinity which is essentially to consider a family of games with an increasing number of minor agents. Now, we define

$$\mathcal{I}_k = \{i \mid \theta_i = k, 1 \leq i \leq N\}, \quad N_k = |\mathcal{I}_k|,$$

where  $N_k$  is the cardinality of index set  $\mathcal{I}_k, 1 \leq k \leq K$ . Let  $\pi_k^{(N)} = \frac{N_k}{N}$  for  $k \in \{1, \dots, K\}$  then  $\pi^{(N)} = (\pi_1^{(N)}, \dots, \pi_K^{(N)})$  is a probability vector to represent the empirical distribution of  $\theta_1, \dots, \theta_N$ . The following assumption gives some statistical properties for  $\theta_i$ . For more details, the reader is referred to [24]:

**(A1)** There exists a probability mass vector  $\pi = (\pi_1, \pi_2, \dots, \pi_K)$  such that  $\lim_{N \rightarrow +\infty} \pi^{(N)} = \pi$  and  $\min_{1 \leq k \leq K} \pi_k > 0$ .

From **(A1)** we know that when  $N \rightarrow +\infty$ , the proportion of  $k$ -type agents becomes stable for each  $k$  and that the number of each type agents tends to infinity. Otherwise, the agents in given type with bounded size should be excluded from consideration when analyzing asymptotic behavior as  $N \rightarrow +\infty$ . Throughout the paper we make the convention that  $N$  is suitable large such that  $\min_{1 \leq k \leq K} N_k \geq 1$ .

Now let us specify the admissible control set and cost functionals of our linear-quadratic-Gaussian mixed game with input control constraint. We call  $u_0$  an admissible control for the major agent if  $u_0 \in \mathcal{U}_{ad}^0$ , where  $\mathcal{U}_{ad}^0 := \{u(\cdot) \mid u(\cdot) \in L_{\mathcal{F}}^2(0, T; \Gamma_0)\}$ . Here  $\Gamma_0 \subset \mathbb{R}^m$  is a nonempty closed convex set. Moreover, for each  $1 \leq i \leq N$ , we also define decentralized admissible control  $u_i$  for the minor agent  $\mathcal{A}_i$  as  $u_i \in \mathcal{U}_{ad}^i$ , where for a nonempty closed convex set  $\Gamma_{\theta_i} \subset \mathbb{R}^m$ ,  $\mathcal{U}_{ad}^i := \{u_i(\cdot) \mid u_i(\cdot) \in L_{\mathcal{F}^i}^2(0, T; \Gamma_{\theta_i})\}$ .

**Remark 3.1** We give the following typical examples for the closed convex constraint set  $\Gamma$ :  $\Gamma^1 = \mathbb{R}_+^m$  represents that the control can only take positive values, it connects with the mean-variance portfolio selection problem with no-shorting, see [6, 23]. The linear subspace  $\Gamma^2 = (\mathbb{R}e_i)^\perp$  (where  $(e_1, e_2, \dots, e_m)$  is the canonical basis of  $\mathbb{R}^m$ ) represents that the control can only take from a hyperplane, it is used to deal with that in the investment theory, each manager has access to the whole market except some fixed firm who has private information. For more examples of linear constraints and their economic meaning, the reader is referred to [17].  $\Gamma$  can also be some closed cone (i.e.  $\Gamma$  is closed and if  $u \in \Gamma$ , then  $\alpha u \in \Gamma$ , for all  $\alpha \geq 0$ ), e.g.  $\Gamma^3 = \{u \in \mathbb{R}^m : \Upsilon u = 0\}$  or  $\Gamma^4 = \{u \in \mathbb{R}^m : \Upsilon u \leq 0\}$ , where  $\Upsilon \in \mathbb{R}^{n \times m}$ . For investigations on the stochastic LQ problems with conic control constraint, the readers are referred to [15, 23].

Let  $u = (u_0, u_1, \dots, u_N)$  be the set of strategies of all  $N + 1$  agents,  $u_{-0} = (u_1, u_2, \dots, u_N)$  be the control strategies except  $\mathcal{A}_0$  and  $u_{-i} = (u_0, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N)$  be the set of strategies except the  $i$ -th agent  $\mathcal{A}_i$ . We introduce the cost functional of the major agent as

$$\begin{aligned} \mathcal{J}_0(u_0, u_{-0}) = & \frac{1}{2} \mathbb{E} \left[ \int_0^T \left\langle Q_0(t)(x_0(t) - \rho_0 x^{(N)}(t)), x_0(t) - \rho_0 x^{(N)}(t) \right\rangle \right. \\ & \left. + \left\langle R_0(t)u_0(t), u_0(t) \right\rangle dt + \left\langle G_0(x_0(T) - \rho_0 x^{(N)}(T)), x_0(T) - \rho_0 x^{(N)}(T) \right\rangle \right] \end{aligned} \quad (3)$$

and the cost functional of the minor agent  $\mathcal{A}_i$  as

$$\begin{aligned} \mathcal{J}_i(u_i, u_{-i}) = & \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \left\langle Q(x_i - \rho x^{(N)} - (1-\rho)x_0(t)), x_i - \rho x^{(N)} - (1-\rho)x_0(t) \right\rangle + \left\langle R_{\theta_i} u_i, u_i \right\rangle \right) dt \right. \\ & \left. + \left\langle G(x_i(T) - \rho x^{(N)}(T) - (1-\rho)x_0(T)), x_i(T) - \rho x^{(N)}(T) - (1-\rho)x_0(T) \right\rangle \right]. \end{aligned} \quad (4)$$

We mention that for notational brevity, the time argument is suppressed in above equation as well as in the sequel when necessary.

We impose the following assumptions:

**(A2)** The coefficients of the states satisfy that, for  $1 \leq i \leq N$ ,

$$\begin{aligned} A_0(\cdot), A_{\theta_i}(\cdot), C_0(\cdot), C(\cdot), F_0^1(\cdot), F_1(\cdot), F_0^2(\cdot), F_2(\cdot), H(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times n}), \\ B_0(\cdot), B(\cdot), D_0(\cdot), D_{\theta_i}(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times m}), b_0(\cdot), b(\cdot), \sigma_0(\cdot), \sigma(\cdot) \in L^\infty(0, T; \mathbb{R}^n). \end{aligned}$$

**(A3)** The coefficients of cost functionals satisfy that, for  $1 \leq i \leq N$ ,

$$\begin{aligned} Q_0(\cdot), Q(\cdot) \in L^\infty(0, T; \mathcal{S}^n), R_0(\cdot), R_{\theta_i}(\cdot) \in L^\infty(0, T; \mathcal{S}^m), G_0, G \in \mathcal{S}^n, \\ Q_0(\cdot) \geq 0, Q(\cdot) \geq 0, R_0(\cdot) > 0, R_{\theta_i}(\cdot) > 0, G_0 \geq 0, G \geq 0, \rho_0, \rho \in [0, 1]. \end{aligned}$$

Here  $L^\infty(0, T; \mathcal{H})$  denotes the space of uniformly bounded functions mapping from  $[0, T]$  to  $\mathcal{H}$ . It follows that, under assumptions **(A2)** and **(A3)**, the system (1) and (2) admits a unique solution  $x_0, x_i(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))$  for given admissible control  $u_0$  and  $u_i$ . Now, let us formulate the LQG mixed games with control constraint.

**Problem (CC).** Find a strategies set  $\bar{u} = (\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$  where  $\bar{u}_i(\cdot) \in \mathcal{U}_{ad}^i$ ,  $0 \leq i \leq N$ , such that

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) = \inf_{u_i(\cdot) \in \mathcal{U}_{ad}^i} \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot)), \quad 0 \leq i \leq N.$$

We call  $\bar{u}$  an optimal strategies set for Problem **(CC)**.

For comparison, we present also the definition of  $\varepsilon$ -Nash equilibrium.

**Definition 3.1** A set of strategies  $\bar{u}_i(\cdot) \in \mathcal{U}_{ad}^i$ ,  $0 \leq i \leq N$ , is called an  $\varepsilon$ -Nash equilibrium with respect to costs  $\mathcal{J}^i$ ,  $0 \leq i \leq N$ , if there exists a  $\varepsilon = \varepsilon(N) \geq 0$ ,  $\lim_{N \rightarrow +\infty} \varepsilon(N) = 0$ , such that for any  $0 \leq i \leq N$ , we have

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) \leq \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot)) + \varepsilon,$$

when any alternative strategy  $u_i \in \mathcal{U}_{ad}^i$  is applied by  $\mathcal{A}_i$ .

**Remark 3.2** If  $\varepsilon = 0$ , Definition 3.1 reduces to the usual exact Nash equilibrium.

### 3.1 Stochastic optimal control problem of the major agent

To study Problem (CC), as explained in the introduction, instead of studying the centralized optimization strategies which are complicate especially as the number of the agents tends to infinity, we investigate the decentralized strategies via the limiting problem with the help of frozen limiting state-average. To this end, we need figure out the representation of limiting process using heuristic arguments. Based on it, we can find the decentralized strategies by consistency condition. We formalize the auxiliary limiting mixed game via the approximation of the average state  $x^{(N)}$ . Since  $\pi_k^{(N)} \approx \pi_k$  for large  $N$  and

$$x^{(N)} = \frac{1}{N} \sum_{k=1}^K \sum_{i \in \mathcal{I}_k} x_i = \sum_{k=1}^K \pi_k^{(N)} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} x_i,$$

we may approximate  $x^{(N)}$  by  $\sum_{k=1}^K \pi_k m_k$ , where  $m_k \in \mathbb{R}^n$  is used to approximate  $\frac{1}{N_k} \sum_{i \in \mathcal{I}_k} x_i$ . Denote  $m = (m'_1, m'_2, \dots, m'_K)'$ , which is called the set of aggregate quantities. Replacing  $x^{(N)}$  of (1) and (3) by  $\sum_{k=1}^K \pi_k m_k$ , the major agent's dynamics is given by

$$\begin{aligned} dz_0(t) = & [A_0(t)z_0(t) + B_0(t)u_0(t) + F_0^1(t) \sum_{k=1}^K \pi_k m_k(t) + b_0(t)]dt \\ & + [C_0(t)z_0(t) + D_0(t)u_0(t) + F_0^2(t) \sum_{k=1}^K \pi_k m_k(t) + \sigma_0(t)]dW_0(t), \quad z_0(0) = x_0 \in \mathbb{R}^n, \end{aligned} \quad (5)$$

and the limiting cost functional is

$$\begin{aligned} J_0(u_0) = & \frac{1}{2} \mathbb{E} \left[ \int_0^T \left\langle Q_0(t)(z_0(t) - \rho_0 \sum_{k=1}^K \pi_k m_k(t)), z_0(t) - \rho_0 \sum_{k=1}^K \pi_k m_k(t) \right\rangle \right. \\ & \left. + \langle R_0(t)u_0(t), u_0(t) \rangle dt + \left\langle G_0(z_0(T) - \rho_0 \sum_{k=1}^K \pi_k m_k(T)), z_0(T) - \rho_0 \sum_{k=1}^K \pi_k m_k(T) \right\rangle \right]. \end{aligned} \quad (6)$$

For simplicity, let  $\otimes$  be the Kronecker product of two matrix (see [18]) and we denote  $F_0^{1,\pi} := \pi \otimes F_0^1$ ,  $F_0^{2,\pi} := \pi \otimes F_0^2$ ,  $\rho_0^\pi := \pi \otimes \rho_0 I_{n \times n}$ . Then (5) and (6) become respectively to

$$\begin{aligned} dz_0(t) = & [A_0(t)z_0(t) + B_0(t)u_0(t) + F_0^{1,\pi}(t)m(t) + b_0(t)]dt \\ & + [C_0(t)z_0(t) + D_0(t)u_0(t) + F_0^{2,\pi}(t)m(t) + \sigma_0(t)]dW_0(t), \quad z_0(0) = x_0 \in \mathbb{R}^n, \end{aligned} \quad (7)$$

and

$$\begin{aligned} J_0(u_0) = & \frac{1}{2} \mathbb{E} \left[ \int_0^T \left\langle Q_0(t)(z_0(t) - \rho_0^\pi m(t)), z_0(t) - \rho_0^\pi m(t) \right\rangle \right. \\ & \left. + \langle R_0(t)u_0(t), u_0(t) \rangle dt + \left\langle G_0(z_0(T) - \rho_0^\pi m(T)), z_0(T) - \rho_0^\pi m(T) \right\rangle \right]. \end{aligned}$$



We define the following auxiliary stochastic optimal control problem for major agent with infinite population:

**Problem (LCC-Major).** For major agent  $\mathcal{A}_0$ , find  $u_0^*(\cdot) \in \mathcal{U}_{ad}^0$  satisfying

$$J_0(u_0^*(\cdot)) = \inf_{u_0(\cdot) \in \mathcal{U}_{ad}^0} J_0(u_0(\cdot)).$$

Then  $u_0^*(\cdot)$  is called a decentralized optimal control for Problem **(LCC-Major)**.

Now, similar to [21], we would like to apply SMP to above limiting LQG problem **(LCC-Major)** with input constraint. We introduce the following first order adjoint equation:

$$\begin{cases} dp_0(t) = -[A'_0(t)p_0(t) - Q_0(t)(z_0(t) - \rho_0^\pi m(t)) + C'_0(t)q_0(t)]dt + q_0(t)dW_0(t), \\ p_0(T) = -G_0(z_0(T) - \rho_0^\pi m(T)), \end{cases} \quad (8)$$

as well as the Hamiltonian function

$$\begin{aligned} H_0(t, p, q, x, u) = & \langle p, A_0x + B_0u + F_0^{1,\pi}m + b_0 \rangle + \langle q, C_0x + D_0u + F_0^{2,\pi}m + \sigma_0 \rangle \\ & - \frac{1}{2} \langle Q_0(x - \rho_0^\pi m), x - \rho_0^\pi m \rangle - \frac{1}{2} \langle R_0u, u \rangle. \end{aligned}$$

Since  $\Gamma_0$  is a closed convex set, for optimal control  $u_0^*$ , related optimal state  $z_0^*$  and related solution  $(p_0^*, q_0^*)$  to (8), the SMP reads as the following local form

$$\left\langle \frac{\partial H_0}{\partial u}(t, p_0^*, q_0^*, z_0^*, u_0^*), u - u_0^* \right\rangle \leq 0, \quad \text{for all } u \in \Gamma_0, \text{ a.e. } t \in [0, T], \mathbb{P} - a.s. \quad (9)$$

Similar to the argument in page 5 of [21], using the well-known results of convex analysis (see Theorem 5.2 of [10] or Theorem 4.1 of [21]), we obtain that (9) is equivalent to

$$u_0^*(t) = \mathbf{P}_{\Gamma_0}[R_0^{-1}(t)(B'_0(t)p_0^*(t) + D'_0(t)q_0^*(t))], \quad \text{a.e. } t \in [0, T], \mathbb{P} - a.s. \quad (10)$$

where  $\mathbf{P}_{\Gamma_0}[\cdot]$  is the projection mapping from  $\mathbb{R}^m$  to its closed convex subset  $\Gamma_0$  under the norm  $\|\cdot\|_{R_0}$  (where  $\|x\|_{R_0}^2 = \langle \langle x, x \rangle \rangle := \langle R_0^{\frac{1}{2}}x, R_0^{\frac{1}{2}}x \rangle$ ). Finally, by substituting (10) in (7) and (8), we get the following Hamiltonian system for the major agent:

$$\begin{cases} dz_0 = \left( A_0z_0 + B_0\mathbf{P}_{\Gamma_0}[R_0^{-1}(B'_0p_0 + D'_0q_0)] + F_0^{1,\pi}m + b_0 \right) dt \\ \quad + \left( C_0z_0 + D_0\mathbf{P}_{\Gamma_0}[R_0^{-1}(B'_0p_0 + D'_0q_0)] + F_0^{2,\pi}m + \sigma_0 \right) dW_0(t), \\ dp_0 = -(A'_0p_0 - Q_0(z_0 - \rho_0^\pi m) + C'_0q_0)dt + q_0dW_0(t), \\ z_0(0) = x_0, \quad p_0(T) = -G_0(z_0(T) - \rho_0^\pi m(T)). \end{cases} \quad (11)$$

### 3.2 Stochastic optimal control problem for minor agent

By denoting  $F^{1,\pi} := \pi \otimes F_1$ ,  $F^{2,\pi} := \pi \otimes F_2$ ,  $\rho^\pi := \pi \otimes \rho I_{n \times n}$ , the limiting state of minor agent  $\mathcal{A}_i$  is

$$\begin{cases} dz_i = \left( A_{\theta_i}z_i + Bu_i + F^{1,\pi}m + b \right) dt + \left( Cz_i + D_{\theta_i}u_i + F^{2,\pi}m + Hz_0 + \sigma \right) dW_i(t), \\ z_i(0) = x. \end{cases}$$

The limiting cost functional is given by:

$$\begin{aligned}
J_i(u_i) = & \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \left\langle Q \left( z_i - \rho \sum_{k=1}^K \pi_k m_k - (1-\rho) z_0 \right), z_i - \rho \sum_{k=1}^K \pi_k m_k - (1-\rho) z_0 \right\rangle + \left\langle R_{\theta_i} u_i, u_i \right\rangle \right) dt \right. \\
& \left. + \left\langle G \left( z_i(T) - \rho \sum_{k=1}^K \pi_k m_k(T) - (1-\rho) z_0(T) \right), z_i(T) - \rho \sum_{k=1}^K \pi_k m_k(T) - (1-\rho) z_0(T) \right\rangle \right], \tag{12}
\end{aligned}$$

or equivalently by

$$\begin{aligned}
J_i(u_i) = & \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \left\langle Q \left( z_i - \rho^\pi m - (1-\rho) z_0 \right), z_i - \rho^\pi m - (1-\rho) z_0 \right\rangle + \left\langle R_{\theta_i} u_i, u_i \right\rangle \right) dt \right. \\
& \left. + \left\langle G \left( z_i(T) - \rho^\pi m(T) - (1-\rho) z_0(T) \right), z_i(T) - \rho^\pi m(T) - (1-\rho) z_0(T) \right\rangle \right],
\end{aligned}$$

and the related limiting stochastic optimal control problem for the minor agents is:

**Problem (LCC-Minor).** For each minor agent  $\mathcal{A}_i$ , find  $u_i^*(\cdot) \in \mathcal{U}_{ad}^i$  satisfying

$$J_i(u_i^*(\cdot)) = \inf_{u_i(\cdot) \in \mathcal{U}_{ad}^i} J_i(u_i(\cdot)).$$

Then  $u_i^*(\cdot)$  is called a decentralized optimal control for Problem **(LCC-Minor)**.

Similar to the major agent, we obtain the following Hamiltonian system for minor agent  $\mathcal{A}_i$ ,

$$\left\{ \begin{aligned} dz_i &= \left( A_{\theta_i} z_i + B \mathbf{P}_{\Gamma_{\theta_i}} [R_{\theta_i}^{-1} (B' p_i + D'_{\theta_i} q_i)] + F_1^\pi m + b \right) dt \\ &+ \left( C z_i + D_{\theta_i} \mathbf{P}_{\Gamma_{\theta_i}} [R_{\theta_i}^{-1} (B' p_i + D'_{\theta_i} q_i)] + F_2^\pi m + H z_0 + \sigma \right) dW_i(t), \\ dp_i &= - \left( A'_{\theta_i} p_i - Q \left( z_i - \rho^\pi m - (1-\rho) z_0 \right) + C' q_i \right) dt + q_i dW_i(t) + q_{i,0} dW_0(t), \\ z_i(0) &= x, \quad p_i(T) = -G \left( z_i(T) - \rho^\pi m(T) - (1-\rho) z_0(T) \right). \end{aligned} \right. \tag{13}$$

Here,  $\mathbf{P}_{\Gamma_{\theta_i}}[\cdot]$  is the projection mapping from  $\mathbb{R}^m$  to its closed convex subset  $\Gamma_{\theta_i}$  under the norm  $\|\cdot\|_{R_{\theta_i}}$ . We mention that the limiting minor agent's state  $z_i$  depends also on the limiting major agent's state  $z_0$ , it makes that  $z_i$  is  $\mathcal{F}^i$ -adapted, thus  $q_{i,0} dW_0(t)$  appears in the adjoint equation.

### 3.3 Consistency condition system for mixed game

Let us first focus on the  $k$ -type minor agent. When  $i \in \mathcal{I}_k = \{i \mid \theta_i = k\}$ , we denote  $A_{\theta_i} = A_k$ ,  $D_{\theta_i} = D_k$ ,  $R_{\theta_i} = R_k$  and  $\Gamma_{\theta_i} = \Gamma_k$ . We would like to approximate  $x_i$  by  $z_i$  when  $N \rightarrow +\infty$ , thus  $m_k$  should satisfy the consistency condition (noticing that Assumption **(A1)** implies that  $N_k \rightarrow \infty$  if  $N \rightarrow \infty$ )

$$m_k(\cdot) = \lim_{N \rightarrow +\infty} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} z_i(\cdot).$$

Recall that for  $i, j \in \mathcal{I}_k$ ,  $z_i$  and  $z_j$  are identically distributed, and conditional independent (under  $\mathbb{E}(\cdot | \mathcal{F}^{W_0})$ ). Thus by conditional strong law of large number, we have (the convergence is in the sense of almost surely, see e.g. [36])

$$m_k(\cdot) = \lim_{N \rightarrow +\infty} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} z_i(\cdot) = \mathbb{E}(z_i(\cdot) | \mathcal{F}^{W_0}), \tag{14}$$

where  $z_i$  is given by (13) with  $A_{\theta_i} = A_k, D_{\theta_i} = D_k, R_{\theta_i} = R_k, \Gamma_{\theta_i} = \Gamma_k$ . By combining (11), (13) and (14), we get the following consistency condition system or Nash certainty equivalence principle of  $k$ -type minor agent, for  $1 \leq k \leq K$ : (As mentioned before, for notational brevity, the time argument is suppressed in following equations except  $\mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0})$  to emphasise its dependence on conditional expectation under  $\mathcal{F}_t^{W_0}$ )

$$\left\{ \begin{array}{l} d\alpha_k = \left( A_k\alpha_k + B\mathbf{P}_{\Gamma_k} [R_k^{-1}(B'\beta_k + D'_k\gamma_k)] + F_1 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) + b \right) dt \\ + \left( C\alpha_k + D_k\mathbf{P}_{\Gamma_k} [R_k^{-1}(B'\beta_k + D'_k\gamma_k)] + F_2 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) + H\alpha_0 + \sigma \right) dW_k(t), \\ d\beta_k = -(A'_k\beta_k - Q(\alpha_k - \rho \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) - (1-\rho)\alpha_0) + C'\gamma_k) dt + \gamma_k dW_k(t) + \gamma_{k,0} dW_0(t), \\ \alpha_k(0) = x, \quad \beta_k(T) = -G(\alpha_k(T) - \rho \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(T)|\mathcal{F}_T^{W_0}) - (1-\rho)\alpha_0(T)), \end{array} \right. \quad (15)$$

where  $\alpha_0$  satisfies the following FBSDE which is coupled with all  $k$ -type minor agents:

$$\left\{ \begin{array}{l} d\alpha_0 = \left( A_0\alpha_0 + B_0\mathbf{P}_{\Gamma_0} [R_0^{-1}(B'_0\beta_0 + D'_0\gamma_0)] + F_0^1 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) + b_0 \right) dt \\ + \left( C_0\alpha_0 + D_0\mathbf{P}_{\Gamma_0} [R_0^{-1}(B'_0\beta_0 + D'_0\gamma_0)] + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) + \sigma_0 \right) dW_0(t), \\ d\beta_0 = -(A'_0\beta_0 - Q_0(\alpha_0 - \rho_0 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0})) + C'_0\gamma_0) dt + \gamma_0 dW_0(t), \\ \alpha_0(0) = x_0, \quad \beta_0(T) = -G_0(\alpha_0(T) - \rho_0 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(T)|\mathcal{F}_T^{W_0})). \end{array} \right. \quad (16)$$

We consider together the major agent and all kinds of minor agents, i.e. (16) and (15) for all  $1 \leq k \leq K$ , then there arise  $2K+2$  fully coupled equations including  $K+1$  forward equations and  $K+1$  backward equations. Such fully coupled equations are called consistency condition system. Once we can solve it, then  $m_k = \mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0})$  which depends on the conditional distribution of  $\alpha_k$ , this allows us, in (15), to use arbitrary Brownian motion  $W_k$  which is independent of  $W_0$ . Finally, let us introduce the following notation which will be used in the following sections

$$\Phi(t) := \sum_{i=1}^K \pi_i m_i = \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}). \quad (17)$$

## 4 Existence and uniqueness of consistency condition system: local time horizon case

This section aims to establish the well-posedness of consistency condition system (15)-(16) in small time duration using the method of contraction mapping. Similar to the classical results on FBSDEs, see for example Chapter 1 Section 5 of Ma and Yong [35], we need introduce the following additional assumption:

**(A4)** We suppose  $R_0^{-1}(\cdot), R_{\theta_i}^{-1}(\cdot) \in L^\infty(0, T; \mathcal{S}^m)$  and  $M_0|D|^2 < 1$ , where  $|D| := \max_{0 \leq k \leq K} |D_k|$  and  $M_0 := \max \{ |G_0|^2(1 + \rho_0^2), |G|^2(1 + \rho^2 + (1 - \rho)^2) \}$ .

For simplicity, we denote

$$\varphi_0(p, q) := \mathbf{P}_{\Gamma_0} [R_0^{-1}(B'_0 p + D'_0 q)], \quad \varphi_{\theta_i}(p, q) := \mathbf{P}_{\Gamma_{\theta_i}} [R_{\theta_i}^{-1}(B'_i p + D'_i q)].$$

We have the following theorem:

**Theorem 4.1** *Assume (A1)-(A4), then there exists a  $T_0 > 0$ , such that for any  $T \in (0, T_0]$ , the system (15)-(16) has a unique solution  $(\alpha_0, \beta_0, \gamma_0, \alpha_k, \beta_k, \gamma_k, \gamma_{k,0})$ ,  $1 \leq k \leq K$ , satisfying*

$$\begin{aligned} \alpha_0, \beta_0 &\in L^2_{\mathcal{F}^{W_0}}(\Omega; C(0, T; \mathbb{R}^n)), \quad \alpha_k, \beta_k \in L^2_{\mathcal{F}^k}(\Omega; C(0, T; \mathbb{R}^n)), \\ \gamma_0 &\in L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^n), \quad \gamma_k, \gamma_{k,0} \in L^2_{\mathcal{F}^k}(0, T; \mathbb{R}^n), \quad 1 \leq k \leq K. \end{aligned} \quad (18)$$

**Proof** Let  $T_0 \in (0, 1]$  be undetermined and  $0 < T \leq T_0$ . We denote

$$\begin{aligned} \mathcal{N}[0, T] &:= L^2_{\mathcal{F}^{W_0}}(\Omega; C(0, T; \mathbb{R}^n)) \times L^2_{\mathcal{F}^1}(\Omega; C(0, T; \mathbb{R}^n)) \times \dots \times L^2_{\mathcal{F}^K}(\Omega; C(0, T; \mathbb{R}^n)) \\ &\quad \times L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^1}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}^K}(0, T; \mathbb{R}^n) \\ &\quad \times L^2_{\mathcal{F}^1}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}^K}(0, T; \mathbb{R}^n). \end{aligned}$$

For  $(Y_0, Y_1, \dots, Y_K, Z_0, Z_1, \dots, Z_K, \Upsilon_{1,0}, \dots, \Upsilon_{K,0}) \in \mathcal{N}[0, T]$ , we introduce the following norm:

$$\begin{aligned} &\|(Y_0, Y_1, \dots, Y_K, Z_0, Z_1, \dots, Z_K, \Upsilon_{1,0}, \dots, \Upsilon_{K,0})\|_{\mathcal{N}[0, T]}^2 \\ &:= \sup_{t \in [0, T]} \mathbb{E} \left\{ \sum_{k=0}^K |Y_k(t)|^2 + \sum_{k=0}^K \int_0^T |Z_k(s)|^2 ds + \sum_{k=1}^K \int_0^T |\Upsilon_{k,0}(s)|^2 ds \right\}. \end{aligned} \quad (19)$$

Let  $\overline{\mathcal{N}}[0, T]$  be the completion of  $\mathcal{N}[0, T]$  in  $L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^1}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}^K}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^1}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}^K}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^1}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}^K}(0, T; \mathbb{R}^n)$  under norm (19). Now for any

$$(Y_0^j, Y_1^j, \dots, Y_K^j, Z_0^j, Z_1^j, \dots, Z_K^j, \Upsilon_{1,0}^j, \dots, \Upsilon_{K,0}^j) \in \mathcal{N}[0, T], \quad j = 1, 2,$$

we solve respectively the following system including  $K + 1$  SDEs, for  $1 \leq k \leq K$ :

$$\left\{ \begin{aligned} d\alpha_0^j &= \left( A_0 \alpha_0^j + B_0 \varphi_0(Y_0^j, Z_0^j) + F_0^1 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t) | \mathcal{F}_t^{W_0}] + b_0 \right) dt \\ &\quad + \left( C_0 \alpha_0^j + D_0 \varphi_0(Y_0^j, Z_0^j) + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t) | \mathcal{F}_t^{W_0}] + \sigma_0 \right) dW_0(t) \\ d\alpha_k^j &= \left( A_k \alpha_k^j + B_k \varphi_k(Y_k^j, Z_k^j) + F_1 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t) | \mathcal{F}_t^{W_0}] + b \right) dt \\ &\quad + \left( C_k \alpha_k^j + D_k \varphi_k(Y_k^j, Z_k^j) + F_2 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t) | \mathcal{F}_t^{W_0}] + H \alpha_0^j + \sigma \right) dW_k(t) \\ \alpha_0^j(0) &= x_0, \quad \alpha_k^j(0) = x. \end{aligned} \right. \quad (20)$$

Then (20) admits a unique solution for  $j = 1, 2$ ,

$$(\alpha_0^j, \alpha_1^j, \dots, \alpha_K^j) \in L^2_{\mathcal{F}^{W_0}}(\Omega; C(0, T; \mathbb{R}^n)) \times L^2_{\mathcal{F}^1}(\Omega; C(0, T; \mathbb{R}^n)) \times \dots \times L^2_{\mathcal{F}^K}(\Omega; C(0, T; \mathbb{R}^n)).$$

Indeed, (20) is a  $n(K + 1)$ -dimensional SDE with the mean-field term  $\sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i(t) | \mathcal{F}_t^{W_0}]$ . We can prove the well-posedness of such SDEs system by noticing  $\mathbb{E}|\mathbb{E}[\alpha_i(t) | \mathcal{F}_t^{W_0}]|^2 \leq \mathbb{E}|\alpha_i(t)|^2$  and

by constructing a fixed point using the classical contraction mapping method, we omit the proof here. Now let us denote for  $0 \leq k \leq K$ ,

$$\begin{aligned}\hat{\alpha}_k &:= \alpha_k^1 - \alpha_k^2, & \hat{\varphi}_k &:= \varphi_k(Y_k^1, Z_k^1) - \varphi_k(Y_k^2, Z_k^2) \\ \hat{Y}_k &:= Y_k^1 - Y_k^2, & \hat{Z}_k &:= Z_k^1 - Z_k^2, & \hat{\Upsilon}_{k,0} &:= \Upsilon_{k,0}^1 - \Upsilon_{k,0}^2.\end{aligned}$$

Applying Itô's formula, we obtain

$$\begin{aligned}d|\hat{\alpha}_0|^2 &= 2\left\langle \hat{\alpha}_0, A_0\hat{\alpha}_0 + B_0\hat{\varphi}_0 + F_0^1 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] \right\rangle dt \\ &\quad + \left| C_0\hat{\alpha}_0 + D_0\hat{\varphi}_0 + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] \right|^2 dt \\ &\quad + 2\left\langle \hat{\alpha}_0, C_0\hat{\alpha}_0 + D_0\hat{\varphi}_0 + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] \right\rangle dW_0(t)\end{aligned}$$

and

$$\begin{aligned}d|\hat{\alpha}_k|^2 &= 2\left\langle \hat{\alpha}_k, A_k\hat{\alpha}_k + B\hat{\varphi}_k + F_1 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] \right\rangle dt \\ &\quad + \left| C\hat{\alpha}_k + D_k\hat{\varphi}_k + F_2 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] + H\hat{\alpha}_0 \right|^2 \\ &\quad + 2\left\langle \hat{\alpha}_k, C\hat{\alpha}_k + D_k\hat{\varphi}_k + F_2 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] + H\hat{\alpha}_0 \right\rangle dW_k(t).\end{aligned}$$

Thus by using **(A2)**-**(A3)**,  $\mathbb{E}|\mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}]| \leq \mathbb{E}|\hat{\alpha}_i(s)|$ ,  $\mathbb{E}|\mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}]|^2 \leq \mathbb{E}|\hat{\alpha}_i(s)|^2$  as well as that  $\varphi_k$  is Lipschitz with Lipschitz constant 1 (see Proposition 4.2 of [21]), we have

$$\begin{aligned}\mathbb{E}|\hat{\alpha}_0(t)|^2 &\leq 2\mathbb{E} \int_0^t \left( |A_0||\hat{\alpha}_0|^2 + |B_0||\hat{\alpha}_0||\hat{\varphi}_0| + |F_0^1||\hat{\alpha}_0| \sum_{i=1}^K |\mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}]| \right) ds \\ &\quad + \mathbb{E} \int_0^t \left| C_0\hat{\alpha}_0 + D_0\hat{\varphi}_0 + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}] \right|^2 ds \\ &\leq C_\varepsilon \mathbb{E} \int_0^t \sum_{i=0}^K |\hat{\alpha}_i|^2 ds + \mathbb{E} \int_0^t (|D_0|^2 + \varepsilon)(|\hat{Y}_0|^2 + |\hat{Z}_0|^2) ds\end{aligned}\tag{21}$$

and

$$\begin{aligned}\mathbb{E}|\hat{\alpha}_k(t)|^2 &\leq 2\mathbb{E} \int_0^t \left( |A_k||\hat{\alpha}_k|^2 + |B||\hat{\alpha}_k||\hat{\varphi}_k| + |F_1||\hat{\alpha}_k| \sum_{i=1}^K |\mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}]| \right) ds \\ &\quad + \mathbb{E} \int_0^t \left| C\hat{\alpha}_k + D_k\hat{\varphi}_k + \sum_{i=1}^K F_2 \mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}] + H\hat{\alpha}_0 \right|^2 ds \\ &\leq C_\varepsilon \mathbb{E} \int_0^t \sum_{i=0}^K |\hat{\alpha}_i|^2 ds + \mathbb{E} \int_0^t (|D_k|^2 + \varepsilon)(|\hat{Y}_k|^2 + |\hat{Z}_k|^2) ds,\end{aligned}\tag{22}$$

where  $C_\varepsilon$  is a constant independent of  $T$ , which may vary line by line. Adding up (21) and (22) for  $1 \leq k \leq K$ , we have

$$\mathbb{E} \sum_{i=0}^K |\hat{\alpha}_i(t)|^2 \leq C_\varepsilon \mathbb{E} \int_0^t \sum_{i=0}^K |\hat{\alpha}_i(s)|^2 ds + \mathbb{E} \int_0^t \sum_{i=0}^K (|D_i|^2 + \varepsilon)(|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds,$$



Similarly, by applying Itô's formula to  $|\hat{\beta}_k(t)|^2$ ,  $1 \leq k \leq K$ , we have

$$\begin{aligned} & \mathbb{E} \left| \hat{\beta}_k(t) \right|^2 + \mathbb{E} \int_t^T |\hat{\gamma}_k|^2 ds + \mathbb{E} \int_t^T |\hat{\gamma}_{k,0}|^2 ds \\ &= \mathbb{E} \left| \hat{\beta}_k(T) \right|^2 + 2\mathbb{E} \int_t^T \left\langle \hat{\beta}_k, A'_k \hat{Y}_k - Q \left( \hat{\alpha}_k - \rho \sum_{i=1}^K \pi_i \mathbb{E} [\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}] - (1-\rho)\hat{\alpha}_0 \right) - C' \hat{Z}_k \right\rangle ds. \end{aligned} \quad (26)$$

Noticing that  $\rho \in [0, 1]$ , we have

$$\begin{aligned} \mathbb{E} \left| \hat{\beta}_k(T) \right|^2 &\leq |G|^2 \left| \hat{\alpha}_k(T) - \rho \sum_{i=1}^K \pi_i \mathbb{E} [\hat{\alpha}_i(T) | \mathcal{F}_T^{W_0}] - (1-\rho)\hat{\alpha}_0(T) \right|^2 \\ &\leq |G|^2 (1 + \rho^2 + (1-\rho)^2) \left( \mathbb{E} \left| \hat{\alpha}_k(T) - \rho \mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}] \right|^2 + \sum_{i=0, i \neq k}^K \mathbb{E} |\hat{\alpha}_i(T)|^2 \right) \\ &\leq |G|^2 (1 + \rho^2 + (1-\rho)^2) \mathbb{E} \sum_{i=0}^K |\hat{\alpha}_i(T)|^2, \end{aligned}$$

where we have used the fact that

$$\begin{aligned} & \mathbb{E} \left| \hat{\alpha}_k(T) - \rho \mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}] \right|^2 \\ &= \mathbb{E} |\hat{\alpha}_k(T)|^2 + \rho^2 \mathbb{E} |\mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}]|^2 - 2\rho \mathbb{E} \left[ \hat{\alpha}_k(T) \mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}] \right] \\ &= \mathbb{E} |\hat{\alpha}_k(T)|^2 + \rho^2 \mathbb{E} |\mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}]|^2 - 2\rho \mathbb{E} \left( \mathbb{E} \left[ \hat{\alpha}_k(T) \mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}] \right] | \mathcal{F}_T^{W_0} \right) \\ &= \mathbb{E} |\hat{\alpha}_k(T)|^2 + (\rho^2 - 2\rho) \mathbb{E} |\mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}]|^2 \leq \mathbb{E} |\hat{\alpha}_k(T)|^2. \end{aligned}$$

Thus, (26) yields that

$$\begin{aligned} & \mathbb{E} \left| \hat{\beta}_k(t) \right|^2 + \mathbb{E} \int_t^T |\hat{\gamma}_k|^2 ds + \mathbb{E} \int_t^T |\hat{\gamma}_{k,0}|^2 ds \\ &= \mathbb{E} \left| \hat{\beta}_k(T) \right|^2 + 2\mathbb{E} \int_t^T \left\langle \hat{\beta}_k, A'_k \hat{Y}_k - Q \left( \hat{\alpha}_k - \rho \sum_{i=1}^K \pi_i \mathbb{E} [\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}] - (1-\rho)\hat{\alpha}_0 \right) - C' \hat{Z}_k \right\rangle ds \\ &\leq |G|^2 (1 + \rho^2 + (1-\rho)^2) \mathbb{E} \sum_{i=0}^K |\hat{\alpha}_i(T)|^2 + C_\varepsilon \mathbb{E} \int_t^T |\beta_k|^2 ds \\ &\quad + \mathbb{E} \int_t^T \sum_{i=0}^K |\alpha_i|^2 ds + \varepsilon \mathbb{E} \int_t^T (|\hat{Y}_k|^2 + |\hat{Z}_k|^2) ds. \end{aligned}$$

Substituting (23) into above inequality, we have

$$\begin{aligned} & \mathbb{E} \left| \hat{\beta}_k(t) \right|^2 + \mathbb{E} \int_t^T |\hat{\gamma}_k|^2 ds + \mathbb{E} \int_t^T |\hat{\gamma}_{k,0}|^2 ds \\ &\leq (|G|^2 (1 + \rho^2 + (1-\rho)^2) + T) e^{C_\varepsilon T} \mathbb{E} \sum_{i=0}^K \int_0^T (|D_i|^2 + \varepsilon) (|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds \\ &\quad + C_\varepsilon \mathbb{E} \int_t^T |\hat{\beta}_k|^2 ds + \varepsilon \mathbb{E} \int_t^T (|\hat{Y}_k|^2 + |\hat{Z}_k|^2) ds. \end{aligned} \quad (27)$$

Adding up (25) and (27) for  $1 \leq k \leq K$ , we obtain (recall  $|D|^2 := \max_{0 \leq k \leq K} |D_k|^2$  and  $M_0 := \max \{|G_0|^2(1 + \rho_0^2), |G|^2(1 + \rho^2 + (1 - \rho)^2)\}$ )

$$\begin{aligned}
& \mathbb{E} \sum_{i=0}^K |\hat{\beta}_i|^2 + \mathbb{E} \sum_{i=0}^K \int_t^T |\hat{\gamma}_i|^2 ds + \mathbb{E} \sum_{i=1}^K \int_t^T |\hat{\gamma}_{i,0}|^2 ds \\
& \leq (M_0 + T) e^{C_\varepsilon T} \mathbb{E} \sum_{i=0}^K \int_0^T (|D_i|^2 + \varepsilon) (|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds \\
& \quad + C_\varepsilon \mathbb{E} \int_t^T \sum_{i=0}^K |\hat{\beta}_i|^2 ds + \varepsilon \mathbb{E} \int_0^T \sum_{i=0}^K (|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds \\
& \leq C_\varepsilon \mathbb{E} \int_t^T \sum_{i=0}^K |\hat{\beta}_i|^2 ds + [(M_0 + T) e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon] \mathbb{E} \sum_{i=0}^K \int_0^T (|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds.
\end{aligned}$$

The Gronwall's inequality yields that

$$\begin{aligned}
& \mathbb{E} \sum_{i=0}^K |\hat{\beta}_i|^2 + \mathbb{E} \sum_{i=0}^K \int_t^T |\hat{\gamma}_i|^2 ds + \mathbb{E} \sum_{i=1}^K \int_t^T |\hat{\gamma}_{i,0}|^2 ds \\
& \leq e^{C_\varepsilon T} [(M_0 + T) e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon] \mathbb{E} \sum_{i=0}^K \int_0^T (|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds \\
& \leq e^{C_\varepsilon T} T [(M_0 + T) e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon] \sup_{0 \leq t \leq T} \mathbb{E} \sum_{i=0}^K |\hat{Y}_i(t)|^2 \\
& \quad + e^{C_\varepsilon T} [(M_0 + T) e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon] \mathbb{E} \sum_{i=0}^K \int_0^T |\hat{Z}_i|^2 ds \\
& \leq e^{C_\varepsilon T} (T + 1) [(M_0 + T) e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon] \\
& \quad \cdot \|(\hat{Y}_0, \hat{Y}_1, \dots, \hat{Y}_K, \hat{Z}_0, \hat{Z}_1, \dots, \hat{Z}_K, \hat{\Upsilon}_{1,0}, \dots, \hat{\Upsilon}_{K,0})\|_{\overline{\mathcal{N}}[0,T]} \\
& = e^{C_\varepsilon T} (T + 1) [M_0 e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon + T e^{C_\varepsilon T} (|D|^2 + \varepsilon)] \\
& \quad \cdot \|(\hat{Y}_0, \hat{Y}_1, \dots, \hat{Y}_K, \hat{Z}_0, \hat{Z}_1, \dots, \hat{Z}_K, \hat{\Upsilon}_{1,0}, \dots, \hat{\Upsilon}_{K,0})\|_{\overline{\mathcal{N}}[0,T]}
\end{aligned} \tag{28}$$

Noticing that assumption **(A4)** holds, by first choosing  $\varepsilon > 0$  small enough such that  $M_0(|D|^2 + \varepsilon) + \varepsilon < 1$ , then choosing  $T > 0$  small enough, we obtain from (28) that

$$\begin{aligned}
& \|(\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_K, \hat{\gamma}_0, \hat{\gamma}_1, \dots, \hat{\gamma}_K, \hat{\gamma}_{1,0}, \dots, \hat{\gamma}_{K,0})\|_{\overline{\mathcal{N}}[0,T]} \\
& \leq \delta \|(\hat{Y}_0, \hat{Y}_1, \dots, \hat{Y}_K, \hat{Z}_0, \hat{Z}_1, \dots, \hat{Z}_K, \hat{\Upsilon}_{1,0}, \dots, \hat{\Upsilon}_{K,0})\|_{\overline{\mathcal{N}}[0,T]},
\end{aligned}$$

for some  $0 < \delta < 1$ . This means that the mapping  $\mathcal{T} : \overline{\mathcal{N}}[0, T] \rightarrow \overline{\mathcal{N}}[0, T]$  is contractive. By the contraction mapping theorem, there exists a unique fixed point

$$(\beta_0, \beta_1, \dots, \beta_K, \gamma_0, \gamma_1, \dots, \gamma_K, \gamma_{1,0}, \dots, \gamma_{K,0}) \in \overline{\mathcal{N}}[0, T].$$

Moreover, classical BSDE theory allows us to show that

$$(\beta_0, \beta_1, \dots, \beta_K, \gamma_0, \gamma_1, \dots, \gamma_K, \gamma_{1,0}, \dots, \gamma_{K,0}) \in \mathcal{N}[0, T].$$

Let  $\alpha_k$ ,  $0 \leq k \leq K$ , be the corresponding solution of (20). Then, one can obtain that the system (15)-(16) has a unique solution  $(\alpha_0, \beta_0, \gamma_0, \alpha_k, \beta_k, \gamma_k, \gamma_{k,0})$ ,  $1 \leq k \leq K$ , such that (18) holds.  $\square$



## 5 Existence and uniqueness of consistency condition system-global time horizon case

The section aims to establish the well-posedness of consistency condition system (15)-(16) for arbitrary  $T$ , we first study one general kind of conditional mean field forward-backward stochastic differential equations (MF-FBSDE) by using the method of Pardoux and Tang [39].

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  is a complete, filtered probability space satisfying usual conditions and  $\{W_i(t), 0 \leq i \leq d\}_{0 \leq t \leq T}$  is a  $d+1$ -dimensional Brownian motion on this space. Let  $\mathcal{F}_t$  be the natural filtration generated by  $\{W_i(s), 0 \leq i \leq d, 0 \leq s \leq t\}$  and augmented by  $\mathcal{N}_{\mathbb{P}}$  (the class of  $\mathbb{P}$ -null sets of  $\mathcal{F}$ ). Let  $\mathcal{F}_t^{W_0}$  be the augmentation of  $\sigma\{W_0(s), 0 \leq s \leq t\}$  by  $\mathcal{N}_{\mathbb{P}}$ . We consider the following general conditional MF-FBSDE:

$$\begin{cases} dX(s) = b(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s))ds + \sigma(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s))dW(s), \\ -dY(s) = f(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s))ds - Z(s)dW(s), \quad s \in [0, T], \\ X(0) = x, \quad Y(T) = g(X(T), \mathbb{E}[X(T)|\mathcal{F}_T^{W_0}]), \end{cases} \quad (29)$$

where the adapted processes  $X, Y, Z$  take their values in  $\mathbb{R}^n, \mathbb{R}^l$  and  $\mathbb{R}^{l \times (d+1)}$ , respectively. The coefficients  $b, \sigma$  and  $f$  are defined on  $\Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l \times (d+1)}$ , such that  $b(\cdot, \cdot, x, m, y, z)$ ,  $\sigma(\cdot, \cdot, x, m, y, z)$  and  $f(\cdot, \cdot, x, m, y, z)$  are  $\{\mathcal{F}_t\}$ -progressively measurable processes, for all fixed  $(x, m, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l \times (d+1)}$ . The coefficient  $g$  is defined on  $\Omega \times \mathbb{R}^n \times \mathbb{R}^n$  and  $g(\cdot, x, m)$  is  $\mathcal{F}_T$ -measurable, for all fixed  $(x, m) \in \mathbb{R}^n \times \mathbb{R}^n$ . Moreover, the functions  $b, \sigma, f$  and  $g$  are continuous w.r.t.  $(x, m, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l \times (d+1)}$  and satisfy the following assumption

( $H_1$ ) There exist  $\lambda_1, \lambda_2 \in \mathbb{R}$  and positive constants  $k_0, k_i, i = 1, 2, \dots, 12$  such that for all  $t, x, x_1, x_2, m, m_1, m_2, y, y_1, y_2, z, z_1, z_2$  a.s.

- (i)  $\langle b(t, x_1, m, y, z) - b(t, x_2, m, y, z), x_1 - x_2 \rangle \leq \lambda_1 |x_1 - x_2|^2$ ,
- (ii)  $|b(t, x, m_1, y_1, z_1) - b(t, x, m_2, y_2, z_2)| \leq k_1 |m_1 - m_2| + k_2 |y_1 - y_2| + k_3 |z_1 - z_2|$ ,
- (iii)  $|b(t, x, m, y, z)| \leq |b(t, 0, m, y, z)| + k_0(1 + |x|)$ ,
- (iv)  $\langle f(t, x, m, y_1, z) - f(t, x, m, y_2, z), y_1 - y_2 \rangle \leq \lambda_2 |y_1 - y_2|^2$ ,
- (v)  $|f(t, x_1, m_1, y, z_1) - f(t, x_1, m_2, y, z_2)| \leq k_4 |x_1 - x_2| + k_5 |m_1 - m_2| + k_6 |z_1 - z_2|$ ,
- (vi)  $|f(t, x, m, y, z)| \leq |f(t, x, m, 0, z)| + k_0(1 + |y|)$ ,
- (vii)  $|\sigma(t, x_1, m_1, y_1, z_1) - \sigma(t, x_2, m_2, y_2, z_2)|^2$   
 $\leq k_7^2 |x_1 - x_2|^2 + k_8^2 |m_1 - m_2|^2 + k_9^2 |y_1 - y_2|^2 + k_{10}^2 |z_1 - z_2|^2$ ,
- (viii)  $|g(x_1, m_1) - g(x_2, m_2)|^2 \leq k_{11}^2 |x_1 - x_2| + k_{12}^2 |m_1 - m_2|$ .

( $H_2$ ) It holds that

$$\mathbb{E} \int_0^T (|b(s, 0, 0, 0, 0)|^2 + |\sigma(s, 0, 0, 0, 0)|^2 + |f(s, 0, 0, 0, 0)|^2) ds + \mathbb{E}|g(0, 0)|^2 < +\infty.$$

**Remark 5.1** From the mean-field structure of (29), sometimes the following conditions holds:

( $H_1$ ) - ( $i'$ ): There exist  $\lambda_1, \hat{k}_1 \in \mathbb{R}$ , such that for all  $t, y, z$  and process  $X_1, X_2$ , a.s.

$$\begin{aligned} & \mathbb{E} \langle b(t, X_1, \mathbb{E}[X_1(t)|\mathcal{F}_t^{W_0}], y, z) - b(t, X_2, \mathbb{E}[X_2(t)|\mathcal{F}_t^{W_0}], y, z), X_1 - X_2 \rangle \\ & \leq (\lambda_1 + \hat{k}_1) \mathbb{E}|X_1 - X_2|^2. \end{aligned}$$

For example, if  $b(t, x, m, y, z) = \lambda_1 x + \hat{k}_1 m + b_1(y, z)$ , then it obviously satisfies above assumption. Indeed, our mean-filed FBSDE satisfies this assumption.

Let  $\mathcal{H}$  be an Euclidean space. Recall that  $L_{\mathcal{F}}^2(0, T; \mathcal{H})$ , denotes the Hilbert space of  $\mathcal{H}$ -valued  $\{\mathcal{F}_s\}$ -progressively measurable processes  $\{u(s), s \in [0, T]\}$  such that

$$\|u\| := \left( E \int_0^T |u(s)|^2 ds \right)^{1/2} < \infty.$$

For  $\lambda \in \mathbb{R}$ , we define an equivalent norm on  $L_{\mathcal{F}}^2(0, T; \mathcal{H})$ :

$$\|u\|_{\lambda} := \left( E \int_0^T e^{-\lambda s} |u(s)|^2 ds \right)^{1/2}.$$

Now let us consider MF-FBSDE (29), its fully-coupled structure arises difficulties for establishing its wellposedness. Similar to [39], when the coupling is weak enough, MF-FBSDE (29) should be solvable. The following is the main results on MF-FBSDE (29) and the proof is give in the appendix.

**Theorem 5.1** *Suppose that assumption  $(H_1)$  and  $(H_2)$  hold. Then there exists a  $\delta_0 > 0$ , which depends on  $k_i, \lambda_1, \lambda_2, T$ , for  $i = 1, 4, 5, 6, 7, 8, 11, 12$  such that when  $k_2, k_3, k_9, k_{10} \in [0, \delta_0)$ , there exists a unique adapted solution  $(X, Y, Z) \in L_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m \times (d+1)})$  to MF-FBSDE (29). Further, if  $2(\lambda_1 + \lambda_2) < -2k_1 - k_6^2 - k_7^2 - k_8^2$ , there exists a  $\delta_1 > 0$ , which depends on  $k_i, \lambda_1, \lambda_2$ , for  $i = 1, 4, 5, 6, 7, 8, 11, 12$  and is independent of  $T$ , such that when  $k_2, k_3, k_9, k_{10} \in [0, \delta_1)$ , there exists a unique adapted solution  $(X, Y, Z)$  to MF-FBSDE (29).*

**Remark 5.2** *If in additional  $(H_1) - (i')$  holds (see Remark 5.1), by repeating the above discussion, one can show that if  $2(\lambda_1 + \lambda_2) < -2\hat{k}_1 - k_6^2 - k_7^2 - k_8^2$ , there exists a  $\delta_1 > 0$ , which depends on  $\hat{k}_1, k_i, \lambda_1, \lambda_2$ , for  $i = 4, 5, 6, 7, 8, 11, 12$  and is independent of  $T$ , such that when  $k_2, k_3, k_9, k_{10} \in [0, \delta_1)$ , there exists a unique adapted solution  $(X, Y, Z)$  to MF-FBSDE (29).*

Now let us apply Theorem 5.1 to obtain the well-posedness of consistency condition system (15)-(16) which is

$$\left\{ \begin{array}{l} d\alpha_k = \left( A_k \alpha_k + B \mathbf{P}_{\Gamma_k} [R_k^{-1} (B' \beta_k + D'_k \gamma_k)] + F_1 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) + b \right) dt \\ + \left( C \alpha_k + D_k \mathbf{P}_{\Gamma_k} [R_k^{-1} (B' \beta_k + D'_k \gamma_k)] + F_2 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) + H \alpha_0 + \sigma \right) dW_k(t), \\ d\beta_k = -(A'_k \beta_k - Q(\alpha_k - \rho \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) - (1 - \rho) \alpha_0) + C' \gamma_k) dt + \gamma_k dW_k(t) + \gamma_{k,0} dW_0(t), \\ \alpha_k(0) = x, \quad \beta_k(T) = -G(\alpha_k(T) - \rho \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(T) | \mathcal{F}_T^{W_0}) - (1 - \rho) \alpha_0(T)), \end{array} \right.$$

where  $\alpha_0$  satisfies the following FBSDE which is coupled with all  $k$ -type minor agents:

$$\left\{ \begin{array}{l} d\alpha_0 = \left( A_0 \alpha_0 + B_0 \mathbf{P}_{\Gamma_0} [R_0^{-1} (B'_0 \beta_0 + D'_0 \gamma_0)] + F_0^1 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) + b_0 \right) dt \\ + \left( C_0 \alpha_0 + D_0 \mathbf{P}_{\Gamma_0} [R_0^{-1} (B'_0 \beta_0 + D'_0 \gamma_0)] + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) + \sigma_0 \right) dW_0(t), \\ d\beta_0 = -(A'_0 \beta_0 - Q_0(\alpha_0 - \rho_0 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0})) + C'_0 \gamma_0) dt + \gamma_0 dW_0(t), \\ \alpha_0(0) = x_0, \quad \beta_0(T) = -G_0(\alpha_0(T) - \rho_0 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(T) | \mathcal{F}_T^{W_0})). \end{array} \right.$$

Recall that

$$\varphi_0(p, q) = \mathbf{P}_{\Gamma_0} [R_0^{-1}(B'_0 p + D'_0 q)], \quad \varphi_k(p, q) = \mathbf{P}_{\Gamma_k} [R_k^{-1}(B'_k p + D'_k q)].$$

If we denote

$$\begin{aligned} W &= (W_0, W_1, \dots, W_K)', \quad \Pi = (0, \pi_1, \dots, \pi_K), \quad \alpha = (\alpha'_0, \alpha'_1, \dots, \alpha'_K)', \quad \beta = (\beta'_0, \beta'_1, \dots, \beta'_K)', \\ \mathbb{X} &= (x'_0, x', \dots, x')', \quad \mathbb{E}(\alpha(t) | \mathcal{F}_t^{W_0}) = (\mathbb{E}(\alpha_0(t) | \mathcal{F}_t^{W_0})', \mathbb{E}(\alpha_1(t) | \mathcal{F}_t^{W_0})', \dots, \mathbb{E}(\alpha_K(t) | \mathcal{F}_t^{W_0})')', \\ \Phi(\beta, \gamma) &= (\varphi_0(\beta_0, \gamma_0), \varphi_1(\beta_1, \gamma_1), \dots, \varphi_K(\beta_K, \gamma_K))', \quad \rho_0^\Pi := \Pi \otimes \rho I_{n \times n}, \quad \rho^\Pi := \Pi \otimes \rho I_{n \times n}, \\ F_0^{1, \Pi} &= \Pi \otimes F_0^1, \quad F_0^{2, \Pi} := \Pi \otimes F_0^2, \quad F^{1, \Pi} := \Pi \otimes F_1, \quad F^{2, \Pi} := \Pi \otimes F_2, \end{aligned}$$

$$\gamma = \begin{pmatrix} \gamma_0 & 0 & 0 & \dots & 0 \\ \gamma_{1,0} & \gamma_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots \\ \gamma_{K,0} & 0 & 0 & \dots & \gamma_K \end{pmatrix}, \quad \mathbb{B}_0 = \begin{pmatrix} b_0 \\ b \\ \vdots \\ b \end{pmatrix}, \quad \mathbb{D}_0 = \begin{pmatrix} \sigma_0 & 0 & 0 & \dots \\ 0 & \sigma & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma \end{pmatrix},$$

$$\mathbb{A} = \begin{pmatrix} A_0 & 0 & 0 & \dots \\ 0 & A_1 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_K \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} B_0 & 0 & 0 & \dots \\ 0 & B & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B \end{pmatrix}, \quad \mathbb{R}^{-1} = \begin{pmatrix} R_0 & 0 & 0 & \dots \\ 0 & R_1 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_K \end{pmatrix},$$

$$\mathbb{Q} = \begin{pmatrix} Q_0 & 0 & 0 & \dots \\ Q(1-\rho) & Q & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ Q(1-\rho) & 0 & \dots & Q \end{pmatrix}, \quad \mathbb{G} = \begin{pmatrix} G_0 & 0 & 0 & \dots \\ G(1-\rho) & G & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ G(1-\rho) & 0 & \dots & G \end{pmatrix}, \quad \mathbb{F}_1^\Pi = \begin{pmatrix} F_0^{1, \Pi} \\ F^{1, \Pi} \\ \vdots \\ F^{1, \Pi} \end{pmatrix}, \quad \mathbb{F}_2^\Pi = \begin{pmatrix} F_0^{2, \Pi} \\ F^{2, \Pi} \\ \vdots \\ F^{2, \Pi} \end{pmatrix},$$

$$\mathbb{Q}^\Pi = \begin{pmatrix} Q_0 \rho_0^\Pi \\ Q \rho^\Pi \\ \vdots \\ Q \rho^\Pi \end{pmatrix}, \quad \mathbb{G}^\Pi = \begin{pmatrix} G_0 \rho_0^\Pi \\ G \rho^\Pi \\ \vdots \\ G \rho^\Pi \end{pmatrix}, \quad \mathbb{H} = \begin{pmatrix} 0 \\ H \\ \vdots \\ H \end{pmatrix}, \quad \mathbb{H}(\alpha) = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & H\alpha_0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H\alpha_0 \end{pmatrix},$$

$$\mathbb{F}_2^\Pi(\mathbb{E}(\alpha(t) | \mathcal{F}_t^{W_0})) = \begin{pmatrix} F_0^{2, \Pi} \mathbb{E}(\alpha(t) | \mathcal{F}_t^{W_0}) & 0 & 0 & \dots \\ 0 & F^{2, \Pi} \mathbb{E}(\alpha(t) | \mathcal{F}_t^{W_0}) & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F^{2, \Pi} \mathbb{E}(\alpha(t) | \mathcal{F}_t^{W_0}) \end{pmatrix},$$

$$\mathbb{C} = \begin{pmatrix} C_0 & 0 & 0 & \dots \\ 0 & C & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C \end{pmatrix}, \quad \mathbb{C}(\alpha) = \begin{pmatrix} C_0 \alpha_0 & 0 & 0 & \dots \\ 0 & C\alpha & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C\alpha \end{pmatrix}, \quad \mathbb{C}(\gamma) = \begin{pmatrix} C'_0 \gamma_0 \\ C' \gamma_1 \\ \vdots \\ C' \gamma_K \end{pmatrix},$$

$$\mathbb{D} = \begin{pmatrix} D_0 & 0 & 0 & \dots \\ 0 & D_1 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_K \end{pmatrix}, \quad \mathbb{D}(\beta, \gamma) = \begin{pmatrix} D_0 \varphi_0(\beta_0, \gamma_0) & 0 & 0 & \dots \\ 0 & D_1 \varphi_1(\beta_1, \gamma_1) & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_K \varphi_K(\beta_K, \gamma_K) \end{pmatrix}.$$

Using above notations, the system (15)-(16) can be written as

$$\begin{cases} d\alpha = \left( \mathbb{A}\alpha + \mathbb{B}\Phi(\beta, \gamma) + \mathbb{F}_1^\Pi \mathbb{E}(\alpha(t) | \mathcal{F}_t^{W_0}) + \mathbb{B}_0 \right) dt \\ \quad + \left( \mathbb{C}(\alpha) + \mathbb{D}(\beta, \gamma) + \mathbb{F}_2^\Pi \left( \mathbb{E}(\alpha(t) | \mathcal{F}_t^{W_0}) \right) + \mathbb{H}(\alpha) + \mathbb{D}_0 \right) dW(t), \\ d\beta = - \left( \mathbb{A}'\beta - \mathbb{Q}\alpha + \mathbb{Q}^\Pi \mathbb{E}(\alpha(t) | \mathcal{F}_t^{W_0}) + \mathbb{C}(\gamma) \right) dt + \gamma dW(t), \\ \alpha(0) = \mathbb{X}, \quad \beta(T) = -\mathbb{G}\alpha(T) + \mathbb{G}^\Pi \mathbb{E}(\alpha(T) | \mathcal{F}_T^{W_0}). \end{cases} \quad (30)$$

Now let  $\lambda^*$  be the largest eigenvalue of the symmetric matrix  $\frac{1}{2}(\mathbb{A} + \mathbb{A}')$ . Recalling that the projection operator is Lipschitz continuous with Lipschitz constant 1, then by comparing (30) with (29), one can check that the coefficients of Assumption  $(H_1)$  can be chosen as following

$$\begin{aligned} \lambda_1 = \lambda_2 = \lambda^*, \quad k_0 = \|\mathbb{A}\|, \quad k_1 = \|\mathbb{F}_1^\Pi\|, \quad k_2 = k_3 = \|\mathbb{R}^{-1}\| \|\mathbb{B}\| (\|\mathbb{B}\| + \|\mathbb{D}\|), \\ k_4 = \|\mathbb{Q}\|, \quad k_5 = \|\mathbb{Q}^\Pi\|, \quad k_6 = \|\mathbb{C}\|, \quad k_7^2 = 4(\|\mathbb{C}\| + \|\mathbb{H}\|)^2, \quad k_8^2 = 4\|\mathbb{F}_2^\Pi\|^2, \\ k_9 = k_{10} = \|\mathbb{R}^{-1}\| \|\mathbb{D}\| (\|\mathbb{B}\| + \|\mathbb{D}\|), \quad k_{11}^2 = 2\|\mathbb{G}\|^2, \quad k_{12}^2 = 2\|\mathbb{G}^\Pi\|^2. \end{aligned}$$

Thus by applying Theorem 5.1, we obtain the following global wellposedness of (30).

**Theorem 5.2** *Suppose that*

$$4\lambda^* < -2\|\mathbb{F}_1^\Pi\| - \|\mathbb{C}\|^2 - 4(\|\mathbb{C}\| + \|\mathbb{H}\|)^2 - 4\|\mathbb{F}_2^\Pi\|^2,$$

*then there exists a  $\delta_1 > 0$ , which depends on  $\lambda^*$ ,  $\|\mathbb{F}_1^\Pi\|$ ,  $\|\mathbb{Q}\|$ ,  $\|\mathbb{Q}^\Pi\|$ ,  $\|\mathbb{C}\|$ ,  $\|\mathbb{H}\|$ ,  $\|\mathbb{F}_2^\Pi\|$ ,  $\|\mathbb{G}\|$ ,  $\|\mathbb{G}^\Pi\|$ , and is independent of  $T$ , such that when  $\|\mathbb{R}^{-1}\|, \|\mathbb{B}\|, \|\mathbb{D}\| \in [0, \delta_1)$ , there exists a unique adapted solution  $(\alpha, \beta, \gamma)$  to consistency condition system (15)-(16).*

**Remark 5.3** *Let  $\lambda_{\mathbb{F}_1^\Pi}^*$  be the largest eigenvalue of  $\frac{1}{2}(\mathbb{F}_1^\Pi + (\mathbb{F}_1^\Pi)')$ . Noticing Remark 5.1, one can check that  $(H_1) - (i')$  holds with  $\hat{k}_1 = \lambda_{\mathbb{F}_1^\Pi}^*$ . Thus, from Remark 5.2, we have that if*

$$4\lambda^* < -2\lambda_{\mathbb{F}_1^\Pi}^* - \|\mathbb{C}\|^2 - 4(\|\mathbb{C}\| + \|\mathbb{H}\|)^2 - 4\|\mathbb{F}_2^\Pi\|^2,$$

*then there exists a  $\delta_1 > 0$ , which depends on  $\lambda^*$ ,  $\lambda_{\mathbb{F}_1^\Pi}^*$ ,  $\|\mathbb{Q}\|$ ,  $\|\mathbb{Q}^\Pi\|$ ,  $\|\mathbb{C}\|$ ,  $\|\mathbb{H}\|$ ,  $\|\mathbb{F}_2^\Pi\|$ ,  $\|\mathbb{G}\|$ ,  $\|\mathbb{G}^\Pi\|$ , and is independent of  $T$ , such that when  $\|\mathbb{R}^{-1}\|, \|\mathbb{B}\|, \|\mathbb{D}\| \in [0, \delta_1)$ , there exists a unique adapted solution  $(\alpha, \beta, \gamma)$  to consistency condition system (15)-(16).*

## 6 $\varepsilon$ -Nash Equilibrium for Problem (CC)

In Section 2, we characterized the decentralized strategies  $\{\bar{u}_i^i, 0 \leq i \leq N\}$  of **Problem (CC)** through the auxiliary **Problem (LCC)** and consistency condition system. Now, we turn to verify the  $\varepsilon$ -Nash equilibrium of these decentralized strategies. Here, we proceed our verification based on the assumptions of local time horizon case (Section 4). Note that it can also be verified based on global horizon case (Section 5) without essential difficulties. For major agent  $\mathcal{A}_0$  and minor agent  $\mathcal{A}_i$ , the decentralized states  $\check{x}_t^0$  and  $\check{x}_t^i$  are given respectively by

$$\begin{cases} d\check{x}_0 = \left[ A_0\check{x}_0 + B_0\varphi_0(\bar{p}_0, \bar{q}_0) + F_0^1\check{x}^{(N)} + b_0 \right] dt + \left[ C_0\check{x}_0 + D_0\varphi_0(\bar{p}_0, \bar{q}_0) + F_0^2\check{x}^{(N)} + \sigma_0 \right] dW_0(t), \\ d\check{x}_i = \left[ A_{\theta_i}\check{x}_i + B\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_1\check{x}^{(N)} + b \right] dt + \left[ C\check{x}_i + D_{\theta_i}\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_2\check{x}^{(N)} + H\check{x}_0 + \sigma \right] dW_i(t), \\ \check{x}_0(0) = x_0, \quad \check{x}_i(0) = x, \end{cases} \quad (31)$$

where  $\check{x}^{(N)} = \frac{1}{N} \sum_{i=1}^N \check{x}^i$  and the processes  $(\bar{p}_0, \bar{q}_0, \bar{p}_i, \bar{q}_i)$  are solved by

$$\left\{ \begin{aligned} d\bar{x}_0 &= \left( A_0 \bar{x}_0 + B_0 \varphi_0(\bar{p}_0, \bar{q}_0) + F_0^1 \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0}) + b_0 \right) dt \\ &\quad + \left( C_0 \bar{x}_0 + D_0 \varphi_0(\bar{p}_0, \bar{q}_0) + F_0^2 \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0}) + \sigma_0 \right) dW_0(t), \\ d\bar{p}_0 &= -(A'_0 \bar{p}_0 - Q_0(\bar{x}_0 - \rho_0 \sum_{i=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0})) + C'_0 \bar{q}_0) dt + \bar{q}_0 dW_0(t), \\ d\bar{x}_i &= \left( A_{\theta_i} \bar{x}_i + B_{\theta_i} \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_1 \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0}) + b \right) dt \\ &\quad + \left( C_{\theta_i} \bar{x}_i + D_{\theta_i} \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_2 \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0}) + H \bar{x}_0 + \sigma \right) dW_i(t), \\ d\bar{p}_i &= -(A'_{\theta_i} \bar{p}_i - Q(\bar{x}_i - \rho \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0})) - (1 - \rho) \bar{x}_0) + C'_i \bar{q}_i) dt + \bar{q}_i dW_i(t) + \bar{q}_{i,0} dW_0(t), \\ \bar{x}_0(0) &= x_0, \quad \bar{p}_0(T) = -G_0(\bar{x}_0(T) - \rho_0 \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(T) | \mathcal{F}_T^{W_0})), \\ \bar{x}_i(0) &= x, \quad \bar{p}_i(T) = -G(\bar{x}_i(T) - \rho \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(T) | \mathcal{F}_T^{W_0}) - (1 - \rho) \bar{x}_0(T)). \end{aligned} \right. \quad (32)$$

Here we recall that

$$\varphi_0(p, q) := \mathbf{P}_{\Gamma_0} [R_0^{-1}(B'_0 p + D'_0 q)], \quad \varphi_{\theta_i}(p, q) := \mathbf{P}_{\Gamma_{\theta_i}} [R_{\theta_i}^{-1}(B'_{\theta_i} p + D'_{\theta_i} q)],$$

and  $\alpha_k$ ,  $1 \leq k \leq K$  are given by (15) and (16). We mention that (32) gives also the dynamics of the limiting state  $(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_N)$  and one can check easily that  $(\bar{x}_0, \bar{p}_0, \bar{q}_0) = (\alpha_0, \beta_0, \gamma_0)$ . Now, we would like to show that for  $\bar{u}_0 = \varphi_0(\bar{p}_0, \bar{q}_0)$  and  $\bar{u}_i = \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i)$ ,  $1 \leq i \leq N$ ,  $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$  is an  $\varepsilon$ -Nash equilibrium of **Problem (CC)**. Let us first present following several lemmas.

**Lemma 6.1** *Under (A1)-(A4), there exists a constant  $M$  independent of  $N$ , such that*

$$\sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |\check{x}^i(t)|^2 \leq M.$$

**Proof** From Theorem 4.1, we know that on a small time interval the system of fully coupled FBSDE (15)-(16) has a unique solution (for the global case, see Theorem 5.2 and Remark 5.3)

$$(\alpha_0, \beta_0, \gamma_0) \in L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^{n \times 3}) \text{ and } (\alpha_k, \beta_k, \gamma_k, \gamma_{k,0}) \in L^2_{\mathcal{F}^k}(0, T; \mathbb{R}^{n \times 4}), \quad 1 \leq k \leq K.$$

Then, the classical results on FBSDEs yields that (32) also has a unique solution

$$(\bar{x}_0, \bar{p}_0, \bar{q}_0) \in L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^{n \times 3}) \text{ and } (\bar{x}_i, \bar{p}_i, \bar{q}_i, \bar{q}_{i,0}) \in L^2_{\mathcal{F}^i}(0, T; \mathbb{R}^{n \times 4}), \quad 1 \leq i \leq N.$$

(Indeed, FBSDEs (32) has a unique solution for arbitrary  $T$  by using the Theorem 2.1 of [21] and the results of [22, 40]). Thus, SDEs system (31) has also a unique solution

$$(\check{x}_0, \check{x}_1, \dots, \check{x}_N) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^n).$$

Moreover, since  $\{W_i\}_{i=1}^N$  is  $N$ -dimensional Brownian motion whose components are independent identically distributed, we have that for the  $k$ -type minor agents, under the conditional expectation  $\mathbb{E}(\cdot|\mathcal{F}^{W_0})$ , for each  $1 \leq k \leq K$ , the processes  $(\bar{x}_i, \bar{p}_i, \bar{q}_i)$ ,  $i \in \mathcal{I}_k$ , are independent identically distributed. We also note that for each  $1 \leq k \leq K$ ,  $\check{x}_i$ ,  $i \in \mathcal{I}_k$ , are identically distributed. Noticing that  $(\bar{p}_0, \bar{q}_0) \in L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^n)$ , and  $(\bar{p}_i, \bar{q}_i) \in L^2_{\mathcal{F}^i}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^i}(0, T; \mathbb{R}^n)$ ,  $1 \leq i \leq N$ , then the Lipschitz property of the projection onto the convex set yields that  $\varphi_{\theta_0}(\bar{p}_0, \bar{q}_0) := \varphi_0(\bar{p}_0, \bar{q}_0) = \mathbf{P}_{\Gamma_0} \left( R_0^{-1} (B^T \bar{p}_0 + D^T \bar{q}_0) \right) \in L^2_{\mathcal{F}^{W_0}}(0, T; \Gamma_0)$  and  $\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) := \mathbf{P}_{\Gamma_{\theta_i}} \left( R_{\theta_i}^{-1} (B^T \bar{p}_i + D^T \bar{q}_i) \right) \in L^2_{\mathcal{F}^i}(0, T; \Gamma_{\theta_i})$ ,  $1 \leq i \leq N$ . Moreover, there exists a constant  $M$  independent of  $N$  which may vary line by line in the following, such that for all  $0 \leq i \leq N$ ,  $0 \leq k \leq K$ ,

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} (|\alpha_k(t)|^2 + |\beta_k(t)|^2 + |\bar{x}_i(t)|^2 + |\bar{p}_i(t)|^2) \\ & + \mathbb{E} \int_0^T (|\gamma_k(t)|^2 + |\bar{q}_i(t)|^2 + |\varphi_{\theta_i}(\bar{p}_i(t), \bar{q}_i(t))|^2) \leq M. \end{aligned} \quad (33)$$

From (31), by using Burkholder-Davis-Gundy (BDG) inequality, it follows that for any  $t \in [0, T]$ ,

$$\mathbb{E} \sup_{0 \leq s \leq t} |\check{x}_0(s)|^2 \leq M + M \mathbb{E} \int_0^t [|\check{x}_0(s)|^2 + |\check{x}^{(N)}(s)|^2] ds \leq M + M \mathbb{E} \int_0^t \left[ |\check{x}_0(s)|^2 + \frac{1}{N} \sum_{i=1}^N |\check{x}_i(s)|^2 \right] ds,$$

and by Gronwall's inequality, we obtain

$$\mathbb{E} \sup_{0 \leq s \leq t} |\check{x}_0(s)|^2 \leq M + M \mathbb{E} \int_0^t \frac{1}{N} \sum_{i=1}^N |\check{x}_i(s)|^2 ds, \quad (34)$$

Similarly, from (31) again, we have

$$\mathbb{E} \sup_{0 \leq s \leq t} |\check{x}_i(s)|^2 \leq M + M \mathbb{E} \int_0^t \left[ |\check{x}_0(s)|^2 + |\check{x}_i(s)|^2 + \frac{1}{N} \sum_{i=1}^N |\check{x}_i(s)|^2 \right] ds.$$

Then using (34), we get

$$\mathbb{E} \sup_{0 \leq s \leq t} |\check{x}_i(s)|^2 \leq M + M \mathbb{E} \int_0^t \left[ |\check{x}_i(s)|^2 + \frac{1}{N} \sum_{i=1}^N |\check{x}_i(s)|^2 \right] ds. \quad (35)$$

Thus

$$\mathbb{E} \sup_{0 \leq s \leq t} \sum_{i=1}^N |\check{x}_i(s)|^2 \leq \mathbb{E} \sum_{i=1}^N \sup_{0 \leq s \leq t} |\check{x}_i(s)|^2 \leq MN + 2M \mathbb{E} \int_0^t \left[ \sum_{i=1}^N |\check{x}_i(s)|^2 \right] ds.$$

By Gronwall's inequality, it is easy to obtain  $\mathbb{E} \sup_{0 \leq t \leq T} \sum_{i=1}^N |\check{x}_i(t)|^2 = O(N)$ , for any  $1 \leq i \leq N$ . Then, substituting this estimate to (35) and using Gronwall's inequality once again, we have  $\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_i(t)|^2 \leq M$ , for any  $1 \leq i \leq N$ . By applying this estimate to (34), we get  $\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_0(t)|^2 \leq M$ . We complete the proof.  $\square$

Now, we recall that

$$\check{x}^{(N)} = \frac{1}{N} \sum_{k=1}^K \sum_{i \in \mathcal{I}_k} \check{x}_i = \sum_{k=1}^K \pi_k^{(N)} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \check{x}_i, \quad \text{and} \quad \Phi(t) = \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0}),$$

then we have

**Lemma 6.2** Under (A1)-(A4), there exists a constant  $M$  independent of  $N$  such that

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \check{x}^{(N)}(t) - \Phi(t) \right|^2 \leq M \left( \frac{1}{N} + \varepsilon_N^2 \right), \text{ where } \varepsilon_N = \sup_{1 \leq k \leq K} |\pi_k^{(N)} - \pi_k|.$$

**Proof** For each fixed  $1 \leq k \leq K$ , we consider the  $k$ -type minor agents. We denote  $\check{x}^{(k)} := \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \check{x}_i$ . Let us add up  $N_k$  states of all  $k$ -type minor agents and then divide by  $N_k$ , we have

$$\begin{aligned} d\check{x}^{(k)} &= \left[ A_k \check{x}^{(k)} + \frac{B}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i, \bar{q}_i) + F_1 \check{x}^{(N)} + b_0 \right] dt \\ &\quad + \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \left[ C \check{x}_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 \check{x}^{(N)} + H \check{x}_0 + \sigma_0 \right] dW_i(t), \quad \check{x}^{(k)}(0) = x. \end{aligned} \quad (36)$$

Recall  $m_k = \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0})$ , and taking conditional expectation  $\mathbb{E}(\cdot | \mathcal{F}_t^{W_0})$  on the first equation of (15) and noticing that  $\mathbb{E}(\varphi_k(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) = \mathbb{E}(\varphi_k(\beta_k, \gamma_k) | \mathcal{F}_t^{W_0})$ , for any  $i \in \mathcal{I}_k$  and  $\alpha_k(s)$  is  $\mathcal{F}_s^k$ -adapted, then we have

$$dm_k = \left( A_k m_k + B \mathbb{E}(\varphi_k(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) + F_1 \Phi + b \right) dt, \quad m_k(0) = x. \quad (37)$$

From (36) and (37), by denoting  $\Delta_k(t) := \check{x}^{(k)}(t) - m_k(t)$ , we have

$$\begin{aligned} d\Delta_k &= \left[ A_k \Delta_k + F_1 (\check{x}^{(N)} - \Phi) + \frac{B}{N_k} \left( \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i, \bar{q}_i) - \mathbb{E}(\varphi_k(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) \right) \right] dt \\ &\quad + \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \left[ C \check{x}_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 \check{x}^{(N)} + H \check{x}_0 + \sigma_0 \right] dW_i(t), \quad \Delta(0) = 0. \end{aligned}$$

The inequality  $(x + y)^2 \leq 2x^2 + 2y^2$  and Cauchy-Schwartz inequality yield that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |\Delta_k(s)|^2 &\leq M \mathbb{E} \int_0^t \left[ |\Delta_k(s)|^2 + |\check{x}^{(N)}(s) - \Phi(s)|^2 \right] ds \\ &\quad + M \mathbb{E} \int_0^t \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0}) \right|^2 ds \\ &\quad + \frac{2}{N_k^2} \mathbb{E} \sup_{0 \leq r \leq t} \left| \int_0^r \sum_{i \in \mathcal{I}_k} \left[ C \check{x}_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 \check{x}^{(N)} + H \check{x}_0 + \sigma_0 \right] dW_i(s) \right|^2. \end{aligned}$$

From BDG inequality, we obtain

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |\Delta_k(s)|^2 &\leq M \mathbb{E} \int_0^t \left[ |\Delta_k(s)|^2 + |\check{x}^{(N)}(s) - \Phi(s)|^2 \right] ds \\ &\quad + M \mathbb{E} \int_0^t \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0}) \right|^2 ds \\ &\quad + \frac{M}{N_k^2} \mathbb{E} \sum_{i \in \mathcal{I}_k} \int_0^t \left| C \check{x}_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 (\check{x}^{(N)} - \Phi) + F_2 \Phi + H \check{x}_0 + \sigma_0 \right|^2 ds. \end{aligned} \quad (38)$$

Let us first focus on the second term of the right hand-side of (38). Since for each fixed  $s \in [0, T]$ , under the conditional expectation  $\mathbb{E}(\cdot | \mathcal{F}_s^{W_0})$ , for each  $1 \leq k \leq K$ , the processes  $(\bar{x}_i(s), \bar{p}_i(s), \bar{q}_i(s))$ ,  $i \in \mathcal{I}_k$ , are independent identically distributed, if we denote  $\mu(s) = \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0})$ , then  $\mu$  does not depend on  $i$  and moreover we have

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s) \right|^2 = \frac{1}{N_k^2} \mathbb{E} \left| \sum_{i \in \mathcal{I}_k} [\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s)] \right|^2 \\ &= \frac{1}{N_k^2} \mathbb{E} \left( \sum_{i \in \mathcal{I}_k} |\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s)|^2 + \sum_{i, j \in \mathcal{I}_k, j \neq i} \langle \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s), \varphi_k(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \rangle \right). \end{aligned}$$

Since  $(\bar{p}_i(s), \bar{q}_i(s))$ ,  $i \in \mathcal{I}_k$ , are independent under  $\mathbb{E}(\cdot | \mathcal{F}_s^{W_0})$ , we have

$$\begin{aligned} & \mathbb{E} \sum_{i, j \in \mathcal{I}_k, j \neq i} \langle \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s), \varphi_k(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \rangle \\ &= \mathbb{E} \left[ \sum_{i, j \in \mathcal{I}_k, j \neq i} \mathbb{E} \left( \langle \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s), \varphi_k(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \rangle \middle| \mathcal{F}_s^{W_0} \right) \right] \\ &= \mathbb{E} \left[ \sum_{i, j \in \mathcal{I}_k, j \neq i} \left\langle \mathbb{E} \left( \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s) \middle| \mathcal{F}_s^{W_0} \right), \mathbb{E} \left( \varphi_k(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \middle| \mathcal{F}_s^{W_0} \right) \right\rangle \right] = 0. \end{aligned}$$

Then, due to the fact that  $(\bar{p}_i, \bar{q}_i)$ ,  $1 \leq i \leq N$  are identically distributed, we obtain

$$\begin{aligned} & \mathbb{E} \int_0^t \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0}) \right|^2 ds \\ &= \frac{1}{N_k^2} \int_0^t \mathbb{E} \sum_{i \in \mathcal{I}_k} |\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s)|^2 ds = \frac{1}{N_k} \int_0^t \mathbb{E} |\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s)|^2 ds \leq \frac{M}{N_k}, \end{aligned}$$

where the last equality due to (33). Now we focus on the third term of the right hand-side of (38), using (33), Lemma 6.1 and that  $(\check{x}_i(s), \bar{p}_i(s), \bar{q}_i(s))$ ,  $i \in \mathcal{I}_k$ , are identically distributed, it follows

$$\begin{aligned} & \frac{M}{N_k^2} \sum_{i \in \mathcal{I}_k} \mathbb{E} \int_0^t \left| C\check{x}_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2(\check{x}^{(N)} - \Phi) + F_2\Phi + H\check{x}_0 + \sigma \right|^2 ds \\ &\leq \frac{M}{N_k^2} \sum_{i \in \mathcal{I}_k} \mathbb{E} \int_0^t \left( |\check{x}_i(s)|^2 + |\varphi_k(\bar{p}_i, \bar{q}_i)|^2 + |\check{x}^{(N)}(s) - \Phi(s)|^2 + |\Phi(s)|^2 + |\check{x}_0(s)|^2 + |\sigma(s)|^2 \right) ds \\ &\leq \frac{M}{N_k} \mathbb{E} \int_0^t \mathbb{E} |\check{x}^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N_k}. \end{aligned}$$

Therefore, from above analysis, we get from (38) that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\Delta_k(s)|^2 \leq M \mathbb{E} \int_0^t \left[ |\Delta_k(s)|^2 + |\check{x}^{(N)}(s) - \Phi(s)|^2 \right] ds + \frac{M}{N_k},$$

and Gronwall's inequality yields that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\Delta_k(s)|^2 \leq M \mathbb{E} \int_0^t |\check{x}^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N_k}. \quad (39)$$



Since

$$\begin{aligned}
\check{x}^{(N)}(s) - \Phi(s) &= \sum_{k=1}^K \left[ \pi_k^{(N)} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \check{x}_i(s) - \pi_k \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right] \\
&= \sum_{k=1}^K \left[ \pi_k^{(N)} \left( \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \check{x}_i(s) - \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right) + (\pi_k^{(N)} - \pi_k) \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right] \\
&= \sum_{k=1}^K \pi_k^{(N)} \Delta_k(s) + \sum_{k=1}^K (\pi_k^{(N)} - \pi_k) \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}),
\end{aligned}$$

by using (33),(39) and  $\pi_k^{(N)} = \frac{N_k}{N} \leq 1$  we obtain that for any  $t \in [0, T]$ ,

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} |\check{x}^{(N)}(s) - \Phi(s)|^2 &\leq \mathbb{E} \sum_{k=1}^K \sup_{0 \leq s \leq t} \pi_k^{(N)} |\Delta_k(s)|^2 + M \varepsilon_N^2 \\
&\leq M \mathbb{E} \int_0^t |\check{x}^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N} + M \varepsilon_N^2.
\end{aligned}$$

Finally, by using Gronwall's inequality, we complete the proof.  $\square$

**Lemma 6.3** *Under the assumptions of (A1)-(A4), we have*

$$\sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_i(t) - \bar{x}_i(t)|^2 \leq M \left( \frac{1}{N} + \varepsilon_N^2 \right).$$

**Proof** On the one hand, from both the first equation of (31) and (32), we have

$$\begin{cases} d(\check{x}_0 - \bar{x}_0) = \left[ A_0(\check{x}_0 - \bar{x}_0) + F_0^1(\check{x}^{(N)} - \Phi) \right] dt + \left[ C_0(\check{x}_0 - \bar{x}_0) + F_0^2(\check{x}^{(N)} - \Phi) \right] dW_i(t), \\ \check{x}_0(0) - \bar{x}_0(0) = 0. \end{cases}$$

The classical estimate for the SDE yields that

$$\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_0(t) - \bar{x}_0(t)|^2 \leq M \mathbb{E} \int_0^T |\check{x}^{(N)}(s) - \Phi(s)|^2 ds,$$

where  $M$  is a constant independent of  $N$ . Noticing Lemma 6.2, we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_0(t) - \bar{x}_0(t)|^2 \leq M \left( \frac{1}{N} + \varepsilon_N^2 \right). \quad (40)$$

On the other hand, from the second equation of (31) and the third equation of (32), we have that for  $1 \leq i \leq N$ ,

$$\begin{cases} d(\check{x}_i - \bar{x}_i) = \left[ A_{\theta_i}(\check{x}_i - \bar{x}_i) + F_1(\check{x}^{(N)} - \Phi) \right] dt + \left[ C(\check{x}_i - \bar{x}_i) + F_2(\check{x}^{(N)} - \Phi) + H(\check{x}_0 - \bar{x}_0) \right] dW_i(t), \\ \check{x}_i(0) - \bar{x}_i(0) = 0. \end{cases}$$

The classical estimate for the SDE yields that

$$\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_i(t) - \bar{x}_i(t)|^2 \leq M \mathbb{E} \int_0^T \left( |\check{x}^{(N)}(s) - \Phi(s)|^2 + |\check{x}_0(s) - \bar{x}_0(s)|^2 \right) ds.$$

Noticing Lemma 6.2 and (40), we obtain  $\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_i(t) - \bar{x}_i(t)|^2 \leq M \left( \frac{1}{N} + \varepsilon_N^2 \right)$ . Thus, considering also (40) we complete the proof.  $\square$

**Lemma 6.4** *Under the assumptions of (A1)-(A4), for all  $0 \leq i \leq N$ , we have*

$$\left| \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - J_i(\bar{u}_i) \right| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).$$

**Proof** Let us first consider the major agent. Recall (3), (6) and (17), we have

$$\begin{aligned} \mathcal{J}_0(\bar{u}_0, \bar{u}_{-0}) - J_0(\bar{u}_0) &= \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \left\langle Q_0(\check{x}_0 - \rho_0 \check{x}^{(N)}), \check{x}_0 - \rho_0 \check{x}^{(N)} \right\rangle - \left\langle Q_0(\bar{x}_0 - \rho_0 \Phi), \bar{x}_0 - \rho_0 \Phi \right\rangle \right) dt \right. \\ &\quad \left. + \left\langle G_0(\check{x}_0(T) - \rho_0 \check{x}^{(N)}(T)), \check{x}_0(T) - \rho_0 \check{x}^{(N)}(T) \right\rangle - \left\langle G_0(\bar{x}_0(T) - \rho_0 \Phi(T)), \bar{x}_0(T) - \rho_0 \Phi(T) \right\rangle \right]. \end{aligned} \quad (41)$$

From (33), we have  $\mathbb{E} \sup_{0 \leq t \leq T} |\bar{x}_0(t)|^2 \leq M$  and  $\mathbb{E} \sup_{0 \leq t \leq T} |\alpha_i(t)|^2 \leq M$ , for any  $0 \leq i \leq N$ . Noticing that  $\Phi(t) = \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0})$ , it is not hard to check that  $\mathbb{E} \sup_{0 \leq t \leq T} |\Phi(t)|^2 \leq M$ . Now, from such estimates and Lemma 6.2, Lemma 6.3 as well as

$$\begin{aligned} &\langle Q_0(a-b), a-b \rangle - \langle Q_0(c-d), c-d \rangle \\ &= \langle Q_0(a-b - (c-d)), a-b - (c-d) \rangle + 2\langle Q_0(a-b - (c-d)), c-d \rangle, \end{aligned}$$

we have

$$\begin{aligned} &\left| \mathbb{E} \left[ \int_0^T \left( \left\langle Q_0(\check{x}_0 - \rho_0 \check{x}^{(N)}), \check{x}_0 - \rho_0 \check{x}^{(N)} \right\rangle - \left\langle Q_0(\bar{x}_0 - \rho_0 \Phi), \bar{x}_0 - \rho_0 \Phi \right\rangle \right) dt \right] \right| \\ &\leq M \int_0^T \mathbb{E} \left| \check{x}_0 - \rho_0 \check{x}^{(N)} - (\bar{x}_0 - \rho_0 \Phi) \right|^2 dt + M \int_0^T \mathbb{E} \left| \check{x}_0 - \rho_0 \check{x}^{(N)} - (\bar{x}_0 - \rho_0 \Phi) \right| |\bar{x}_0 - \rho_0 \Phi| dt \\ &\leq M \int_0^T \mathbb{E} |\check{x}_0 - \bar{x}_0|^2 dt + M \int_0^T \mathbb{E} \left| \check{x}^{(N)} - \Phi \right|^2 dt \\ &\quad + M \int_0^T \left( \mathbb{E} \left| \check{x}_0 - \rho_0 \check{x}^{(N)} - (\bar{x}_0 - \rho_0 \Phi) \right|^2 \right)^{\frac{1}{2}} \left( \mathbb{E} |\bar{x}_0 - \rho_0 \Phi|^2 \right)^{\frac{1}{2}} dt \\ &\leq M \int_0^T \mathbb{E} |\check{x}_0 - \bar{x}_0|^2 dt + M \int_0^T \mathbb{E} \left| \check{x}^{(N)} - \Phi \right|^2 dt + M \int_0^T \left( \mathbb{E} |\check{x}_0 - \bar{x}_0|^2 + \mathbb{E} \left| \check{x}^{(N)} - \Phi \right|^2 \right)^{\frac{1}{2}} dt \\ &= O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right). \end{aligned}$$

Similar argument allows us to show that

$$\begin{aligned} &\left| \mathbb{E} \left[ \left\langle G_0(\check{x}_0(T) - \rho_0 \check{x}^{(N)}(T)), \check{x}_0(T) - \rho_0 \check{x}^{(N)}(T) \right\rangle - \left\langle G_0(\bar{x}_0(T) - \rho_0 \Phi(T)), \bar{x}_0(T) - \rho_0 \Phi(T) \right\rangle \right] \right| \\ &= O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right). \end{aligned}$$

Thus, the proof for the major agent is completed by noticing (41). Let us now focus on the minor agents, for  $1 \leq i \leq N$ , recalling (4), (12) and (17), we have

$$\begin{aligned} \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) &= \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \left\langle Q(\check{x}_i - \rho \check{x}^{(N)} - (1-\rho)\check{x}_0), \check{x}_i - \rho \check{x}^{(N)} - (1-\rho)\check{x}_0 \right\rangle + \langle R_{\theta_i} \bar{u}_i, \bar{u}_i \rangle \right) dt \right. \\ &\quad \left. + \left\langle G(\check{x}_i(T) - \rho \check{x}^{(N)}(T) - (1-\rho)\check{x}_0(T)), \check{x}_i(T) - \rho \check{x}^{(N)}(T) - (1-\rho)\check{x}_0(T) \right\rangle \right] \end{aligned}$$

and

$$J_i(\bar{u}_i) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle Q(\bar{x}_i - \rho\Phi - (1-\rho)\bar{x}_0), \bar{x}_i - \rho\Phi - (1-\rho)\bar{x}_0 \rangle dt + \langle R_{\theta_i} \bar{u}_i, \bar{u}_i \rangle \right) dt \right. \\ \left. + \langle G(\bar{x}_i(T) - \rho\Phi(T) - (1-\rho)\bar{x}_0(T)), \bar{x}_i(T) - \rho\Phi(T) - (1-\rho)\bar{x}_0(T) \rangle \right].$$

From (33), we have  $\mathbb{E} \sup_{0 \leq t \leq T} |\bar{x}_i(t)|^2 \leq M$ . Using such estimate,  $\mathbb{E} \sup_{0 \leq t \leq T} |\Phi(t)|^2 \leq M$ , Lemmas 6.2, 6.3, similar to the major agent, it follows that  $\left| \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - J_i(\bar{u}_i) \right| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right)$ .  $\square$

## 6.1 Major agent's perturbation

In this subsection, we will prove that the control strategies set  $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N)$  is an  $\varepsilon$ -Nash equilibrium of **Problem (CC)** for the major agent, i.e.  $\exists \varepsilon = \varepsilon(N) \geq 0, \lim_{N \rightarrow \infty} \varepsilon(N) = 0$  s.t

$$\mathcal{J}_0(\bar{u}_0(\cdot), \bar{u}_{-0}(\cdot)) \leq \mathcal{J}_0(u_0(\cdot), \bar{u}_{-0}(\cdot)) + \varepsilon, \quad \text{for any } u_0 \in \mathcal{U}_{ad}^0.$$

Let us consider that the major agent  $\mathcal{A}_0$  uses an alternative strategy  $u_0$  and each minor agent  $\mathcal{A}_i$  uses the control  $\bar{u}_i = \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i)$ , where  $(\bar{p}_i, \bar{q}_i)$  are solved from (32). Then the realized state system with major agent's perturbation is, for  $1 \leq i \leq N$ ,

$$\begin{cases} dy_0 = [A_0 y_0 + B_0 u_0 + F_0^1 y^{(N)} + b_0] dt + [C_0 y_0 + D_0 u_0 + F_0^2 y^{(N)} + \sigma_0] dW_0(t), \\ dy_i = [A_{\theta_i} y_i + B \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_1 y^{(N)} + b_0] dt \\ \quad + [C y_i + D_{\theta_i} \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_2 y^{(N)} + H y_0 + \sigma_0] dW_i(t), \\ y_0(0) = x_0, \quad y_i(0) = x, \end{cases} \quad (42)$$

where  $y^{(N)} = \frac{1}{N} \sum_{i=1}^N y_i$ . The well-posedness of above SDEs system is easy to obtain. To prove  $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$  is an  $\varepsilon$ -Nash equilibrium for the major agent, we need to show that for possible alternative control  $u_0$ ,  $\inf_{u_0 \in \mathcal{U}_{ad}^0} \mathcal{J}_0(u_0, \bar{u}_{-0}) \geq \mathcal{J}_0(\bar{u}_0, \bar{u}_{-0}) - \varepsilon$ . Then we only need to consider the perturbation  $u_0 \in \mathcal{U}_{ad}^0$  such that  $\mathcal{J}_0(u_0, \bar{u}_{-0}) \leq \mathcal{J}_0(\bar{u}_0, \bar{u}_{-0})$ . Thus, noticing  $Q_0 \geq 0$  and  $G_0 \geq 0$ , from Lemma 6.4, we have

$$\mathbb{E} \int_0^T \langle R_0 u_0(t), u_0(t) \rangle dt \leq \mathcal{J}_0(u_0, \bar{u}_{-0}) \leq \mathcal{J}_0(\bar{u}_0, \bar{u}_{-0}) \leq J_0(\bar{u}_0) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right),$$

which implies that (noting **(A4)**),  $\mathbb{E} \int_0^T |u_0(t)|^2 dt \leq M$ , where  $M$  is a constant independent of  $N$ . Then similar to Lemma 6.1, we can show that

$$\sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |y_i(t)|^2 \leq M. \quad (43)$$

**Lemma 6.5** *Under the assumptions of **(A1)**-**(A4)**, we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| y^{(N)}(t) - \Phi(t) \right|^2 = O\left(\frac{1}{N} + \varepsilon_N^2\right).$$

**Proof** For each fixed  $1 \leq k \leq K$ , we consider the  $k$ -type minor agents. We denote  $y^{(k)} := \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} y_i$ . As there are  $N_k$  minor agents of the  $k$ -type, let us add up their states and then

divided by  $N_k$ , it follows that

$$dy^{(k)} = \left[ A_k y^{(k)} + \frac{B}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i, \bar{q}_i) + F_1 y^{(N)} + b_0 \right] dt \\ + \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \left[ C y_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 y^{(N)} + H y_0 + \sigma_0 \right] dW_i(t), \quad y^{(k)}(0) = x.$$

Recall (37) and if we denote  $\tilde{\Delta}_k(t) := y^{(k)}(t) - m_k(t)$ , it follows that

$$d\tilde{\Delta}_k = \left[ A_k \tilde{\Delta}_k + \frac{B}{N_k} \left( \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i, \bar{q}_i) - \mathbb{E}(\varphi_k(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) \right) + F_1 (y^{(N)} - \Phi) \right] dt \\ + \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \left[ C y_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 y^{(N)} + H y_0 + \sigma_0 \right] dW_i(t), \quad \tilde{\Delta}(0) = 0.$$

Similar to the argument in the proof of Lemma 6.2, we can show that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{\Delta}_k(s)|^2 \leq M \mathbb{E} \int_0^t \left[ |\tilde{\Delta}_k(s)|^2 + |y^{(N)}(s) - \Phi(s)|^2 \right] ds \\ + M \mathbb{E} \int_0^t \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0}) \right|^2 ds \\ + \frac{M}{N_k^2} \mathbb{E} \sum_{i \in \mathcal{I}_k} \int_0^t \left| C y_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 (y^{(N)} - \Phi) + F_2 \Phi + H y_0 + \sigma_0 \right|^2 ds, \quad (44)$$

and

$$\mathbb{E} \int_0^t \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0}) \right|^2 ds \leq \frac{M}{N_k}.$$

Using (33), (43) and the fact that  $(y_i(s), \bar{p}_i(s), \bar{q}_i(s))$ ,  $i \in \mathcal{I}_k$ , are identically distributed, we have

$$\frac{M}{N_k^2} \sum_{i \in \mathcal{I}_k} \mathbb{E} \int_0^t \left| C y_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 (y^{(N)} - \Phi) + F_2 \Phi + H y_0 + \sigma_0 \right|^2 ds \\ = \frac{M}{N_k} \mathbb{E} \int_0^t \mathbb{E} |y^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N_k}.$$

Therefore, we get from (44) that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{\Delta}_k(s)|^2 \leq M \mathbb{E} \int_0^t \left[ |\tilde{\Delta}_k(s)|^2 + |y^{(N)}(s) - \Phi(s)|^2 \right] ds + \frac{M}{N_k},$$

and Gronwall's inequality yields that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{\Delta}_k(s)|^2 \leq M \mathbb{E} \int_0^t \left[ |y^{(N)}(s) - \Phi(s)|^2 \right] ds.$$

Similar to the proof of Lemma 6.2 again, and using **(A1)**, we have for any  $t \in [0, T]$ ,

$$\mathbb{E} \sup_{0 \leq s \leq t} |y^{(N)}(s) - \Phi(s)|^2 \leq \mathbb{E} \sum_{k=1}^K \sup_{0 \leq s \leq t} \pi_k^{(N)} |\Delta_k(s)|^2 + M \varepsilon_N^2 \\ \leq M \mathbb{E} \int_0^t \left[ |y^{(N)}(s) - \Phi(s)|^2 \right] ds + \frac{M}{N} + M \varepsilon_N^2.$$

Finally, Gronwall's inequality allows us to complete the proof.  $\square$

Now, we introduce the following system of the decentralized limiting state with the major's perturbation control, for  $1 \leq i \leq N$ :

$$\begin{cases} d\bar{y}_0 = [A_0\bar{y}_0 + B_0u_0 + F_0^1\Phi + b_0] dt + [C_0\bar{y}_0 + D_0u_0 + F_0^2\Phi + \sigma_0] dW_0(t) \\ d\bar{y}_i = [A_{\theta_i}\bar{y}_i + B\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_1\Phi + b_0] dt \\ \quad + [C\bar{y}_i + D_{\theta_i}\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_2\Phi + H\bar{y}_0 + \sigma_0] dW_i(t) \\ \bar{y}_0(0) = x_0, \quad \bar{y}_i(0) = x. \end{cases} \quad (45)$$

**Lemma 6.6** *Under the assumptions of (A1)-(A4), we have*

$$\sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |y_i(t) - \bar{y}_i(t)|^2 = O\left(\frac{1}{N} + \varepsilon_N^2\right).$$

**Proof** From both the first equation of (42) and (45), we obtain

$$\begin{cases} d(y_0 - \bar{y}_0) = [A(y_0 - \bar{y}_0) + F_0^1(y^{(N)} - \Phi)] dt + [C(y_0 - \bar{y}_0) + F_0^2(y^{(N)} - \Phi)] dW_0(t), \\ y_0(0) - \bar{y}_0(0) = 0. \end{cases}$$

With the help of classical estimates of SDE and Lemma 6.5, it is easy to obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} |y_0(t) - \bar{y}_0(t)|^2 = O\left(\frac{1}{N} + \varepsilon_N^2\right). \quad (46)$$

Now, for any  $1 \leq i \leq N$ , from both the second equation of (42) and (45), we get

$$\begin{aligned} d(y_i - \bar{y}_i) &= [A_{\theta_i}(y_i - \bar{y}_i) + F_1(y^{(N)} - \Phi)] dt \\ &\quad + [C(y_i - \bar{y}_i) + F_2(y^{(N)} - \Phi) + H(y_0 - \bar{y}_0)] dW_i(t), \quad y_i(0) - \bar{y}_i(0) = 0. \end{aligned}$$

The classical estimates of SDE, Lemma 6.5 and (46) allow us to complete the proof.  $\square$

**Lemma 6.7** *Under (A1)-(A4), for the major agent's perturbation control  $u_0$ , we have*

$$|\mathcal{J}_0(u_0, \bar{u}_{-0}) - J_0(u_0)| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).$$

**Proof** Recall (3), (6) and (17), we have

$$\begin{aligned} \mathcal{J}_0(u_0, \bar{u}_{-0}) - J_0(u_0) &= \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle Q_0(y_0 - \rho_0 y^{(N)}), y_0 - \rho_0 y^{(N)} \rangle - \langle Q_0(\bar{y}_0 - \rho_0 \Phi), \bar{y}_0 - \rho_0 \Phi \rangle \right) dt \right. \\ &\quad \left. + \langle G_0(y_0(T) - \rho_0 y^{(N)}(T)), y_0(T) - \rho_0 y^{(N)}(T) \rangle - \langle G_0(\bar{y}_0(T) - \rho_0 \Phi(T)), \bar{y}_0(T) - \rho_0 \Phi(T) \rangle \right]. \end{aligned} \quad (47)$$

Similar to Lemma 6.4, by using Lemmas 6.5, 6.6 and  $\mathbb{E} \left( |\bar{y}_0(t)|^2 + |\Phi(t)|^2 \right) \leq M$ , we have

$$\begin{aligned} &\left| \mathbb{E} \int_0^T \left( \langle Q_0(y_0 - \rho_0 y^{(N)}), y_0 - \rho_0 y^{(N)} \rangle - \langle Q_0(\bar{y}_0 - \rho_0 \Phi), \bar{y}_0 - \rho_0 \Phi \rangle \right) dt \right| \\ &\leq M \int_0^T \mathbb{E} |y_0 - \bar{y}_0|^2 dt + M \int_0^T \mathbb{E} |y^{(N)} - \Phi|^2 dt + M \int_0^T \left( \mathbb{E} |y_0 - \bar{y}_0|^2 + \mathbb{E} |y^{(N)} - \Phi|^2 \right)^{\frac{1}{2}} dt \\ &= O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right), \end{aligned}$$

and

$$\left| \mathbb{E} \left[ \left\langle G_0(y_0(T) - \rho_0 y^{(N)}(T)), y_0(T) - \rho_0 y^{(N)}(T) \right\rangle - \left\langle G_0(\bar{y}_0(T) - \rho_0 \Phi(T)), \bar{y}_0(T) - \rho_0 \Phi(T) \right\rangle \right] \right| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).$$

The proof is completed by noticing (47).  $\square$

**Theorem 6.1** *Under the assumptions of (A1)-(A4), then the strategies set  $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$  is an  $\varepsilon$ -Nash equilibrium of **Problem (CC)** for the major agent.*

**Proof** Combining Lemma 6.4 and Lemma 6.7, we have

$$\mathcal{J}_0(\bar{u}_0, \bar{u}_{-0}) \leq J_0(\bar{u}_0) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right) \leq J_0(u_0) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right) \leq \mathcal{J}_0(u_0, \bar{u}_{-0}) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right),$$

where the second inequality comes from the fact that  $J_0(\bar{u}_0) = \inf_{u_0 \in \mathcal{U}_{ad}^0} J_0(u_0)$ . Consequently, Theorem 6.1 holds with  $\varepsilon = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right)$ .  $\square$

## 6.2 Minor agent's perturbation

Now, let us consider the following case: a given minor agent  $\mathcal{A}_i$  uses an alternative strategy  $u_i \in \mathcal{U}_{ad}^i$ , the major agent uses  $\bar{u}_0 = \varphi_0(\bar{p}_0, \bar{q}_0)$  while other minor agents  $\mathcal{A}_j$  use the control  $\bar{u}_j = \varphi_{\theta_j}(\bar{p}_j, \bar{q}_j)$ ,  $j \neq i$ ,  $1 \leq j \leq N$ , where  $(\bar{p}_j, \bar{q}_j)$ ,  $0 \leq j \leq N$ ,  $j \neq i$ , are solved from (32). Then the realized state system with the minor agent's perturbation is, for  $1 \leq j \leq N$ ,  $j \neq i$ ,

$$\begin{cases} dl_0 = \left[ A_0 l_0 + B_0 \varphi_0(\bar{p}_0, \bar{q}_0) + F_0^1 l^{(N)} + b_0 \right] dt + \left[ C_0 l_0 + D_0 \varphi_0(\bar{p}_0, \bar{q}_0) + F_0^2 l^{(N)} + \sigma_0 \right] dW_0(t) \\ dl_i = \left[ A_{\theta_i} l_i + B u_i + F_1 l^{(N)} + b_0 \right] dt + \left[ C l_i + D_{\theta_i} u_i + F_2 l^{(N)} + H l_0 + \sigma_0 \right] dW_i(t), \\ dl_j = \left[ A_{\theta_j} l_j + B \varphi_{\theta_j}(\bar{p}_j, \bar{q}_j) + F_1 l^{(N)} + b_0 \right] dt + \left[ C l_j + D_{\theta_j} \varphi_{\theta_j}(\bar{p}_j, \bar{q}_j) + F_2 l^{(N)} + H l_0 + \sigma_0 \right] dW_i(t), \\ l_0(0) = x_0, \quad l_i(0) = l_j(0) = x, \end{cases} \quad (48)$$

where  $l^{(N)} = \frac{1}{N} \sum_{i=1}^N l^i$ . The well-posedness of above SDEs system is easily to obtain. Similar to the argument of major agent, to prove  $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$  is an  $\varepsilon$ -Nash equilibrium for the minor agent, noticing  $Q \geq 0$ ,  $G \geq 0$ ,  $R_{\theta_i} > 0$  and Lemma 6.4, we only need to consider the perturbation  $u_i \in \mathcal{U}_{ad}^i$  satisfying

$$\mathbb{E} \int_0^T |u_i(t)|^2 dt \leq M, \quad (49)$$

where  $M$  is a constant independent of  $N$ . Similar to Lemma 6.1, we can show that

$$\sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |l_i(t)|^2 \leq M. \quad (50)$$

We first present the following lemma

**Lemma 6.8** *Under the assumptions of (A1)-(A4), we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| l^{(N)}(t) - \Phi(t) \right|^2 = O\left(\frac{1}{N} + \varepsilon_N^2\right).$$

**Proof** We know that for each fixed  $i$ , there exists a unique  $1 \leq \bar{k} \leq K$ , such that  $i \in \mathcal{I}_{\bar{k}}$ . Let us denote  $l^{(k)} := \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} l_i$ ,  $1 \leq k \leq K$ . We first consider the  $k$ -type minor agents, where  $k \neq \bar{k}$ . Adding up their states and then divided by  $N_k$ , we have for  $k \neq \bar{k}$ ,

$$dl^{(k)} = \left[ A_k l^{(k)} + \frac{B}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i, \bar{q}_i) + F_1 l^{(N)} + b_0 \right] dt + \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \left[ Cl_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 l^{(N)} + Hl_0 + \sigma_0 \right] dW_i(t), \quad l^{(k)}(0) = x.$$

Similar to the proof of Lemma 6.2, for  $m_k = \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0})$ , we have

$$\mathbb{E} \sup_{0 \leq s \leq t} |l^{(k)}(s) - m_k(s)|^2 \leq M \mathbb{E} \int_0^t \left[ |l^{(N)}(s) - \Phi(s)|^2 \right] ds + \frac{M}{N_k}. \quad (51)$$

Now let us focus on the  $\bar{k}$ -type minor agents, we have

$$dl^{(\bar{k})} = \left[ A_{\bar{k}} l^{(\bar{k})} + \frac{B}{N_{\bar{k}}} u_i + \frac{B}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j, \bar{q}_j) + F_1 l^{(N)} + b_0 \right] dt + \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}} \left[ Cl_j + F_2 l^{(N)} + Hl_0 + \sigma_0 \right] dW_j(t) + \frac{1}{N_{\bar{k}}} D_{\bar{k}} u_i dW_i(t) + \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} D_{\bar{k}} \varphi_{\bar{k}}(\bar{p}_j, \bar{q}_j) dW_j(t), \quad l^{(\bar{k})}(0) = x.$$

Recalling (37) and if we denote  $\Xi := l^{(\bar{k})} - m_{\bar{k}}$ , it follows that

$$d\Xi = \left[ A\Xi + F_1(l^{(N)} - \Phi) + \frac{1}{N_{\bar{k}}} Bu_i + \frac{B}{N_{\bar{k}}} \left( \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j, \bar{q}_j) - \mathbb{E}(\varphi_{\bar{k}}(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) \right) \right] dt + \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}} \left[ Cl_j + F_2 l^{(N)} + Hl_0 + \sigma_0 \right] dW_j(t), + \frac{1}{N_{\bar{k}}} D_{\bar{k}} u_i dW_i(t) + \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} D_{\bar{k}} \varphi_{\bar{k}}(\bar{p}_j, \bar{q}_j) dW_j(t), \quad \Pi(0) = 0.$$

By Cauchy-Schwartz inequality and BDG inequality, we obtain that for any  $t \in [0, T]$ ,

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |\Xi(s)|^2 &\leq M \mathbb{E} \int_0^t \left( |\Xi(s)|^2 + \frac{1}{N_{\bar{k}}^2} |u_i(s)|^2 + |l^{(N)}(s) - \Phi(s)|^2 \right) ds \\ &\quad + M \mathbb{E} \int_0^t \left[ \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j, \bar{q}_j) - \mathbb{E}(\varphi_{\bar{k}}(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) \right]^2 ds \\ &\quad + \frac{M}{N_{\bar{k}}^2} \mathbb{E} \sum_{j \in \mathcal{I}_{\bar{k}}} \int_0^t \left| F_2(l^{(N)}(s) - \Phi(s)) + F_2\Phi(s) + Cl_j(s) + Hl_0(s) + \sigma(s) \right|^2 ds \\ &\quad + \frac{M}{N_{\bar{k}}^2} \mathbb{E} \int_0^t |u_i(s)|^2 ds + \frac{M}{N_{\bar{k}}^2} \mathbb{E} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \int_0^t |\varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s))|^2 ds. \end{aligned} \quad (52)$$

On the one hand, since for each fixed  $s \in [0, T]$ , under the conditional expectation  $\mathbb{E}(\cdot | \mathcal{F}_s^{W_0})$ , the processes  $(\bar{p}_i(s), \bar{q}_i(s))$ ,  $i \in \mathcal{I}_{\bar{k}}$ , are independent identically distributed. If we denote  $\mu(s) = \mathbb{E}(\varphi_{\bar{k}}(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0})$ , then  $\mu$  does not depend on  $i$ . Moreover,

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \right|^2 \\ & \leq 2 \mathbb{E} \left| \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \frac{N_{\bar{k}} - 1}{N_{\bar{k}}} \mu(s) \right|^2 + 2 \mathbb{E} \left| \frac{1}{N_{\bar{k}}} \mu(s) \right|^2 \\ & = 2 \frac{(N_{\bar{k}} - 1)^2}{N_{\bar{k}}^2} \mathbb{E} \left| \frac{1}{N_{\bar{k}} - 1} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \right|^2 + \frac{2}{N_{\bar{k}}^2} \mathbb{E} |\mu(s)|^2. \end{aligned}$$

Then, due to (33) and the fact that  $(\bar{p}_i(s), \bar{q}_i(s))$ ,  $i \in \mathcal{I}_{\bar{k}}$ , are independent identically distributed under  $\mathbb{E}(\cdot | \mathcal{F}_s^{W_0})$ , similar to the proof of Lemma 6.2, we can obtain

$$\begin{aligned} & \int_0^t \mathbb{E} \left| \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \right|^2 ds \\ & \leq 2 \frac{(N_{\bar{k}} - 1)^2}{N_{\bar{k}}^2} \int_0^t \mathbb{E} \left| \frac{1}{N_{\bar{k}} - 1} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \right|^2 ds + \frac{2}{N_{\bar{k}}^2} \int_0^t \mathbb{E} |\mu(s)|^2 ds \\ & = 2 \frac{N_{\bar{k}} - 1}{N_{\bar{k}}^2} \int_0^t \mathbb{E} |\varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s)|^2 ds + \frac{2}{N_{\bar{k}}^2} \int_0^t \mathbb{E} |\mu(s)|^2 ds \leq \frac{M}{N_{\bar{k}}}. \end{aligned}$$

On the other hand, due to (49) and (50), we get

$$\begin{aligned} & \frac{M}{N_{\bar{k}}^2} \mathbb{E} \int_0^t |u_i(s)|^2 ds + \frac{M}{N_{\bar{k}}^2} \mathbb{E} \sum_{j=1}^N \int_0^t \left| F_2(l^{(N)}(s) - \Phi(s)) + F_2\Phi(s) + Cl_j(s) + Hl_0(s) + \sigma(s) \right|^2 ds \\ & \leq \frac{M}{N_{\bar{k}}} \mathbb{E} \int_0^t |l^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N_{\bar{k}}}. \end{aligned}$$

Moreover, since  $(\bar{p}_i(s), \bar{q}_i(s))$ ,  $i \in \mathcal{I}_{\bar{k}}$ , are identically distributed under  $\mathbb{E}(\cdot | \mathcal{F}_s^{W_0})$ , we have  $\frac{M}{N_{\bar{k}}^2} \mathbb{E} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \int_0^t |\varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s))|^2 ds \leq \frac{M}{N_{\bar{k}}}$ . Therefore, from above estimates, we get from (52) that, for any  $t \in [0, T]$ ,

$$\mathbb{E} \sup_{0 \leq s \leq T} |\Xi(s)|^2 \leq M \mathbb{E} \int_0^t [|\Xi(s)|^2 + |l^{(N)}(s) - \Phi(s)|^2] ds + \frac{M}{N_{\bar{k}}},$$

which yields, by using Gronwall's inequality, that

$$\mathbb{E} \sup_{0 \leq s \leq t} |l^{(\bar{k})}(s) - m_{\bar{k}}(s)|^2 \leq M \mathbb{E} \int_0^t [ |l^{(N)}(s) - \Phi(s)|^2 ] ds + \frac{M}{N_{\bar{k}}}. \quad (53)$$

Consequently, noticing (51) and (53), we have for each  $1 \leq k \leq K$ ,

$$\mathbb{E} \sup_{0 \leq s \leq t} |l^{(k)}(s) - m_k(s)|^2 \leq M \mathbb{E} \int_0^t [ |l^{(N)}(s) - \Phi(s)|^2 ] ds + \frac{M}{N_k}. \quad (54)$$



Since

$$\begin{aligned}
l^{(N)}(s) - \Phi(s) &= \sum_{k=1}^K \left[ \pi_k^{(N)} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} l_i(s) - \pi_k \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right] \\
&= \sum_{k=1}^K \left[ \pi_k^{(N)} \left( \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} l_i(s) - \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right) + \left( \pi_k^{(N)} - \pi_k \right) \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right] \\
&= \sum_{k=1}^K \pi_k^{(N)} (l^{(k)}(s) - m_k(s)) + \sum_{k=1}^K \left( \pi_k^{(N)} - \pi_k \right) \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}),
\end{aligned}$$

then by (33), (54) and  $\pi_k^{(N)} = \frac{N_k}{N} \leq 1$ , we have for any  $t \in [0, T]$ ,

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} |l^{(N)}(s) - \Phi(s)|^2 &\leq \mathbb{E} \sum_{k=1}^K \sup_{0 \leq s \leq t} \pi_k^{(N)} |l^{(k)}(s) - m_k(s)|^2 + M \varepsilon_N^2 \\
&\leq M \mathbb{E} \int_0^t \left[ |l^{(N)}(s) - \Phi(s)|^2 \right] ds + \frac{M}{N} + M \varepsilon_N^2.
\end{aligned}$$

Finally, by using Gronwall's inequality, we complete the proof.  $\square$

Now, we introduce the following system of decentralized limiting state with the perturbation strategy of minor agent  $\mathcal{A}_i$ : for  $1 \leq j \leq N$ ,  $j \neq i$ ,

$$\begin{cases} d\bar{l}_0 = [A_0 \bar{l}_0 + B_0 \varphi_0(\bar{p}_0, \bar{q}_0) + F_0^1 \Phi + b_0] dt + [C_0 \bar{l}_0 + D_0 \varphi_0(\bar{p}_0, \bar{q}_0) + F_0^2 \Phi + \sigma_0] dW_0(t) \\ d\bar{l}_i = [A_{\theta_i} \bar{l}_i + B u_i + F_1 \Phi + b_0] dt + [C \bar{l}_i + D_{\theta_i} u_i + F_2 \Phi + H \bar{l}_0 + \sigma_0] dW_i(t), \\ d\bar{l}_j = [A_{\theta_j} \bar{l}_j + B \varphi_{\theta_j}(\bar{p}_j, \bar{q}_j) + F_1 \Phi + b_0] dt + [C \bar{l}_j + D_{\theta_j} \varphi_{\theta_j}(\bar{p}_j, \bar{q}_j) + F_2 \Phi + H \bar{l}_0 + \sigma_0] dW_i(t), \\ \bar{l}_0(0) = x_0, \quad \bar{l}_i(0) = \bar{l}_j(0) = x. \end{cases} \quad (55)$$

**Lemma 6.9** *Under the assumptions of (A1)-(A4), we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} \left( \left| l_0(t) - \bar{l}_0(t) \right|^2 + \left| l_i(t) - \bar{l}_i(t) \right|^2 \right) = O\left( \frac{1}{N} + \varepsilon_N^2 \right). \quad (56)$$

**Proof** From both the first equation of (48) and (55), we obtain

$$\begin{cases} d(l_0 - \bar{l}_0) = [A_0(l_0 - \bar{l}_0) + F_0^1(l^{(N)} - \Phi)] dt + [C(l_0 - \bar{l}_0) + F_0^2(l^{(N)} - \Phi)] dW_0(t), \\ l_0(0) - \bar{l}_0(0) = 0. \end{cases}$$

With the help of classical estimates of SDE and Lemma 6.8, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| l_0(t) - \bar{l}_0(t) \right|^2 = O\left( \frac{1}{N} + \varepsilon_N^2 \right). \quad (57)$$

Now, from both the second equation of (48) and (55), we obtain

$$\begin{aligned} d(l_i - \bar{l}_i) &= [A_{\theta_i}(l_i - \bar{l}_i) + F_1(l^{(N)} - \Phi)] dt \\ &\quad + [C(l_i - \bar{l}_i) + F_2(l^{(N)} - \Phi) + H(l_0 - \bar{l}_0)] dW_0(t), \quad l_i(0) - \bar{l}_i(0) = 0. \end{aligned}$$

From the classical estimates of SDE, Lemma 6.8 and (57), it is easy to obtain (56).  $\square$

**Lemma 6.10** *Under the assumptions of (A1)-(A4), for each  $1 \leq i \leq N$ , for the minor agent  $A_i$ 's perturbation control  $u_i$ , we have*

$$\left| \mathcal{J}_i(u_i, \bar{u}_{-i}) - J_0(u_i) \right| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).$$

**Proof** Recall (4), (12) and (17), we have

$$\begin{aligned} & \mathcal{J}_i(u_i, \bar{u}_{-i}) - J_i(u_i) \\ &= \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \left\langle Q(l_i - \rho l^{(N)}) - (1 - \rho)l_0, l_i - \rho l^{(N)} - (1 - \rho)l_0 \right\rangle \right. \right. \\ & \quad \left. \left. - \left\langle Q(\bar{l}_i - \rho \Phi - (1 - \rho)\bar{l}_0), \bar{l}_i - \rho \Phi - (1 - \rho)\bar{l}_0 \right\rangle \right) dt \right. \\ & \quad \left. + \left\langle G(l_i(T) - \rho l^{(N)}(T) - (1 - \rho)l_0(T)), l_i(T) - \rho l^{(N)}(T) - (1 - \rho)l_0(T) \right\rangle \right. \\ & \quad \left. - \left\langle G(\bar{l}_i(T) - \rho \Phi(T) - (1 - \rho)\bar{l}_0(T)), \bar{l}_i(T) - \rho \Phi(T) - (1 - \rho)\bar{l}_0(T) \right\rangle \right]. \end{aligned} \quad (58)$$

Similar to the proof of Lemma 6.4, by using Lemma 6.8, 6.9 and  $\mathbb{E} \left( |\bar{l}_0(t)|^2 + |\bar{l}_i(t)|^2 + |\Phi(t)|^2 \right) \leq M$ , we have

$$\begin{aligned} & \left| \mathbb{E} \int_0^T \left( \left\langle Q(l_i - \rho l^{(N)} - (1 - \rho)l_0), l_i - \rho l^{(N)} - (1 - \rho)l_0 \right\rangle \right. \right. \\ & \quad \left. \left. - \left\langle Q(\bar{l}_i - \rho \Phi) - (1 - \rho)\bar{l}_0, \bar{l}_i - \rho \Phi - (1 - \rho)\bar{l}_0 \right\rangle \right) dt \right| \\ & \leq M \int_0^T \mathbb{E} |l_i - \bar{l}_i|^2 dt + M \int_0^T \mathbb{E} |l_0 - \bar{l}_0|^2 dt + M \int_0^T \mathbb{E} |l^{(N)} - \Phi|^2 dt \\ & \quad + M \int_0^T \left( \mathbb{E} |l_i - \bar{l}_i|^2 + \mathbb{E} |l_0 - \bar{l}_0|^2 + \mathbb{E} |l^{(N)} - \Phi|^2 \right)^{\frac{1}{2}} dt = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right) \end{aligned}$$

and

$$\begin{aligned} & \left| \mathbb{E} \left[ \left\langle G(l_i(T) - \rho l^{(N)}(T) - (1 - \rho)l_0(T)), l_i(T) - \rho l^{(N)}(T) - (1 - \rho)l_0(T) \right\rangle \right. \right. \\ & \quad \left. \left. - \left\langle G(\bar{l}_i(T) - \rho \Phi(T) - (1 - \rho)\bar{l}_0(T)), \bar{l}_i(T) - \rho \Phi(T) - (1 - \rho)\bar{l}_0(T) \right\rangle \right] \right| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right). \end{aligned}$$

The proof is completed by noticing (58).  $\square$

**Theorem 6.2** *Under the assumptions of (A1)-(A4),  $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$  is an  $\varepsilon$ -Nash equilibrium of Problem (CC) for minor agents.*

**Proof** For each  $1 \leq i \leq N$ , combining Lemma 6.4 and Lemma 6.10, we have

$$\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) \leq J_i(\bar{u}_i) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right) \leq J_i(u_i) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right) \leq \mathcal{J}_i(u_i, \bar{u}_{-i}) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right),$$

where the second inequality comes from the fact that  $J_i(\bar{u}_i) = \inf_{u_i \in \mathcal{U}_{ad}^i} J_i(u_i)$ . Consequently, Theorem 6.2 holds with  $\varepsilon = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right)$ .  $\square$

By combining Theorems 6.1, 6.2, we obtain the following main result of this paper:

**Theorem 6.3** *Under the assumptions of (A1)-(A4),  $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$  is an  $\varepsilon$ -Nash equilibrium of Problem (CC), where  $\bar{u}_0 = \varphi_0(\bar{p}_0, \bar{q}_0)$ ,  $\bar{u}_i = \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i)$ ,  $1 \leq i \leq N$  for*

$$\varphi_0(p, q) := \mathbf{P}_{\Gamma_0} [R_0^{-1}(B'_0 p + D'_0 q)], \quad \varphi_{\theta_i}(p, q) := \mathbf{P}_{\Gamma_{\theta_i}} [R_{\theta_i}^{-1}(B'_i p + D'_i q)].$$

## 7 Appendix

We give this appendix to prove Theorem 5.1. The fully-coupled structure of MF-FBSDE (29) arises difficulties for establishing its wellposedness. Motivated by Pardoux and Tang [39] Theorem 3.1, we can establish the wellposedness of MF-FBSDE (29) for arbitrary time duration when it is weakly coupled.

Let us first note that for a given  $(Y(\cdot), Z(\cdot)) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m \times (d+1)})$ , the forward equation in the MF-FBSDE (29) has a unique solution  $X(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ , thus we introduce a map  $\mathcal{M}_1 : L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m \times (d+1)}) \rightarrow L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ , through

$$\begin{aligned} X(t) = & x + \int_0^t b(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s)) ds \\ & + \int_0^t \sigma(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s)) dW(s). \end{aligned} \quad (59)$$

We mention that the wellposedness of (59) can be established by using the contraction mapping method under assumption  $(H_1)$ ,  $(H_2)$ , although it has the term  $\mathbb{E}[X_s|\mathcal{F}_s^{W_0}]$ . We omit the proof. Moreover, with the help of BDG inequality, it follows that  $\mathbb{E} \sup_{t \in [0, T]} |X(t)|^2 < \infty$ .

**Lemma 7.1** *Let  $X_i$  be the solution of (59) corresponding to  $(Y_i(\cdot), Z_i(\cdot)) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m \times (d+1)})$ ,  $i = 1, 2$ . Then for all  $\lambda \in \mathbb{R}$ ,  $K_1, K_2 > 0$ , we have*

$$\begin{aligned} & e^{-\lambda t} \mathbb{E} |X_1(t) - X_2(t)|^2 + \bar{\lambda}_1 \int_0^t e^{-\lambda s} \mathbb{E} |X_1(s) - X_2(s)|^2 ds \\ & \leq (k_2 K_1 + k_9^2) \int_0^t e^{-\lambda s} \mathbb{E} |Y_1(s) - Y_2(s)|^2 ds + (k_3 K_2 + k_{10}^2) \int_0^t e^{-\lambda s} \mathbb{E} |Z_1(s) - Z_2(s)|^2 ds, \end{aligned} \quad (60)$$

where  $\bar{\lambda}_1 := \lambda - 2\lambda_1 - k_2 K_1^{-1} - k_3 K_2^{-1} - 2k_1 - k_7^2 - k_8^2$ . Moreover,

$$\begin{aligned} e^{-\lambda t} \mathbb{E} |X_1(t) - X_2(t)|^2 & \leq (k_2 K_1 + k_9^2) \int_0^t e^{-\bar{\lambda}_1(t-s)} e^{-\lambda s} \mathbb{E} |Y_1(s) - Y_2(s)|^2 ds \\ & \quad + (k_3 K_2 + k_{10}^2) \int_0^t e^{-\bar{\lambda}_1(t-s)} e^{-\lambda s} \mathbb{E} |Z_1(s) - Z_2(s)|^2 ds. \end{aligned} \quad (61)$$

**Proof** We denote  $\bar{X} := X_1 - X_2$ ,  $\bar{Y} := Y_1 - Y_2$ ,  $\bar{Z} := Z_1 - Z_2$ ,  $\bar{b} := b(X_1, \mathbb{E}[X_1|\mathcal{F}^{W_0}], Y_1, Z_1) - b(X_2, \mathbb{E}[X_2|\mathcal{F}^{W_0}], Y_2, Z_2)$ ,  $\bar{\sigma} := \sigma(X_1, \mathbb{E}[X_1|\mathcal{F}^{W_0}], Y_1, Z_1) - \sigma(X_2, \mathbb{E}[X_2|\mathcal{F}^{W_0}], Y_2, Z_2)$ . Applying Itô's formula to  $e^{-\lambda t} \mathbb{E} |X_1(t) - X_2(t)|^2$  and taking expectation, we obtain

$$e^{-\lambda t} \mathbb{E} |\bar{X}(t)|^2 = -\lambda \int_0^t e^{-\lambda s} \mathbb{E} |\bar{X}(s)|^2 ds + 2\mathbb{E} \int_0^t e^{-\lambda s} \langle \bar{X}(s), \bar{b}(s) \rangle ds + \mathbb{E} \int_0^t e^{-\lambda s} |\bar{\sigma}(s)|^2 ds. \quad (62)$$

Noticing that

$$\begin{aligned} & 2\langle \bar{X}(s), \bar{b}(s) \rangle \\ & = 2\langle \bar{X}(s), b(s, X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) - b(X_2(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) \rangle \\ & \quad + 2\langle \bar{X}(s), b(s, X_2(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) - b(X_2(s), \mathbb{E}[X_2(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_2(s)) \rangle \\ & \leq 2\lambda_1 |\bar{X}(s)|^2 + 2|\bar{X}(s)| (k_1 |\mathbb{E}[\bar{X}(s)|\mathcal{F}_s^{W_0}]| + k_2 |\bar{Y}(s)| + k_3 |\bar{Z}(s)|) \\ & \leq (2\lambda_1 + k_2 K_1^{-1} + k_3 K_2^{-1}) |\bar{X}(s)|^2 + 2k_1 \mathbb{E}[|\bar{X}(s)|^2 |\mathcal{F}_s^{W_0}] + k_2 K_1 |\bar{Y}(s)|^2 + k_3 K_2 |\bar{Z}(s)|^2, \end{aligned}$$

and

$$\begin{aligned} |\bar{\sigma}(s)|^2 & \leq k_7^2 |\bar{X}(s)|^2 + k_8^2 \mathbb{E}[\bar{X}(s)|\mathcal{F}_s^{W_0}]^2 + k_9^2 |\bar{Y}(s)|^2 + k_{10}^2 |\bar{Z}(s)|^2 \\ & \leq k_7^2 |\bar{X}(s)|^2 + k_8^2 \mathbb{E}[|\bar{X}(s)|^2 |\mathcal{F}_s^{W_0}] + k_9^2 |\bar{Y}(s)|^2 + k_{10}^2 |\bar{Z}(s)|^2. \end{aligned}$$

Then, from (62) and  $\mathbb{E}[\mathbb{E}[|\bar{X}(s)|^2|\mathcal{F}_s^{W_0}]] = \mathbb{E}|\bar{X}(s)|^2$ , we can obtain (60).

Now, we apply Itô's formula to  $e^{-\bar{\lambda}_1(t-s)}e^{-\lambda s}\mathbb{E}|X_1(s) - X_2(s)|^2$  for  $s \in [0, t]$  and taking expectation, it follows that

$$\begin{aligned} e^{-\lambda t}\mathbb{E}|\bar{X}(t)|^2 &= -(\lambda - \bar{\lambda}_1) \int_0^t e^{-\bar{\lambda}_1(t-s)}e^{-\lambda s}\mathbb{E}|\bar{X}(s)|^2 ds + 2\mathbb{E} \int_0^t e^{-\bar{\lambda}_1(t-s)}e^{-\lambda s}\langle \bar{X}(s), \bar{b}(s) \rangle ds \\ &\quad + \mathbb{E} \int_0^t e^{-\bar{\lambda}_1(t-s)}e^{-\lambda s}|\bar{\sigma}(s)|^2 ds. \end{aligned} \quad (63)$$

From above estimates and (63), one can prove (61).  $\square$

**Remark 7.1** By integrating both sides of (61) on  $[0, T]$  and using  $\frac{1-e^{-\bar{\lambda}_1(T-s)}}{\bar{\lambda}_1} \leq \frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1}$ ,  $\forall s \in [0, T]$ , we have

$$\|X_1 - X_2\|_{\lambda}^2 \leq \frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1} [(k_2 K_1 + k_9^2)\|Y_1 - Y_2\|_{\lambda}^2 + (k_3 K_2 + k_{10}^2)\|Z_1 - Z_2\|_{\lambda}^2]. \quad (64)$$

Let  $t = T$  in (61) and noticing that  $e^{-\bar{\lambda}_1(T-s)} \leq 1 \vee e^{-\bar{\lambda}_1 T}$ ,  $\forall s \in [0, T]$ , thus

$$e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 \leq [1 \vee e^{-\bar{\lambda}_1 T}] [(k_2 K_1 + k_9^2)\|Y_1 - Y_2\|_{\lambda}^2 + (k_3 K_2 + k_{10}^2)\|Z_1 - Z_2\|_{\lambda}^2]. \quad (65)$$

In particular, if  $\bar{\lambda}_1 > 0$ , we have

$$e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 \leq (k_2 K_1 + k_9^2)\|Y_1 - Y_2\|_{\lambda}^2 + (k_3 K_2 + k_{10}^2)\|Z_1 - Z_2\|_{\lambda}^2. \quad (66)$$

Similarly, for a given  $X(\cdot) \in L_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$ , the backward equation in the MF-FBSDE (29) has a unique solution  $(Y(\cdot), Z(\cdot)) \in L_{\mathcal{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m \times (d+1)})$ , thus we can introduce another map  $\mathcal{M}_2 : L_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \rightarrow L_{\mathcal{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m \times (d+1)})$ , through

$$Y(t) = g(X(T), \mathbb{E}[X(T)|\mathcal{F}_T^{W_0}]) + \int_t^T f(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s)) ds - \int_t^T Z(s) dW(s). \quad (67)$$

The wellposedness of (67) under assumption  $(H_1), (H_2)$  is referred to Darling and Pardoux [16] Theorem 3.4 or Buckdahn and Nie [12] Lemma 2.2. Moreover, we have  $\mathbb{E} \sup_{t \in [0, T]} |Y(t)|^2 < \infty$ .

**Lemma 7.2** Let  $(Y_i(\cdot), Z_i(\cdot))$  be the solution of (67) corresponding to  $X_i \in L_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$ ,  $i = 1, 2$ . Then for all  $\lambda \in \mathbb{R}$ ,  $K_3, K_4 > 0$ , we have

$$\begin{aligned} &e^{-\lambda t}\mathbb{E}|Y_1(t) - Y_2(t)|^2 + \bar{\lambda}_2 \int_t^T e^{-\lambda s}\mathbb{E}|Y_1(s) - Y_2(s)|^2 ds \\ &\quad + (1 - k_6 K_4) \int_t^T e^{-\lambda s}\mathbb{E}|Z_1(s) - Z_2(s)|^2 ds \\ &\leq (k_{11}^2 + k_{12}^2)e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5)K_3 \int_t^T e^{-\lambda s}\mathbb{E}|X_1(s) - X_2(s)|^2 ds, \end{aligned} \quad (68)$$

where  $\bar{\lambda}_2 := -\lambda - 2\lambda_2 - (k_4 + k_5)K_3^{-1} - k_6 K_4^{-1}$ . Moreover,

$$\begin{aligned} &e^{-\lambda t}\mathbb{E}|Y_1(t) - Y_2(t)|^2 + (1 - k_6 K_4) \int_t^T e^{-\bar{\lambda}_2(s-t)}e^{-\lambda s}\mathbb{E}|Z_1(s) - Z_2(s)|^2 ds \\ &\leq (k_{11}^2 + k_{12}^2)e^{-\bar{\lambda}_2(T-t)}e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 \\ &\quad + (k_4 + k_5)K_3 \int_t^T e^{-\bar{\lambda}_2(s-t)}e^{-\lambda s}\mathbb{E}|X_1(s) - X_2(s)|^2 ds. \end{aligned} \quad (69)$$

**Proof** We denote  $\bar{X} := X_1 - X_2$ ,  $\bar{Y} := Y_1 - Y_2$ ,  $\bar{Z} := Z_1 - Z_2$ ,  $\bar{f} := f(X_1, \mathbb{E}[X_1|\mathcal{F}^{W_0}], Y_1, Z_1) - f(X_2, \mathbb{E}[X_2|\mathcal{F}^{W_0}], Y_2, Z_2)$ . Applying Itô's formula to  $e^{-\lambda t} \mathbb{E}|Y_1(t) - Y_2(t)|^2$  and taking expectation, we obtain

$$\begin{aligned} & e^{-\lambda t} \mathbb{E}|\bar{Y}(t)|^2 - \lambda \int_t^T e^{-\lambda s} \mathbb{E}|\bar{Y}(s)|^2 ds + \mathbb{E} \int_t^T e^{-\lambda s} |\bar{Z}(s)|^2 ds \\ &= e^{-\lambda T} \mathbb{E}|\bar{Y}(T)|^2 + 2\mathbb{E} \int_t^T e^{-\lambda s} \langle \bar{Y}(s), \bar{f}(s) \rangle ds. \end{aligned} \quad (70)$$

Noticing that

$$\begin{aligned} & 2\langle \bar{Y}(s), \bar{f}(s) \rangle \\ &= 2\langle \bar{Y}(s), f(s, X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) - f(X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_1(s)) \rangle \\ & \quad + 2\langle \bar{Y}(s), f(s, X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_1(s)) - f(X_2(s), \mathbb{E}[X_2(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_2(s)) \rangle \\ &\leq 2\lambda_2 |\bar{Y}(s)|^2 + 2|\bar{Y}(s)| (k_4 |\bar{X}(s)| + k_5 |\mathbb{E}[\bar{X}(s)|\mathcal{F}_s^{W_0}]| + k_6 |\bar{Z}(s)|) \\ &\leq (2\lambda_2 + k_4 K_3^{-1} + k_5 K_3^{-1} + k_6 K_4^{-1}) |\bar{Y}(s)|^2 + k_4 K_3 |\bar{X}(s)|^2 + k_5 K_3 \mathbb{E}[|\bar{X}(s)|^2 | \mathcal{F}_s^{W_0}] + k_6 K_4 |\bar{Z}(s)|^2, \end{aligned}$$

and

$$\begin{aligned} |\bar{Y}(T)|^2 &= |g(X_1(T), \mathbb{E}[X_1(T)|\mathcal{F}_T^{W_0}]) - g(X_2(T), \mathbb{E}[X_2(T)|\mathcal{F}_T^{W_0}])|^2 \\ &\leq k_{11}^2 |\bar{X}(T)|^2 + k_{12}^2 \mathbb{E}[|\bar{X}(s)|^2 | \mathcal{F}_s^{W_0}]. \end{aligned}$$

Then, from (70) and  $\mathbb{E}[\mathbb{E}[|\bar{X}(s)|^2 | \mathcal{F}_s^{W_0}]] = \mathbb{E}|\bar{X}(s)|^2$ , we can obtain (68).

Now, we apply Itô's formula to  $e^{-\bar{\lambda}_2(s-t)} e^{-\lambda s} \mathbb{E}|Y_1(s) - Y_2(s)|^2$  for  $s \in [t, T]$  and taking expectation, it follows that

$$\begin{aligned} & e^{-\lambda t} \mathbb{E}|\bar{Y}(t)|^2 - (\lambda + \bar{\lambda}_2) \int_t^T e^{-\bar{\lambda}_2(s-t)} e^{-\lambda s} \mathbb{E}|\bar{Y}(s)|^2 ds + \mathbb{E} \int_t^T e^{-\bar{\lambda}_2(s-t)} e^{-\lambda s} |\bar{Z}(s)|^2 ds \\ &= e^{-\bar{\lambda}_2(T-t)} e^{-\lambda T} \mathbb{E}|\bar{Y}(s)|^2 + 2\mathbb{E} \int_t^T e^{-\bar{\lambda}_2(s-t)} e^{-\lambda s} \langle \bar{Y}(s), \bar{f}(s) \rangle ds. \end{aligned} \quad (71)$$

From above estimates and (71), one can prove (69).  $\square$

**Remark 7.2** Now we choose  $K_4$  satisfying  $0 < K_4 \leq k_6^{-1}$ , then by integrating both sides of (69) on  $[0, T]$  and using  $\frac{1-e^{-\bar{\lambda}_2 s}}{\bar{\lambda}_2} \leq \frac{1-e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2}$ ,  $\forall s \in [0, T]$ , we have

$$\|Y_1 - Y_2\|_\lambda^2 \leq \frac{1 - e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2} \left[ (k_{11}^2 + k_{12}^2) e^{-\lambda T} \mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5) K_3 \|X_1 - X_2\|_\lambda^2 \right]. \quad (72)$$

Let  $t = 0$  in (69) and noticing that  $1 \wedge e^{-\bar{\lambda}_2 T} \leq e^{-\bar{\lambda}_2 s} \leq 1 \vee e^{-\bar{\lambda}_2 T}$ ,  $\forall s \in [0, T]$ , thus

$$\|Z_1 - Z_2\|_\lambda^2 \leq \frac{(k_{11}^2 + k_{12}^2) e^{-\bar{\lambda}_2 T} e^{-\lambda T} \mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5) K_3 (1 \vee e^{-\bar{\lambda}_2 T}) \|X_1 - X_2\|_\lambda^2}{(1 - k_6 K_4) (1 \wedge e^{-\bar{\lambda}_2 T})}. \quad (73)$$

On the other hand, if  $\bar{\lambda}_2 > 0$ , let  $t = 0$  in (68), we have

$$\|Z_1 - Z_2\|_\lambda^2 \leq \frac{(k_{11}^2 + k_{12}^2) e^{-\lambda T} \mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5) K_3 \|X_1 - X_2\|_\lambda^2}{1 - k_6 K_4}. \quad (74)$$

Now, we present the proof of Theorem 5.1.

**Proof of Theorem 5.1** We define  $\mathcal{M} := \mathcal{M}_2 \circ \mathcal{M}_1$ , recalling that  $\mathcal{M}_1$  is defined through (59) and  $\mathcal{M}_2$  is defined through (67). Thus  $\mathcal{M}$  maps  $L_{\mathcal{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m \times (d+1)})$  into itself. To

prove the theorem, it is only need to show that  $\mathcal{M}$  is a contraction mapping for some equivalent norm  $\|\cdot\|_\lambda$ . In fact, for  $(Y_i, Z_i) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m \times (d+1)})$ , let  $X_i := \mathcal{M}_1(Y_i, Z_i)$  and  $(\bar{Y}_i, \bar{Z}_i) := \mathcal{M}((Y_i, Z_i))$ , from (64), (65), (72) and (73) based on Lemma 7.1, 7.2, we have

$$\begin{aligned}
& \|\bar{Y}_1 - \bar{Y}_2\|_\lambda^2 + \|\bar{Z}_1 - \bar{Z}_2\|_\lambda^2 \\
& \leq \left[ \frac{1-e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2} + \frac{1 \vee e^{-\bar{\lambda}_2 T}}{(1-k_6 K_4)(1 \wedge e^{-\bar{\lambda}_2 T})} \right] \\
& \quad \times \left[ (k_{11}^2 + k_{12}^2) e^{-\lambda T} \mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5) K_3 \|X_1 - X_2\|_\lambda^2 \right] \\
& \leq \left[ \frac{1-e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2} + \frac{1 \vee e^{-\bar{\lambda}_2 T}}{(1-k_6 K_4)(1 \wedge e^{-\bar{\lambda}_2 T})} \right] \\
& \quad \times \left[ (k_{11}^2 + k_{12}^2) (1 \vee e^{-\bar{\lambda}_1 T}) + (k_4 + k_5) K_3 \frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1} \right] \\
& \quad \times \left[ (k_2 K_1 + k_9^2) \|Y_1 - Y_2\|_\lambda^2 + (k_3 K_2 + k_{10}^2) \|Z_1 - Z_2\|_\lambda^2 \right].
\end{aligned}$$

Recalling that  $\bar{\lambda}_1 := \lambda - 2\lambda_1 - k_2 K_1^{-1} - k_3 K_2^{-1} - 2k_1 - k_7^2 - k_8^2$  and  $\bar{\lambda}_2 := -\lambda - 2\lambda_2 - (k_4 + k_5) K_3^{-1} - k_6 K_4^{-1}$ . Then by choosing suitable  $\lambda$  (e.g. we can easily choose  $\lambda$  big enough such that  $\bar{\lambda}_1 \geq 1$  and  $\bar{\lambda}_2 \leq 0$ ), the first assertion is immediate.

Now let us prove the second assertion. Since  $2(\lambda_1 + \lambda_2) < -2k_1 - k_6^2 - k_7^2 - k_8^2$ , we can choose  $\lambda \in \mathbb{R}$ ,  $0 < K_4 \leq k_6^{-1}$  and sufficient large  $K_1, K_2, K_3$  such that

$$\bar{\lambda}_1 > 0, \quad \bar{\lambda}_2 > 0, \quad 1 - K_4 k_6 > 0.$$

Then from (64), (66), (72) and (74), we have

$$\begin{aligned}
& \|\bar{Y}_1 - \bar{Y}_2\|_\lambda^2 + \|\bar{Z}_1 - \bar{Z}_2\|_\lambda^2 \\
& \leq \left[ \frac{1}{\bar{\lambda}_2} + \frac{1}{1-k_6 K_4} \right] \times \left[ (k_{11}^2 + k_{12}^2) e^{-\lambda T} \mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5) K_3 \|X_1 - X_2\|_\lambda^2 \right] \\
& \leq \left[ \frac{1}{\bar{\lambda}_2} + \frac{1}{1-k_6 K_4} \right] \times \left[ k_{11}^2 + k_{12}^2 + (k_4 + k_5) K_3 \frac{1}{\bar{\lambda}_1} \right] \\
& \quad \times \left[ (k_2 K_1 + k_9^2) \|Y_1 - Y_2\|_\lambda^2 + (k_3 K_2 + k_{10}^2) \|Z_1 - Z_2\|_\lambda^2 \right].
\end{aligned}$$

This completes the second assertion.  $\square$

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