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Linear-Quadratic-Gaussian Mixed Mean-field Games with Heterogeneous Input Constraints

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Abstract

We consider a class of linear-quadratic-Gaussian mean-field games with a major agent and considerable heterogeneous minor agents in the presence of mean-field interactions. The individual admissible controls are constrained in closed convex subsets Γ_k of \mathbb{R}^m . The decentralized strategies for individual agents and consistency condition system are represented in an unified manner through a class of mean-field forward-backward stochastic differential equations involving projection operators on Γ_k . The well-posedness of consistency system is established in both the local and global cases by the contraction mapping and discounting method respectively. Related ε -Nash equilibrium property is also verified.

Key words: ε -Nash equilibrium, Forward-backward stochastic differential equation, Input constraint, Projection operator, Linear-quadratic mixed mean-field games.

AMS Subject Classification: 60H10, 60H30, 91A10, 91A23, 91A25, 93E20

1 Introduction

Mean-field games (MFGs) for stochastic large-population systems have been well-studied because of their wide applications in various fields, such as economics, engineering, social sciences, and operational research. Large-population systems are distinguished by a large number of agents (or players), such that the individual influence of any single agent on the overall population is negligible, but the effects of their statistical behaviors cannot be ignored at the population scale. Mathematically, all agents are weakly-coupled in their dynamics or cost functionals through the state-average (in a linear state case) or the general empirical measure (in a nonlinear state case), which characterizes the statistical effect generated by the population from a macroscopic perspective. Because of these features, when the number of agents is sufficiently high, complicated

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coupling features arise and it is unrealistic for a given agent to obtain all other agents' information. Consequently, for an agent to design centralized strategies based on the information of all peers in a large-population system is intractable. Alternatively, one reasonable and practical direction is to transform a high-dimensional and weakly-coupled problem to a low-dimensional and decoupled one, thus the complexity in both analysis and computation can be reduced. For this purpose, one method is to investigate relevant decentralized strategies based only on local information: that is, the relevant strategies are designed only upon the individual state of the given agent, together with some mass-effect quantities, which are computed off-line.

In this context, motivated by the theory of the propagation of chaos, Lasry and Lions [30, 31, 32] proposed distributed closed-loop strategies for each agent to solve the limiting problem, which were formulated as a coupled nonlinear forward-backward system consisting of a Hamilton-Jacobi-Bellman (HJB) equation and a Fokker-Planck equation. Moreover, the limiting problem enabled the design of approximate Nash equilibrium strategies. Independently, Caines, Huang and Malhamé [28] developed a similar program called the Nash certainty equivalence (NCE) principle, which was motivated by the analysis of large communications networks. In principle, the MFGs procedure consists of the following four main steps (see [5, 13, 25, 27, 32]): (1) A limiting mass-effect term is introduced, which comes from the asymptotic mass-effect behavior when the agent number N tends to infinity. This limiting term should be treated as an exogenous and undetermined "frozen" term at this moment; (2) Through replacing the mass-effect term with the frozen limiting term, related limiting optimization problem can be introduced. Thus, the initial highly-coupled problem can be decoupled and only parameterized by this generic frozen limit. Subsequently, using standard control techniques (see [44]), an HJB equation can be obtained because of the dynamic programming principle (DPP), or a Hamiltonian system can be obtained because of the stochastic maximum principle (SMP); the obtained equations can characterize the decentralized optimal strategies; (3) A consistency condition is established to guarantee that the set of decentralized optimal strategies collectively replicates the mass-effect; (4) the derived decentralized strategies are shown to be ε -Nash equilibrium, which justifies the aforementioned scheme for finding the approximate Nash equilibrium.

For further analysis details of MFGs, readers are referred to [1, 5, 13, 19, 25, 42, 43]. That substantial literatures exists studying linear-quadratic-Gaussian (LQG)-MFG is remarkable. For example, [26] studied LQG MFGs using the common Riccati equation approach; [8] adopted SMP with the help of adjoint equations; [2] and [33] studied ergodic LQG MFGs; [27] studied LQG MFGs with nonuniform agents through state-aggregation by empirical distribution. For further research, readers are referred to [3, 7, 41] and the references therein.

All aforementioned mentioned works examine standard MFGs, which (except [28]) require that all the agents be statistically identical and that the individual influence on the overall population of a single agent be negligible as the number of the agents tends to infinity. However, in the real world, some models exist in which a major agent has a significant influence on other agents (called minor agents) no matter how numerous the minor-agents may be. Such interactions exists in numerous socio-economic problems (e.g., [29, 45]). This type of games involving agents with different power levels is usually called a mixed type games. Compared with MFGs with only minor agents, a distinctive feature of mixed type MFGs is that the mean field behavior of the minor agents is affected by the major agent; thus, it is a random process, and the influence of the major agent on the minor agents is not negligible in the limiting problem. To deal with such new features, conditional distribution with respect to the major agent's information flow is introduced (see [37, 14]).

We now discuss some works on MFGs with a major agent and minor agents that are related to our paper. To the best of our knowledge, the first LQG MFG with a major agent and minor agents was studied by [24], in which the minor agents were from a total K classes. In the study of [38], the authors examined mean-field LQG mixed games with continuum-parameterized minor agents. [37] investigated nonlinear stochastic dynamic systems with major and minor agents. [11] studied nonlinear stochastic differential games, involving a major agent and a large number of collectively acting minor agents, as two-person zero-sum stochastic differential games of the type feedback control against feedback control, the limiting behaviors of the saddle point controls were also studied. For further research, readers are referred to [4, 14] and the references therein.

In this study, we investigated a class of LQG MFGs with major agents and minor agents in the presence of control constraints. In all of the aforementioned papers about linear quadratic (LQ) control problems, the control was unconstrained, and the (feedback) control was constructed from DPP or SMP, which is automatically admissible. Whereas, if we impose constraints on the admissible control, the whole LQ approach fails to apply, (see e.g., [15, 23]). We emphasize that the LQ control problems with control constraints have wide applications in finance and economics. For example, the mean-variance problem with prohibiting short-selling can be transferred to LQ control problems with positive control, (see e.g., [6, 23]). The optimal investment problems where the agents have relative performance, (i.e., their portfolio constraints take different forms), can also be tackled through approach of LQ control problems with input constraint, (see e.g, [20, 17]). Remark 3.1 of the current paper provides several other constraint sets $\Gamma \subset \mathbb{R}^m$ as well as their applications. For an investigation of LQ problems with positive controls or a more general study where the control is constrained in a given convex cone, readers are referred to [9] for a deterministic case and [15, 23, 34] for a stochastic case.

To the best of our knowledge, the current study is the first to examine constrained LQG MFGs with major agents and large numbers of minor agents. In addition to the control constraint being fully new, our study also has other novel features when compared with other relevant studies: in [24, 38], the diffusion term simply takes a constant; however, this study considers the mean-field LQG mixed games in which the diffusion term depends on the major agent's and the minor agent's states, as well as the individual control strategy. This creates additional difficulties, especially when applying the general SMP; in [37, 11], a nonlinear stochastic differential games was studied; however, in this study, we put ourselves in a LQ mean-field framework with individual controls constrained in a closed convex set; thus, we can explicitly present the optimal strategies through a projection operator. Moreover, we use SMP to obtain the optimal strategies through Hamiltonian systems that are fully coupled forward-backward stochastic differential equations (FBSDEs); this is different from [37], which used DPP and a verification theorem to characterize the optimal strategies. Here, we connect the consistency condition to a new type of conditional mean field forward-backward stochastic differential equations (MF-FBSDEs) involving projection operators. We establish its well-posedness under suitable conditions using a fixed point theorem, both in local cases and global cases. Unlike our previous paper [21], we now focus on the mixed game, which is more realistic and more difficult. In this situation, the consistency condition is a conditional MF-FBSDE that does not satisfy the usual monotonicity condition of [22]. Moreover, we require an additional subtle analysis to analyze the major agent's influence and establish the approximate Nash equilibrium. Finally, motivated by [17], we believe that our results can be applied to solve the optimal investment problems caused by major agent and N minor agents.

The remainder of this paper is structured as follows: In Section 2, we formulate the LQG MFGs with a control constraint involving a major agent and minor agents. Decentralized strate-

gies are derived through a FBSDE with projection operators. A consistency condition is also established using some fully coupled FBSDEs that come from the SMP. In Section 3, we prove the well-posedness of fully coupled conditional MF-FBSDEs, which characterize the consistency condition in the local time horizon case. In Section 4, we ascertain the wellposedness of the global time case. In Section 5, we verify the ε -Nash equilibrium of the decentralized strategies.

The main contributions of this paper can be summarized as follows:

- To introduce and analyze a new class of LQ mixed MFGs using SMP. In our setting, both the major agent and minor agents are constrained in their control inputs.
- The diffusion terms of major and minor agents are dependent on their states and control variables.
- The consistency condition system or NCE is represented through a new type of conditional mean-filed type FBSDE with projection operators.
- The existence and uniqueness of such an NCE system are established in a local case (i.e., small time horizon) using the contraction mapping method, and in a global case (i.e., arbitrary time horizon) using the discounting method.

2 Notations and terminology

Consider a finite time horizon $[0, T]$ for fixed $T > 0$. We assume $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ is a complete, filtered probability space satisfying usual conditions and $\{W_i(t), 0 \leq i \leq N\}_{0 \leq t \leq T}$ is a $(N+1)$ -dimensional Brownian motion on this space. Let \mathcal{F}_t be the natural filtration generated by $\{W_i(s), 0 \leq i \leq N, 0 \leq s \leq t\}$ and augmented by $\mathcal{N}_{\mathbb{P}}$ (the class of \mathbb{P} -null sets of \mathcal{F}). Let $\mathcal{F}_t^{W_0}, \mathcal{F}_t^{W_i}, \mathcal{F}_t^i$ be respectively the augmentation of $\sigma\{W_0(s), 0 \leq s \leq t\}$, $\sigma\{W_i(s), 0 \leq s \leq t\}$, $\sigma\{W_0(s), W_i(s), 0 \leq s \leq t\}$ by $\mathcal{N}_{\mathbb{P}}$. Here, $\{\mathcal{F}_t^{W_0}\}_{0 \leq t \leq T}$ stands for the information of the major agent, while $\{\mathcal{F}_t^{W_i}\}_{0 \leq t \leq T}$ represents the individual information of i -th minor agent.

Throughout the paper, x' denotes the transpose of a vector or a matrix x , \mathcal{S}^n denotes the set of symmetric $n \times n$ matrices with real elements. For a matrix $M \in \mathbb{R}^{n \times d}$, we define the norm $|M| := \sqrt{\text{Tr}(M'M)}$. If $M \in \mathcal{S}^n$ is positive (semi) definite, we write $M > (\geq) 0$. Let \mathcal{H} be a given Hilbert space, the set of \mathcal{H} -valued continuous functions is denoted by $C(0, T; \mathcal{H})$. If $N(\cdot) \in C(0, T; \mathcal{S}^n)$ and $N(t) > (\geq) 0$ for every $t \in [0, T]$, we say that $N(\cdot)$ is positive (semi) definite, which is denoted by $N(\cdot) > (\geq) 0$. Now, for a given Hilbert space \mathcal{H} and a filtration $\{\mathcal{G}_t\}_{0 \leq t \leq T}$, we also introduce the following spaces which will be used in this paper:

$$L_{\mathcal{G}_T}^2(\Omega; \mathcal{H}) := \{\xi : \Omega \rightarrow \mathcal{H} \mid \xi \text{ is } \mathcal{G}_T\text{-measurable such that } \mathbb{E}|\xi|^2 < \infty\},$$

$$L_{\mathcal{G}}^2(0, T; \mathcal{H}) := \left\{x(\cdot) : [0, T] \times \Omega \rightarrow \mathcal{H} \mid x(\cdot) \text{ is } \mathcal{G}_t\text{-progressively measurable process such that}$$

$$\mathbb{E} \int_0^T |x(t)|^2 dt < \infty\right\},$$

$$L_{\mathcal{G}}^2(\Omega; C(0, T; \mathcal{H})) := \left\{x(\cdot) : [0, T] \times \Omega \rightarrow \mathcal{H} \mid x(\cdot) \text{ is } \mathcal{G}\text{-adapted continuous process such that}$$

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |x(t)|^2 \right] < \infty\right\}.$$

3 LQG mixed games with control constraint

We consider a linear-quadratic-Gaussian mixed mean-field game involving a major agent \mathcal{A}_0 and a heterogeneous large-population with N individual minor agents $\{\mathcal{A}_i : 1 \leq i \leq N\}$. Unlike other works of LQG mixed games, our control domain is constrained in a closed convex set of the full Euclidean space, and more details of constraints will be given later. The states x_0 for major agent \mathcal{A}_0 and x_i for each minor agent \mathcal{A}_i are modeled by the following controlled linear stochastic differential equations (SDEs) with mean-field coupling:

$$\begin{aligned} dx_0(t) = & [A_0(t)x_0(t) + B_0(t)u_0(t) + F_0^1(t)x^{(N)}(t) + b_0(t)]dt \\ & + [C_0(t)x_0(t) + D_0(t)u_0(t) + F_0^2(t)x^{(N)}(t) + \sigma_0(t)]dW_0(t), \quad x_0(0) = x_0 \in \mathbb{R}^n, \end{aligned} \quad (1)$$

and

$$\begin{aligned} dx_i(t) = & [A_{\theta_i}(t)x_i(t) + B(t)u_i(t) + F_1(t)x^{(N)}(t) + b(t)]dt \\ & + [C(t)x_i(t) + D_{\theta_i}(t)u_i(t) + F_2(t)x^{(N)}(t) + Hx_0(t) + \sigma(t)]dW_i(t), \quad x_i(0) = x \in \mathbb{R}^n, \end{aligned} \quad (2)$$

where $x^{(N)}(\cdot) = \frac{1}{N} \sum_{i=1}^N x_i(\cdot)$ is the state-average of minor agents. Recalling that \mathcal{F}_t^i is the individual decentralized information while \mathcal{F}_t is the centralized information driven by all Brownian motion components. We point out that the heterogeneous noise W_i is specific for individual agent \mathcal{A}_i whereas $x_i(t)$ is adapted to \mathcal{F}_t instead of \mathcal{F}_t^i due to the coupling state-average $x^{(N)}$. The coefficients of (1) and (2) are deterministic matrix-valued functions with appropriate dimensions. The number θ_i is a parameter of agent \mathcal{A}_i to model a heterogeneous population of minor agents, for more explanations, see [24]. For sake of notations, in (2), we only set the coefficients $A(\cdot)$ and $D(\cdot)$ (see also $R(\cdot)$ in (4)) to be dependent on θ_i . Similar analysis can be proceeded in case that all other coefficients depend also on θ_i . In this paper, we assume that θ_i takes values from a finite set $\Theta := \{1, 2, \dots, K\}$ which means that totally K types of minor agents are considered. We call \mathcal{A}_i a k -type minor agent if $\theta_i = k \in \Theta$.

In this paper, we are interested in the asymptotic behavior as N tends to infinity which is essentially to consider a family of games with an increasing number of minor agents. To describe the related large population system, let us first define

$$\mathcal{I}_k = \{i \mid \theta_i = k, 1 \leq i \leq N\}, \quad N_k = |\mathcal{I}_k|,$$

where N_k is the cardinality of index set $\mathcal{I}_k, 1 \leq k \leq K$. Let $\pi_k^{(N)} = \frac{N_k}{N}$ for $k \in \{1, \dots, K\}$ then $\pi^{(N)} = (\pi_1^{(N)}, \dots, \pi_K^{(N)})$ is a probability vector to represent the empirical distribution of $\theta_1, \dots, \theta_N$. The following assumption gives some statistical properties for θ_i . For more details, the reader is referred to [24]:

(A1) There exists a probability mass vector $\pi = (\pi_1, \pi_2, \dots, \pi_K)$ such that $\lim_{N \rightarrow +\infty} \pi^{(N)} = \pi$ and $\min_{1 \leq k \leq K} \pi_k > 0$.

From **(A1)** we know that when $N \rightarrow +\infty$, the proportion of k -type agents becomes stable for each k and that the number of each type agents tends to infinity. Otherwise, the agents in given type with bounded size should be excluded from consideration when analyzing asymptotic behavior as $N \rightarrow +\infty$. Throughout the paper we make the convention that N is suitable large such that $\min_{1 \leq k \leq K} N_k \geq 1$.

Now let us specify the admissible control set and cost functionals of our linear-quadratic-Gaussian mixed game with input control constraint. We call u_0 an admissible control for the major agent if $u_0 \in \mathcal{U}_{ad}^0$, where $\mathcal{U}_{ad}^0 := \{u(\cdot) \mid u(\cdot) \in L_{\mathcal{F}}^2(0, T; \Gamma_0)\}$. Here $\Gamma_0 \subset \mathbb{R}^m$ is a nonempty closed convex set. Moreover, for each $1 \leq i \leq N$, we also define decentralized admissible control u_i for the minor agent \mathcal{A}_i as $u_i \in \mathcal{U}_{ad}^i$, where for a nonempty closed convex set $\Gamma_{\theta_i} \subset \mathbb{R}^m$, $\mathcal{U}_{ad}^i := \{u_i(\cdot) \mid u_i(\cdot) \in L_{\mathcal{F}_i}^2(0, T; \Gamma_{\theta_i})\}$.

Remark 3.1 We give the following typical examples for the closed convex constraint set Γ : $\Gamma^1 = \mathbb{R}_+^m$ represents that the control can only take positive values, it connects with the mean-variance portfolio selection problem with no-shorting, see [6, 23]. The linear subspace $\Gamma^2 = (\mathbb{R}e_i)^\perp$ (where (e_1, e_2, \dots, e_m) is the canonical basis of \mathbb{R}^m) represents that the control can only take from a hyperplane, it is used to deal with that in the investment theory, each manager has access to the whole market except some fixed firm who has private information. For more examples of linear constraints and their economic meaning, the reader is referred to [17]. Γ can also be some closed cone (i.e. Γ is closed and if $u \in \Gamma$, then $\alpha u \in \Gamma$, for all $\alpha \geq 0$), e.g. $\Gamma^3 = \{u \in \mathbb{R}^m : \Upsilon u = 0\}$ or $\Gamma^4 = \{u \in \mathbb{R}^m : \Upsilon u \leq 0\}$, where $\Upsilon \in \mathbb{R}^{n \times m}$. For investigations on the stochastic LQ problems with conic control constraint, the readers are referred to [15, 23].

Let $u = (u_0, u_1, \dots, u_N)$ be the set of strategies of all $N + 1$ agents, $u_{-0} = (u_1, u_2, \dots, u_N)$ be the control strategies except \mathcal{A}_0 and $u_{-i} = (u_0, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N)$ be the set of strategies except the i -th agent \mathcal{A}_i . We introduce the cost functional of the major agent as

$$\begin{aligned} \mathcal{J}_0(u_0, u_{-0}) &= \frac{1}{2} \mathbb{E} \left[\int_0^T \left\langle Q_0(t)(x_0(t) - \rho_0 x^{(N)}(t)), x_0(t) - \rho_0 x^{(N)}(t) \right\rangle \right. \\ &\quad \left. + \left\langle R_0(t)u_0(t), u_0(t) \right\rangle dt + \left\langle G_0(x_0(T) - \rho_0 x^{(N)}(T)), x_0(T) - \rho_0 x^{(N)}(T) \right\rangle \right] \end{aligned} \quad (3)$$

and the cost functional of the minor agent \mathcal{A}_i as

$$\begin{aligned} \mathcal{J}_i(u_i, u_{-i}) &= \frac{1}{2} \mathbb{E} \left[\int_0^T \left(\left\langle Q(x_i - \rho x^{(N)} - (1-\rho)x_0(t)), x_i - \rho x^{(N)} - (1-\rho)x_0(t) \right\rangle + \left\langle R_{\theta_i} u_i, u_i \right\rangle \right) dt \right. \\ &\quad \left. + \left\langle G(x_i(T) - \rho x^{(N)}(T) - (1-\rho)x_0(T)), x_i(T) - \rho x^{(N)}(T) - (1-\rho)x_0(T) \right\rangle \right]. \end{aligned} \quad (4)$$

We mention that for notational brevity, the time argument is suppressed in above equation as well as in the sequel when necessary.

We impose the following assumptions:

(A2) The coefficients of the states satisfy that, for $1 \leq i \leq N$,

$$\begin{aligned} A_0(\cdot), A_{\theta_i}(\cdot), C_0(\cdot), C(\cdot), F_0^1(\cdot), F_1(\cdot), F_0^2(\cdot), F_2(\cdot), H(\cdot) &\in L^\infty(0, T; \mathbb{R}^{n \times n}), \\ B_0(\cdot), B(\cdot), D_0(\cdot), D_{\theta_i}(\cdot) &\in L^\infty(0, T; \mathbb{R}^{n \times m}), b_0(\cdot), b(\cdot), \sigma_0(\cdot), \sigma(\cdot) \in L^\infty(0, T; \mathbb{R}^n). \end{aligned}$$

(A3) The coefficients of cost functionals satisfy that, for $1 \leq i \leq N$,

$$\begin{aligned} Q_0(\cdot), Q(\cdot) &\in L^\infty(0, T; \mathcal{S}^n), R_0(\cdot), R_{\theta_i}(\cdot) \in L^\infty(0, T; \mathcal{S}^m), G_0, G \in \mathcal{S}^n, \\ Q_0(\cdot) \geq 0, Q(\cdot) \geq 0, R_0(\cdot) > 0, R_{\theta_i}(\cdot) > 0, G_0 \geq 0, G \geq 0, \rho_0, \rho &\in [0, 1]. \end{aligned}$$

Here $L^\infty(0, T; \mathcal{H})$ denotes the space of uniformly bounded functions mapping from $[0, T]$ to \mathcal{H} . It follows that, under assumptions **(A2)** and **(A3)**, the system (1) and (2) admits a unique solution $x_0, x_i(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))$ for given admissible control u_0 and u_i . Now, let us formulate the LQG mixed games with control constraint.

Problem (CC). Find a strategies set $\bar{u} = (\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$ where $\bar{u}_i(\cdot) \in \mathcal{U}_{ad}^i$, $0 \leq i \leq N$, such that

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) = \inf_{u_i(\cdot) \in \mathcal{U}_{ad}^i} \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot)), \quad 0 \leq i \leq N.$$

We call \bar{u} an optimal strategies set for Problem **(CC)**.

For comparison, we present also the definition of ε -Nash equilibrium.

Definition 3.1 A set of strategies $\bar{u}_i(\cdot) \in \mathcal{U}_{ad}^i$, $0 \leq i \leq N$, is called an ε -Nash equilibrium with respect to costs \mathcal{J}^i , $0 \leq i \leq N$, if there exists a $\varepsilon = \varepsilon(N) \geq 0$, $\lim_{N \rightarrow +\infty} \varepsilon(N) = 0$, such that for any $0 \leq i \leq N$, we have

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) \leq \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot)) + \varepsilon,$$

when any alternative strategy $u_i \in \mathcal{U}_{ad}^i$ is applied by \mathcal{A}_i .

Remark 3.2 If $\varepsilon = 0$, Definition 3.1 reduces to the usual exact Nash equilibrium.

3.1 Stochastic optimal control problem of the major agent

As explained in introduction, the centralized optimization strategies to Problem (CC), are rather complicate and infeasible to be applied when the number of the agents tends to infinity. Alternatively, we investigate the decentralized strategies via the limiting problem with the help of frozen limiting state-average. To this end, we first figure out the representation of limiting process using heuristic arguments. Based on it, we can find the decentralized strategies by consistency condition, and then rigorously verify the derived strategies satisfy the approximate Nash equilibrium. We formalize the auxiliary limiting mixed game via the approximation of the average state $x^{(N)}$. Since $\pi_k^{(N)} \approx \pi_k$ for large N and

$$x^{(N)} = \frac{1}{N} \sum_{k=1}^K \sum_{i \in \mathcal{I}_k} x_i = \sum_{k=1}^K \pi_k^{(N)} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} x_i,$$

we may approximate $x^{(N)}$ by $\sum_{k=1}^K \pi_k m_k$, where $m_k \in \mathbb{R}^n$ is used to approximate $\frac{1}{N_k} \sum_{i \in \mathcal{I}_k} x_i$. Denote $m = (m'_1, m'_2, \dots, m'_K)'$, which is called the set of aggregate quantities. Replacing $x^{(N)}$ of (1) and (3) by $\sum_{k=1}^K \pi_k m_k$, the major agent's dynamics is given by

$$\begin{aligned} dz_0(t) = & [A_0(t)z_0(t) + B_0(t)u_0(t) + F_0^1(t) \sum_{k=1}^K \pi_k m_k(t) + b_0(t)]dt \\ & + [C_0(t)z_0(t) + D_0(t)u_0(t) + F_0^2(t) \sum_{k=1}^K \pi_k m_k(t) + \sigma_0(t)]dW_0(t), \quad z_0(0) = x_0 \in \mathbb{R}^n, \end{aligned} \quad (5)$$

and the limiting cost functional is

$$\begin{aligned} J_0(u_0) = & \frac{1}{2} \mathbb{E} \left[\int_0^T \left\langle Q_0(t)(z_0(t) - \rho_0 \sum_{k=1}^K \pi_k m_k(t)), z_0(t) - \rho_0 \sum_{k=1}^K \pi_k m_k(t) \right\rangle \right. \\ & \left. + \langle R_0(t)u_0(t), u_0(t) \rangle dt + \left\langle G_0(z_0(T) - \rho_0 \sum_{k=1}^K \pi_k m_k(T)), z_0(T) - \rho_0 \sum_{k=1}^K \pi_k m_k(T) \right\rangle \right]. \end{aligned} \quad (6)$$

For simplicity, let \otimes be the Kronecker product of two matrix (see [18]) and we denote $F_0^{1,\pi} := \pi \otimes F_0^1$, $F_0^{2,\pi} := \pi \otimes F_0^2$, $\rho_0^\pi := \pi \otimes \rho_0 I_{n \times n}$. Then (5) and (6) become respectively to

$$\begin{aligned} dz_0(t) = & [A_0(t)z_0(t) + B_0(t)u_0(t) + F_0^{1,\pi}(t)m(t) + b_0(t)]dt \\ & + [C_0(t)z_0(t) + D_0(t)u_0(t) + F_0^{2,\pi}(t)m(t) + \sigma_0(t)]dW_0(t), \quad z_0(0) = x_0 \in \mathbb{R}^n, \end{aligned} \quad (7)$$

and

$$\begin{aligned} J_0(u_0) = & \frac{1}{2} \mathbb{E} \left[\int_0^T \left\langle Q_0(t)(z_0(t) - \rho_0^\pi m(t)), z_0(t) - \rho_0^\pi m(t) \right\rangle \right. \\ & \left. + \langle R_0(t)u_0(t), u_0(t) \rangle dt + \left\langle G_0(z_0(T) - \rho_0^\pi m(T)), z_0(T) - \rho_0^\pi m(T) \right\rangle \right]. \end{aligned}$$

We define the following auxiliary stochastic optimal control problem for major agent with infinite population:

Problem (LCC-Major). For major agent \mathcal{A}_0 , find $u_0^*(\cdot) \in \mathcal{U}_{ad}^0$ satisfying

$$J_0(u_0^*(\cdot)) = \inf_{u_0(\cdot) \in \mathcal{U}_{ad}^0} J_0(u_0(\cdot)).$$

Then $u_0^*(\cdot)$ is called a decentralized optimal control for Problem **(LCC-Major)**.

Now, similar to [21], we would like to apply SMP to above limiting LQG problem **(LCC-Major)** with input constraint. We introduce the following first order adjoint equation:

$$\begin{cases} dp_0(t) = -[A'_0(t)p_0(t) - Q_0(t)(z_0(t) - \rho_0^\pi m(t)) + C'_0(t)q_0(t)]dt + q_0(t)dW_0(t), \\ p_0(T) = -G_0(z_0(T) - \rho_0^\pi m(T)), \end{cases} \quad (8)$$

as well as the Hamiltonian function

$$\begin{aligned} H_0(t, p, q, x, u) = & \langle p, A_0x + B_0u + F_0^{1,\pi}m + b_0 \rangle + \langle q, C_0x + D_0u + F_0^{2,\pi}m + \sigma_0 \rangle \\ & - \frac{1}{2} \langle Q_0(x - \rho_0^\pi m), x - \rho_0^\pi m \rangle - \frac{1}{2} \langle R_0u, u \rangle. \end{aligned}$$

Since Γ_0 is a closed convex set, for optimal control u_0^* , related optimal state z_0^* and related solution (p_0^*, q_0^*) to (8), the SMP reads as the following local form

$$\left\langle \frac{\partial H_0}{\partial u}(t, p_0^*, q_0^*, z_0^*, u_0^*), u - u_0^* \right\rangle \leq 0, \quad \text{for all } u \in \Gamma_0, \text{ a.e. } t \in [0, T], \mathbb{P} - a.s. \quad (9)$$

Similar to the argument in page 5 of [21], using the well-known results of convex analysis (see Theorem 5.2 of [10] or Theorem 4.1 of [21]), we obtain that (9) is equivalent to

$$u_0^*(t) = \mathbf{P}_{\Gamma_0}[R_0^{-1}(t)(B'_0(t)p_0^*(t) + D'_0(t)q_0^*(t))], \quad \text{a.e. } t \in [0, T], \mathbb{P} - a.s. \quad (10)$$

where $\mathbf{P}_{\Gamma_0}[\cdot]$ is the projection mapping from \mathbb{R}^m to its closed convex subset Γ_0 under the norm $\|\cdot\|_{R_0}$ (where $\|x\|_{R_0}^2 = \langle \langle x, x \rangle \rangle := \langle R_0^{\frac{1}{2}}x, R_0^{\frac{1}{2}}x \rangle$). Finally, by substituting (10) in (7) and (8), we get the following Hamiltonian system for the major agent:

$$\begin{cases} dz_0 = \left(A_0z_0 + B_0\mathbf{P}_{\Gamma_0}[R_0^{-1}(B'_0p_0 + D'_0q_0)] + F_0^{1,\pi}m + b_0 \right) dt \\ \quad + \left(C_0z_0 + D_0\mathbf{P}_{\Gamma_0}[R_0^{-1}(B'_0p_0 + D'_0q_0)] + F_0^{2,\pi}m + \sigma_0 \right) dW_0(t), \\ dp_0 = -(A'_0p_0 - Q_0(z_0 - \rho_0^\pi m) + C'_0q_0)dt + q_0dW_0(t), \\ z_0(0) = x_0, \quad p_0(T) = -G_0(z_0(T) - \rho_0^\pi m(T)). \end{cases} \quad (11)$$

3.2 Stochastic optimal control problem for minor agent

By denoting $F^{1,\pi} := \pi \otimes F_1$, $F^{2,\pi} := \pi \otimes F_2$, $\rho^\pi := \pi \otimes \rho I_{n \times n}$, the limiting state of minor agent \mathcal{A}_i is

$$\begin{cases} dz_i = \left(A_{\theta_i}z_i + Bu_i + F^{1,\pi}m + b \right) dt + \left(Cz_i + D_{\theta_i}u_i + F^{2,\pi}m + Hz_0 + \sigma \right) dW_i(t), \\ z_i(0) = x. \end{cases}$$

The limiting cost functional is given by:

$$\begin{aligned}
J_i(u_i) = & \frac{1}{2} \mathbb{E} \left[\int_0^T \left(\left\langle Q \left(z_i - \rho \sum_{k=1}^K \pi_k m_k - (1-\rho) z_0 \right), z_i - \rho \sum_{k=1}^K \pi_k m_k - (1-\rho) z_0 \right\rangle + \langle R_{\theta_i} u_i, u_i \rangle \right) dt \right. \\
& \left. + \left\langle G \left(z_i(T) - \rho \sum_{k=1}^K \pi_k m_k(T) - (1-\rho) z_0(T) \right), z_i(T) - \rho \sum_{k=1}^K \pi_k m_k(T) - (1-\rho) z_0(T) \right\rangle \right], \tag{12}
\end{aligned}$$

or equivalently by

$$\begin{aligned}
J_i(u_i) = & \frac{1}{2} \mathbb{E} \left[\int_0^T \left(\left\langle Q \left(z_i - \rho^\pi m - (1-\rho) z_0 \right), z_i - \rho^\pi m - (1-\rho) z_0 \right\rangle + \langle R_{\theta_i} u_i, u_i \rangle \right) dt \right. \\
& \left. + \left\langle G \left(z_i(T) - \rho^\pi m(T) - (1-\rho) z_0(T) \right), z_i(T) - \rho^\pi m(T) - (1-\rho) z_0(T) \right\rangle \right],
\end{aligned}$$

and the related limiting stochastic optimal control problem for the minor agents is:

Problem (LCC-Minor). For each minor agent \mathcal{A}_i , find $u_i^*(\cdot) \in \mathcal{U}_{ad}^i$ satisfying

$$J_i(u_i^*(\cdot)) = \inf_{u_i(\cdot) \in \mathcal{U}_{ad}^i} J_i(u_i(\cdot)).$$

Then $u_i^*(\cdot)$ is called a decentralized optimal control for Problem **(LCC-Minor)**.

Similar to the major agent, we obtain the following Hamiltonian system for minor agent \mathcal{A}_i ,

$$\left\{ \begin{aligned} dz_i &= \left(A_{\theta_i} z_i + B \mathbf{P}_{\Gamma_{\theta_i}} [R_{\theta_i}^{-1} (B' p_i + D'_{\theta_i} q_i)] + F_1^\pi m + b \right) dt \\ &+ \left(C z_i + D_{\theta_i} \mathbf{P}_{\Gamma_{\theta_i}} [R_{\theta_i}^{-1} (B' p_i + D'_{\theta_i} q_i)] + F_2^\pi m + H z_0 + \sigma \right) dW_i(t), \\ dp_i &= - \left(A'_{\theta_i} p_i - Q \left(z_i - \rho^\pi m - (1-\rho) z_0 \right) + C' q_i \right) dt + q_i dW_i(t) + q_{i,0} dW_0(t), \\ z_i(0) &= x, \quad p_i(T) = -G \left(z_i(T) - \rho^\pi m(T) - (1-\rho) z_0(T) \right). \end{aligned} \right. \tag{13}$$

Here, $\mathbf{P}_{\Gamma_{\theta_i}}[\cdot]$ is the projection mapping from \mathbb{R}^m to its closed convex subset Γ_{θ_i} under the norm $\|\cdot\|_{R_{\theta_i}}$. We mention that the limiting minor agent's state z_i depends also on the limiting major agent's state z_0 , it makes that z_i is \mathcal{F}^i -adapted, thus $q_{i,0} dW_0(t)$ appears in the adjoint equation.

3.3 Consistency condition system for mixed game

Let us first focus on the k -type minor agent. When $i \in \mathcal{I}_k = \{i \mid \theta_i = k\}$, we denote $A_{\theta_i} = A_k$, $D_{\theta_i} = D_k$, $R_{\theta_i} = R_k$ and $\Gamma_{\theta_i} = \Gamma_k$. We would like to approximate x_i by z_i when $N \rightarrow +\infty$, thus m_k should satisfy the consistency condition (noticing that Assumption **(A1)** implies that $N_k \rightarrow \infty$ if $N \rightarrow \infty$)

$$m_k(\cdot) = \lim_{N \rightarrow +\infty} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} z_i(\cdot).$$

Recall that for $i, j \in \mathcal{I}_k$, z_i and z_j are identically distributed, and conditional independent (under $\mathbb{E}(\cdot \mid \mathcal{F}^{W_0})$). Thus by conditional strong law of large number, we have (the convergence is in the sense of almost surely, see e.g. [36])

$$m_k(\cdot) = \lim_{N \rightarrow +\infty} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} z_i(\cdot) = \mathbb{E}(z_i(\cdot) \mid \mathcal{F}^{W_0}), \tag{14}$$

where z_i is given by (13) with $A_{\theta_i} = A_k, D_{\theta_i} = D_k, R_{\theta_i} = R_k, \Gamma_{\theta_i} = \Gamma_k$. By combining (11), (13) and (14), we get the following consistency condition system or Nash certainty equivalence principle of k -type minor agent, for $1 \leq k \leq K$: (As mentioned before, for notational brevity, the time argument is suppressed in following equations except $\mathbb{E}(\alpha_k(t)|\mathcal{F}_t^{W_0})$ to emphasise its dependence on conditional expectation under $\mathcal{F}_t^{W_0}$)

$$\left\{ \begin{array}{l} d\alpha_k = \left(A_k \alpha_k + B \mathbf{P}_{\Gamma_k} [R_k^{-1} (B' \beta_k + D'_k \gamma_k)] + F_1 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) + b \right) dt \\ \quad + \left(C \alpha_k + D_k \mathbf{P}_{\Gamma_k} [R_k^{-1} (B' \beta_k + D'_k \gamma_k)] + F_2 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) + H \alpha_0 + \sigma \right) dW_k(t), \\ d\beta_k = -(A'_k \beta_k - Q(\alpha_k - \rho \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0})) - (1 - \rho) \alpha_0) + C' \gamma_k dt + \gamma_k dW_k(t) + \gamma_{k,0} dW_0(t), \\ \alpha_k(0) = x, \quad \beta_k(T) = -G(\alpha_k(T) - \rho \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(T) | \mathcal{F}_T^{W_0}) - (1 - \rho) \alpha_0(T)), \end{array} \right. \quad (15)$$

where α_0 satisfies the following FBSDE which is coupled with all k -type minor agents:

$$\left\{ \begin{array}{l} d\alpha_0 = \left(A_0 \alpha_0 + B_0 \mathbf{P}_{\Gamma_0} [R_0^{-1} (B'_0 \beta_0 + D'_0 \gamma_0)] + F_0^1 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) + b_0 \right) dt \\ \quad + \left(C_0 \alpha_0 + D_0 \mathbf{P}_{\Gamma_0} [R_0^{-1} (B'_0 \beta_0 + D'_0 \gamma_0)] + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) + \sigma_0 \right) dW_0(t), \\ d\beta_0 = -(A'_0 \beta_0 - Q_0(\alpha_0 - \rho_0 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0})) + C'_0 \gamma_0) dt + \gamma_0 dW_0(t), \\ \alpha_0(0) = x_0, \quad \beta_0(T) = -G_0(\alpha_0(T) - \rho_0 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(T) | \mathcal{F}_T^{W_0})). \end{array} \right. \quad (16)$$

We consider together the major agent and all kinds of minor agents, i.e. (16) and (15) for all $1 \leq k \leq K$, then there arise $2K+2$ fully coupled equations including $K+1$ forward equations and $K+1$ backward equations. Such fully coupled equations are called consistency condition system. Once we can solve it, then $m_k = \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0})$ which depends on the conditional distribution of α_k , this allows us, in (15), to use arbitrary Brownian motion W_k which is independent of W_0 . Finally, let us introduce the following notation which will be used in the following sections

$$\Phi(t) := \sum_{i=1}^K \pi_i m_i = \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}). \quad (17)$$

4 Existence and uniqueness of consistency condition system: local time horizon case

This section aims to establish the well-posedness of consistency condition system (15)-(16) in small time duration using the method of contraction mapping. Similar to the classical results on FBSDEs, see for example Chapter 1 Section 5 of Ma and Yong [35], we need introduce the following additional assumption:

(A4) We suppose $R_0^{-1}(\cdot), R_{\theta_i}^{-1}(\cdot) \in L^\infty(0, T; \mathcal{S}^m)$ and $M_0 |D|^2 < 1$, where $|D| := \max_{0 \leq k \leq K} |D_k|$ and $M_0 := \max \{ |G_0|^2 (1 + \rho_0^2), |G|^2 (1 + \rho^2 + (1 - \rho)^2) \}$.

For simplicity, we denote

$$\varphi_0(p, q) := \mathbf{P}_{\Gamma_0} [R_0^{-1} (B'_0 p + D'_0 q)], \quad \varphi_{\theta_i}(p, q) := \mathbf{P}_{\Gamma_{\theta_i}} [R_{\theta_i}^{-1} (B'_i p + D'_i q)].$$

We have the following theorem:

Theorem 4.1 *Assume (A1)-(A4), then there exists a $T_0 > 0$, such that for any $T \in (0, T_0]$, the system (15)-(16) has a unique solution $(\alpha_0, \beta_0, \gamma_0, \alpha_k, \beta_k, \gamma_k, \gamma_{k,0})$, $1 \leq k \leq K$, satisfying*

$$\begin{aligned} \alpha_0, \beta_0 &\in L^2_{\mathcal{F}^{W_0}}(\Omega; C(0, T; \mathbb{R}^n)), & \alpha_k, \beta_k &\in L^2_{\mathcal{F}^k}(\Omega; C(0, T; \mathbb{R}^n)), \\ \gamma_0 &\in L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^n), & \gamma_k, \gamma_{k,0} &\in L^2_{\mathcal{F}^k}(0, T; \mathbb{R}^n), \quad 1 \leq k \leq K. \end{aligned} \quad (18)$$

Proof Let $T_0 \in (0, 1]$ be undetermined and $0 < T \leq T_0$. We denote

$$\begin{aligned} \mathcal{N}[0, T] &:= L^2_{\mathcal{F}^{W_0}}(\Omega; C(0, T; \mathbb{R}^n)) \times L^2_{\mathcal{F}^1}(\Omega; C(0, T; \mathbb{R}^n)) \times \dots \times L^2_{\mathcal{F}^K}(\Omega; C(0, T; \mathbb{R}^n)) \\ &\quad \times L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^1}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}^K}(0, T; \mathbb{R}^n) \\ &\quad \times L^2_{\mathcal{F}^1}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}^K}(0, T; \mathbb{R}^n). \end{aligned}$$

For $(Y_0, Y_1, \dots, Y_K, Z_0, Z_1, \dots, Z_K, \Upsilon_{1,0}, \dots, \Upsilon_{K,0}) \in \mathcal{N}[0, T]$, we introduce the following norm:

$$\begin{aligned} &\| (Y_0, Y_1, \dots, Y_K, Z_0, Z_1, \dots, Z_K, \Upsilon_{1,0}, \dots, \Upsilon_{K,0}) \|_{\mathcal{N}[0, T]}^2 \\ &:= \sup_{t \in [0, T]} \mathbb{E} \left\{ \sum_{k=0}^K |Y_k(t)|^2 + \sum_{k=0}^K \int_0^T |Z_k(s)|^2 ds + \sum_{k=1}^K \int_0^T |\Upsilon_{k,0}(s)|^2 ds \right\}. \end{aligned} \quad (19)$$

Let $\overline{\mathcal{N}}[0, T]$ be the completion of $\mathcal{N}[0, T]$ in $L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^1}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}^K}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^{W_0}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^1}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}^K}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^1}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}^K}(0, T; \mathbb{R}^n)$ under norm (19). Now for any

$$(Y_0^j, Y_1^j, \dots, Y_K^j, Z_0^j, Z_1^j, \dots, Z_K^j, \Upsilon_{1,0}^j, \dots, \Upsilon_{K,0}^j) \in \mathcal{N}[0, T], \quad j = 1, 2,$$

we solve respectively the following system including $K + 1$ SDEs, for $1 \leq k \leq K$:

$$\left\{ \begin{aligned} d\alpha_0^j &= \left(A_0 \alpha_0^j + B_0 \varphi_0(Y_0^j, Z_0^j) + F_0^1 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t) | \mathcal{F}_t^{W_0}] + b_0 \right) dt \\ &\quad + \left(C_0 \alpha_0^j + D_0 \varphi_0(Y_0^j, Z_0^j) + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t) | \mathcal{F}_t^{W_0}] + \sigma_0 \right) dW_0(t) \\ d\alpha_k^j &= \left(A_k \alpha_k^j + B \varphi_k(Y_k^j, Z_k^j) + F_1 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t) | \mathcal{F}_t^{W_0}] + b \right) dt \\ &\quad + \left(C \alpha_k^j + D_k \varphi_k(Y_k^j, Z_k^j) + F_2 \sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i^j(t) | \mathcal{F}_t^{W_0}] + H \alpha_0^j + \sigma \right) dW_k(t) \\ \alpha_0^j(0) &= x_0, \quad \alpha_k^j(0) = x. \end{aligned} \right. \quad (20)$$

Then (20) admits a unique solution for $j = 1, 2$,

$$(\alpha_0^j, \alpha_1^j, \dots, \alpha_K^j) \in L^2_{\mathcal{F}^{W_0}}(\Omega; C(0, T; \mathbb{R}^n)) \times L^2_{\mathcal{F}^1}(\Omega; C(0, T; \mathbb{R}^n)) \times \dots \times L^2_{\mathcal{F}^K}(\Omega; C(0, T; \mathbb{R}^n)).$$

Indeed, (20) is a $n(K + 1)$ -dimensional SDE with the mean-field term $\sum_{i=1}^K \pi_i \mathbb{E}[\alpha_i(t) | \mathcal{F}_t^{W_0}]$. We can prove the well-posedness of such SDEs system by noticing $\mathbb{E}|\mathbb{E}[\alpha_i(t) | \mathcal{F}_t^{W_0}]|^2 \leq \mathbb{E}|\alpha_i(t)|^2$ and

by constructing a fixed point using the classical contraction mapping method, we omit the proof here. Now let us denote for $0 \leq k \leq K$,

$$\begin{aligned}\hat{\alpha}_k &:= \alpha_k^1 - \alpha_k^2, & \hat{\varphi}_k &:= \varphi_k(Y_k^1, Z_k^1) - \varphi_k(Y_k^2, Z_k^2) \\ \hat{Y}_k &:= Y_k^1 - Y_k^2, & \hat{Z}_k &:= Z_k^1 - Z_k^2, & \hat{\Upsilon}_{k,0} &:= \Upsilon_{k,0}^1 - \Upsilon_{k,0}^2.\end{aligned}$$

Applying Itô's formula, we obtain

$$\begin{aligned}d|\hat{\alpha}_0|^2 &= 2\left\langle \hat{\alpha}_0, A_0\hat{\alpha}_0 + B_0\hat{\varphi}_0 + F_0^1 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] \right\rangle dt \\ &\quad + \left| C_0\hat{\alpha}_0 + D_0\hat{\varphi}_0 + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] \right|^2 dt \\ &\quad + 2\left\langle \hat{\alpha}_0, C_0\hat{\alpha}_0 + D_0\hat{\varphi}_0 + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] \right\rangle dW_0(t)\end{aligned}$$

and

$$\begin{aligned}d|\hat{\alpha}_k|^2 &= 2\left\langle \hat{\alpha}_k, A_k\hat{\alpha}_k + B\hat{\varphi}_k + F_1 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] \right\rangle dt \\ &\quad + \left| C\hat{\alpha}_k + D_k\hat{\varphi}_k + F_2 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] + H\hat{\alpha}_0 \right|^2 \\ &\quad + 2\left\langle \hat{\alpha}_k, C\hat{\alpha}_k + D_k\hat{\varphi}_k + F_2 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(t) | \mathcal{F}_t^{W_0}] + H\hat{\alpha}_0 \right\rangle dW_k(t).\end{aligned}$$

Thus by using **(A2)**-**(A3)**, $\mathbb{E}|\mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}]| \leq \mathbb{E}|\hat{\alpha}_i(s)|$, $\mathbb{E}|\mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}]|^2 \leq \mathbb{E}|\hat{\alpha}_i(s)|^2$ as well as that φ_k is Lipschitz with Lipschitz constant 1 (see Proposition 4.2 of [21]), we have

$$\begin{aligned}\mathbb{E}|\hat{\alpha}_0(t)|^2 &\leq 2\mathbb{E} \int_0^t \left(|A_0|\hat{\alpha}_0|^2 + |B_0|\hat{\alpha}_0|\hat{\varphi}_0| + |F_0^1|\hat{\alpha}_0| \sum_{i=1}^K |\mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}]| \right) ds \\ &\quad + \mathbb{E} \int_0^t \left| C_0\hat{\alpha}_0 + D_0\hat{\varphi}_0 + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}] \right|^2 ds \\ &\leq C_\varepsilon \mathbb{E} \int_0^t \sum_{i=0}^K |\hat{\alpha}_i|^2 ds + \mathbb{E} \int_0^t (|D_0|^2 + \varepsilon)(|\hat{Y}_0|^2 + |\hat{Z}_0|^2) ds\end{aligned}\tag{21}$$

and

$$\begin{aligned}\mathbb{E}|\hat{\alpha}_k(t)|^2 &\leq 2\mathbb{E} \int_0^t \left(|A_k|\hat{\alpha}_k|^2 + |B|\hat{\alpha}_k|\hat{\varphi}_k| + |F_1|\hat{\alpha}_k| \sum_{i=1}^K |\mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}]| \right) ds \\ &\quad + \mathbb{E} \int_0^t \left| C\hat{\alpha}_k + D_k\hat{\varphi}_k + \sum_{i=1}^K F_2 \mathbb{E}[\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}] + H\hat{\alpha}_0 \right|^2 ds \\ &\leq C_\varepsilon \mathbb{E} \int_0^t \sum_{i=0}^K |\hat{\alpha}_i|^2 ds + \mathbb{E} \int_0^t (|D_k|^2 + \varepsilon)(|\hat{Y}_k|^2 + |\hat{Z}_k|^2) ds,\end{aligned}\tag{22}$$

where C_ε is a constant independent of T , which may vary line by line. Adding up (21) and (22) for $1 \leq k \leq K$, we have

$$\mathbb{E} \sum_{i=0}^K |\hat{\alpha}_i(t)|^2 \leq C_\varepsilon \mathbb{E} \int_0^t \sum_{i=0}^K |\hat{\alpha}_i(s)|^2 ds + \mathbb{E} \int_0^t \sum_{i=0}^K (|D_i|^2 + \varepsilon)(|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds,$$

Similarly, by applying Itô's formula to $|\hat{\beta}_k(t)|^2$, $1 \leq k \leq K$, we have

$$\begin{aligned} & \mathbb{E} \left| \hat{\beta}_k(t) \right|^2 + \mathbb{E} \int_t^T |\hat{\gamma}_k|^2 ds + \mathbb{E} \int_t^T |\hat{\gamma}_{k,0}|^2 ds \\ &= \mathbb{E} \left| \hat{\beta}_k(T) \right|^2 + 2\mathbb{E} \int_t^T \left\langle \hat{\beta}_k, A'_k \hat{Y}_k - Q \left(\hat{\alpha}_k - \rho \sum_{i=1}^K \pi_i \mathbb{E} [\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}] - (1-\rho)\hat{\alpha}_0 \right) - C' \hat{Z}_k \right\rangle ds. \end{aligned} \quad (26)$$

Noticing that $\rho \in [0, 1]$, we have

$$\begin{aligned} \mathbb{E} \left| \hat{\beta}_k(T) \right|^2 &\leq |G|^2 \left| \hat{\alpha}_k(T) - \rho \sum_{i=1}^K \pi_i \mathbb{E} [\hat{\alpha}_i(T) | \mathcal{F}_T^{W_0}] - (1-\rho)\hat{\alpha}_0(T) \right|^2 \\ &\leq |G|^2 (1 + \rho^2 + (1-\rho)^2) \left(\mathbb{E} \left| \hat{\alpha}_k(T) - \rho \mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}] \right|^2 + \sum_{i=0, i \neq k}^K \mathbb{E} |\hat{\alpha}_i(T)|^2 \right) \\ &\leq |G|^2 (1 + \rho^2 + (1-\rho)^2) \mathbb{E} \sum_{i=0}^K |\hat{\alpha}_i(T)|^2, \end{aligned}$$

where we have used the fact that

$$\begin{aligned} & \mathbb{E} \left| \hat{\alpha}_k(T) - \rho \mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}] \right|^2 \\ &= \mathbb{E} |\hat{\alpha}_k(T)|^2 + \rho^2 \mathbb{E} |\mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}]|^2 - 2\rho \mathbb{E} \left[\hat{\alpha}_k(T) \mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}] \right] \\ &= \mathbb{E} |\hat{\alpha}_k(T)|^2 + \rho^2 \mathbb{E} |\mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}]|^2 - 2\rho \mathbb{E} \left(\mathbb{E} \left[\hat{\alpha}_k(T) \mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}] \right] | \mathcal{F}_T^{W_0} \right) \\ &= \mathbb{E} |\hat{\alpha}_k(T)|^2 + (\rho^2 - 2\rho) \mathbb{E} |\mathbb{E} [\hat{\alpha}_k(T) | \mathcal{F}_T^{W_0}]|^2 \leq \mathbb{E} |\hat{\alpha}_k(T)|^2. \end{aligned}$$

Thus, (26) yields that

$$\begin{aligned} & \mathbb{E} \left| \hat{\beta}_k(t) \right|^2 + \mathbb{E} \int_t^T |\hat{\gamma}_k|^2 ds + \mathbb{E} \int_t^T |\hat{\gamma}_{k,0}|^2 ds \\ &= \mathbb{E} \left| \hat{\beta}_k(T) \right|^2 + 2\mathbb{E} \int_t^T \left\langle \hat{\beta}_k, A'_k \hat{Y}_k - Q \left(\hat{\alpha}_k - \rho \sum_{i=1}^K \pi_i \mathbb{E} [\hat{\alpha}_i(s) | \mathcal{F}_s^{W_0}] - (1-\rho)\hat{\alpha}_0 \right) - C' \hat{Z}_k \right\rangle ds \\ &\leq |G|^2 (1 + \rho^2 + (1-\rho)^2) \mathbb{E} \sum_{i=0}^K |\hat{\alpha}_i(T)|^2 + C_\varepsilon \mathbb{E} \int_t^T |\beta_k|^2 ds \\ &\quad + \mathbb{E} \int_t^T \sum_{i=0}^K |\alpha_i|^2 ds + \varepsilon \mathbb{E} \int_t^T (|\hat{Y}_k|^2 + |\hat{Z}_k|^2) ds. \end{aligned}$$

Substituting (23) into above inequality, we have

$$\begin{aligned} & \mathbb{E} \left| \hat{\beta}_k(t) \right|^2 + \mathbb{E} \int_t^T |\hat{\gamma}_k|^2 ds + \mathbb{E} \int_t^T |\hat{\gamma}_{k,0}|^2 ds \\ &\leq (|G|^2 (1 + \rho^2 + (1-\rho)^2) + T) e^{C_\varepsilon T} \mathbb{E} \sum_{i=0}^K \int_0^T (|D_i|^2 + \varepsilon) (|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds \\ &\quad + C_\varepsilon \mathbb{E} \int_t^T |\beta_k|^2 ds + \varepsilon \mathbb{E} \int_t^T (|\hat{Y}_k|^2 + |\hat{Z}_k|^2) ds. \end{aligned} \quad (27)$$

Adding up (25) and (27) for $1 \leq k \leq K$, we obtain (recall $|D|^2 := \max_{0 \leq k \leq K} |D_k|^2$ and $M_0 := \max \{|G_0|^2(1 + \rho_0^2), |G|^2(1 + \rho^2 + (1 - \rho)^2)\}$)

$$\begin{aligned}
& \mathbb{E} \sum_{i=0}^K \left| \hat{\beta}_i \right|^2 + \mathbb{E} \sum_{i=0}^K \int_t^T |\hat{\gamma}_i|^2 ds + \mathbb{E} \sum_{i=1}^K \int_t^T |\hat{\gamma}_{i,0}|^2 ds \\
& \leq (M_0 + T) e^{C_\varepsilon T} \mathbb{E} \sum_{i=0}^K \int_0^T (|D_i|^2 + \varepsilon) (|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds \\
& \quad + C_\varepsilon \mathbb{E} \int_t^T \sum_{i=0}^K \left| \hat{\beta}_i \right|^2 ds + \varepsilon \mathbb{E} \int_0^T \sum_{i=0}^K (|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds \\
& \leq C_\varepsilon \mathbb{E} \int_t^T \sum_{i=0}^K \left| \hat{\beta}_i \right|^2 ds + [(M_0 + T) e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon] \mathbb{E} \sum_{i=0}^K \int_0^T (|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds.
\end{aligned}$$

The Gronwall's inequality yields that

$$\begin{aligned}
& \mathbb{E} \sum_{i=0}^K \left| \hat{\beta}_i \right|^2 + \mathbb{E} \sum_{i=0}^K \int_t^T |\hat{\gamma}_i|^2 ds + \mathbb{E} \sum_{i=1}^K \int_t^T |\hat{\gamma}_{i,0}|^2 ds \\
& \leq e^{C_\varepsilon T} [(M_0 + T) e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon] \mathbb{E} \sum_{i=0}^K \int_0^T (|\hat{Y}_i|^2 + |\hat{Z}_i|^2) ds \\
& \leq e^{C_\varepsilon T} T [(M_0 + T) e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon] \sup_{0 \leq t \leq T} \mathbb{E} \sum_{i=0}^K |\hat{Y}_i(t)|^2 \\
& \quad + e^{C_\varepsilon T} [(M_0 + T) e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon] \mathbb{E} \sum_{i=0}^K \int_0^T |\hat{Z}_i|^2 ds \\
& \leq e^{C_\varepsilon T} (T + 1) [(M_0 + T) e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon] \\
& \quad \cdot \|(\hat{Y}_0, \hat{Y}_1, \dots, \hat{Y}_K, \hat{Z}_0, \hat{Z}_1, \dots, \hat{Z}_K, \hat{\Upsilon}_{1,0}, \dots, \hat{\Upsilon}_{K,0})\|_{\overline{\mathcal{N}}[0,T]} \\
& = e^{C_\varepsilon T} (T + 1) [M_0 e^{C_\varepsilon T} (|D|^2 + \varepsilon) + \varepsilon + T e^{C_\varepsilon T} (|D|^2 + \varepsilon)] \\
& \quad \cdot \|(\hat{Y}_0, \hat{Y}_1, \dots, \hat{Y}_K, \hat{Z}_0, \hat{Z}_1, \dots, \hat{Z}_K, \hat{\Upsilon}_{1,0}, \dots, \hat{\Upsilon}_{K,0})\|_{\overline{\mathcal{N}}[0,T]}
\end{aligned} \tag{28}$$

Noticing that assumption **(A4)** holds, by first choosing $\varepsilon > 0$ small enough such that $M_0(|D|^2 + \varepsilon) + \varepsilon < 1$, then choosing $T > 0$ small enough, we obtain from (28) that

$$\begin{aligned}
& \|(\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_K, \hat{\gamma}_0, \hat{\gamma}_1, \dots, \hat{\gamma}_K, \hat{\gamma}_{1,0}, \dots, \hat{\gamma}_{K,0})\|_{\overline{\mathcal{N}}[0,T]} \\
& \leq \delta \|(\hat{Y}_0, \hat{Y}_1, \dots, \hat{Y}_K, \hat{Z}_0, \hat{Z}_1, \dots, \hat{Z}_K, \hat{\Upsilon}_{1,0}, \dots, \hat{\Upsilon}_{K,0})\|_{\overline{\mathcal{N}}[0,T]},
\end{aligned}$$

for some $0 < \delta < 1$. This means that the mapping $\mathcal{T} : \overline{\mathcal{N}}[0, T] \rightarrow \overline{\mathcal{N}}[0, T]$ is contractive. By the contraction mapping theorem, there exists a unique fixed point

$$(\beta_0, \beta_1, \dots, \beta_K, \gamma_0, \gamma_1, \dots, \gamma_K, \gamma_{1,0}, \dots, \gamma_{K,0}) \in \overline{\mathcal{N}}[0, T].$$

Moreover, classical BSDE theory allows us to show that

$$(\beta_0, \beta_1, \dots, \beta_K, \gamma_0, \gamma_1, \dots, \gamma_K, \gamma_{1,0}, \dots, \gamma_{K,0}) \in \mathcal{N}[0, T].$$

Let α_k , $0 \leq k \leq K$, be the corresponding solution of (20). Then, one can obtain that the system (15)-(16) has a unique solution $(\alpha_0, \beta_0, \gamma_0, \alpha_k, \beta_k, \gamma_k, \gamma_{k,0})$, $1 \leq k \leq K$, such that (18) holds. \square

5 Existence and uniqueness of consistency condition system-global time horizon case

The section aims to establish the well-posedness of consistency condition system (15)-(16) for arbitrary T , we first study one general kind of conditional mean-field forward-backward stochastic differential equations (MF-FBSDE) by using the discounting method of Pardoux and Tang [39].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete, filtered probability space satisfying usual conditions and $\{W_i(t), 0 \leq i \leq d\}_{0 \leq t \leq T}$ is a $d+1$ -dimensional Brownian motion on this space. Let \mathcal{F}_t be the natural filtration generated by $\{W_i(s), 0 \leq i \leq d, 0 \leq s \leq t\}$ and augmented by $\mathcal{N}_{\mathbb{P}}$ (the class of \mathbb{P} -null sets of \mathcal{F}). Let $\mathcal{F}_t^{W_0}$ be the augmentation of $\sigma\{W_0(s), 0 \leq s \leq t\}$ by $\mathcal{N}_{\mathbb{P}}$. We consider the following general conditional MF-FBSDE:

$$\begin{cases} dX(s) = b(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s))ds + \sigma(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s))dW(s), \\ -dY(s) = f(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s))ds - Z(s)dW(s), \quad s \in [0, T], \\ X(0) = x, \quad Y(T) = g(X(T), \mathbb{E}[X(T)|\mathcal{F}_T^{W_0}]), \end{cases} \quad (29)$$

where the adapted processes X, Y, Z take their values in $\mathbb{R}^n, \mathbb{R}^l$ and $\mathbb{R}^{l \times (d+1)}$, respectively. The coefficients b, σ and f are defined on $\Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l \times (d+1)}$, such that $b(\cdot, \cdot, x, m, y, z)$, $\sigma(\cdot, \cdot, x, m, y, z)$ and $f(\cdot, \cdot, x, m, y, z)$ are $\{\mathcal{F}_t\}$ -progressively measurable processes, for all fixed $(x, m, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l \times (d+1)}$. The coefficient g is defined on $\Omega \times \mathbb{R}^n \times \mathbb{R}^n$ and $g(\cdot, x, m)$ is \mathcal{F}_T -measurable, for all fixed $(x, m) \in \mathbb{R}^n \times \mathbb{R}^n$. Moreover, the functions b, σ, f and g are continuous w.r.t. $(x, m, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l \times (d+1)}$ and satisfy the following assumptions:

(H₁) There exist $\lambda_1, \lambda_2 \in \mathbb{R}$ and positive constants $k_0, k_i, i = 1, 2, \dots, 12$ such that for all $t, x, x_1, x_2, m, m_1, m_2, y, y_1, y_2, z, z_1, z_2$ a.s.

- (i) $\langle b(t, x_1, m, y, z) - b(t, x_2, m, y, z), x_1 - x_2 \rangle \leq \lambda_1 |x_1 - x_2|^2$,
- (ii) $|b(t, x, m_1, y_1, z_1) - b(t, x, m_2, y_2, z_2)| \leq k_1 |m_1 - m_2| + k_2 |y_1 - y_2| + k_3 |z_1 - z_2|$,
- (iii) $|b(t, x, m, y, z)| \leq |b(t, 0, m, y, z)| + k_0(1 + |x|)$,
- (iv) $\langle f(t, x, m, y_1, z) - f(t, x, m, y_2, z), y_1 - y_2 \rangle \leq \lambda_2 |y_1 - y_2|^2$,
- (v) $|f(t, x_1, m_1, y, z_1) - f(t, x_1, m_2, y, z_2)| \leq k_4 |x_1 - x_2| + k_5 |m_1 - m_2| + k_6 |z_1 - z_2|$,
- (vi) $|f(t, x, m, y, z)| \leq |f(t, x, m, 0, z)| + k_0(1 + |y|)$,
- (vii) $|\sigma(t, x_1, m_1, y_1, z_1) - \sigma(t, x_2, m_2, y_2, z_2)|^2$
 $\leq k_7^2 |x_1 - x_2|^2 + k_8^2 |m_1 - m_2|^2 + k_9^2 |y_1 - y_2|^2 + k_{10}^2 |z_1 - z_2|^2$,
- (viii) $|g(x_1, m_1) - g(x_2, m_2)|^2 \leq k_{11}^2 |x_1 - x_2| + k_{12}^2 |m_1 - m_2|$.

(H₂) It holds that

$$\mathbb{E} \int_0^T (|b(s, 0, 0, 0, 0)|^2 + |\sigma(s, 0, 0, 0, 0)|^2 + |f(s, 0, 0, 0, 0)|^2) ds + \mathbb{E}|g(0, 0)|^2 < +\infty.$$

Remark 5.1 From the mean-field structure of (29), sometimes the following conditions holds:

(H₁) - (i') : There exist $\lambda_1, \hat{k}_1 \in \mathbb{R}$, such that for all t, y, z and process X_1, X_2 , a.s.

$$\begin{aligned} & \mathbb{E} \langle b(t, X_1, \mathbb{E}[X_1(t)|\mathcal{F}_t^{W_0}], y, z) - b(t, X_2, \mathbb{E}[X_2(t)|\mathcal{F}_t^{W_0}], y, z), X_1 - X_2 \rangle \\ & \leq (\lambda_1 + \hat{k}_1) \mathbb{E}|X_1 - X_2|^2. \end{aligned}$$

For example, if $b(t, x, m, y, z) = \lambda_1 x + \widehat{k}_1 m + b_1(y, z)$, then it obviously satisfies above assumption. Indeed, our mean-filed FBSDE satisfies this assumption.

Let \mathcal{H} be an Euclidean space. Recall that $L_{\mathcal{F}}^2(0, T; \mathcal{H})$, denotes the Hilbert space of \mathcal{H} -valued $\{\mathcal{F}_s\}$ -progressively measurable processes $\{u(s), s \in [0, T]\}$ such that

$$\|u\| := \left(E \int_0^T |u(s)|^2 ds \right)^{1/2} < \infty.$$

For $\lambda \in \mathbb{R}$, we define an equivalent norm on $L_{\mathcal{F}}^2(0, T; \mathcal{H})$:

$$\|u\|_{\lambda} := \left(E \int_0^T e^{-\lambda s} |u(s)|^2 ds \right)^{1/2}.$$

Now let us consider MF-FBSDE (29), its fully-coupled structure arises difficulties for establishing its wellposedness. Similar to [39], when the coupling is weak enough, MF-FBSDE (29) should be solvable. The following is the main results on MF-FBSDE (29) and the proof is give in the appendix.

Theorem 5.1 *Suppose that assumption (H_1) and (H_2) hold. Then there exists a $\delta_0 > 0$, which depends on $k_i, \lambda_1, \lambda_2, T$, for $i = 1, 4, 5, 6, 7, 8, 11, 12$ such that when $k_2, k_3, k_9, k_{10} \in [0, \delta_0)$, there exists a unique adapted solution $(X, Y, Z) \in L_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m \times (d+1)})$ to MF-FBSDE (29). Further, if $2(\lambda_1 + \lambda_2) < -2k_1 - k_6^2 - k_7^2 - k_8^2$, there exists a $\delta_1 > 0$, which depends on $k_i, \lambda_1, \lambda_2$, for $i = 1, 4, 5, 6, 7, 8, 11, 12$ and is independent of T , such that when $k_2, k_3, k_9, k_{10} \in [0, \delta_1)$, there exists a unique adapted solution (X, Y, Z) to MF-FBSDE (29).*

Remark 5.2 *If in additional $(H_1) - (i')$ holds (see Remark 5.1), by repeating the above discussion, one can show that if $2(\lambda_1 + \lambda_2) < -2\widehat{k}_1 - k_6^2 - k_7^2 - k_8^2$, there exists a $\delta_1 > 0$, which depends on $\widehat{k}_1, k_i, \lambda_1, \lambda_2$, for $i = 4, 5, 6, 7, 8, 11, 12$ and is independent of T , such that when $k_2, k_3, k_9, k_{10} \in [0, \delta_1)$, there exists a unique adapted solution (X, Y, Z) to MF-FBSDE (29).*

Now let us apply Theorem 5.1 to obtain the well-posedness of consistency condition system (15)-(16) which is

$$\left\{ \begin{array}{l} d\alpha_k = \left(A_k \alpha_k + B \mathbf{P}_{\Gamma_k} [R_k^{-1} (B' \beta_k + D'_k \gamma_k)] + F_1 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) + b \right) dt \\ \quad + \left(C \alpha_k + D_k \mathbf{P}_{\Gamma_k} [R_k^{-1} (B' \beta_k + D'_k \gamma_k)] + F_2 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) + H \alpha_0 + \sigma \right) dW_k(t), \\ d\beta_k = - \left(A'_k \beta_k - Q(\alpha_k - \rho \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t) | \mathcal{F}_t^{W_0}) - (1 - \rho) \alpha_0) + C' \gamma_k \right) dt + \gamma_k dW_k(t) + \gamma_{k,0} dW_0(t), \\ \alpha_k(0) = x, \quad \beta_k(T) = -G(\alpha_k(T) - \rho \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(T) | \mathcal{F}_T^{W_0}) - (1 - \rho) \alpha_0(T)), \end{array} \right.$$

where α_0 satisfies the following FBSDE which is coupled with all k -type minor agents:

$$\left\{ \begin{array}{l} d\alpha_0 = \left(A_0\alpha_0 + B_0\mathbf{P}_{\Gamma_0}[R_0^{-1}(B'_0\beta_0 + D'_0\gamma_0)] + F_0^1 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) + b_0 \right) dt \\ \quad + \left(C_0\alpha_0 + D_0\mathbf{P}_{\Gamma_0}[R_0^{-1}(B'_0\beta_0 + D'_0\gamma_0)] + F_0^2 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0}) + \sigma_0 \right) dW_0(t), \\ d\beta_0 = -(A'_0\beta_0 - Q_0(\alpha_0 - \rho_0 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(t)|\mathcal{F}_t^{W_0})) + C'_0\gamma_0) dt + \gamma_0 dW_0(t), \\ \alpha_0(0) = x_0, \quad \beta_0(T) = -G_0(\alpha_0(T) - \rho_0 \sum_{i=1}^K \pi_i \mathbb{E}(\alpha_i(T)|\mathcal{F}_T^{W_0})). \end{array} \right.$$

Recall that

$$\varphi_0(p, q) = \mathbf{P}_{\Gamma_0}[R_0^{-1}(B'_0p + D'_0q)], \quad \varphi_k(p, q) = \mathbf{P}_{\Gamma_k}[R_k^{-1}(B'_kp + D'_kq)].$$

If we denote

$$\begin{aligned} W &= (W_0, W_1, \dots, W_K)', \quad \Pi = (0, \pi_1, \dots, \pi_K), \quad \alpha = (\alpha'_0, \alpha'_1, \dots, \alpha'_K)', \quad \beta = (\beta'_0, \beta'_1, \dots, \beta'_K)', \\ \mathbb{X} &= (x'_0, x'_1, \dots, x'_K)', \quad \mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0}) = (\mathbb{E}(\alpha_0(t)|\mathcal{F}_t^{W_0}), \mathbb{E}(\alpha_1(t)|\mathcal{F}_t^{W_0}), \dots, \mathbb{E}(\alpha_K(t)|\mathcal{F}_t^{W_0}))', \\ \Phi(\beta, \gamma) &= (\varphi_0(\beta_0, \gamma_0), \varphi_1(\beta_1, \gamma_1), \dots, \varphi_K(\beta_K, \gamma_K))', \quad \rho_0^\Pi := \Pi \otimes \rho I_{n \times n}, \quad \rho^\Pi := \Pi \otimes \rho I_{n \times n}, \\ F_0^{1, \Pi} &= \Pi \otimes F_0^1, \quad F_0^{2, \Pi} := \Pi \otimes F_0^2, \quad F^{1, \Pi} := \Pi \otimes F_1, \quad F^{2, \Pi} := \Pi \otimes F_2, \\ \gamma &= \begin{pmatrix} \gamma_0 & 0 & 0 & \dots & 0 \\ \gamma_{1,0} & \gamma_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots \\ \gamma_{K,0} & 0 & 0 & \dots & \gamma_K \end{pmatrix}, \quad \mathbb{B}_0 = \begin{pmatrix} b_0 \\ b \\ \vdots \\ b \end{pmatrix}, \quad \mathbb{D}_0 = \begin{pmatrix} \sigma_0 & 0 & 0 & \dots \\ 0 & \sigma & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma \end{pmatrix}, \\ \mathbb{A} &= \begin{pmatrix} A_0 & 0 & 0 & \dots \\ 0 & A_1 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_K \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} B_0 & 0 & 0 & \dots \\ 0 & B & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B \end{pmatrix}, \quad \mathbb{R}^{-1} = \begin{pmatrix} R_0 & 0 & 0 & \dots \\ 0 & R_1 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_K \end{pmatrix}, \\ \mathbb{Q} &= \begin{pmatrix} Q_0 & 0 & 0 & \dots \\ Q(1-\rho) & Q & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ Q(1-\rho) & 0 & \dots & Q \end{pmatrix}, \quad \mathbb{G} = \begin{pmatrix} G_0 & 0 & 0 & \dots \\ G(1-\rho) & G & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ G(1-\rho) & 0 & \dots & G \end{pmatrix}, \quad \mathbb{F}_1^\Pi = \begin{pmatrix} F_0^{1, \Pi} \\ F^{1, \Pi} \\ \vdots \\ F^{1, \Pi} \end{pmatrix}, \quad \mathbb{F}_2^\Pi = \begin{pmatrix} F_0^{2, \Pi} \\ F^{2, \Pi} \\ \vdots \\ F^{2, \Pi} \end{pmatrix}, \\ \mathbb{Q}^\Pi &= \begin{pmatrix} Q_0 \rho_0^\Pi \\ Q \rho^\Pi \\ \vdots \\ Q \rho^\Pi \end{pmatrix}, \quad \mathbb{G}^\Pi = \begin{pmatrix} G_0 \rho_0^\Pi \\ G \rho^\Pi \\ \vdots \\ G \rho^\Pi \end{pmatrix}, \quad \mathbb{H} = \begin{pmatrix} 0 \\ H \\ \vdots \\ H \end{pmatrix}, \quad \mathbb{H}(\alpha) = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & H\alpha_0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H\alpha_0 \end{pmatrix}, \\ \mathbb{F}_2^\Pi(\mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0})) &= \begin{pmatrix} F_0^{2, \Pi} \mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0}) & 0 & 0 & \dots \\ 0 & F^{2, \Pi} \mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0}) & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F^{2, \Pi} \mathbb{E}(\alpha(t)|\mathcal{F}_t^{W_0}) \end{pmatrix}, \end{aligned}$$

$$\mathbb{C} = \begin{pmatrix} C_0 & 0 & 0 & \dots \\ 0 & C & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C \end{pmatrix}, \quad \mathbb{C}(\alpha) = \begin{pmatrix} C_0\alpha_0 & 0 & 0 & \dots \\ 0 & C\alpha & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C\alpha \end{pmatrix}, \quad \mathbb{C}(\gamma) = \begin{pmatrix} C'_0\gamma_0 \\ C'\gamma_1 \\ \vdots \\ C'\gamma_K \end{pmatrix},$$

$$\mathbb{D} = \begin{pmatrix} D_0 & 0 & 0 & \dots \\ 0 & D_1 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_K \end{pmatrix}, \quad \mathbb{D}(\beta, \gamma) = \begin{pmatrix} D_0\varphi_0(\beta_0, \gamma_0) & 0 & 0 & \dots \\ 0 & D_1\varphi_1(\beta_1, \gamma_1) & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_K\varphi_K(\beta_K, \gamma_K) \end{pmatrix}.$$

Using above notations, the system (15)-(16) can be written as

$$\begin{cases} d\alpha = \left(\mathbb{A}\alpha + \mathbb{B}\Phi(\beta, \gamma) + \mathbb{F}_1^\Pi \mathbb{E}(\alpha(t) | \mathcal{F}_t^{W_0}) + \mathbb{B}_0 \right) dt \\ \quad + \left(\mathbb{C}(\alpha) + \mathbb{D}(\beta, \gamma) + \mathbb{F}_2^\Pi \left(\mathbb{E}(\alpha(t) | \mathcal{F}_t^{W_0}) \right) + \mathbb{H}(\alpha) + \mathbb{D}_0 \right) dW(t), \\ d\beta = - \left(\mathbb{A}'\beta - \mathbb{Q}\alpha + \mathbb{Q}^\Pi \mathbb{E}(\alpha(t) | \mathcal{F}_t^{W_0}) \right) dt + \gamma dW(t), \\ \alpha(0) = \mathbb{X}, \quad \beta(T) = -\mathbb{G}\alpha(T) + \mathbb{G}^\Pi \mathbb{E}(\alpha(T) | \mathcal{F}_T^{W_0}). \end{cases} \quad (30)$$

Now let λ^* be the largest eigenvalue of the symmetric matrix $\frac{1}{2}(\mathbb{A} + \mathbb{A}')$. Recalling that the projection operator is Lipschitz continuous with Lipschitz constant 1, then by comparing (30) with (29), one can check that the coefficients of Assumption (H_1) can be chosen as following

$$\begin{aligned} \lambda_1 = \lambda_2 = \lambda^*, \quad k_0 = \|\mathbb{A}\|, \quad k_1 = \|\mathbb{F}_1^\Pi\|, \quad k_2 = k_3 = \|\mathbb{R}^{-1}\| \|\mathbb{B}\| (\|\mathbb{B}\| + \|\mathbb{D}\|), \\ k_4 = \|\mathbb{Q}\|, \quad k_5 = \|\mathbb{Q}^\Pi\|, \quad k_6 = \|\mathbb{C}\|, \quad k_7^2 = 4(\|\mathbb{C}\| + \|\mathbb{H}\|)^2, \quad k_8^2 = 4\|\mathbb{F}_2^\Pi\|^2, \\ k_9 = k_{10} = \|\mathbb{R}^{-1}\| \|\mathbb{D}\| (\|\mathbb{B}\| + \|\mathbb{D}\|), \quad k_{11}^2 = 2\|\mathbb{G}\|^2, \quad k_{12}^2 = 2\|\mathbb{G}^\Pi\|^2. \end{aligned}$$

Thus by applying Theorem 5.1, we obtain the following global wellposedness of (30).

Theorem 5.2 *Suppose that*

$$4\lambda^* < -2\|\mathbb{F}_1^\Pi\| - \|\mathbb{C}\|^2 - 4(\|\mathbb{C}\| + \|\mathbb{H}\|)^2 - 4\|\mathbb{F}_2^\Pi\|^2,$$

then there exists a $\delta_1 > 0$, which depends on $\lambda^, \|\mathbb{F}_1^\Pi\|, \|\mathbb{Q}\|, \|\mathbb{Q}^\Pi\|, \|\mathbb{C}\|, \|\mathbb{H}\|, \|\mathbb{F}_2^\Pi\|, \|\mathbb{G}\|, \|\mathbb{G}^\Pi\|$, and is independent of T , such that when $\|\mathbb{R}^{-1}\|, \|\mathbb{B}\|, \|\mathbb{D}\| \in [0, \delta_1)$, there exists a unique adapted solution (α, β, γ) to consistency condition system (15)-(16).*

Remark 5.3 *Let $\lambda_{\mathbb{F}_1^\Pi}^*$ be the largest eigenvalue of $\frac{1}{2}(\mathbb{F}_1^\Pi + (\mathbb{F}_1^\Pi)')$. Noticing Remark 5.1, one can check that $(H_1) - (i')$ holds with $\hat{k}_1 = \lambda_{\mathbb{F}_1^\Pi}^*$. Thus, from Remark 5.2, we have that if*

$$4\lambda^* < -2\lambda_{\mathbb{F}_1^\Pi}^* - \|\mathbb{C}\|^2 - 4(\|\mathbb{C}\| + \|\mathbb{H}\|)^2 - 4\|\mathbb{F}_2^\Pi\|^2,$$

then there exists a $\delta_1 > 0$, which depends on $\lambda^, \lambda_{\mathbb{F}_1^\Pi}^*, \|\mathbb{Q}\|, \|\mathbb{Q}^\Pi\|, \|\mathbb{C}\|, \|\mathbb{H}\|, \|\mathbb{F}_2^\Pi\|, \|\mathbb{G}\|, \|\mathbb{G}^\Pi\|$, and is independent of T , such that when $\|\mathbb{R}^{-1}\|, \|\mathbb{B}\|, \|\mathbb{D}\| \in [0, \delta_1)$, there exists a unique adapted solution (α, β, γ) to consistency condition system (15)-(16).*

6 ε -Nash Equilibrium for Problem (CC)

In Section 2, we characterized the decentralized strategies $\{\bar{u}_t^i, 0 \leq i \leq N\}$ of **Problem (CC)** through the auxiliary **Problem (LCC)** and consistency condition system. Now, we turn to

verify the ε -Nash equilibrium of these decentralized strategies. Here, we proceed our verification based on the assumptions of local time horizon case (Section 4). Note that it can also be verified based on global horizon case (Section 5) without essential difficulties. For major agent \mathcal{A}_0 and minor agent \mathcal{A}_i , the decentralized states \check{x}_t^0 and \check{x}_t^i are given respectively by

$$\begin{cases} d\check{x}_0 = \left[A_0\check{x}_0 + B_0\varphi_0(\bar{p}_0, \bar{q}_0) + F_0^1\check{x}^{(N)} + b_0 \right] dt + \left[C_0\check{x}_0 + D_0\varphi_0(\bar{p}_0, \bar{q}_0) + F_0^2\check{x}^{(N)} + \sigma_0 \right] dW_0(t), \\ d\check{x}_i = \left[A_{\theta_i}\check{x}_i + B\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_1\check{x}^{(N)} + b \right] dt + \left[C\check{x}_i + D_{\theta_i}\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_2\check{x}^{(N)} + H\check{x}_0 + \sigma \right] dW_i(t), \\ \check{x}_0(0) = x_0, \quad \check{x}_i(0) = x, \end{cases} \quad (31)$$

where $\check{x}^{(N)} = \frac{1}{N} \sum_{i=1}^N \check{x}^i$ and the processes $(\bar{p}_0, \bar{q}_0, \bar{p}_i, \bar{q}_i)$ are solved by

$$\begin{cases} d\bar{x}_0 = \left(A_0\bar{x}_0 + B_0\varphi_0(\bar{p}_0, \bar{q}_0) + F_0^1 \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0}) + b_0 \right) dt \\ \quad + \left(C_0\bar{x}_0 + D_0\varphi_0(\bar{p}_0, \bar{q}_0) + F_0^2 \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0}) + \sigma_0 \right) dW_0(t), \\ d\bar{p}_0 = - \left(A'\bar{p}_0 - Q_0(\bar{x}_0 - \rho \sum_{i=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0})) + C'\bar{q}_0 \right) dt + \bar{q}_0 dW_0(t), \\ d\bar{x}_i = \left(A_{\theta_i}\bar{x}_i + B\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_1 \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0}) + b \right) dt \\ \quad + \left(C\bar{x}_i + D_{\theta_i}\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_2 \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0}) + H\bar{x}_0 + \sigma \right) dW_i(t), \\ d\bar{p}_i = - \left(A'_{\theta_i}\bar{p}_i - Q(\bar{x}_i - \rho \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0})) - (1-\rho)\bar{x}_0 + C'\bar{q}_i \right) dt + \bar{q}_i dW_i(t) + \bar{q}_{i,0} dW_0(t), \\ \bar{x}_0(0) = x_0, \quad \bar{p}_0(T) = -G_0(\bar{x}_0(T) - \rho \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(T) | \mathcal{F}_T^{W_0})), \\ \bar{x}_i(0) = x, \quad \bar{p}_i(T) = -G(\bar{x}_i(T) - \rho \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(T) | \mathcal{F}_T^{W_0}) - (1-\rho)\bar{x}_0(T)). \end{cases} \quad (32)$$

Here we recall that

$$\varphi_0(p, q) := \mathbf{P}_{\Gamma_0} [R_0^{-1}(B'_0 p + D'_0 q)], \quad \varphi_{\theta_i}(p, q) := \mathbf{P}_{\Gamma_{\theta_i}} [R_{\theta_i}^{-1}(B' p + D'_{\theta_i} q)],$$

and α_k , $1 \leq k \leq K$ are given by (15) and (16). We mention that (32) gives also the dynamics of the limiting state $(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_N)$ and one can check easily that $(\bar{x}_0, \bar{p}_0, \bar{q}_0) = (\alpha_0, \beta_0, \gamma_0)$. Now, we would like to show that for $\bar{u}_0 = \varphi_0(\bar{p}_0, \bar{q}_0)$ and $\bar{u}_i = \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i)$, $1 \leq i \leq N$, $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$ is an ε -Nash equilibrium of **Problem (CC)**. Let us first present following several lemmas.

Lemma 6.1 *Under (A1)-(A4), there exists a constant M independent of N , such that*

$$\sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} \left| \check{x}^i(t) \right|^2 \leq M.$$

Proof From Theorem 4.1, we know that on a small time interval the system of fully coupled FBSDE (15)-(16) has a unique solution (for the global case, see Theorem 5.2 and Remark 5.3)

$$(\alpha_0, \beta_0, \gamma_0) \in L^2_{\mathcal{F}W_0}(0, T; \mathbb{R}^{n \times 3}) \text{ and } (\alpha_k, \beta_k, \gamma_k, \gamma_{k,0}) \in L^2_{\mathcal{F}^k}(0, T; \mathbb{R}^{n \times 4}), \quad 1 \leq k \leq K.$$

Then, the classical results on FBSDEs yields that (32) also has a unique solution

$$(\bar{x}_0, \bar{p}_0, \bar{q}_0) \in L^2_{\mathcal{F}W_0}(0, T; \mathbb{R}^{n \times 3}) \text{ and } (\bar{x}_i, \bar{p}_i, \bar{q}_i, \bar{q}_{i,0}) \in L^2_{\mathcal{F}^i}(0, T; \mathbb{R}^{n \times 4}), \quad 1 \leq i \leq N.$$

(Indeed, FBSDEs (32) has a unique solution for arbitrary T by using the Theorem 2.1 of [21] and the results of [22, 40]). Thus, SDEs system (31) has also a unique solution

$$(\check{x}_0, \check{x}_1, \dots, \check{x}_N) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times \dots \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^n).$$

Moreover, since $\{W_i\}_{i=1}^N$ is N -dimensional Brownian motion whose components are independent identically distributed, we have that for the k -type minor agents, under the conditional expectation $\mathbb{E}(\cdot | \mathcal{F}^{W_0})$, for each $1 \leq k \leq K$, the processes $(\bar{x}_i, \bar{p}_i, \bar{q}_i)$, $i \in \mathcal{I}_k$, are independent identically distributed. We also note that for each $1 \leq k \leq K$, \check{x}_i , $i \in \mathcal{I}_k$, are identically distributed. Noticing that $(\bar{p}_0, \bar{q}_0) \in L^2_{\mathcal{F}W_0}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}W_0}(0, T; \mathbb{R}^n)$, and $(\bar{p}_i, \bar{q}_i) \in L^2_{\mathcal{F}^i}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}^i}(0, T; \mathbb{R}^n)$, $1 \leq i \leq N$, then the Lipschitz property of the projection onto the convex set yields that $\varphi_{\theta_0}(\bar{p}_0, \bar{q}_0) := \varphi_0(\bar{p}_0, \bar{q}_0) = \mathbf{P}_{\Gamma_0} \left(R_0^{-1} (B^T \bar{p}_0 + D^T \bar{q}_0) \right) \in L^2_{\mathcal{F}W_0}(0, T; \Gamma_0)$ and $\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) := \mathbf{P}_{\Gamma_{\theta_i}} \left(R_{\theta_i}^{-1} (B^T \bar{p}_i + D^T \bar{q}_i) \right) \in L^2_{\mathcal{F}^i}(0, T; \Gamma_{\theta_i})$, $1 \leq i \leq N$. Moreover, there exists a constant M independent of N which may vary line by line in the following, such that for all $0 \leq i \leq N$, $0 \leq k \leq K$,

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} (|\alpha_k(t)|^2 + |\beta_k(t)|^2 + |\bar{x}_i(t)|^2 + |\bar{p}_i(t)|^2) \\ & + \mathbb{E} \int_0^T (|\gamma_k(t)|^2 + |\bar{q}_i(t)|^2 + |\varphi_{\theta_i}(\bar{p}_i(t), \bar{q}_i(t))|^2) dt \leq M. \end{aligned} \quad (33)$$

From (31), by using Burkholder-Davis-Gundy (BDG) inequality, it follows that for any $t \in [0, T]$,

$$\mathbb{E} \sup_{0 \leq s \leq t} |\check{x}_0(s)|^2 \leq M + M \mathbb{E} \int_0^t [|\check{x}_0(s)|^2 + |\check{x}^{(N)}(s)|^2] ds \leq M + M \mathbb{E} \int_0^t [|\check{x}_0(s)|^2 + \frac{1}{N} \sum_{i=1}^N |\check{x}_i(s)|^2] ds,$$

and by Gronwall's inequality, we obtain

$$\mathbb{E} \sup_{0 \leq s \leq t} |\check{x}_0(s)|^2 \leq M + M \mathbb{E} \int_0^t \frac{1}{N} \sum_{i=1}^N |\check{x}_i(s)|^2 ds, \quad (34)$$

Similarly, from (31) again, we have

$$\mathbb{E} \sup_{0 \leq s \leq t} |\check{x}_i(s)|^2 \leq M + M \mathbb{E} \int_0^t [|\check{x}_0(s)|^2 + |\check{x}_i(s)|^2 + \frac{1}{N} \sum_{i=1}^N |\check{x}_i(s)|^2] ds.$$

Then using (34), we get

$$\mathbb{E} \sup_{0 \leq s \leq t} |\check{x}_i(s)|^2 \leq M + M \mathbb{E} \int_0^t [|\check{x}_i(s)|^2 + \frac{1}{N} \sum_{i=1}^N |\check{x}_i(s)|^2] ds. \quad (35)$$

Thus

$$\mathbb{E} \sup_{0 \leq s \leq t} \sum_{i=1}^N |\check{x}_i(s)|^2 \leq \mathbb{E} \sum_{i=1}^N \sup_{0 \leq s \leq t} |\check{x}_i(s)|^2 \leq MN + 2M \mathbb{E} \int_0^t \left[\sum_{i=1}^N |\check{x}_i(s)|^2 \right] ds.$$

By Gronwall's inequality, it is easy to obtain $\mathbb{E} \sup_{0 \leq t \leq T} \sum_{i=1}^N |\check{x}_i(t)|^2 = O(N)$, for any $1 \leq i \leq N$.

Then, substituting this estimate to (35) and using Gronwall's inequality once again, we have $\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_i(t)|^2 \leq M$, for any $1 \leq i \leq N$. By applying this estimate to (34), we get

$$\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_0(t)|^2 \leq M. \text{ We complete the proof. } \quad \square$$

Now, we recall that

$$\check{x}^{(N)} = \frac{1}{N} \sum_{k=1}^K \sum_{i \in \mathcal{I}_k} \check{x}_i = \sum_{k=1}^K \pi_k^{(N)} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \check{x}_i, \quad \text{and} \quad \Phi(t) = \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0}),$$

then we have

Lemma 6.2 *Under (A1)-(A4), there exists a constant M independent of N such that*

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \check{x}^{(N)}(t) - \Phi(t) \right|^2 \leq M \left(\frac{1}{N} + \varepsilon_N^2 \right), \quad \text{where } \varepsilon_N = \sup_{1 \leq k \leq K} |\pi_k^{(N)} - \pi_k|.$$

Proof For each fixed $1 \leq k \leq K$, we consider the k -type minor agents. We denote $\check{x}^{(k)} := \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \check{x}_i$. Let us add up N_k states of all k -type minor agents and then divide by N_k , we have

$$\begin{aligned} d\check{x}^{(k)} &= \left[A_k \check{x}^{(k)} + \frac{B}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i, \bar{q}_i) + F_1 \check{x}^{(N)} + b_0 \right] dt \\ &\quad + \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \left[C \check{x}_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 \check{x}^{(N)} + H \check{x}_0 + \sigma_0 \right] dW_i(t), \quad \check{x}^{(k)}(0) = x. \end{aligned} \quad (36)$$

Recall $m_k = \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0})$, and taking conditional expectation $\mathbb{E}(\cdot | \mathcal{F}_t^{W_0})$ on the first equation of (15) and noticing that $\mathbb{E}(\varphi_k(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) = \mathbb{E}(\varphi_k(\beta_k, \gamma_k) | \mathcal{F}_t^{W_0})$, for any $i \in \mathcal{I}_k$ and $\alpha_k(s)$ is \mathcal{F}_s^k -adapted, then we have

$$dm_k = \left(A_k m_k + B \mathbb{E}(\varphi_k(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) + F_1 \Phi + b \right) dt, \quad m_k(0) = x. \quad (37)$$

From (36) and (37), by denoting $\Delta_k(t) := \check{x}^{(k)}(t) - m_k(t)$, we have

$$\begin{aligned} d\Delta_k &= \left[A_k \Delta_k + F_1 (\check{x}^{(N)} - \Phi) + \frac{B}{N_k} \left(\sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i, \bar{q}_i) - \mathbb{E}(\varphi_k(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) \right) \right] dt \\ &\quad + \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \left[C \check{x}_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 \check{x}^{(N)} + H \check{x}_0 + \sigma_0 \right] dW_i(t), \quad \Delta(0) = 0. \end{aligned}$$

The inequality $(x + y)^2 \leq 2x^2 + 2y^2$ and Cauchy-Schwartz inequality yield that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |\Delta_k(s)|^2 &\leq M \mathbb{E} \int_0^t \left[|\Delta_k(s)|^2 + |\check{x}^{(N)}(s) - \Phi(s)|^2 \right] ds \\ &\quad + M \mathbb{E} \int_0^t \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0}) \right|^2 ds \\ &\quad + \frac{2}{N_k^2} \mathbb{E} \sup_{0 \leq r \leq t} \left| \int_0^r \sum_{i \in \mathcal{I}_k} \left[C \check{x}_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 \check{x}^{(N)} + H \check{x}_0 + \sigma_0 \right] dW_i(s) \right|^2. \end{aligned}$$

From BDG inequality, we obtain

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |\Delta_k(s)|^2 &\leq M \mathbb{E} \int_0^t \left[|\Delta_k(s)|^2 + |\check{x}^{(N)}(s) - \Phi(s)|^2 \right] ds \\ &\quad + M \mathbb{E} \int_0^t \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0}) \right|^2 ds \\ &\quad + \frac{M}{N_k^2} \mathbb{E} \sum_{i \in \mathcal{I}_k} \int_0^t \left| C \check{x}_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 (\check{x}^{(N)} - \Phi) + F_2 \Phi + H \check{x}_0 + \sigma_0 \right|^2 ds. \end{aligned} \quad (38)$$

Let us first focus on the second term of the right hand-side of (38). Since for each fixed $s \in [0, T]$, under the conditional expectation $\mathbb{E}(\cdot | \mathcal{F}_s^{W_0})$, for each $1 \leq k \leq K$, the processes $(\bar{x}_i(s), \bar{p}_i(s), \bar{q}_i(s))$, $i \in \mathcal{I}_k$, are independent identically distributed, if we denote $\mu(s) = \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0})$, then μ does not depend on i and moreover we have

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s) \right|^2 = \frac{1}{N_k^2} \mathbb{E} \left| \sum_{i \in \mathcal{I}_k} [\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s)] \right|^2 \\ &= \frac{1}{N_k^2} \mathbb{E} \left(\sum_{i \in \mathcal{I}_k} |\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s)|^2 + \sum_{i, j \in \mathcal{I}_k, j \neq i} \langle \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s), \varphi_k(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \rangle \right). \end{aligned}$$

Since $(\bar{p}_i(s), \bar{q}_i(s))$, $i \in \mathcal{I}_k$, are independent under $\mathbb{E}(\cdot | \mathcal{F}_s^{W_0})$, we have

$$\begin{aligned} & \mathbb{E} \sum_{i, j \in \mathcal{I}_k, j \neq i} \langle \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s), \varphi_k(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \rangle \\ &= \mathbb{E} \left[\sum_{i, j \in \mathcal{I}_k, j \neq i} \mathbb{E} \left(\langle \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s), \varphi_k(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \rangle \middle| \mathcal{F}_s^{W_0} \right) \right] \\ &= \mathbb{E} \left[\sum_{i, j \in \mathcal{I}_k, j \neq i} \left\langle \mathbb{E} \left(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s) \middle| \mathcal{F}_s^{W_0} \right), \mathbb{E} \left(\varphi_k(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \middle| \mathcal{F}_s^{W_0} \right) \right\rangle \right] = 0. \end{aligned}$$

Then, due to the fact that (\bar{p}_i, \bar{q}_i) , $1 \leq i \leq N$ are identically distributed, we obtain

$$\begin{aligned} & \mathbb{E} \int_0^t \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0}) \right|^2 ds \\ &= \frac{1}{N_k^2} \int_0^t \mathbb{E} \sum_{i \in \mathcal{I}_k} |\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s)|^2 ds = \frac{1}{N_k} \int_0^t \mathbb{E} |\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mu(s)|^2 ds \leq \frac{M}{N_k}, \end{aligned}$$

where the last equality due to (33). Now we focus on the third term of the right hand-side of (38), using (33), Lemma 6.1 and that $(\check{x}_i(s), \bar{p}_i(s), \bar{q}_i(s))$, $i \in \mathcal{I}_k$, are identically distributed, it follows

$$\begin{aligned} & \frac{M}{N_k^2} \sum_{i \in \mathcal{I}_k} \mathbb{E} \int_0^t \left| C\check{x}_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2(\check{x}^{(N)} - \Phi) + F_2\Phi + H\check{x}_0 + \sigma_0 \right|^2 ds \\ &\leq \frac{M}{N_k^2} \sum_{i \in \mathcal{I}_k} \mathbb{E} \int_0^t \left(|\check{x}_i(s)|^2 + |\varphi_k(\bar{p}_i, \bar{q}_i)|^2 + |\check{x}^{(N)}(s) - \Phi(s)|^2 + |\Phi(s)|^2 + |\check{x}_0(s)|^2 + |\sigma(s)|^2 \right) ds \\ &\leq \frac{M}{N_k} \mathbb{E} \int_0^t \mathbb{E} |\check{x}^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N_k}. \end{aligned}$$

Therefore, from above analysis, we get from (38) that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\Delta_k(s)|^2 \leq M \mathbb{E} \int_0^t \left[|\Delta_k(s)|^2 + |\check{x}^{(N)}(s) - \Phi(s)|^2 \right] ds + \frac{M}{N_k},$$

and Gronwall's inequality yields that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\Delta_k(s)|^2 \leq M \mathbb{E} \int_0^t |\check{x}^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N_k}. \quad (39)$$

Since

$$\begin{aligned}
\check{x}^{(N)}(s) - \Phi(s) &= \sum_{k=1}^K \left[\pi_k^{(N)} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \check{x}_i(s) - \pi_k \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right] \\
&= \sum_{k=1}^K \left[\pi_k^{(N)} \left(\frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \check{x}_i(s) - \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right) + \left(\pi_k^{(N)} - \pi_k \right) \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right] \\
&= \sum_{k=1}^K \pi_k^{(N)} \Delta_k(s) + \sum_{k=1}^K \left(\pi_k^{(N)} - \pi_k \right) \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}),
\end{aligned}$$

by using (33),(39) and $\pi_k^{(N)} = \frac{N_k}{N} \leq 1$ we obtain that for any $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} |\check{x}^{(N)}(s) - \Phi(s)|^2 &\leq \mathbb{E} \sum_{k=1}^K \sup_{0 \leq s \leq t} \pi_k^{(N)} |\Delta_k(s)|^2 + M \varepsilon_N^2 \\
&\leq M \mathbb{E} \int_0^t |\check{x}^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N} + M \varepsilon_N^2.
\end{aligned}$$

Finally, by using Gronwall's inequality, we complete the proof. \square

Lemma 6.3 *Under the assumptions of (A1)-(A4), we have*

$$\sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_i(t) - \bar{x}_i(t)|^2 \leq M \left(\frac{1}{N} + \varepsilon_N^2 \right).$$

Proof On the one hand, from both the first equation of (31) and (32), we have

$$\begin{cases} d(\check{x}_0 - \bar{x}_0) = \left[A_0(\check{x}_0 - \bar{x}_0) + F_0^1(\check{x}^{(N)} - \Phi) \right] dt + \left[C_0(\check{x}_0 - \bar{x}_0) + F_0^2(\check{x}^{(N)} - \Phi) \right] dW_i(t), \\ \check{x}_0(0) - \bar{x}_0(0) = 0. \end{cases}$$

The classical estimate for the SDE yields that

$$\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_0(t) - \bar{x}_0(t)|^2 \leq M \mathbb{E} \int_0^T |\check{x}^{(N)}(s) - \Phi(s)|^2 ds,$$

where M is a constant independent of N . Noticing Lemma 6.2, we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_0(t) - \bar{x}_0(t)|^2 \leq M \left(\frac{1}{N} + \varepsilon_N^2 \right). \quad (40)$$

On the other hand, from the second equation of (31) and the third equation of (32), we have that for $1 \leq i \leq N$,

$$\begin{cases} d(\check{x}_i - \bar{x}_i) = \left[A_{\theta_i}(\check{x}_i - \bar{x}_i) + F_1(\check{x}^{(N)} - \Phi) \right] dt + \left[C(\check{x}_i - \bar{x}_i) + F_2(\check{x}^{(N)} - \Phi) + H(\check{x}_0 - \bar{x}_0) \right] dW_i(t), \\ \check{x}_i(0) - \bar{x}_i(0) = 0. \end{cases}$$

The classical estimate for the SDE yields that

$$\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_i(t) - \bar{x}_i(t)|^2 \leq M \mathbb{E} \int_0^T \left(|\check{x}^{(N)}(s) - \Phi(s)|^2 + |\check{x}_0(s) - \bar{x}_0(s)|^2 \right) ds.$$

Noticing Lemma 6.2 and (40), we obtain $\mathbb{E} \sup_{0 \leq t \leq T} |\check{x}_i(t) - \bar{x}_i(t)|^2 \leq M \left(\frac{1}{N} + \varepsilon_N^2 \right)$. Thus, considering also (40) we complete the proof. \square

Lemma 6.4 Under the assumptions of **(A1)**-**(A4)**, for all $0 \leq i \leq N$, we have

$$\left| \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - J_i(\bar{u}_i) \right| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).$$

Proof Let us first consider the major agent. Recall (3), (6) and (17), we have

$$\begin{aligned} \mathcal{J}_0(\bar{u}_0, \bar{u}_{-0}) - J_0(\bar{u}_0) &= \frac{1}{2} \mathbb{E} \left[\int_0^T \left(\left\langle Q_0(\check{x}_0 - \rho_0 \check{x}^{(N)}), \check{x}_0 - \rho_0 \check{x}^{(N)} \right\rangle - \left\langle Q_0(\bar{x}_0 - \rho_0 \Phi), \bar{x}_0 - \rho_0 \Phi \right\rangle \right) dt \right. \\ &\quad \left. + \left\langle G_0(\check{x}_0(T) - \rho_0 \check{x}^{(N)}(T)), \check{x}_0(T) - \rho_0 \check{x}^{(N)}(T) \right\rangle - \left\langle G_0(\bar{x}_0(T) - \rho_0 \Phi(T)), \bar{x}_0(T) - \rho_0 \Phi(T) \right\rangle \right]. \end{aligned} \quad (41)$$

From (33), we have $\mathbb{E} \sup_{0 \leq t \leq T} |\bar{x}_0(t)|^2 \leq M$ and $\mathbb{E} \sup_{0 \leq t \leq T} |\alpha_i(t)|^2 \leq M$, for any $0 \leq i \leq N$. Noticing that $\Phi(t) = \sum_{k=1}^K \pi_k \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0})$, it is not hard to check that $\mathbb{E} \sup_{0 \leq t \leq T} |\Phi(t)|^2 \leq M$. Now, from such estimates and Lemma 6.2, Lemma 6.3 as well as

$$\begin{aligned} &\langle Q_0(a - b), a - b \rangle - \langle Q_0(c - d), c - d \rangle \\ &= \langle Q_0(a - b - (c - d)), a - b - (c - d) \rangle + 2 \langle Q_0(a - b - (c - d)), c - d \rangle, \end{aligned}$$

we have

$$\begin{aligned} &\left| \mathbb{E} \left[\int_0^T \left(\left\langle Q_0(\check{x}_0 - \rho_0 \check{x}^{(N)}), \check{x}_0 - \rho_0 \check{x}^{(N)} \right\rangle - \left\langle Q_0(\bar{x}_0 - \rho_0 \Phi), \bar{x}_0 - \rho_0 \Phi \right\rangle \right) dt \right] \right| \\ &\leq M \int_0^T \mathbb{E} \left| \check{x}_0 - \rho_0 \check{x}^{(N)} - (\bar{x}_0 - \rho_0 \Phi) \right|^2 dt + M \int_0^T \mathbb{E} \left| \check{x}_0 - \rho_0 \check{x}^{(N)} - (\bar{x}_0 - \rho_0 \Phi) \right| |\bar{x}_0 - \rho_0 \Phi| dt \\ &\leq M \int_0^T \mathbb{E} |\check{x}_0 - \bar{x}_0|^2 dt + M \int_0^T \mathbb{E} \left| \check{x}^{(N)} - \Phi \right|^2 dt \\ &\quad + M \int_0^T \left(\mathbb{E} \left| \check{x}_0 - \rho_0 \check{x}^{(N)} - (\bar{x}_0 - \rho_0 \Phi) \right|^2 \right)^{\frac{1}{2}} \left(\mathbb{E} |\bar{x}_0 - \rho_0 \Phi|^2 \right)^{\frac{1}{2}} dt \\ &\leq M \int_0^T \mathbb{E} |\check{x}_0 - \bar{x}_0|^2 dt + M \int_0^T \mathbb{E} \left| \check{x}^{(N)} - \Phi \right|^2 dt + M \int_0^T \left(\mathbb{E} |\check{x}_0 - \bar{x}_0|^2 + \mathbb{E} \left| \check{x}^{(N)} - \Phi \right|^2 \right)^{\frac{1}{2}} dt \\ &= O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right). \end{aligned}$$

Similar argument allows us to show that

$$\begin{aligned} &\left| \mathbb{E} \left[\left\langle G_0(\check{x}_0(T) - \rho_0 \check{x}^{(N)}(T)), \check{x}_0(T) - \rho_0 \check{x}^{(N)}(T) \right\rangle - \left\langle G_0(\bar{x}_0(T) - \rho_0 \Phi(T)), \bar{x}_0(T) - \rho_0 \Phi(T) \right\rangle \right] \right| \\ &= O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right). \end{aligned}$$

Thus, the proof for the major agent is completed by noticing (41). Let us now focus on the minor agents, for $1 \leq i \leq N$, recalling (4), (12) and (17), we have

$$\begin{aligned} \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) &= \frac{1}{2} \mathbb{E} \left[\int_0^T \left(\left\langle Q(\check{x}_i - \rho \check{x}^{(N)} - (1 - \rho) \check{x}_0), \check{x}_i - \rho \check{x}^{(N)} - (1 - \rho) \check{x}_0 \right\rangle + \left\langle R_{\theta_i} \bar{u}_i, \bar{u}_i \right\rangle \right) dt \right. \\ &\quad \left. + \left\langle G(\check{x}_i(T) - \rho \check{x}^{(N)}(T) - (1 - \rho) \check{x}_0(T)), \check{x}_i(T) - \rho \check{x}^{(N)}(T) - (1 - \rho) \check{x}_0(T) \right\rangle \right] \end{aligned}$$

and

$$J_i(\bar{u}_i) = \frac{1}{2} \mathbb{E} \left[\int_0^T \left(\langle Q(\bar{x}_i - \rho\Phi - (1-\rho)\bar{x}_0), \bar{x}_i - \rho\Phi - (1-\rho)\bar{x}_0 \rangle dt + \langle R_{\theta_i} \bar{u}_i, \bar{u}_i \rangle \right) dt \right. \\ \left. + \langle G(\bar{x}_i(T) - \rho\Phi(T) - (1-\rho)\bar{x}_0(T)), \bar{x}_i(T) - \rho\Phi(T) - (1-\rho)\bar{x}_0(T) \rangle \right].$$

From (33), we have $\mathbb{E} \sup_{0 \leq t \leq T} |\bar{x}_i(t)|^2 \leq M$. Using such estimate, $\mathbb{E} \sup_{0 \leq t \leq T} |\Phi(t)|^2 \leq M$, Lemmas 6.2, 6.3, similar to the major agent, it follows that $\left| \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - J_i(\bar{u}_i) \right| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right)$. \square

6.1 Major agent's perturbation

In this subsection, we will prove that the control strategies set $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N)$ is an ε -Nash equilibrium of **Problem (CC)** for the major agent, i.e. $\exists \varepsilon = \varepsilon(N) \geq 0, \lim_{N \rightarrow \infty} \varepsilon(N) = 0$ s.t

$$\mathcal{J}_0(\bar{u}_0(\cdot), \bar{u}_{-0}(\cdot)) \leq \mathcal{J}_0(u_0(\cdot), \bar{u}_{-0}(\cdot)) + \varepsilon, \quad \text{for any } u_0 \in \mathcal{U}_{ad}^0.$$

Let us consider that the major agent \mathcal{A}_0 uses an alternative strategy u_0 and each minor agent \mathcal{A}_i uses the control $\bar{u}_i = \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i)$, where (\bar{p}_i, \bar{q}_i) are solved from (32). Then the realized state system with major agent's perturbation is, for $1 \leq i \leq N$,

$$\begin{cases} dy_0 = [A_0 y_0 + B_0 u_0 + F_0^1 y^{(N)} + b_0] dt + [C_0 y_0 + D_0 u_0 + F_0^2 y^{(N)} + \sigma_0] dW_0(t), \\ dy_i = [A_{\theta_i} y_i + B \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_1 y^{(N)} + b_0] dt \\ \quad + [C y_i + D_{\theta_i} \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_2 y^{(N)} + H y_0 + \sigma_0] dW_i(t), \\ y_0(0) = x_0, \quad y_i(0) = x, \end{cases} \quad (42)$$

where $y^{(N)} = \frac{1}{N} \sum_{i=1}^N y_i$. The well-posedness of above SDEs system is easy to obtain. To prove $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$ is an ε -Nash equilibrium for the major agent, we need to show that for possible alternative control u_0 , $\inf_{u_0 \in \mathcal{U}_{ad}^0} \mathcal{J}_0(u_0, \bar{u}_{-0}) \geq \mathcal{J}_0(\bar{u}_0, \bar{u}_{-0}) - \varepsilon$. Then we only need to consider the perturbation $u_0 \in \mathcal{U}_{ad}^0$ such that $\mathcal{J}_0(u_0, \bar{u}_{-0}) \leq \mathcal{J}_0(\bar{u}_0, \bar{u}_{-0})$. Thus, noticing $Q_0 \geq 0$ and $G_0 \geq 0$, from Lemma 6.4, we have

$$\mathbb{E} \int_0^T \langle R_0 u_0(t), u_0(t) \rangle dt \leq \mathcal{J}_0(u_0, \bar{u}_{-0}) \leq \mathcal{J}_0(\bar{u}_0, \bar{u}_{-0}) \leq J_0(\bar{u}_0) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right),$$

which implies that (noting **(A4)**), $\mathbb{E} \int_0^T |u_0(t)|^2 dt \leq M$, where M is a constant independent of N . Then similar to Lemma 6.1, we can show that

$$\sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |y_i(t)|^2 \leq M. \quad (43)$$

Lemma 6.5 *Under the assumptions of **(A1)**-**(A4)**, we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| y^{(N)}(t) - \Phi(t) \right|^2 = O\left(\frac{1}{N} + \varepsilon_N^2\right).$$

Proof For each fixed $1 \leq k \leq K$, we consider the k -type minor agents. We denote $y^{(k)} := \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} y_i$. As there are N_k minor agents of the k -type, let us add up their states and then

divided by N_k , it follows that

$$dy^{(k)} = \left[A_k y^{(k)} + \frac{B}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i, \bar{q}_i) + F_1 y^{(N)} + b_0 \right] dt \\ + \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \left[C y_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 y^{(N)} + H y_0 + \sigma_0 \right] dW_i(t), \quad y^{(k)}(0) = x.$$

Recall (37) and if we denote $\tilde{\Delta}_k(t) := y^{(k)}(t) - m_k(t)$, it follows that

$$d\tilde{\Delta}_k = \left[A_k \tilde{\Delta}_k + \frac{B}{N_k} \left(\sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i, \bar{q}_i) - \mathbb{E}(\varphi_k(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) \right) + F_1 (y^{(N)} - \Phi) \right] dt \\ + \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \left[C y_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 y^{(N)} + H y_0 + \sigma_0 \right] dW_i(t), \quad \tilde{\Delta}_k(0) = 0.$$

Similar to the argument in the proof of Lemma 6.2, we can show that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{\Delta}_k(s)|^2 \leq M \mathbb{E} \int_0^t \left[|\tilde{\Delta}_k(s)|^2 + |y^{(N)}(s) - \Phi(s)|^2 \right] ds \\ + M \mathbb{E} \int_0^t \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0}) \right|^2 ds \\ + \frac{M}{N_k^2} \mathbb{E} \sum_{i \in \mathcal{I}_k} \int_0^t \left| C y_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 (y^{(N)} - \Phi) + F_2 \Phi + H y_0 + \sigma_0 \right|^2 ds, \quad (44)$$

and

$$\mathbb{E} \int_0^t \left| \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i(s), \bar{q}_i(s)) - \mathbb{E}(\varphi_k(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0}) \right|^2 ds \leq \frac{M}{N_k}.$$

Using (33), (43) and the fact that $(y_i(s), \bar{p}_i(s), \bar{q}_i(s))$, $i \in \mathcal{I}_k$, are identically distributed, we have

$$\frac{M}{N_k^2} \sum_{i \in \mathcal{I}_k} \mathbb{E} \int_0^t \left| C y_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 (y^{(N)} - \Phi) + F_2 \Phi + H y_0 + \sigma_0 \right|^2 ds \\ = \frac{M}{N_k} \mathbb{E} \int_0^t \mathbb{E} |y^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N_k}.$$

Therefore, we get from (44) that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{\Delta}_k(s)|^2 \leq M \mathbb{E} \int_0^t \left[|\tilde{\Delta}_k(s)|^2 + |y^{(N)}(s) - \Phi(s)|^2 \right] ds + \frac{M}{N_k},$$

and Gronwall's inequality yields that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{\Delta}_k(s)|^2 \leq M \mathbb{E} \int_0^t \left[|y^{(N)}(s) - \Phi(s)|^2 \right] ds.$$

Similar to the proof of Lemma 6.2 again, and using **(A1)**, we have for any $t \in [0, T]$,

$$\mathbb{E} \sup_{0 \leq s \leq t} |y^{(N)}(s) - \Phi(s)|^2 \leq \mathbb{E} \sum_{k=1}^K \sup_{0 \leq s \leq t} \pi_k^{(N)} |\Delta_k(s)|^2 + M \varepsilon_N^2 \\ \leq M \mathbb{E} \int_0^t \left[|y^{(N)}(s) - \Phi(s)|^2 \right] ds + \frac{M}{N} + M \varepsilon_N^2.$$

Finally, Gronwall's inequality allows us to complete the proof. \square

Now, we introduce the following system of the decentralized limiting state with the major's perturbation control, for $1 \leq i \leq N$:

$$\begin{cases} d\bar{y}_0 = [A_0\bar{y}_0 + B_0u_0 + F_0^1\Phi + b_0] dt + [C_0\bar{y}_0 + D_0u_0 + F_0^2\Phi + \sigma_0] dW_0(t) \\ d\bar{y}_i = [A_{\theta_i}\bar{y}_i + B\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_1\Phi + b_0] dt \\ \quad + [C\bar{y}_i + D_{\theta_i}\varphi_{\theta_i}(\bar{p}_i, \bar{q}_i) + F_2\Phi + H\bar{y}_0 + \sigma_0] dW_i(t) \\ \bar{y}_0(0) = x_0, \quad \bar{y}_i(0) = x. \end{cases} \quad (45)$$

Lemma 6.6 *Under the assumptions of (A1)-(A4), we have*

$$\sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |y_i(t) - \bar{y}_i(t)|^2 = O\left(\frac{1}{N} + \varepsilon_N^2\right).$$

Proof From both the first equation of (42) and (45), we obtain

$$\begin{cases} d(y_0 - \bar{y}_0) = [A(y_0 - \bar{y}_0) + F_0^1(y^{(N)} - \Phi)] dt + [C(y_0 - \bar{y}_0) + F_0^2(y^{(N)} - \Phi)] dW_0(t), \\ y_0(0) - \bar{y}_0(0) = 0. \end{cases}$$

With the help of classical estimates of SDE and Lemma 6.5, it is easy to obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} |y_0(t) - \bar{y}_0(t)|^2 = O\left(\frac{1}{N} + \varepsilon_N^2\right). \quad (46)$$

Now, for any $1 \leq i \leq N$, from both the second equation of (42) and (45), we get

$$\begin{aligned} d(y_i - \bar{y}_i) &= [A_{\theta_i}(y_i - \bar{y}_i) + F_1(y^{(N)} - \Phi)] dt \\ &\quad + [C(y_i - \bar{y}_i) + F_2(y^{(N)} - \Phi) + H(y_0 - \bar{y}_0)] dW_i(t), \quad y_i(0) - \bar{y}_i(0) = 0. \end{aligned}$$

The classical estimates of SDE, Lemma 6.5 and (46) allow us to complete the proof. \square

Lemma 6.7 *Under (A1)-(A4), for the major agent's perturbation control u_0 , we have*

$$|\mathcal{J}_0(u_0, \bar{u}_{-0}) - J_0(u_0)| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).$$

Proof Recall (3), (6) and (17), we have

$$\begin{aligned} \mathcal{J}_0(u_0, \bar{u}_{-0}) - J_0(u_0) &= \frac{1}{2} \mathbb{E} \left[\int_0^T \left(\langle Q_0(y_0 - \rho_0 y^{(N)}), y_0 - \rho_0 y^{(N)} \rangle - \langle Q_0(\bar{y}_0 - \rho_0 \Phi), \bar{y}_0 - \rho_0 \Phi \rangle \right) dt \right. \\ &\quad \left. + \langle G_0(y_0(T) - \rho_0 y^{(N)}(T)), y_0(T) - \rho_0 y^{(N)}(T) \rangle - \langle G_0(\bar{y}_0(T) - \rho_0 \Phi(T)), \bar{y}_0(T) - \rho_0 \Phi(T) \rangle \right]. \end{aligned} \quad (47)$$

Similar to Lemma 6.4, by using Lemmas 6.5, 6.6 and $\mathbb{E} (|\bar{y}_0(t)|^2 + |\Phi(t)|^2) \leq M$, we have

$$\begin{aligned} &\left| \mathbb{E} \int_0^T \left(\langle Q_0(y_0 - \rho_0 y^{(N)}), y_0 - \rho_0 y^{(N)} \rangle - \langle Q_0(\bar{y}_0 - \rho_0 \Phi), \bar{y}_0 - \rho_0 \Phi \rangle \right) dt \right| \\ &\leq M \int_0^T \mathbb{E} |y_0 - \bar{y}_0|^2 dt + M \int_0^T \mathbb{E} |y^{(N)} - \Phi|^2 dt + M \int_0^T \left(\mathbb{E} |y_0 - \bar{y}_0|^2 + \mathbb{E} |y^{(N)} - \Phi|^2 \right)^{\frac{1}{2}} dt \\ &= O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right), \end{aligned}$$

and

$$\left| \mathbb{E} \left[\left\langle G_0(y_0(T) - \rho_0 y^{(N)}(T)), y_0(T) - \rho_0 y^{(N)}(T) \right\rangle - \left\langle G_0(\bar{y}_0(T) - \rho_0 \Phi(T)), \bar{y}_0(T) - \rho_0 \Phi(T) \right\rangle \right] \right| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).$$

The proof is completed by noticing (47). \square

Theorem 6.1 *Under the assumptions of (A1)-(A4), then the strategies set $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$ is an ε -Nash equilibrium of **Problem (CC)** for the major agent.*

Proof Combining Lemma 6.4 and Lemma 6.7, we have

$$\mathcal{J}_0(\bar{u}_0, \bar{u}_{-0}) \leq J_0(\bar{u}_0) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right) \leq J_0(u_0) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right) \leq \mathcal{J}_0(u_0, \bar{u}_{-0}) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right),$$

where the second inequality comes from the fact that $J_0(\bar{u}_0) = \inf_{u_0 \in \mathcal{U}_{ad}^0} J_0(u_0)$. Consequently, Theorem 6.1 holds with $\varepsilon = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right)$. \square

6.2 Minor agent's perturbation

Now, let us consider the following case: a given minor agent \mathcal{A}_i uses an alternative strategy $u_i \in \mathcal{U}_{ad}^i$, the major agent uses $\bar{u}_0 = \varphi_0(\bar{p}_0, \bar{q}_0)$ while other minor agents \mathcal{A}_j use the control $\bar{u}_j = \varphi_{\theta_j}(\bar{p}_j, \bar{q}_j)$, $j \neq i$, $1 \leq j \leq N$, where (\bar{p}_j, \bar{q}_j) , $0 \leq j \leq N$, $j \neq i$, are solved from (32). Then the realized state system with the minor agent's perturbation is, for $1 \leq j \leq N$, $j \neq i$,

$$\begin{cases} dl_0 = \left[A_0 l_0 + B_0 \varphi_0(\bar{p}_0, \bar{q}_0) + F_0^1 l^{(N)} + b_0 \right] dt + \left[C_0 l_0 + D_0 \varphi_0(\bar{p}_0, \bar{q}_0) + F_0^2 l^{(N)} + \sigma_0 \right] dW_0(t) \\ dl_i = \left[A_{\theta_i} l_i + B u_i + F_1 l^{(N)} + b_0 \right] dt + \left[C l_i + D_{\theta_i} u_i + F_2 l^{(N)} + H l_0 + \sigma_0 \right] dW_i(t), \\ dl_j = \left[A_{\theta_j} l_j + B \varphi_{\theta_j}(\bar{p}_j, \bar{q}_j) + F_1 l^{(N)} + b_0 \right] dt + \left[C l_j + D_{\theta_j} \varphi_{\theta_j}(\bar{p}_j, \bar{q}_j) + F_2 l^{(N)} + H l_0 + \sigma_0 \right] dW_i(t), \\ l_0(0) = x_0, \quad l_i(0) = l_j(0) = x, \end{cases} \quad (48)$$

where $l^{(N)} = \frac{1}{N} \sum_{i=1}^N l^i$. The well-posedness of above SDEs system is easily to obtain. Similar to the argument of major agent, to prove $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$ is an ε -Nash equilibrium for the minor agent, noticing $Q \geq 0$, $G \geq 0$, $R_{\theta_i} > 0$ and Lemma 6.4, we only need to consider the perturbation $u_i \in \mathcal{U}_{ad}^i$ satisfying

$$\mathbb{E} \int_0^T |u_i(t)|^2 dt \leq M, \quad (49)$$

where M is a constant independent of N . Similar to Lemma 6.1, we can show that

$$\sup_{0 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |l_i(t)|^2 \leq M. \quad (50)$$

We first present the following lemma

Lemma 6.8 *Under the assumptions of (A1)-(A4), we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| l^{(N)}(t) - \Phi(t) \right|^2 = O\left(\frac{1}{N} + \varepsilon_N^2\right).$$

Proof We know that for each fixed i , there exists a unique $1 \leq \bar{k} \leq K$, such that $i \in \mathcal{I}_{\bar{k}}$. Let us denote $l^{(k)} := \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} l_i$, $1 \leq k \leq K$. We first consider the k -type minor agents, where $k \neq \bar{k}$. Adding up their states and then divided by N_k , we have for $k \neq \bar{k}$,

$$dl^{(k)} = \left[A_k l^{(k)} + \frac{B}{N_k} \sum_{i \in \mathcal{I}_k} \varphi_k(\bar{p}_i, \bar{q}_i) + F_1 l^{(N)} + b_0 \right] dt + \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} \left[Cl_i + D_k \varphi_k(\bar{p}_i, \bar{q}_i) + F_2 l^{(N)} + Hl_0 + \sigma_0 \right] dW_i(t), \quad l^{(k)}(0) = x.$$

Similar to the proof of Lemma 6.2, for $m_k = \mathbb{E}(\alpha_k(t) | \mathcal{F}_t^{W_0})$, we have

$$\mathbb{E} \sup_{0 \leq s \leq t} |l^{(k)}(s) - m_k(s)|^2 \leq M \mathbb{E} \int_0^t \left[|l^{(N)}(s) - \Phi(s)|^2 \right] ds + \frac{M}{N_k}. \quad (51)$$

Now let us focus on the \bar{k} -type minor agents, we have

$$dl^{(\bar{k})} = \left[A_{\bar{k}} l^{(\bar{k})} + \frac{B}{N_{\bar{k}}} u_i + \frac{B}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j, \bar{q}_j) + F_1 l^{(N)} + b_0 \right] dt + \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}} \left[Cl_j + F_2 l^{(N)} + Hl_0 + \sigma_0 \right] dW_j(t) + \frac{1}{N_{\bar{k}}} D_{\bar{k}} u_i dW_i(t) + \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} D_{\bar{k}} \varphi_{\bar{k}}(\bar{p}_j, \bar{q}_j) dW_j(t), \quad l^{(\bar{k})}(0) = x.$$

Recalling (37) and if we denote $\Xi := l^{(\bar{k})} - m_{\bar{k}}$, it follows that

$$d\Xi = \left[A\Xi + F_1(l^{(N)} - \Phi) + \frac{1}{N_{\bar{k}}} Bu_i + \frac{B}{N_{\bar{k}}} \left(\sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j, \bar{q}_j) - \mathbb{E}(\varphi_{\bar{k}}(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) \right) \right] dt + \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}} \left[Cl_j + F_2 l^{(N)} + Hl_0 + \sigma_0 \right] dW_j(t), + \frac{1}{N_{\bar{k}}} D_{\bar{k}} u_i dW_i(t) + \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} D_{\bar{k}} \varphi_{\bar{k}}(\bar{p}_j, \bar{q}_j) dW_j(t), \quad \Pi(0) = 0.$$

By Cauchy-Schwartz inequality and BDG inequality, we obtain that for any $t \in [0, T]$,

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |\Xi(s)|^2 &\leq M \mathbb{E} \int_0^t \left(|\Xi(s)|^2 + \frac{1}{N_{\bar{k}}^2} |u_i(s)|^2 + |l^{(N)}(s) - \Phi(s)|^2 \right) ds \\ &\quad + M \mathbb{E} \int_0^t \left| \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j, \bar{q}_j) - \mathbb{E}(\varphi_{\bar{k}}(\bar{p}_i, \bar{q}_i) | \mathcal{F}_t^{W_0}) \right|^2 ds \\ &\quad + \frac{M}{N_{\bar{k}}^2} \mathbb{E} \sum_{j \in \mathcal{I}_{\bar{k}}} \int_0^t \left| F_2(l^{(N)}(s) - \Phi(s)) + F_2\Phi(s) + Cl_j(s) + Hl_0(s) + \sigma(s) \right|^2 ds \\ &\quad + \frac{M}{N_{\bar{k}}^2} \mathbb{E} \int_0^t |u_i(s)|^2 ds + \frac{M}{N_{\bar{k}}^2} \mathbb{E} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \int_0^t |\varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s))|^2 ds. \end{aligned} \quad (52)$$

On the one hand, since for each fixed $s \in [0, T]$, under the conditional expectation $\mathbb{E}(\cdot | \mathcal{F}_s^{W_0})$, the processes $(\bar{p}_i(s), \bar{q}_i(s))$, $i \in \mathcal{I}_{\bar{k}}$, are independent identically distributed. If we denote $\mu(s) = \mathbb{E}(\varphi_{\bar{k}}(\bar{p}_i(s), \bar{q}_i(s)) | \mathcal{F}_s^{W_0})$, then μ does not depend on i . Moreover,

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \right|^2 \\ & \leq 2\mathbb{E} \left| \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \frac{N_{\bar{k}} - 1}{N_{\bar{k}}} \mu(s) \right|^2 + 2\mathbb{E} \left| \frac{1}{N_{\bar{k}}} \mu(s) \right|^2 \\ & = 2 \frac{(N_{\bar{k}} - 1)^2}{N_{\bar{k}}^2} \mathbb{E} \left| \frac{1}{N_{\bar{k}} - 1} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \right|^2 + \frac{2}{N_{\bar{k}}^2} \mathbb{E} |\mu(s)|^2. \end{aligned}$$

Then, due to (33) and the fact that $(\bar{p}_i(s), \bar{q}_i(s))$, $i \in \mathcal{I}_{\bar{k}}$, are independent identically distributed under $\mathbb{E}(\cdot | \mathcal{F}_s^{W_0})$, similar to the proof of Lemma 6.2, we can obtain

$$\begin{aligned} & \int_0^t \mathbb{E} \left| \frac{1}{N_{\bar{k}}} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \right|^2 ds \\ & \leq 2 \frac{(N_{\bar{k}} - 1)^2}{N_{\bar{k}}^2} \int_0^t \mathbb{E} \left| \frac{1}{N_{\bar{k}} - 1} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s) \right|^2 ds + \frac{2}{N_{\bar{k}}^2} \int_0^t \mathbb{E} |\mu(s)|^2 ds \\ & = 2 \frac{N_{\bar{k}} - 1}{N_{\bar{k}}^2} \int_0^t \mathbb{E} |\varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s)) - \mu(s)|^2 ds + \frac{2}{N_{\bar{k}}^2} \int_0^t \mathbb{E} |\mu(s)|^2 ds \leq \frac{M}{N_{\bar{k}}}. \end{aligned}$$

On the other hand, due to (49) and (50), we get

$$\begin{aligned} & \frac{M}{N_{\bar{k}}^2} \mathbb{E} \int_0^t |u_i(s)|^2 ds + \frac{M}{N_{\bar{k}}^2} \mathbb{E} \sum_{j=1}^N \int_0^t \left| F_2(l^{(N)}(s) - \Phi(s)) + F_2\Phi(s) + Cl_j(s) + Hl_0(s) + \sigma(s) \right|^2 ds \\ & \leq \frac{M}{N_{\bar{k}}} \mathbb{E} \int_0^t |l^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N_{\bar{k}}}. \end{aligned}$$

Moreover, since $(\bar{p}_i(s), \bar{q}_i(s))$, $i \in \mathcal{I}_{\bar{k}}$, are identically distributed under $\mathbb{E}(\cdot | \mathcal{F}_s^{W_0})$, we have $\frac{M}{N_{\bar{k}}^2} \mathbb{E} \sum_{j \in \mathcal{I}_{\bar{k}}, j \neq i} \int_0^t |\varphi_{\bar{k}}(\bar{p}_j(s), \bar{q}_j(s))|^2 ds \leq \frac{M}{N_{\bar{k}}}$. Therefore, from above estimates, we get from (52) that, for any $t \in [0, T]$,

$$\mathbb{E} \sup_{0 \leq s \leq T} |\Xi(s)|^2 \leq M \mathbb{E} \int_0^t |\Xi(s)|^2 + |l^{(N)}(s) - \Phi(s)|^2 ds + \frac{M}{N_{\bar{k}}},$$

which yields, by using Gronwall's inequality, that

$$\mathbb{E} \sup_{0 \leq s \leq t} |l^{(\bar{k})}(s) - m_{\bar{k}}(s)|^2 \leq M \mathbb{E} \int_0^t \left[|l^{(N)}(s) - \Phi(s)|^2 \right] ds + \frac{M}{N_{\bar{k}}}. \quad (53)$$

Consequently, noticing (51) and (53), we have for each $1 \leq k \leq K$,

$$\mathbb{E} \sup_{0 \leq s \leq t} |l^{(k)}(s) - m_k(s)|^2 \leq M \mathbb{E} \int_0^t \left[|l^{(N)}(s) - \Phi(s)|^2 \right] ds + \frac{M}{N_k}. \quad (54)$$

Since

$$\begin{aligned}
l^{(N)}(s) - \Phi(s) &= \sum_{k=1}^K \left[\pi_k^{(N)} \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} l_i(s) - \pi_k \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right] \\
&= \sum_{k=1}^K \left[\pi_k^{(N)} \left(\frac{1}{N_k} \sum_{i \in \mathcal{I}_k} l_i(s) - \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right) + \left(\pi_k^{(N)} - \pi_k \right) \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}) \right] \\
&= \sum_{k=1}^K \pi_k^{(N)} (l^{(k)}(s) - m_k(s)) + \sum_{k=1}^K \left(\pi_k^{(N)} - \pi_k \right) \mathbb{E}(\alpha_k(s) | \mathcal{F}_s^{W_0}),
\end{aligned}$$

then by (33), (54) and $\pi_k^{(N)} = \frac{N_k}{N} \leq 1$, we have for any $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} |l^{(N)}(s) - \Phi(s)|^2 &\leq \mathbb{E} \sum_{k=1}^K \sup_{0 \leq s \leq t} \pi_k^{(N)} |l^{(k)}(s) - m_k(s)|^2 + M \varepsilon_N^2 \\
&\leq M \mathbb{E} \int_0^t [|l^{(N)}(s) - \Phi(s)|^2] ds + \frac{M}{N} + M \varepsilon_N^2.
\end{aligned}$$

Finally, by using Gronwall's inequality, we complete the proof. \square

Now, we introduce the following system of decentralized limiting state with the perturbation strategy of minor agent \mathcal{A}_i : for $1 \leq j \leq N$, $j \neq i$,

$$\begin{cases} d\bar{l}_0 = [A_0 \bar{l}_0 + B_0 \varphi_0(\bar{p}_0, \bar{q}_0) + F_0^1 \Phi + b_0] dt + [C_0 \bar{l}_0 + D_0 \varphi_0(\bar{p}_0, \bar{q}_0) + F_0^2 \Phi + \sigma_0] dW_0(t) \\ d\bar{l}_i = [A_{\theta_i} \bar{l}_i + B u_i + F_1 \Phi + b_0] dt + [C \bar{l}_i + D_{\theta_i} u_i + F_2 \Phi + H \bar{l}_0 + \sigma_0] dW_i(t), \\ d\bar{l}_j = [A_{\theta_j} \bar{l}_j + B \varphi_{\theta_j}(\bar{p}_j, \bar{q}_j) + F_1 \Phi + b_0] dt + [C \bar{l}_j + D_{\theta_j} \varphi_{\theta_j}(\bar{p}_j, \bar{q}_j) + F_2 \Phi + H \bar{l}_0 + \sigma_0] dW_i(t), \\ \bar{l}_0(0) = x_0, \quad \bar{l}_i(0) = \bar{l}_j(0) = x. \end{cases} \quad (55)$$

Lemma 6.9 *Under the assumptions of (A1)-(A4), we have*

$$\mathbb{E} \sup_{0 \leq t \leq T} \left(|l_0(t) - \bar{l}_0(t)|^2 + |l_i(t) - \bar{l}_i(t)|^2 \right) = O\left(\frac{1}{N} + \varepsilon_N^2\right). \quad (56)$$

Proof From both the first equation of (48) and (55), we obtain

$$\begin{cases} d(l_0 - \bar{l}_0) = [A_0(l_0 - \bar{l}_0) + F_0^1(l^{(N)} - \Phi)] dt + [C(l_0 - \bar{l}_0) + F_0^2(l^{(N)} - \Phi)] dW_0(t), \\ l_0(0) - \bar{l}_0(0) = 0. \end{cases}$$

With the help of classical estimates of SDE and Lemma 6.8, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} |l_0(t) - \bar{l}_0(t)|^2 = O\left(\frac{1}{N} + \varepsilon_N^2\right). \quad (57)$$

Now, from both the second equation of (48) and (55), we obtain

$$\begin{aligned} d(l_i - \bar{l}_i) &= [A_{\theta_i}(l_i - \bar{l}_i) + F_1(l^{(N)} - \Phi)] dt \\ &\quad + [C(l_i - \bar{l}_i) + F_2(l^{(N)} - \Phi) + H(l_0 - \bar{l}_0)] dW_0(t), \quad l_i(0) - \bar{l}_i(0) = 0. \end{aligned}$$

From the classical estimates of SDE, Lemma 6.8 and (57), it is easy to obtain (56). \square

Lemma 6.10 *Under the assumptions of (A1)-(A4), for each $1 \leq i \leq N$, for the minor agent \mathcal{A}_i 's perturbation control u_i , we have*

$$\left| \mathcal{J}_i(u_i, \bar{u}_{-i}) - J_0(u_i) \right| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).$$

Proof Recall (4), (12) and (17), we have

$$\begin{aligned} & \mathcal{J}_i(u_i, \bar{u}_{-i}) - J_i(u_i) \\ &= \frac{1}{2} \mathbb{E} \left[\int_0^T \left(\left\langle Q(l_i - \rho l^{(N)}) - (1 - \rho)l_0, l_i - \rho l^{(N)} - (1 - \rho)l_0 \right\rangle \right. \right. \\ & \quad \left. \left. - \left\langle Q(\bar{l}_i - \rho\Phi - (1 - \rho)\bar{l}_0), \bar{l}_i - \rho\Phi - (1 - \rho)\bar{l}_0 \right\rangle \right) dt \right. \\ & \quad \left. + \left\langle G(l_i(T) - \rho l^{(N)}(T) - (1 - \rho)l_0(T)), l_i(T) - \rho l^{(N)}(T) - (1 - \rho)l_0(T) \right\rangle \right. \\ & \quad \left. - \left\langle G(\bar{l}_i(T) - \rho\Phi(T) - (1 - \rho)\bar{l}_0(T)), \bar{l}_i(T) - \rho\Phi(T) - (1 - \rho)\bar{l}_0(T) \right\rangle \right]. \end{aligned} \quad (58)$$

Similar to the proof of Lemma 6.4, by using Lemma 6.8, 6.9 and $\mathbb{E} \left(|\bar{l}_0(t)|^2 + |\bar{l}_i(t)|^2 + |\Phi(t)|^2 \right) \leq M$, we have

$$\begin{aligned} & \left| \mathbb{E} \int_0^T \left(\left\langle Q(l_i - \rho l^{(N)} - (1 - \rho)l_0), l_i - \rho l^{(N)} - (1 - \rho)l_0 \right\rangle \right. \right. \\ & \quad \left. \left. - \left\langle Q(\bar{l}_i - \rho\Phi) - (1 - \rho)\bar{l}_0, \bar{l}_i - \rho\Phi - (1 - \rho)\bar{l}_0 \right\rangle \right) dt \right| \\ & \leq M \int_0^T \mathbb{E} |l_i - \bar{l}_i|^2 dt + M \int_0^T \mathbb{E} |l_0 - \bar{l}_0|^2 dt + M \int_0^T \mathbb{E} |l^{(N)} - \Phi|^2 dt \\ & \quad + M \int_0^T \left(\mathbb{E} |l_i - \bar{l}_i|^2 + \mathbb{E} |l_0 - \bar{l}_0|^2 + \mathbb{E} |l^{(N)} - \Phi|^2 \right)^{\frac{1}{2}} dt = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right) \end{aligned}$$

and

$$\begin{aligned} & \left| \mathbb{E} \left[\left\langle G(l_i(T) - \rho l^{(N)}(T) - (1 - \rho)l_0(T)), l_i(T) - \rho l^{(N)}(T) - (1 - \rho)l_0(T) \right\rangle \right. \right. \\ & \quad \left. \left. - \left\langle G(\bar{l}_i(T) - \rho\Phi(T) - (1 - \rho)\bar{l}_0(T)), \bar{l}_i(T) - \rho\Phi(T) - (1 - \rho)\bar{l}_0(T) \right\rangle \right] \right| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right). \end{aligned}$$

The proof is completed by noticing (58). \square

Theorem 6.2 *Under the assumptions of (A1)-(A4), $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$ is an ε -Nash equilibrium of Problem (CC) for minor agents.*

Proof For each $1 \leq i \leq N$, combining Lemma 6.4 and Lemma 6.10, we have

$$\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) \leq J_i(\bar{u}_i) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right) \leq J_i(u_i) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right) \leq \mathcal{J}_i(u_i, \bar{u}_{-i}) + O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right),$$

where the second inequality comes from the fact that $J_i(\bar{u}_i) = \inf_{u_i \in \mathcal{U}_{ad}^i} J_i(u_i)$. Consequently, Theorem 6.2 holds with $\varepsilon = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right)$. \square

By combining Theorems 6.1, 6.2, we obtain the following main result of this paper:

Theorem 6.3 *Under the assumptions of (A1)-(A4), $(\bar{u}_0, \bar{u}_1, \dots, \bar{u}_N)$ is an ε -Nash equilibrium of Problem (CC), where $\bar{u}_0 = \varphi_0(\bar{p}_0, \bar{q}_0)$, $\bar{u}_i = \varphi_{\theta_i}(\bar{p}_i, \bar{q}_i)$, $1 \leq i \leq N$ for*

$$\varphi_0(p, q) := \mathbf{P}_{\Gamma_0} [R_0^{-1} (B'_0 p + D'_0 q)], \quad \varphi_{\theta_i}(p, q) := \mathbf{P}_{\Gamma_{\theta_i}} [R_{\theta_i}^{-1} (B' p + D'_{\theta_i} q)].$$

7 Appendix

We give this appendix to prove Theorem 5.1. The fully-coupled structure of MF-FBSDE (29) arises difficulties for establishing its wellposedness. Motivated by Pardoux and Tang [39] Theorem 3.1, we can establish the wellposedness of MF-FBSDE (29) for arbitrary time duration when it is weakly coupled.

Let us first note that for a given $(Y(\cdot), Z(\cdot)) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m \times (d+1)})$, the forward equation in the MF-FBSDE (29) has a unique solution $X(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$, thus we introduce a map $\mathcal{M}_1 : L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m \times (d+1)}) \rightarrow L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$, through

$$\begin{aligned} X(t) = & x + \int_0^t b(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s)) ds \\ & + \int_0^t \sigma(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s)) dW(s). \end{aligned} \quad (59)$$

We mention that the wellposedness of (59) can be established by using the contraction mapping method under assumption $(H_1), (H_2)$, although it has the term $\mathbb{E}[X_s|\mathcal{F}_s^{W_0}]$. We omit the proof. Moreover, with the help of BDG inequality, it follows that $\mathbb{E} \sup_{t \in [0, T]} |X(t)|^2 < \infty$.

Lemma 7.1 *Let X_i be the solution of (59) corresponding to $(Y_i(\cdot), Z_i(\cdot)) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m \times (d+1)})$, $i = 1, 2$. Then for all $\lambda \in \mathbb{R}$, $K_1, K_2 > 0$, we have*

$$\begin{aligned} & e^{-\lambda t} \mathbb{E} |X_1(t) - X_2(t)|^2 + \bar{\lambda}_1 \int_0^t e^{-\lambda s} \mathbb{E} |X_1(s) - X_2(s)|^2 ds \\ & \leq (k_2 K_1 + k_9^2) \int_0^t e^{-\lambda s} \mathbb{E} |Y_1(s) - Y_2(s)|^2 ds + (k_3 K_2 + k_{10}^2) \int_0^t e^{-\lambda s} \mathbb{E} |Z_1(s) - Z_2(s)|^2 ds, \end{aligned} \quad (60)$$

where $\bar{\lambda}_1 := \lambda - 2\lambda_1 - k_2 K_1^{-1} - k_3 K_2^{-1} - 2k_1 - k_7^2 - k_8^2$. Moreover,

$$\begin{aligned} e^{-\lambda t} \mathbb{E} |X_1(t) - X_2(t)|^2 & \leq (k_2 K_1 + k_9^2) \int_0^t e^{-\bar{\lambda}_1(t-s)} e^{-\lambda s} \mathbb{E} |Y_1(s) - Y_2(s)|^2 ds \\ & \quad + (k_3 K_2 + k_{10}^2) \int_0^t e^{-\bar{\lambda}_1(t-s)} e^{-\lambda s} \mathbb{E} |Z_1(s) - Z_2(s)|^2 ds. \end{aligned} \quad (61)$$

Proof We denote $\bar{X} := X_1 - X_2$, $\bar{Y} := Y_1 - Y_2$, $\bar{Z} := Z_1 - Z_2$, $\bar{b} := b(X_1, \mathbb{E}[X_1|\mathcal{F}^{W_0}], Y_1, Z_1) - b(X_2, \mathbb{E}[X_2|\mathcal{F}^{W_0}], Y_2, Z_2)$, $\bar{\sigma} := \sigma(X_1, \mathbb{E}[X_1|\mathcal{F}^{W_0}], Y_1, Z_1) - \sigma(X_2, \mathbb{E}[X_2|\mathcal{F}^{W_0}], Y_2, Z_2)$. Applying Itô's formula to $e^{-\lambda t} \mathbb{E} |X_1(t) - X_2(t)|^2$ and taking expectation, we obtain

$$e^{-\lambda t} \mathbb{E} |\bar{X}(t)|^2 = -\lambda \int_0^t e^{-\lambda s} \mathbb{E} |\bar{X}(s)|^2 ds + 2\mathbb{E} \int_0^t e^{-\lambda s} \langle \bar{X}(s), \bar{b}(s) \rangle ds + \mathbb{E} \int_0^t e^{-\lambda s} |\bar{\sigma}(s)|^2 ds. \quad (62)$$

Noticing that

$$\begin{aligned} & 2\langle \bar{X}(s), \bar{b}(s) \rangle \\ & = 2\langle \bar{X}(s), b(s, X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) - b(X_2(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) \rangle \\ & \quad + 2\langle \bar{X}(s), b(s, X_2(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) - b(X_2(s), \mathbb{E}[X_2(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_2(s)) \rangle \\ & \leq 2\lambda_1 |\bar{X}(s)|^2 + 2|\bar{X}(s)| (k_1 |\mathbb{E}[\bar{X}(s)|\mathcal{F}_s^{W_0}]| + k_2 |\bar{Y}(s)| + k_3 |\bar{Z}(s)|) \\ & \leq (2\lambda_1 + k_2 K_1^{-1} + k_3 K_2^{-1}) |\bar{X}(s)|^2 + 2k_1 \mathbb{E}[|\bar{X}(s)|^2 | \mathcal{F}_s^{W_0}] + k_2 K_1 |\bar{Y}(s)|^2 + k_3 K_2 |\bar{Z}(s)|^2, \end{aligned}$$

and

$$\begin{aligned} |\bar{\sigma}(s)|^2 & \leq k_7^2 |\bar{X}(s)|^2 + k_8^2 |\mathbb{E}[\bar{X}(s)|\mathcal{F}_s^{W_0}]|^2 + k_9^2 |\bar{Y}(s)|^2 + k_{10}^2 |\bar{Z}(s)|^2 \\ & \leq k_7^2 |\bar{X}(s)|^2 + k_8^2 \mathbb{E}[|\bar{X}(s)|^2 | \mathcal{F}_s^{W_0}] + k_9^2 |\bar{Y}(s)|^2 + k_{10}^2 |\bar{Z}(s)|^2. \end{aligned}$$

Then, from (62) and $\mathbb{E}[\mathbb{E}[|\bar{X}(s)|^2|\mathcal{F}_s^{W_0}]] = \mathbb{E}|\bar{X}(s)|^2$, we can obtain (60).

Now, we apply Itô's formula to $e^{-\bar{\lambda}_1(t-s)}e^{-\lambda s}\mathbb{E}|X_1(s) - X_2(s)|^2$ for $s \in [0, t]$ and taking expectation, it follows that

$$\begin{aligned} e^{-\lambda t}\mathbb{E}|\bar{X}(t)|^2 &= -(\lambda - \bar{\lambda}_1) \int_0^t e^{-\bar{\lambda}_1(t-s)}e^{-\lambda s}\mathbb{E}|\bar{X}(s)|^2 ds + 2\mathbb{E} \int_0^t e^{-\bar{\lambda}_1(t-s)}e^{-\lambda s}\langle \bar{X}(s), \bar{b}(s) \rangle ds \\ &\quad + \mathbb{E} \int_0^t e^{-\bar{\lambda}_1(t-s)}e^{-\lambda s}|\bar{\sigma}(s)|^2 ds. \end{aligned} \tag{63}$$

From above estimates and (63), one can prove (61). \square

Remark 7.1 *By integrating both sides of (61) on $[0, T]$ and using $\frac{1-e^{-\bar{\lambda}_1(T-s)}}{\bar{\lambda}_1} \leq \frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1}$, $\forall s \in [0, T]$, we have*

$$\|X_1 - X_2\|_\lambda^2 \leq \frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1} [(k_2 K_1 + k_9^2)\|Y_1 - Y_2\|_\lambda^2 + (k_3 K_2 + k_{10}^2)\|Z_1 - Z_2\|_\lambda^2]. \tag{64}$$

Let $t = T$ in (61) and noticing that $e^{-\bar{\lambda}_1(T-s)} \leq 1 \vee e^{-\bar{\lambda}_1 T}$, $\forall s \in [0, T]$, thus

$$e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 \leq [1 \vee e^{-\bar{\lambda}_1 T}] [(k_2 K_1 + k_9^2)\|Y_1 - Y_2\|_\lambda^2 + (k_3 K_2 + k_{10}^2)\|Z_1 - Z_2\|_\lambda^2]. \tag{65}$$

In particular, if $\bar{\lambda}_1 > 0$, we have

$$e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 \leq (k_2 K_1 + k_9^2)\|Y_1 - Y_2\|_\lambda^2 + (k_3 K_2 + k_{10}^2)\|Z_1 - Z_2\|_\lambda^2. \tag{66}$$

Similarly, for a given $X(\cdot) \in L_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$, the backward equation in the MF-FBSDE (29) has a unique solution $(Y(\cdot), Z(\cdot)) \in L_{\mathcal{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m \times (d+1)})$, thus we can introduce another map $\mathcal{M}_2 : L_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \rightarrow L_{\mathcal{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m \times (d+1)})$, through

$$Y(t) = g(X(T), \mathbb{E}[X(T)|\mathcal{F}_T^{W_0}]) + \int_t^T f(s, X(s), \mathbb{E}[X(s)|\mathcal{F}_s^{W_0}], Y(s), Z(s)) ds - \int_t^T Z(s) dW(s). \tag{67}$$

The wellposedness of (67) under assumption (H_1) , (H_2) is referred to Darling and Pardoux [16] Theorem 3.4 or Buckdahn and Nie [12] Lemma 2.2. Moreover, we have $\mathbb{E} \sup_{t \in [0, T]} |Y(t)|^2 < \infty$.

Lemma 7.2 *Let $(Y_i(\cdot), Z_i(\cdot))$ be the solution of (67) corresponding to $X_i \in L_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$, $i = 1, 2$. Then for all $\lambda \in \mathbb{R}$, $K_3, K_4 > 0$, we have*

$$\begin{aligned} &e^{-\lambda t}\mathbb{E}|Y_1(t) - Y_2(t)|^2 + \bar{\lambda}_2 \int_t^T e^{-\lambda s}\mathbb{E}|Y_1(s) - Y_2(s)|^2 ds \\ &\quad + (1 - k_6 K_4) \int_t^T e^{-\lambda s}\mathbb{E}|Z_1(s) - Z_2(s)|^2 ds \\ &\leq (k_{11}^2 + k_{12}^2)e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5)K_3 \int_t^T e^{-\lambda s}\mathbb{E}|X_1(s) - X_2(s)|^2 ds, \end{aligned} \tag{68}$$

where $\bar{\lambda}_2 := -\lambda - 2\lambda_2 - (k_4 + k_5)K_3^{-1} - k_6 K_4^{-1}$. Moreover,

$$\begin{aligned} &e^{-\lambda t}\mathbb{E}|Y_1(t) - Y_2(t)|^2 + (1 - k_6 K_4) \int_t^T e^{-\bar{\lambda}_2(s-t)}e^{-\lambda s}\mathbb{E}|Z_1(s) - Z_2(s)|^2 ds \\ &\leq (k_{11}^2 + k_{12}^2)e^{-\bar{\lambda}_2(T-t)}e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 \\ &\quad + (k_4 + k_5)K_3 \int_t^T e^{-\bar{\lambda}_2(s-t)}e^{-\lambda s}\mathbb{E}|X_1(s) - X_2(s)|^2 ds. \end{aligned} \tag{69}$$

Proof We denote $\bar{X} := X_1 - X_2$, $\bar{Y} := Y_1 - Y_2$, $\bar{Z} := Z_1 - Z_2$, $\bar{f} := f(X_1, \mathbb{E}[X_1|\mathcal{F}^{W_0}], Y_1, Z_1) - f(X_2, \mathbb{E}[X_2|\mathcal{F}^{W_0}], Y_2, Z_2)$. Applying Itô's formula to $e^{-\lambda t}\mathbb{E}|Y_1(t) - Y_2(t)|^2$ and taking expectation, we obtain

$$\begin{aligned} & e^{-\lambda t}\mathbb{E}|\bar{Y}(t)|^2 - \lambda \int_t^T e^{-\lambda s}\mathbb{E}|\bar{Y}(s)|^2 ds + \mathbb{E} \int_t^T e^{-\lambda s}|\bar{Z}(s)|^2 ds \\ &= e^{-\lambda T}\mathbb{E}|\bar{Y}(T)|^2 + 2\mathbb{E} \int_t^T e^{-\lambda s}\langle \bar{Y}(s), \bar{f}(s) \rangle ds. \end{aligned} \quad (70)$$

Noticing that

$$\begin{aligned} & 2\langle \bar{Y}(s), \bar{f}(s) \rangle \\ &= 2\langle \bar{Y}(s), f(s, X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_1(s), Z_1(s)) - f(X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_1(s)) \rangle \\ & \quad + 2\langle \bar{Y}(s), f(s, X_1(s), \mathbb{E}[X_1(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_1(s)) - f(X_2(s), \mathbb{E}[X_2(s)|\mathcal{F}_s^{W_0}], Y_2(s), Z_2(s)) \rangle \\ &\leq 2\lambda_2|\bar{Y}(s)|^2 + 2|\bar{Y}(s)|(k_4|\bar{X}(s)| + k_5|\mathbb{E}[\bar{X}(s)|\mathcal{F}_s^{W_0}]| + k_6|\bar{Z}(s)|) \\ &\leq (2\lambda_2 + k_4K_3^{-1} + k_5K_3^{-1} + k_6K_4^{-1})|\bar{Y}(s)|^2 + k_4K_3|\bar{X}(s)|^2 + k_5K_3\mathbb{E}[|\bar{X}(s)|^2|\mathcal{F}_s^{W_0}] + k_6K_4|\bar{Z}(s)|^2, \end{aligned}$$

and

$$\begin{aligned} |\bar{Y}(T)|^2 &= |g(X_1(T), \mathbb{E}[X_1(T)|\mathcal{F}_T^{W_0}]) - g(X_2(T), \mathbb{E}[X_2(T)|\mathcal{F}_T^{W_0}])|^2 \\ &\leq k_{11}^2|\bar{X}(T)|^2 + k_{12}^2\mathbb{E}[|\bar{X}(s)|^2|\mathcal{F}_s^{W_0}]. \end{aligned}$$

Then, from (70) and $\mathbb{E}[\mathbb{E}[|\bar{X}(s)|^2|\mathcal{F}_s^{W_0}]] = \mathbb{E}|\bar{X}(s)|^2$, we can obtain (68).

Now, we apply Itô's formula to $e^{-\bar{\lambda}_2(s-t)}e^{-\lambda s}\mathbb{E}|Y_1(s) - Y_2(s)|^2$ for $s \in [t, T]$ and taking expectation, it follows that

$$\begin{aligned} & e^{-\lambda t}\mathbb{E}|\bar{Y}(t)|^2 - (\lambda + \bar{\lambda}_2) \int_t^T e^{-\bar{\lambda}_2(s-t)}e^{-\lambda s}\mathbb{E}|\bar{Y}(s)|^2 ds + \mathbb{E} \int_t^T e^{-\bar{\lambda}_2(s-t)}e^{-\lambda s}|\bar{Z}(s)|^2 ds \\ &= e^{-\bar{\lambda}_2(T-t)}e^{-\lambda T}\mathbb{E}|\bar{Y}(s)|^2 + 2\mathbb{E} \int_t^T e^{-\bar{\lambda}_2(s-t)}e^{-\lambda s}\langle \bar{Y}(s), \bar{f}(s) \rangle ds. \end{aligned} \quad (71)$$

From above estimates and (71), one can prove (69). \square

Remark 7.2 Now we choose K_4 satisfying $0 < K_4 \leq k_6^{-1}$, then by integrating both sides of (69) on $[0, T]$ and using $\frac{1-e^{-\bar{\lambda}_2 s}}{\bar{\lambda}_2} \leq \frac{1-e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2}$, $\forall s \in [0, T]$, we have

$$\|Y_1 - Y_2\|_{\lambda}^2 \leq \frac{1 - e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2} \left[(k_{11}^2 + k_{12}^2)e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5)K_3\|X_1 - X_2\|_{\lambda}^2 \right]. \quad (72)$$

Let $t = 0$ in (69) and noticing that $1 \wedge e^{-\bar{\lambda}_2 T} \leq e^{-\bar{\lambda}_2 s} \leq 1 \vee e^{-\bar{\lambda}_2 T}$, $\forall s \in [0, T]$, thus

$$\|Z_1 - Z_2\|_{\lambda}^2 \leq \frac{(k_{11}^2 + k_{12}^2)e^{-\bar{\lambda}_2 T}e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5)K_3(1 \vee e^{-\bar{\lambda}_2 T})\|X_1 - X_2\|_{\lambda}^2}{(1 - k_6K_4)(1 \wedge e^{-\bar{\lambda}_2 T})}. \quad (73)$$

On the other hand, if $\bar{\lambda}_2 > 0$, let $t = 0$ in (68), we have

$$\|Z_1 - Z_2\|_{\lambda}^2 \leq \frac{(k_{11}^2 + k_{12}^2)e^{-\lambda T}\mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5)K_3\|X_1 - X_2\|_{\lambda}^2}{1 - k_6K_4}. \quad (74)$$

Now, we present the proof of Theorem 5.1.

Proof of Theorem 5.1 We define $\mathcal{M} := \mathcal{M}_2 \circ \mathcal{M}_1$, recalling that \mathcal{M}_1 is defined through (59) and \mathcal{M}_2 is defined through (67). Thus \mathcal{M} maps $L_{\mathcal{F}}^2(0, T; \mathbb{R}^m) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m \times (d+1)})$ into itself. To

prove the theorem, it is only need to show that \mathcal{M} is a contraction mapping for some equivalent norm $\|\cdot\|_\lambda$. In fact, for $(Y_i, Z_i) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m \times (d+1)})$, let $X_i := \mathcal{M}_1(Y_i, Z_i)$ and $(\bar{Y}_i, \bar{Z}_i) := \mathcal{M}((Y_i, Z_i))$, from (64), (65), (72) and (73) based on Lemma 7.1, 7.2, we have

$$\begin{aligned}
& \|\bar{Y}_1 - \bar{Y}_2\|_\lambda^2 + \|\bar{Z}_1 - \bar{Z}_2\|_\lambda^2 \\
& \leq \left[\frac{1-e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2} + \frac{1 \vee e^{-\bar{\lambda}_2 T}}{(1-k_6 K_4)(1 \wedge e^{-\bar{\lambda}_2 T})} \right] \\
& \quad \times [(k_{11}^2 + k_{12}^2)e^{-\lambda T} \mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5)K_3 \|X_1 - X_2\|_\lambda^2] \\
& \leq \left[\frac{1-e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2} + \frac{1 \vee e^{-\bar{\lambda}_2 T}}{(1-k_6 K_4)(1 \wedge e^{-\bar{\lambda}_2 T})} \right] \\
& \quad \times \left[(k_{11}^2 + k_{12}^2)(1 \vee e^{-\bar{\lambda}_1 T}) + (k_4 + k_5)K_3 \frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1} \right] \\
& \quad \times [(k_2 K_1 + k_9^2) \|Y_1 - Y_2\|_\lambda^2 + (k_3 K_2 + k_{10}^2) \|Z_1 - Z_2\|_\lambda^2].
\end{aligned}$$

Recalling that $\bar{\lambda}_1 := \lambda - 2\lambda_1 - k_2 K_1^{-1} - k_3 K_2^{-1} - 2k_1 - k_7^2 - k_8^2$ and $\bar{\lambda}_2 := -\lambda - 2\lambda_2 - (k_4 + k_5)K_3^{-1} - k_6 K_4^{-1}$. Then by choosing suitable λ (e.g. we can easily choose λ big enough such that $\bar{\lambda}_1 \geq 1$ and $\bar{\lambda}_2 \leq 0$), the first assertion is immediate.

Now let us prove the second assertion. Since $2(\lambda_1 + \lambda_2) < -2k_1 - k_6^2 - k_7^2 - k_8^2$, we can choose $\lambda \in \mathbb{R}$, $0 < K_4 \leq k_6^{-1}$ and sufficient large K_1, K_2, K_3 such that

$$\bar{\lambda}_1 > 0, \quad \bar{\lambda}_2 > 0, \quad 1 - K_4 k_6 > 0.$$

Then from (64), (66), (72) and (74), we have

$$\begin{aligned}
& \|\bar{Y}_1 - \bar{Y}_2\|_\lambda^2 + \|\bar{Z}_1 - \bar{Z}_2\|_\lambda^2 \\
& \leq \left[\frac{1}{\bar{\lambda}_2} + \frac{1}{1-k_6 K_4} \right] \times [(k_{11}^2 + k_{12}^2)e^{-\lambda T} \mathbb{E}|X_1(T) - X_2(T)|^2 + (k_4 + k_5)K_3 \|X_1 - X_2\|_\lambda^2] \\
& \leq \left[\frac{1}{\bar{\lambda}_2} + \frac{1}{1-k_6 K_4} \right] \times \left[k_{11}^2 + k_{12}^2 + (k_4 + k_5)K_3 \frac{1}{\bar{\lambda}_1} \right] \\
& \quad \times [(k_2 K_1 + k_9^2) \|Y_1 - Y_2\|_\lambda^2 + (k_3 K_2 + k_{10}^2) \|Z_1 - Z_2\|_\lambda^2].
\end{aligned}$$

This completes the second assertion. \square

References

- [1] S. Ahuja. Wellposedness of mean field games with common noise under a weak monotonicity conditions. *SIAM J. Control Optim.*, **54**, 30-48 (2016).
- [2] M. Bardi. Explicit solutions of some linear-quadratic mean field games. *Netw. Heterog. Media.* **7**, 243-261 (2012).
- [3] M. Bardi, F. Priuli. Linear-quadratic N-person and mean-field games with ergodic cost. *SIAM J. Control Optim.* **52**, 3022-3052 (2014).
- [4] A. Bensoussan, M. Chau, S. Yam. Mean Field Games with a Dominating Player. *Appl. Math. Optim.* **74**, 91-128 (2016).
- [5] A. Bensoussan, J. Frehse, S. Yam. Mean Field Games and Mean Field Type Control Theory. Springer, New York, 2013.
- [6] A. Bensoussan, K. Wong, and S. Yam, S. Yung. Time-Consistent Portfolio Selection under Short-Selling Prohibition: From Discrete to Continuous Setting. *SIAM J. Financial Math.* **5**, 153-190 (2014).

- [7] A. Bensoussan, K. Sung, S. Yam. Linear-quadratic time-inconsistent mean field games. *Dyn. Games Appl.* **3**, 537-552 (2013).
- [8] A. Bensoussan, K. Sung, and S. Yam, S. Yung. Linear quadratic mean field games. *J. Optim. Theory Appl.* **169**, 496-529 (2016).
- [9] R. Brammer. Controllability in linear autonomous systems with positive controllers. *SIAM J. Control Optim.*, **10**, 339-353 (1972).
- [10] H. Brezis. Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York Dordrecht Heidelberg London, 2010.
- [11] R. Buckdahn, J. Li, S. Peng. Nonlinear stochastic differential games involving a major player and a large number of collectively acting minor agents. *SIAM J. Control Optim.*, **52**, 451-492 (2014).
- [12] R. Buckdahn and T. Nie. Generalized Hamilton-Jacobi-Bellman equations with Dirichlet boundary condition and stochastic exit time optimal control problem, *SIAM. J. Control Optim.* 54, 602-631 (2016).
- [13] R. Carmona and F. Delarue. Probabilistic analysis of mean-field games. *SIAM J. Control Optim.*, **51**, 2705-2734 (2013).
- [14] R. Carmona, X. Zhu. A probabilistic approach to mean field games with major and minor players. *Ann. Appl. Probab.*, **26(3)**, 1535-1580 (2016).
- [15] X. Chen, X. Zhou. Stochastic linear quadratic control with conic control constraints on an infinite time horizon. *SIAM J. Control Optim.*, **43**, 1120-1150 (2004).
- [16] R. Darling and E. Pardoux. Backward SDE with random terminal time and applications to semilinear elliptic PDE, *Ann. Probab.* 25, 1135-1159 (1997).
- [17] G. Espinosa, N. Touzi. Optimal investment under relative performance concerns. *Math. Finance*, **25**, 221-257 (2015).
- [18] A. Graham. Kronecker products and Matrix Calculus: with applications. Ellis Horwood, Chichester, UK, 1981.
- [19] O. Gueant, J. M. Lasry, P. L. Lions. Mean field games and applications. *Paris-Princeton Lectures on Mathematical Finance 2010. Lecture Notes in Mathematics*, **2003**, 205-266 (2011).
- [20] Y. Hu, P. Imkeller, M. Müller. Utility maximization in incomplete markets. *Ann. Appl. Probab.*, **15**, 1691-1712 (2005).
- [21] Y. Hu, J. Huang, X. Li. Linear Quadratic Mean Field Game with Control Input Constraint. *ESAIM Control Optim. Calc. Var.* In press. arXiv:1610.05895 [math.OC] (2016).
- [22] Y. Hu, S. Peng. Solution of forward-backward stochastic differential equations. *Probab. Theory Relat. Fields*, **103**, 273-283 (1995).
- [23] Y. Hu, X. Zhou. Constrained stochastic LQ control with random coefficients and applications to portfolio selection. *SIAM J. Control Optim.*, **44**, 444-466 (2005).
- [24] M. Huang. Large population LQG games involving a major player: The Nash certainty equivalence principle. *SIAM J. Control Optim.*, **48**, 3318-3353 (2010).

- [25] M. Huang, P. Caines and R. Malhamé. Distributed multi-agent decision-making with partial observations: asymptotic Nash equilibria. *Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems*, Kyoto, Japan (2006).
- [26] M. Huang, P. Caines, R. Malhamé. An invariance principle in large population stochastic dynamic games. *J. Syst. Sci. Complex.* **20(2)**, 162C172 (2007)
- [27] M. Huang, P. Caines, and R. Malhamé. Large-population cost-coupled LQG problems with non-uniform agents: Individual-mass behavior and decentralized ϵ -Nash equilibria. *IEEE Trans. Automat. Control*, **52**, 1560-1571 (2007).
- [28] M. Huang, R. Malhamé and P. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.*, **6**, 221-251 (2006).
- [29] A. Kizilkale, S. Mannor, P. Caines. Larger scale real-time bidding in the smart grid: A mean field framework. *Proceeding of the 51st IEEE CDC*, Mani, HI, 3680-3687 (2012).
- [30] J. Lasry and P. Lions. Jeux à champ moyen.I. Le cas stationnaire. *Comptes Rendus Mathématique*, **343**, 619-625 (2006).
- [31] J. Lasry and P. Lions. Jeux à champ moyen.II. Horizon fini et contrôle optimal. *Comptes Rendus Mathématique*, **343**, 679-684 (2006).
- [32] J. M. Lasry and P. L. Lions. Mean field games. *Japan J. Math.*, **2**, 229-260 (2007).
- [33] T. Li and J. F. Zhang. Asymptotically optimal decentralized control for large population stochastic multiagent systems. *IEEE Trans. Automat. Control*, **53**, 1643-1660 (2008).
- [34] X. Li, X. Zhou, A. Lim. Dynamic mean-variance portfolio selection with no-shorting constraints. *SIAM J. Control Optim.* **40**, 1540-1555 (2002).
- [35] J. Ma and J. Yong. Forward-backward stochastic differential equations and their applications. Springer-Verlag, Berlin Heidelberg, 1999.
- [36] D. Majerek, W. Nowak, W. Zięba. Conditional strong law of large number. *Int. J. Pure Appl. Math.*, **20**, 143-156 (2005).
- [37] M. Nourian, P. Caines, ϵ -Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents. *SIAM J. Control Optim.* **51**, 3302-3331 (2013).
- [38] S. L. Nguyen, M. Huang. Linear-quadratic-Gaussian mixed games with continuum-parameterized minor players. *SIAM J. Control Optim.* **50**, 2907-2937 (2012).
- [39] E. Pardoux and S. Tang. Forward-backward stochastic differential equations and quasilinear parabolic PDEs, *Probab. Theory Relat. Fields.* 114, 123-150 (1999).
- [40] S. Peng, Z. Wu. Fully coupled forward-backward stochastic differential equations and applications to optimal control. *SIAM J. Control Optim.* **37**, 825-843 (1999).
- [41] F. Priuli. Linear-quadratic N-person and mean-field games: infinite horizon games with discounted cost and singular limits. *Dyn. Games Appl.* **5**, 397-419 (2014).
- [42] H. Tembine, Q. Zhu, T. Basar. Risk-sensitive mean-field stochastic differential games. *IEEE Trans. Automat. Control*, **59**, 835-850 (2014).

- [43] B. Wang, J. Zhang. Mean field games for large-population multiagent systems with markov jump parameters. *SIAM J. Control Optim.*, **50**, 2308-2334 (2012).
- [44] J. Yong, X. Zhou. Stochastic controls: Hamiltonian systems and HJB equations. Springer-Verlag, New York ,1999.
- [45] J. Yong. A leader-follower stochastic linear quadratic differential games. *SIAM. J. Control Optim.* **41**, 1015-1041 (2002).