On p-rationality of number fi elds. Applications-PARI/GP programs

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ON p-RATIONALITY OF NUMBER FIELDS APPLICATIONS – PARI/GP PROGRAMS

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Abstract. Let K be a number field. We prove that its ray class group modulo p^2 (resp. 8) if $p > 2$ (resp. $p = 2$) characterizes its p-rationality. Then we give two short, very fast PARI Programs (§§ 3.1, 3.2) testing if K (defined by an irreducible monic polynomial) is p-rational or not. For quadratic fields we verify some densities related to Cohen–Lenstra–Martinet ones and analyse Greenberg's conjecture on the existence of p-rational fields with Galois groups $(\mathbb{Z}/2\mathbb{Z})^t$ needed for the construction of some Galois representations with open image. We give examples for $p = 3$, $t = 5$ and $t = 6$ (§§ 5.1, 5.2) and illustrate other approaches (Pitoun–Varescon, Barbulescu–Ray). We conclude about the existence of imaginary quadratic fields, p-rational for all $p \geq 2$ (Angelakis–Stevenhagen on the concept of "minimal absolute abelian Galois group") which may enlighten a conjecture of p-rationality (Hajir–Maire) giving large Iwasawa μ-invariants of some uniform pro-p-groups.

Résumé Soit K un corps de nombres. Nous montrons que son corps de classes de rayon modulo p^2 (resp. 8) si $p > 2$ (resp. $p = 2$) caractérise sa p-rationalité. Puis nous donnons deux courts programmes PARI (§§ 3.1, 3.2) trs rapides testant si K (défini par un polynôme irréductible unitaire) est p-rationnel ou non. Pour les corps quadratiques nous vérifions certaines densités en relation avec celles de Cohen–Lenstra–Martinet et nous analysons la conjecture de Greenberg sur l'existence de corps p-rationnels de groupes de Galois $(\mathbb{Z}/2\mathbb{Z})^t$ nécessaires pour la construction de certaines représentations galoisiennes d'image ouverte. Nous donnons des exemples pour $p = 3$, $t = 5$ et $t = 6$ (§§ 5.1, 5.2) et illustrons d'autres approches (Pitoun–Varescon, Barbulescu–Ray). Nous concluons sur l'existence de corps quadratiques imaginaires p-rationnels pour tout $p \geq 2$ (Angelakis–Stevenhagen sur le concept de "groupe de Galois abélien absolu minimal") qui peut éclairer une conjecture de p-rationalité (Hajir–Maire) donnant de grands invariants μ d'Iwasawa relatifs à certains pro-p-groupes uniformes.

1. Definition and properties of p-rationality

Let K be a number field and let $p \geq 2$ be a fixed prime number. We denote by $\mathcal{C}\ell_K$ the p-class group of K in the ordinary sense and by E_K the group of p-principal global units $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p}|p} \mathfrak{p}}$ of K.

Let us describe the diagram of the so called *abelian p-ramification theory* (from [10, § III.2 (c), Fig. 2.2], [11, Section 3]), in which $\widetilde{K}$ is the compositum of the $\mathbb{Z}_p$-extensions of K, H_K the p-Hilbert class field, $H_K^{\rm pr}$ the maximal abelian p-ramified (i.e., unramified outside p) pro-p-extension of K, then $\mathrm{Gal}(H_K^{\rm bp}/\widetilde{K})$ is the Bertrandias–Payan module [19, § 2, Diagramme 4].

See [2] and [19, Diagram 2] for a related context with logarithmic class groups.

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Let

$$U_K := \bigoplus_{\mathfrak{p}\,|\,p} U_{\mathfrak{p}}^1$$

be the $\mathbb{Z}_p$-module of p-principal local units of K, where each

$$U_{\mathfrak{p}}^1 := \{u \in K_{\mathfrak{p}}^\times,\ u \equiv 1 \pmod{\overline{\mathfrak{p}}}\}$$

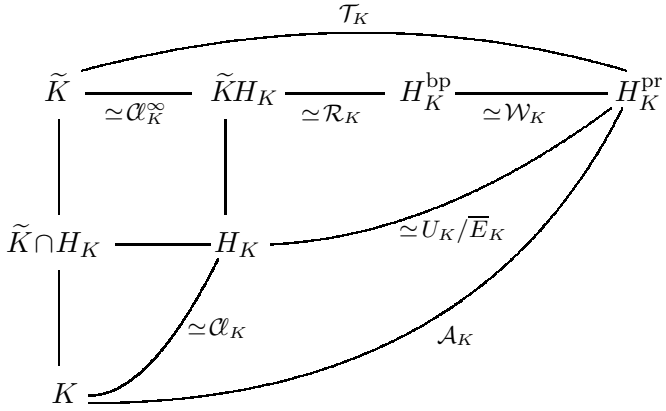
is the group of $\overline{\mathfrak{p}}$-principal units of the completion $K_{\mathfrak{p}}$ of K at $\mathfrak{p} \mid p$, where $\overline{\mathfrak{p}}$ is the maximal ideal of the ring of integers of $K_{\mathfrak{p}}$. For any field k, let μ_k be the p-group of roots of unity of k.

Then put

$$W_K := \mathrm{tor}_{\mathbb{Z}_p}\big(U_K\big) = \bigoplus_{\mathfrak{p}\,|\,p} \mu_{K_{\mathfrak{p}}} \ \text{ and } \ \mathcal{W}_K := W_K/\mu_K.$$

Let $\overline{E}_K$ be the closure in U_K of the diagonal image of E_K; this gives in the diagram

$$\mathrm{Gal}(H_K^{\rm pr}/H_K) \simeq U_K/\overline{E}_K :$$



Put ([10, Chapter III, § (b) & Theorem 2.5]):

$$\mathcal{T}_K := \mathrm{tor}_{\mathbb{Z}_p}(\mathrm{Gal}(H_K^{\rm pr}/K));$$

in the viewpoint of Artin symbol, $\mathcal{T}_K$ is the kernel of the "logarithmic" map:

$$\mathcal{A}_K \xrightarrow{\mathrm{Art}^{-1}} \mathcal{I}_K/\mathcal{P}_{K,\infty} \xrightarrow{\mathrm{Log}} \mathbb{Z}_p\mathrm{Log}(I_K) \subseteq \Big(\bigoplus_{\mathfrak{p}\,|\,p} K_{\mathfrak{p}}\Big)\Big/\mathbb{Q}_p \log(E_K),$$

where $\mathcal{I}_K := I_K \otimes \mathbb{Z}_p$, I_K being the group of prime to p ideals of K, and $\mathcal{P}_{K,\infty}$ the group of infinitesimal principal ideals; log is the p-adic logarithm, and $\mathrm{Log}(\mathfrak{a}) := \frac{1}{m}\log(\alpha) \pmod{\mathbb{Q}_p \log(E_K)}$ for any relation in I_K of the form:

$$\mathfrak{a}^m = (\alpha),\ \ m \in \mathbb{Z},\ \alpha \in K^\times.$$

Let $\mathcal{C}\ell_K^\infty$ be the subgroup of $\mathcal{C}\ell_K$ corresponding, by class field theory, to $\mathrm{Gal}(H_K/\widetilde{K} \cap H_K)$.

We have, under the Leopoldt conjecture, the exact sequence defining $\mathcal{R}_K$ ([10, Lemma III.4.2.4] or [11, Lemma 3.1 & § 5]):

$$1 \to \mathcal{W}_K \longrightarrow \mathrm{tor}_{\mathbb{Z}_p}\big(U_K/\overline{E}_K\big) \xrightarrow{\log} \mathrm{tor}_{\mathbb{Z}_p}\big(\log\big(U_K\big)/\log(\overline{E}_K)\big) =: \mathcal{R}_K \to 0.$$

The group $\mathcal{R}_K$ is called the *normalized p-adic regulator of K* and makes sense for any number field.

Definition 1.1. *The field K is said to be p-rational if the Leopoldt conjecture is satisfied for p in K and if the torsion group $\mathcal{T}_K$ is trivial.*

Theorem 1.2. *(*[9, Théorème et Définition 4.1] *or* [10, Theorem IV.3.5]*). For a number field K, each of the following properties is equivalent to its p-rationality (where $2\,r_2$ is the number of complex embeddings of K):*

(i) $\mathcal{A}_K := \mathrm{Gal}(H_K^{\mathrm{pr}}/K) \simeq \mathbb{Z}_p^{r_2+1}$,

(ii) the Galois group $\mathcal{G}_K$ of the maximal p-ramified pro-p-extension of K is a free pro-p-group on r_2+1 generators (i.e., $\mathrm{H}^2(\mathcal{G}_K, \mathbb{Z}/p\,\mathbb{Z}) = 1$),

(iii) we have the following alternative:

- *either $\mu_p \subset K$, the set of p-places of K is a singleton $\{\mathfrak{p}\}$, and $\mathfrak{p}$ generates the p-class group of K (in the restricted sense for $p = 2$),*
- *or $\mu_p \not\subset K$ (whence $p \neq 2$), no prime ideal $\mathfrak{p} \mid p$ of K is totally split in $K(\mu_p)/K$ and the ω-components of the p-classes of the $\mathfrak{P} \mid p$ in $K(\mu_p)$ generate the ω-component of the p-class group of $K(\mu_p)$, where ω is the Teichmüller character.*

We can give, for $p = 2$ and $p = 3$, more elaborate statements as follows:

Example 1.3. From [13, § III.2, Corollary to Theorem 2] using properties of the "regular kernel" of the $\mathrm{K}_2(K)$ of a number field, or [10, Example IV.3.5.1] from properties of the groups $\mathcal{T}_K$, we can characterize the 2-rationality of 2-extensions of $\mathbb{Q}$, independently of (iii):

The set of abelian 2-rational 2-extensions of $\mathbb{Q}$ are all the subfields of the compositum $\mathbb{Q}(\mu_{2^\infty}) \cdot \mathbb{Q}(\sqrt{-\ell})$, $\ell \equiv 3 \pmod 8$ prime, and all the subfields of the compositum $\mathbb{Q}(\mu_{2^\infty}) \cdot \mathbb{Q}\Big(\sqrt{\sqrt{\ell}\,(a - \sqrt{\ell})/2}\Big)$, $\ell = a^2 + b^2 \equiv 5 \pmod 8$ prime, a odd.

For quadratic fields this gives, for any $\ell \equiv \pm 3 \pmod 8$:

$$K \in \{\mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{2}),\ \mathbb{Q}(\sqrt{-2}),\ \mathbb{Q}(\sqrt{\ell}),\ \mathbb{Q}(\sqrt{-\ell}),\ \mathbb{Q}(\sqrt{2\ell}),\ \mathbb{Q}(\sqrt{-2\ell})\}.$$

Example 1.4. In the same way, the set of abelian 3-rational 3-extensions of $\mathbb{Q}$ are all the subfields of the compositum of any cubic field of prime conductor $\ell = \frac{a^2+27\,b^2}{4} \equiv 4$ or $7 \pmod 9$ (for which a defining monic polynomial is $x^3 + x^2 - \frac{\ell-1}{3}\,x - \frac{\ell\,(a+3)-1}{27}$, $a \equiv 1 \pmod 3$), with the cyclotomic $\mathbb{Z}_3$-extension.

Example 1.5. Consider the prime $p = 3$ and a quadratic field $K = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}$, $K \neq \mathbb{Q}(\sqrt{-3})$; in this case the ω-component of the 3-class group of $K(\mu_3) = K(\sqrt{-3})$ is isomorphic to the 3-class group of the mirror field $K^* := \mathbb{Q}(\sqrt{-3\,d})$. Moreover the 3-class of $\mathfrak{p} \mid 3$ in $\mathbb{Q}(\sqrt{-3\,d})$ is trivial since 3 is ramified or inert in this extension under the non-splitting assumption in $K(\mu_3)/K$. Thus in that case, the 3-rationality of K is equivalent to fulfill the following two conditions for $K \neq \mathbb{Q}(\sqrt{-3})$:

(i) The 3-class group of $K^* = \mathbb{Q}(\sqrt{-3\,d})$ is trivial,

(ii) we have $d \equiv \pm 1 \pmod 3$ (i.e., ramification of 3 in K^*) or $d \equiv 3 \pmod 9$ (i.e., inertia of 3 in K^*).

If $d \equiv 6 \pmod 9$, the prime ideal above 3 in K splits in $K(\sqrt{-3})$ so that the non-3-rationality of K comes from the factor $\bigoplus_{\mathfrak{p}\,\mid\,p} \mu_{K_\mathfrak{p}} \big/ \mu_K = \mu_{K_\mathfrak{p}} \simeq \mathbb{Z}/3\,\mathbb{Z}$ for the unique $\mathfrak{p} \mid 3$ in K.

Remarks 1.6. *(a) Under Leopoldt's conjecture, the transfer homomorphisms $\mathcal{T}_k \to \mathcal{T}_K$ are injective in any extension K/k of number fields* [10, Theorem IV.2.1]*; so if K is p-rational, all the subfields of K are p-rational.*

If $p \nmid [K:\mathbb{Q}]$, the norms $\mathrm{N}_{K/k} : \mathcal{T}_K \to \mathcal{T}_k$ (which correspond to the restrictions $\mathcal{T}_K \to \mathcal{T}_k$ by class field theory) are surjective for all $k \subseteq K$.

When $K/\mathbb{Q}$ is Galois and $p \nmid [K:\mathbb{Q}]$, the reciprocal about the p-rationalities of subfields of K is true for some Galois groups $G := \mathrm{Gal}(K/\mathbb{Q})$ and some families of subfields. This occurs for instance when $K/\mathbb{Q}$ is abelian with the family of all maximal cyclic subfields of K: indeed, as $p \nmid [K:\mathbb{Q}]$, $\mathcal{T}_K \simeq \bigoplus_\chi \mathcal{T}_K^{e_\chi}$, where χ runs trough the set of rational characters of G, e_χ being the corresponding idempotent; then $\mathcal{T}_K^{e_\chi}$ is isomorphic to a submodule of $\mathcal{T}_{k_\chi}$ where k_χ (cyclic) is the subfield of K fixed by the kernel of χ.

For a compositum K of quadratic fields, this means that, for $p > 2$, K is p-rational if and only if all the quadratic subfields of K are p-rational.

(b) When K is a p-extension of a p-rational field k, K is p-rational if and only if the extension K/k is p-primitively ramified. This notion, defined first in [13, Chapter III, § 1 & § 2] *form our Crelle's papers on p-ramification and in* [9, Section 1]*, contains the case of p-ramification and some other explicit cases as in Examples 1.3 and 1.4; this notion has been extensively applied in* [10, Theorem IV.3.3, Definition IV.3.4]*,* [19, 20, 22, 23].

(c) In [16]*, abelian ℓ-extensions ($\ell \neq p$ prime) are used for applications to continuous Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{Z}_p)$ with open image for which one needs the condition (ii) of Theorem 1.2 (i.e., the p-rationality), and where some properties given in the above references are proved again.*

(d) We have conjectured in [12] *that K is p-rational for all $p \gg 0$, but in practical applications one works with small primes; so, care has been taken to consider also the case $p = 2$ in the programs. We ignore what kind of heuristics (in the meaning of many works such as* [4, 5, 6, 7, 27] *on class groups, Tate-Shafarevich groups, ...) are relevant for the $\mathcal{T}_K$ when p varies. This should be very interesting since the $\mathcal{T}_K$ mix classes and units.*

From the general schema, $\mathcal{T}_K = 1$ if and only if the three invariants $\mathcal{W}_K$, $\mathcal{R}_K$ and $\mathcal{C}\ell_K^\infty$, are trivial. Thus, in some cases, it may be possible to test each of these trivialities, depending on the knowledge of the field K; for instance, assuming the p-class group trivial and p unramified, the computation of $\mathcal{W}_K$ is purely local and that of the normalized p-adic regulator $\mathcal{R}_K$, closely related to the classical p-adic regulator, may be given in a specific program since it is the most unpredictable invariant. But a field given by means of a polynomial P may be more mysterious regarding these three factors.

2. General theoretical test of p-rationality

2.1. **Test using a suitable ray class group.** In [26] and [28, Theorem 3.11 & Corollary 4.1], are given analogous methods for the general computation of the structure of $\mathcal{T}_K$, but here we need only to characterize the triviality (or not) of $\mathcal{T}_K$ in the relation:

$$\mathcal{A}_K := \mathrm{Gal}(H_K^{\mathrm{pr}}/K) \simeq \mathbb{Z}_p^r \times \mathcal{T}_K,$$

where $r = r_2 + 1$, $2\,r_2$ being the number of complex embeddings of K. As $\mathcal{T}_K$ is a direct factor in $\mathcal{A}_K$, the structure of the whole Galois group $\mathcal{A}_K$ may be analized at a finite step by means of the Galois group of a suitable ray class field $K(p^n)$ of modulus (p^n). Since PARI gives the structure of ray

class groups $\mathcal{C}\ell_K(p^n) := \mathrm{Gal}(K(p^n)/K)$, the test of p-rationality is obtained for n large enough as follows: if $\mathcal{C}\ell_K(p^n)$ has a p-rank such that:

$$\mathrm{rk}_p(\mathcal{C}\ell_K(p^n)) \geq r+1,$$

then K is not p-rational since $\mathrm{Gal}(\widetilde{K}/K)$ has p-rank r. The minimal n_0 needed for the test is given as a consequence of the following result:

Theorem 2.1. *For any $\mathfrak{p} \mid p$ in K and any $j \geq 1$, let $U_{\mathfrak{p}}^j$ be the group of local units $1+\overline{\mathfrak{p}}^j$, where $\overline{\mathfrak{p}}$ is the maximal ideal of the ring of integers of $K_{\mathfrak{p}}$.*

For a modulus of the form (p^n), $n \geq 0$, let $\mathcal{C}\ell_K(p^n)$ be the corresponding ray class group. Then for $m \geq n \geq 0$, we have the inequalities:

$$0 \leq \mathrm{rk}_p(\mathcal{C}\ell_K(p^m)) - \mathrm{rk}_p(\mathcal{C}\ell_K(p^n)) \leq \sum_{\mathfrak{p}\mid p} \mathrm{rk}_p\big((U_{\mathfrak{p}}^1)^p\, U_{\mathfrak{p}}^{n\cdot e_{\mathfrak{p}}}/(U_{\mathfrak{p}}^1)^p\, U_{\mathfrak{p}}^{m\cdot e_{\mathfrak{p}}}\big),$$

where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$ in $K/\mathbb{Q}$.

Proof. From [10, Theorem I.4.5 & Corollary I.4.5.4] taking for T the set of p-places and for S the set of real infinite places (ordinary sense). □

Corollary 2.2. *We have $\mathrm{rk}_p(\mathcal{C}\ell_K(p^m)) = \mathrm{rk}_p(\mathcal{C}\ell_K(p^n)) = \mathrm{rk}_p(\mathcal{A}_K)$ for all $m \geq n \geq n_0$, where $n_0 = 3$ for $p = 2$ and $n_0 = 2$ for $p > 2$.*

Thus K is p-rational if and only if $\mathrm{rk}_p(\mathcal{C}\ell_K(p^{n_0})) = r$, where $r = r_2 + 1$.

Proof. It is sufficient to get, for some fixed $n \geq 0$:

$$(U_{\mathfrak{p}}^1)^p\, U_{\mathfrak{p}}^{n\cdot e_{\mathfrak{p}}} = (U_{\mathfrak{p}}^1)^p, \text{ for all } \mathfrak{p} \mid p,$$

hence $U_{\mathfrak{p}}^{n\cdot e_{\mathfrak{p}}} \subseteq (U_{\mathfrak{p}}^1)^p$ for all $\mathfrak{p} \mid p$; indeed, we then have:

$$\mathrm{rk}_p(\mathcal{C}\ell_K(p^n)) = \mathrm{rk}_p(\mathcal{C}\ell_K(p^m)) = r + \mathrm{rk}_p(\mathcal{T}_K) \text{ as } m \to \infty,$$

giving $\mathrm{rk}_p(\mathcal{C}\ell_K(p^n)) = r + \mathrm{rk}_p(\mathcal{T}_K)$ for such n.

The condition $U_{\mathfrak{p}}^{n\cdot e_{\mathfrak{p}}} \subseteq (U_{\mathfrak{p}}^1)^p$ is fulfilled as soon as $n \cdot e_{\mathfrak{p}} > \frac{p \cdot e_{\mathfrak{p}}}{p-1}$, hence:

$$n > \frac{p}{p-1}$$

(see [8, Chap. I, § 5.8, Corollary 2] or [29, Proposition 5.7]; a fact also used in [17, Proposition 1.13]), whence the value of n_0; furthermore, $\mathcal{C}\ell_K(p^{n_0})$ gives the exact p-rank of $\mathcal{T}_K$. □

2.2. **Basic invariants of K with PARI** [25]**.** The reader whishing to test only p-rationalities may go directly to Subsections 3.1, 3.2.

The examples shall be given with the following polynomial defining K:

$$P = x^3 - 5\,x + 3$$

(recall that for PARI, P must be monic and in $\mathbb{Z}[x]$), for which one gives the classical information (test of irreducibility, Galois group of the Galois closure of K, discriminant of K) which are the following, with the PARI responses:

```
polisirreducible(x^3-5*x+3)
   1
polgalois(x^3-5*x+3)
   [6,-1,1,"S3"]
factor(nfdisc(x^3-5*x+3))
   [257 1]
```

showing that P is irreducible, that the Galois closure of K is the diedral group of order 6, and that the discriminant of K is the prime 257.

Then K is precised by the signature $[r_1, r_2]$, the structure of the whole class group and the fundamental units.

```
{P=x^3-5*x+3;p=2;K=bnfinit(P,1);C7=component(K,7);C8=component(K,8);
Sign=component(C7,2);print("Signature of K: ",Sign);
print("Structure of the class group: ",component(C8,1));
print("Fundamental system of units: ",component(C8,5))}
[3, 0],  [1, [], []],  [x-1, x^2+2*x-2]
```

giving a totally real field, a trivial class group and the two fundamental units. From $K = bnfinit(P, 1)$ and $C8 = component(K, 8)$, the regulator of K is given by $component(C8, 1)$. The next program gives, for information, the decomposition of some primes p:

```
{P=x^3-5*x+3;K=bnfinit(P,1);
forprime(p=2,13,print(p," ",idealfactor(K,p)))}
2  Mat([[2,[2,0,0]~,1,3,1],1])
3  [[3,[0,0,1]~,1,1,[1,1,-1]~],1;[3,[4,1,-1]~,1,2,[0,0,1]~],1]
5  [[5,[2,0,1]~,1,1,[2,1,2]~],1;[5,[2,1,-3]~,1,2,[2,0,1]~],1]
7  [[7,[1,0,1]~,1,1,[-1,1,-2]~],1;[7,[6,1,-2]~,1,2,[1,0,1]~],1]
11 Mat([[11,[11,0,0]~,1,3,1],1])
13 Mat([[13,[13,0,0]~,1,3,1],1])
```

showing that $2, 11, 13$ are inert, that $3, 5, 7$ split into two prime ideals with residue degrees 1 and 2, respectively. Taking $p = 257$, one obtains:

```
257 [[257,[-101,0,1]~,1,1,[-81,1,100]~],1;
                                    [257,[-78,0,1]~,2,1,[-86,1,77]~],2]
```

which splits into $\mathfrak{p}_1 \cdot \mathfrak{p}_2^2$.

To obtain a polynomial from a compositum of known fields (e.g., quadratic fields) one uses by induction the instruction *polcompositum*:

```
{P1=x^2-2;P2=x^2+5;P3=x^2-7;P=polcompositum(P1,P2);P=component(P,1);
P=polcompositum(P,P3);P=component(P,1);print(P)}
x^8-16*x^6+344*x^4+2240*x^2+19600
```

The instruction giving the structure of $\mathcal{C}\ell_K(p^n)$ is the following (we compute the structure of the ray class groups with modulus p^n, up to $n = 5$, to see the stabilization of the p-ranks; this would give the group invariants of $\mathcal{T}_K$, as is done in [26, 28]):

```
{K=bnfinit(x^3-5*x+3,1);p=2;for(n=0,5,Hpn=bnrinit(K,p^n);
print(n," ",component(Hpn,5)))}
p=2                  p=3                  p=257
0  [1,[]]            0  [1,[]]            0  [1,[]]
1  [1,[]]            1  [1,[]]            1  [128,[128]]
2  [2,[2]]           2  [3,[3]]           2  [32896,[32896]]
3  [4,[2,2]]         3  [9,[9]]           3  [8454272,[8454272]]
4  [8,[4,2]]         4  [27,[27]]         4  [2172747904,[2172747904]]
5  [16,[8,2]]        5  [81,[81]]         5  [558396211328,[558396211328]]
```

(where $32896 = 2^7 \cdot 257$, $8454272 = 2^7 \cdot 257^2$, $2172747904 = 2^7 \cdot 257^3, \ldots$):

Thus K is p-rational for $p = 3$ and 257, but not for $p = 2$.

Now we give the case of the first irregular prime $p = 37$ for which we know that the pth cyclotomic field is not p-rational (indeed, $\mathcal{T}_K = 1$ is equivalent to $\mathcal{C}\ell_K = 1$ for the pth cyclotomic fields [9, Théorème & Définition 2.1]). So we must see that the p-ranks are equal to $19 + 1$, at least for $n \geq 2$:

```
{p=37;K=bnfinit(polcyclo(p),1);
for(n=0,2,Hpn=bnrinit(K,p^n);print(n," ",component(Hpn,5)))}
0 [37,[37]]
1 [624931990990842127748277129373,[1369,37,37,37,37,37,37,37,37,37,
   37,37,37,37,37,37,37,37]]
2 [39053999336377798632089821318184581900671365508469737937312 9,
  [50653,1369,1369,1369,1369,1369,1369,1369,1369,1369,1369,1369,
  1369,1369,1369,1369,1369,37,37,37]]
```

where $1369 = 37^2, 50653 = 37^3$.

3. Full general PARI programs testing p-rationality

We bring together some instructions given in the previous section and recall that, in the programs, $n = 2$ (resp. 3) if $p \neq 2$ (resp. $p = 2$).

3.1. **General program with main invariants and test of p-rationality.** The reader has to introduce an *irreducible monic polynomial* $P \in \mathbb{Z}[x]$ and a *prime number* $p \geq 2$. For $p = 2$ the p-rationality is in the ordinary sense.

```
=====================================================================
  {P=x^3-5*x+3;p=2;K=bnfinit(P,1);
  Sign=component(component(K,7),2);print("Signature of K: ",Sign);
  print("Galois group of the Galois closure of K: ",polgalois(P));
  print("Discriminant: ",factor(component (component(K,7), 3)));
  print("Structure of the class group: ",component(component(K,8),1));
  print("Fundamental system of units: ",component(component(K,8),5));
  r=component(Sign,2)+1;n=2;if(p==2,n=3);
  print(p,"-rank of the compositum of the Z_",p,"-extensions: ",r);
  Hpn=component(component(bnrinit(K,p^n),5),2);L=listcreate;
  e=component(matsize(Hpn),2);R=0;for(k=1,e,c=component(Hpn,e-k+1);
  if(Mod(c,p)==0,R=R+1;listinsert(L,p^valuation(c,p),1)));
  print("Structure of the ray class group mod ",p,"^n: ",L);
  if(R>r,print("rk(T)=",R-r," K is not ",p,"-rational"));
  if(R==r,print("rk(T)=",R-r," K is ",p,"-rational"))}
=====================================================================
```

(i) With the above data, K is not 2-rational nor 293-rational (up to $p \leq 10^5$).

(ii) The field $K = \mathbb{Q}(\sqrt{-161})$ is not 2-rational (one may prove that the 2-Hilbert class field is linearly disjoint from $\widetilde{K}$ [10, Example III.6.7]).

(iii) The field $K = \mathbb{Q}(\sqrt{69})$ is not 3-rational (comes from $\mathcal{R}_K \neq 1$).

(iv) The quartic cyclic fields K defined by $P = x^4 + 5\,x^2 + 5$ (conductor 5) and by $P = x^4 + 13\,x^2 + 13$ (conductor 13) are 2-rational (see Example 1.3).

3.2. **Test of p-rationality (simplified programs).** Since some parts of the first program are useless and some computations intricate (e.g., Galois groups in large degrees), one may use the following simplified program to test only the p-rationality.

We test the 3-rationality of $K = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{5}, \sqrt{11}, \sqrt{97})$ given in [16] (P is computed from the instruction *polcompositum*; the program takes $19\,min$, $26.663\,ms$ and need *allocatemem*(800000000)):

```
Programme I (define the polynomial P and the prime p):
```

```
=====================================================================
{P=x^32-1824*x^30+1504544*x^28-743642240*x^26+246039201472*x^24
-57656224594432*x^22+9874427075761664*x^20-1257037661975865344*x^18
+119781806181353182720*x^16-8534335878932228562944*x^14
+450658848166023111041024*x^12-173301719525678332193996 80*x^10
+47154718860591087610657 1776*x^8-867848478392950825453 9710464*x^6
+10067847337562884434828 3158528*x^4-658128522558747992210233884672*x^2
+19959184335189573840658 60304896;K=bnfinit(P,1);p=3;n=2;if(p==2,n=3);
Kpn=bnrinit(K,p^n);S=component(component(Kpn,1),7);
r=component(component(S,2),2)+1;
print(p,"-rank of the compositum of the Z_",p,"-extensions: ",r);
Hpn=component(component(Kpn,5),2);L=listcreate;
e=component(matsize(Hpn),2);R=0;for(k=1,e,c=component(Hpn,e-k+1);
if(Mod(c,p)==0,R=R+1;listinsert(L,p^valuation(c,p),1)));
print("Structure of the ",p,"-ray class group: ",L);
if(R>r,print("rk(T)=",R-r," K is not ",p,"-rational"));
if(R==r,print("rk(T)=",R-r," K is ",p,"-rational"))}
=====================================================================
3-rank of the compositum of the Z_3-extensions: 17
Structure of the 3-ray class group:List([3,3,3,3,3,3,3,3,3,3,3,3,3,3,9,9,9])
K is 3-rational
```

If one whises to test the p-rationality of K for p varying in an interval $[b, B]$, it is necessary to compute first the data $bnfinit(P, 1)$ which is independent of p and takes lots of time. Then the tests for p-rationalities are very fast.

This gives the following writing where we give the p-structure of $\mathcal{C}\ell_K(p^{n_0})$ up to $p \leq 100$, only for the non-p-rational cases:

Programme II (define P and the interval [b,B] of primes p):

```
=====================================================================
{P=x^32-1824*x^30+1504544*x^28-743642240*x^26+246039201472*x^24
-57656224594432*x^22+9874427075761664*x^20-1257037661975865344*x^18
+119781806181353182720*x^16-8534335878932228562944*x^14
+450658848166023111041024*x^12-173301719525678332193996 80*x^10
+47154718860591087610657 1776*x^8-867848478392950825453 9710464*x^6
+10067847337562884434828 3158528*x^4-658128522558747992210233884672*x^2
+19959184335189573840658 60304896;K=bnfinit(P,1);b=2;B=100;
r=component(component(component(K,7),2),2)+1;
print("p-rank of the compositum of the Z_p-extensions: ",r);
forprime(p=b,B,n=2;if(p==2,n=3);
Kpn=bnrinit(K,p^n);Hpn=component(component(Kpn,5),2);
L=listcreate;e=component(matsize(Hpn),2);
R=0;for(k=1,e,c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;
listinsert(L,p^valuation(c,p),1)));
print("Structure of the ",p,"-ray class group: ",L);
if(R>r,print("rk(T)=",R-r," K is not ",p,"-rational"));
if(R==r,print("rk(T)=",R-r," K is ",p,"-rational")))}
=====================================================================
p-rank of the compositum of the Z_p-extensions:17

List([2,2,2,2,2,4,4,4,4,4,4,8,8,8,8,8,8,8,16,16,16,32,32,64,128])
rk(T)=9 K not 2-rational
List([7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7])
rk(T)=1 K not 7-rational
List([11,11,11,11,11,11,11,11,11,11,11,11,121,121,121,121,121,121,121,1331])
rk(T)=2 K not 11-rational
List([13,13,13,13,13,13,13,13,13,13,13,13,13,13,13,13,13,13])
```

```
rk(T)=1 K not 13-rational
List([17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17,17])
rk(T)=1 K not 17-rational
List([19,19,19,19,19,19,19,19,19,19,19,19,19,19,19,19,19,19])
rk(T)=1 K not 19-rational
List([29,29,29,29,29,29,29,29,29,29,29,29,29,29,29,29,29,29])
rk(T)=1 K not  29-rational
List([31,31,31,31,31,31,31,31,31,31,31,31,31,31,31,31,31,31])
rk(T)=1 K not 31-rational
List([43,43,43,43,43,43,43,43,43,43,43,43,43,43,43,43,43,43])
rk(T)=1 K not 43-rational
List([73,73,73,73,73,73,73,73,73,73,73,73,73,73,73,73,73,73])
rk(T)=1 K not 73-rational
```

The p-rationality holds for $3, 5, 23, 37, 41, 47, 53, 59, 61, 67, 71, 79, 83, 89, 97$ and we find that K is not p-rational up to 10^5 for the primes:

$$p \in \{2, 7, 11, 13, 17, 19, 29, 31, 43, 73, 163, 191, 263, 409, 571, 643, 1049, 2671, 3331, 3917, 6673, 8941, 28477, 36899, 39139, 85601, 99149, 134339, 203393, 231901, 283979, 353711, 363719\}.$$

The above cases of non-p-rationality are numerous, because of the important number of maximal cyclic subfields of K, and are essentially due to some units; for instance, for the unit $\varepsilon = 1 + \sqrt{2} \in E_K$ and $p = 31$, we get:

$$\varepsilon^{30} = 152139002499 + 107578520350\sqrt{2} \equiv 1 \pmod{31^2},$$

which means $\mathcal{R}_K = \frac{1}{31}\log(\varepsilon) \equiv 0 \pmod{31}$.

We must note that, once the important instruction $K = bnfinit(P, 1)$ is done by PARI, the execution time for various p is much shorter.

4. Heuristics on Greenberg's conjecture (with $p = 3$)

4.1. **Generalities.** A conjecture, in relation with the construction of continuous Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ with open image, is the following, stated in the case of a compositum of quadratic fields [16, Conjecture 4.2.1]:

Conjecture 4.1. *For any odd prime p and for any $t \geq 1$ there exists a p-rational field K such that* $\mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z})^t$.

For the link between n and t and more information, see [16, Propositions 6.1.1 & 6.2.2]. For simplicity, we shall discuss this conjecture for $p = 3$ (for other cases, see [3]); then the 3-rationality of a compositum K of t quadratic fields is equivalent to the following condition (from Example 1.5):

For all the quadratic subfields $k =: \mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{-3})$ of K, the 3-class group of the mirror field $k^* := \mathbb{Q}(\sqrt{-3 \cdot d})$ is trivial and 3 does not split in $k^*/\mathbb{Q}$.

This needs $2^t - 1$ conditions of 3-rationality.

4.2. **Technical conditions.** The case $t = 1$ is clear and gives infinitely many 3-rational fields if we refer to classical heuristics [4, 5]. When $t \geq 3$ we must assume that $\mathbb{Q}(\sqrt{-3})$ is not contained in K, otherwise, from Theorem 1.2 (iii), if K_0 is the inertia field of 3 in $K/\mathbb{Q}$, then $[K : K_0] = 2$, $[K_0 : \mathbb{Q}] \geq 4$, and necessarily 3 splits in part in $K_0/\mathbb{Q}$ (the factor $\mathcal{W}_K$ is non-trivial).

For the biquadratic field $\mathbb{Q}(\sqrt{d}, \sqrt{-3})$, $d \not\equiv 0 \pmod 3$, the 3-rationality holds if and only if $d \equiv -1 \pmod 3$ and the 3-class group of the imaginary

field $\mathbb{Q}(\sqrt{d})$ or $\mathbb{Q}(\sqrt{-3\,d})$ is trivial (using Scholz's theorem). In the sequel, we shall assume that K does not contain $\mathbb{Q}(\sqrt{-3})$ and that $t \geq 3$.

Let $K = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_t})$ be such that $\mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z})^t$, $t \geq 3$. We denote by $k = \mathbb{Q}(\sqrt{d})$ any quadratic subfield of K and by k^* the "mirror field" $\mathbb{Q}(\sqrt{-3 \cdot d})$. We then have the obvious lemma:

Lemma 4.2. *The conditions of non-splitting of* 3 *in all the extensions* $k^*/\mathbb{Q}$, $k = \mathbb{Q}(\sqrt{d}) \subseteq K$, *are satisfied if and only if for all the integers* d *we have* $3 \nmid d$ *or* $d \equiv 3 \pmod 9$.

Corollary 4.3. *(i) If* 3 *is unramified in* $K/\mathbb{Q}$, *then* $K = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_t})$ *with* $d_i \equiv \pm 1 \pmod 3$, *for* $i = 1, \ldots, t$.

(ii) If 3 *is ramified in* $K/\mathbb{Q}$ *then* K *is a direct compositum of the form* $K_0\,\mathbb{Q}(\sqrt{3\,d_t})$, *where* 3 *is unramified in* K_0 *and* $d_t \not\equiv 0 \pmod 3$, *whence* $K = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_{t-1}}, \sqrt{3\,d_t})$ *with* $d_i \equiv 1 \pmod 3$ *for* $i = 1, \ldots, t$.

Case (ii) comes from the fact that for any $d \in \langle d_1, \ldots, d_{t-1} \rangle \cdot \mathbb{Q}^{\times 2}$, $d \not\equiv 0 \pmod 3$, the consideration of the subfield $k = \mathbb{Q}(\sqrt{3\,d_t\,d})$ needs, for $k^* = \mathbb{Q}(\sqrt{-d_t\,d})$, the condition $d_t\,d \equiv 1 \pmod 3$, whence the hypothesis as soon as $t \geq 3$.

So whatever the form of K, the results depend on heuristics on the 3-class groups of the $2^t - 1$ fields k^* when random integers d_i are given with the above conditions.

A first heuristic is to assume that a 3-class group does not depend on the above assumption of decomposition of the prime 3; this is natural since only genera theory (i.e., when $p = 2$) is concerned with such problem. Of course, all the $2^t - 1$ integers d are not random, but the classical studies of repartition of class groups show that the two phenomena (an integer d and a 3-class group) are independent.

4.3. **Densities of 3-rational quadratic fields.** We give now densities of 3-rational quadratic fields $K = \mathbb{Q}(\sqrt{d})$. There are two algorithms for this: to use the general Program I of § 3.2 or to use the characterization given by Example 1.5. It is easy to see that Program I takes the same time for real or imaginary fields and that the characterization using the computation of class numbers of the mirror fields K^* is faster for real K.

We shall compare with the heuristics of Cohen–Lenstra–Martinet (see [4, 5]).

So we obtain the following programs in which N (resp. N_3) is the number of squarefree integers d (resp. of 3-rational fields $\mathbb{Q}(\sqrt{d})$) in the interval:

4.3.1. *Real case for* K. The mirror field of $K = \mathbb{Q}(\sqrt{d})$ is $K^* = \mathbb{Q}(\sqrt{-3\,d})$ and we must have $d \equiv 3 \pmod 9$ when $3 \mid d$, otherwise 3 splits in $K(\mu_3)/K$ and $\mathcal{W}_K \neq 1$.

```
{N=0;N3=0;for(d=1,10^3,if(core(d)!=d || Mod(d,9)==-3,next);N=N+1;
v=valuation(d,3);D=coredisc(-3^(1-2*v)*d);h=qfbclassno(D);
if(Mod(h,3)!=0,N3=N3+1));print(N3," ",N," ",N3/N+0.0)}
N3=315551, N=531938, N3/N=0.59321
```

If we restrict ourselves to integers $d \equiv 3 \pmod 9$, we obtain $N_3 = 180717$, $N = 303961$ and the proportion 0.59454. With the condition $3 \nmid d$ (corresponding to the non-ramification of 3 in K) we obtain the similar proportion 0.59185.

This is consistent with the heuristics of Cohen–Martinet, about 3-class groups (see [5, Section 2, § 1.1 (b)] or [4, § 9 (b)]), giving the probability 0.439874 for $3 \mid \#\mathcal{C}l_{k^*}$, whence 0.560126 for the contrary.

4.3.2. *Imaginary case for K.* The program uses the computation of the ray class group of modulus 9 (see Program I of § 3.2):

```
{N=0;N3=0;for(dd=1,10^6,d=-dd;if(core(d)!=d || Mod(d,9)==-3,next);N=N+1;
p=3;n=2;r=2;P=x^2-d;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);
Hpn=component(component(Kpn,5),2);L=listcreate;e=component(matsize(Hpn),2);
R=0;for(k=1,e,c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;
listinsert(L,p^valuation(c,p),1)));if(R==r,N3=N3+1));
print(N3," ",N," ",N3/N+0.0)}
N3=462125,N=531934,N3/N=0.868
```

This is in accordance with the probabilities about 3-class groups in the real case giving the probability 0.841 for $3 \nmid \#\mathcal{C}l_{k^*}$ (see [5, Section 2, § 1.2 (b)]).

Thus, there are approximatively 60% of 3-rational real quadratic fields and around 86% of 3-rational imaginary quadratic fields. So we do not need more precise information, but we must only assume that 3-rational quadratic fields are uniformly distributed whatever the interval of range of the discriminants and that the probabilities are around 0.60 (resp. 0.86) for our reasonnings. For more precise numerical results and heuristics, see [3, § 5], [28, Section 5] and the very complete work on imaginary quadratic fields [27] and all $p \geq 2$.

4.4. **Analyse of Greenberg's conjecture.** We denote by K_t, $t \geq 1$, a compositum of one of the two forms:

$$K_t = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_t}), \text{ with } 3 \nmid d_i \text{ for all } i,$$

$$K_t = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_{t-1}}, \sqrt{3 \cdot d_t}), \text{ with } d_i \equiv 1 \pmod 3 \text{ for all } i.$$

From the above heuristics, we can say that for $t = 1$ there are reasonably infinitely many such 3-rational K_1 with density P_1 less than 0.6 or 0.86.

Now take $t \geq 3$; since K_t is 3-rational if and only if its $2^t - 1$ quadratic subfields are 3-rational, the probability may be assumed to be as follows, under the assumption that all the generators $d_1, \ldots, d_t$ are random under the necessary local conditions of Corollary 4.3:

- If K_t is real, the probability is:

$$P_t \approx 0.60^{2^t - 1}$$

(giving $1.33 \cdot 10^{-7}$ for $t = 5$, $1.06 \cdot 10^{-14}$ for $t = 6$ and $1.12 \cdot 10^{-227}$ for $t = 10$).

- If K_t is imaginary, we have $2^{t-1} - 1$ real quadratic subfields and 2^{t-1} imaginary ones, so that the probability is :

$$P_t \approx 0.60^{2^{t-1}-1} \cdot 0.86^{2^{t-1}}$$

(giving $4.2 \cdot 10^{-5}$ for $t = 5$, $1.06 \cdot 10^{-9}$ for $t = 6$ and $1.25 \cdot 10^{-147}$ for $t = 10$).

This explains the great difficulties to find numerical examples, for $t > 6$ with $p = 3$, in a reasonable interval $[b, B]$ for $d_1, \ldots, d_t$. Recall the imaginary

example with $t = 6$ given in [16, § 4.2]: $\mathbb{Q}(\sqrt{-1}, \sqrt{13}, \sqrt{145}, \sqrt{209}, \sqrt{269}, \sqrt{373})$ (we shall give another similar example in § 5.2).

Naturally, when p increases, it is easier to find examples with larger t but the problem is similar. See [3] for some examples with $p \geq 5$ with quadratic fields (e.g., Table 1 for $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{11}, \sqrt{47}, \sqrt{97})$ with $p = 5$) and cubic fields, then [11, 15, 18] for p-adic regulators. In the framwork of Borel–Cantelli classical heuristic, the conjecture of Greenberg may be true.

5. Computations for compositum of quadratic fields

5.1. **Research of real 3-rational compositum of 5 quadratic fields.** For practical reasons, we write two programs depending on the ramification of 3 in the fields $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3}, \sqrt{d_4}, \sqrt{d_5})$. The two programs verify, from the data, that $\mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/2\,\mathbb{Z})^5$.

5.1.1. *Case where* 3 *is possibly ramified.* The program computes a list of integers $d > 0$ such that $d \equiv 1 \pmod 3$ or $d \equiv 3 \pmod 9$ (see Corollary 4.3) and such that the $\mathbb{Q}(\sqrt{d})$ are 3-rational (then 3 is possibly unramified in some solutions K, but in that case $d_i \equiv 1 \pmod 3$ for all i).

```
{L3=listcreate;N3=0;b=1;B=300;for(d=b,B,
if(core(d)!=d || Mod(d,3)==-1 || Mod(d,9)==-3,next);v=valuation(d,3);
D=coredisc(-3^(1-2*v)*d);h=qfbclassno(D);if(Mod(h,3)!=0,N3=N3+1;
listinsert(L3,d,1)));for(k=1,10^7,a1=random(N3)+1;a2=random(N3)+1;
a3=random(N3)+1;a4=random(N3)+1;a5=random(N3)+1;
d1=component(L3,a1);d2=component(L3,a2);d3=component(L3,a3);
d4=component(L3,a4);d5=component(L3,a5);TT=0;
for(e1=0,1,for(e2=0,1,for(e3=0,1,for(e4=0,1,for(e5=0,1,
dd=d1^e1*d2^e2*d3^e3*d4^e4*d5^e5;dd=core(dd);
if(dd==1 & [e1,e2,e3,e4,e5]!=[0,0,0,0,0],TT=1;break(5)))))));
if(TT==0,T=0;for(e1=0,1,for(e2=0,1,for(e3=0,1,for(e4=0,1,for(e5=0,1,
dd=d1^e1*d2^e2*d3^e3*d4^e4*d5^e5;dd=core(dd);
if(Mod(dd,3)!=0,D=coredisc(-3*dd));if(Mod(dd,3)==0,D=coredisc(-dd/3));
h=qfbclassno(D);if(Mod(h,3)==0,T=1;break(5)))))));
if(T==0,print(d1," ",d2," ",d3," ",d4," ",d5))))}
```

One obtains the following distinct examples:

```
[d1,d2,d3,d4,d5]=
[118,178,31,46,211],[274,291,66,118,262],[22,193,13,262,163],
[37,274,31,211,46],[31,130,166,129,246],[298,13,7,111,210],
[201,157,57,55,219],[255,282,165,298,118],[19,211,61,166,217],
[39,30,129,111,166],[187,246,39,145,31],[66,265,246,219,157],
[55,219,217,102,205],[193,262,163,10,127],[37,61,273,205,21],
[255,115,259,193,7],[31,187,145,246,39]
```

5.1.2. *Case where* 3 *is unramified.* In this case there is no condition on the $d_i \not\equiv 0 \pmod 3$.

```
{L3=listcreate;N3=0;b=2;B=300;for(d=b,B,if(core(d)!=d||Mod(d,3)==0,next);
D=coredisc(-3*d);h=qfbclassno(D);if(Mod(h,3)!=0,N3=N3+1;listinsert(L3,d,1)));
for(k=1,10^7,a1=random(N3)+1;a2=random(N3)+1;a3=random(N3)+1;a4=random(N3)+1;
a5=random(N3)+1;d1=component(L3,a1);d2=component(L3,a2);d3=component(L3,a3);
d4=component(L3,a4);d5=component(L3,a5);TT=0;
for(e1=0,1,for(e2=0,1,for(e3=0,1,for(e4=0,1,for(e5=0,1,
dd=d1^e1*d2^e2*d3^e3*d4^e4*d5^e5;dd=core(dd);
if(dd==1 & [e1,e2,e3,e4,e5]!=[0,0,0,0,0],TT=1;break(5)))))));
if(TT==0,T=0;for(e1=0,1,for(e2=0,1,for(e3=0,1,for(e4=0,1,for(e5=0,1,
dd=d1^e1*d2^e2*d3^e3*d4^e4*d5^e5;dd=core(dd);D=coredisc(-3*dd);
```

```
h=qfbclassno(D);if(Mod(h,3)==0,T=1;break(5)))))));
if(T==0,print(d1," ",d2," ",d3," ",d4," ",d5))))}
```

One obtains the following examples:

```
[d1,d2,d3,d4,d5]=
[91,230,209,194,221],[149,335,301,55,31],[145,157,230,209,194],
[35,193,301,149,263],[226,239,130,158,259],[190,259,143,70,290],
[266,59,17,10,70],[194,11,283,187,31],[107,227,34,130,230],
[13,17,215,31,61],[145,133,218,34,230],[22,215,221,31,161],
[86,149,170,146,145],[158,259,226,146,47],[146,26,269,166,190],
[46,170,38,133,187],[190,119,97,14,263],[203,227,221,194,143],
[239,89,262,53,166],[13,262,193,163,286]
```

In a larger interval, the examples become more rare since the number of discriminants decrease; between $b = 10^3$ and $B = 10^3 + 350$ we get the solutions (when 3 can ramify):

```
[d1,d2,d3,d4,d5]=
[1245,1303,1218,1291,1123],[1177,1003,1309,1173,1054],
[1321,1065,1173,1231,1207],[1155,1326,1111,1105,1093],
[1173,1281,1254,1327,1209],[1030,1174,1177,1218,1015]
```

5.1.3. *Direct verifications.* To verify, one may compute directly the 3-class number and the 3-adic logarithm of the fundamental unit ε of $\mathbb{Q}(\sqrt{d})$ for each quadratic subfield of K; the computation of the normalized regulator (using [11, Proposition 5.2]) is equivalent to that of $\alpha := \dfrac{\varepsilon^q - 1}{3}$ when 3 is unramified (where $q = 2$ or 8), and $\alpha := \dfrac{\varepsilon^2 - 1}{\sqrt{d}}$ when $3 \mid d$, and α must be a 3-adic unit (which can be seen taking its norm):

```
{L3=[110,170,161,38,14];d1=component(L3,1);d2=component(L3,2);
d3=component(L3,3);d4=component(L3,4);d5=component(L3,5);T=0;
for(e1=0,1,for(e2=0,1,for(e3=0,1,for(e4=0,1,for(e5=0,1,
dd=d1^e1*d2^e2*d3^e3*d4^e4*d5^e5;dd=core(dd);if(dd==1,next);
D=coredisc(dd);h=qfbclassno(D);eps=quadunit(D);q=8;if(Mod(dd,3)!=-1,q=2);
E=eps^q-1;if(Mod(D,3)!=0,No=norm(E)/9);if(Mod(D,3)==0,No=norm(E)/3);
No=Mod(No,3);print(dd," ",Mod(h,9)," ",No))))))}
```

giving, for $\mathbb{Q}(\sqrt{110}, \sqrt{170}, \sqrt{161}, \sqrt{38}, \sqrt{14})$ (computation of h modulo 9 for information instead of modulo 3):

d	h mod 9	Regulator	d	h mod 9	Regulator
14	Mod(1,9)	Mod(1,3)	385	Mod(2,9)	Mod(2,3)
38	Mod(1,9)	Mod(1,3)	1045	Mod(4,9)	Mod(2,3)
133	Mod(1,9)	Mod(2,3)	14630	Mod(8,9)	Mod(1,3)
161	Mod(1,9)	Mod(1,3)	17710	Mod(8,9)	Mod(2,3)
46	Mod(1,9)	Mod(2,3)	1265	Mod(2,9)	Mod(1,3)
6118	Mod(4,9)	Mod(2,3)	168245	Mod(8,9)	Mod(1,3)
437	Mod(1,9)	Mod(1,3)	48070	Mod(8,9)	Mod(2,3)
170	Mod(4,9)	Mod(1,3)	187	Mod(2,9)	Mod(2,3)
595	Mod(4,9)	Mod(2,3)	2618	Mod(4,9)	Mod(1,3)
1615	Mod(4,9)	Mod(2,3)	7106	Mod(4,9)	Mod(1,3)
22610	Mod(8,9)	Mod(1,3)	24871	Mod(8,9)	Mod(2,3)
27370	Mod(7,9)	Mod(2,3)	30107	Mod(8,9)	Mod(1,3)
1955	Mod(4,9)	Mod(1,3)	8602	Mod(4,9)	Mod(2,3)
260015	Mod(5,9)	Mod(1,3)	1144066	Mod(7,9)	Mod(2,3)
74290	Mod(8,9)	Mod(2,3)	81719	Mod(8,9)	Mod(1,3)
110	Mod(2,9)	Mod(1,3)			

5.2. **Research of imaginary 3-rational compositum.** The programs written in § 5.1 are valid for any interval $[b, B]$ in $\mathbb{Z}$ and we find, as expected, more solutions that in the real case for $t = 5$.

Give only an example since the calculations are different for imaginary quadratic subfields.

Let $K = \mathbb{Q}(\sqrt{-1}, \sqrt{7}, \sqrt{10}, \sqrt{13}, \sqrt{37})$; then a monic polynomial defining K is:

```
x^32-1056*x^30+480032*x^28-124184704*x^26+20397674176*x^24
-2244202678784*x^22+169874210289152*x^20-8952078632101888*x^18
+329969412292171264*x^16-8547361273484173312*x^14
+156988254584490745856*x^12-2053956312746026434560*x^10
+19066991321006131953664*x^8-123357558863823312388096*x^6
+535739176635907164471296*x^4-1443775880343438717616128*x^2
+198417786002481599748 5056
```

and the Program I of § 3.2 confirms the 3-rationality.

We also find a new example with $t = 6$:

$$K = \mathbb{Q}(\sqrt{-2}, \sqrt{-5}, \sqrt{7}, \sqrt{17}, \sqrt{-19}, \sqrt{59}).$$

Such examples are very rare and it is probably hopeless to find a numerical example with $t = 7$ in a reasonable interval of discriminants.

To verify the 3-rationalities, we use the program of the above subsection, noting that for imaginary quadratic subfields k there is no unit ε and that its 3-class group may be non-trivial, but in this case the 3-Hilbert class field must be contained in $\widetilde{k}$ which gives interesting examples of such phenomena; indeed, this fact is not obvious without explicit knowledge of generators of the 3-classes and we can use directly the Program I of p-rationality for each k such that $h \equiv 0 \pmod 3$.

```
{L3=[70,59,-118,-19,17,-14];d1=component(L3,1);d2=component(L3,2);
d3=component(L3,3);d4=component(L3,4);d5=component(L3,5);d6=component(L3,6);
T=0;for(e1=0,1,for(e2=0,1,for(e3=0,1,for(e4=0,1,for(e5=0,1,for(e6=0,1,
dd=d1^e1*d2^e2*d3^e3*d4^e4*d5^e5*d6^e6;dd=core(dd);
if(dd==1,next);D=coredisc(dd);h=qfbclassno(D);
if(D>0,eps=quadunit(D);q=8;if(Mod(dd,3)!=-1,q=2);E=eps^q-1;
if(Mod(D,3)!=0,No=norm(E)/9);if(Mod(D,3)==0,No=norm(E)/3);No=Mod(No,9));
if(D<0,No=X);print(dd," ",3^valuation(h,3)," ",No)))))))}
```

d	h	No	d	h	No
-14	1	X	-5	1	X
17	1	Mod(1,3)	1190	1	Mod(1,3)
-238	1	X	-85	1	X
-19	1	X	-1330	3	X
266	1	Mod(1,3)	95	1	Mod(1,3)
-323	1	X	-22610	1	X
4522	1	Mod(2,3)	1615	1	Mod(2,3)
-118	3	X	-2065	3	X
413	1	Mod(1,3)	590	1	Mod(1,3)
-2006	3	X	-35105	1	X
7021	1	Mod(2,3)	10030	1	Mod(2,3)
2242	1	Mod(2,3)	39235	1	Mod(2,3)
-7847	1	X	-11210	1	X
38114	1	Mod(1,3)	666995	1	Mod(1,3)
-133399	3	X	-190570	1	X
59	1	Mod(1,3)	4130	1	Mod(1,3)
-826	3	X	-295	1	X

```
1003      1     Mod(2,3)          70210     1     Mod(2,3)
-14042    1     X                 -5015     1     X
-1121     1     X                 -78470    9     X
15694     1     Mod(2,3)          5605      1     Mod(2,3)
-19057    1     X                 -1333990  1     X
266798    1     Mod(1,3)          95285     1     Mod(1,3)
-2        1     X                 -35       1     X
7         1     Mod(2,3)          10        1     Mod(2,3)
-34       1     X                 -595      1     X
119       1     Mod(1,3)          170       1     Mod(1,3)
38        1     Mod(1,3)          665       1     Mod(1,3)
-133      1     X                 -190      1     X
646       1     Mod(2,3)          11305     1     Mod(2,3)
-2261     9     X                 -3230     9     X
70        1     Mod(2,3)
```

From this table, we deduce that the 3-class group of K is isomorphic to $(\mathbb{Z}/9\,\mathbb{Z})^3 \times (\mathbb{Z}/3\,\mathbb{Z})^6$ and that the 3-Hilbert class field of K is contained in the compositum of the 16 independent $\mathbb{Z}_3$-extensions of K distinct from the cyclotomic one (i.e., the compositum of the 16 "anti-cyclotomic" ones).

6. Number fields p-rational for all $p \geq 2$

As soon as there exist units of infinite order in K, the p-rationality is a very difficult question and *a fortiori* the existence of such fields, p-rational for all p. So it remains to consider the fields such that $r_1 + r_2 - 1 = 0$, which characterizes $K = \mathbb{Q}$ (which is p-rational for all p) and the imaginary quadratic fields for which we recall some history.

6.1. **The p-rationality of imaginary quadratic fields.** After a work by Onabe [24] (with a correction to be made in the case $p = 2$), the subject was studied by Angelakis and Stevenhagen [1, Theorem 4.4] in a different (but essentially equivalent) setting. Indeed, it is immediate to see that to have a *minimal absolute abelian Galois group, isomorphic to* $\widehat{\mathbb{Z}}^2 \times \prod_{n\geq 1} \mathbb{Z}/n\,\mathbb{Z}$, is equivalent, for the imaginary quadratic field K, to be p-rational for all p, what we have explained and generalized for any number field in [14, § 1.1].

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field and let p be any prime number. Then K is p-rational if and only if $\mathcal{W}_K = 1$ and the p-Hilbert class field H_K is contained in $\widetilde{K}$.

For $p = 2$, the conditions are given in Example 1.3 about complex and real abelian fields of degree a power of 2. For $p = 3$, $\mathcal{W}_K \simeq \mathbb{Z}/3\mathbb{Z}$ for $-d \equiv -3 \pmod 9$ and $-d \neq -3$. Then for $p > 3$, we get $\mathcal{W}_K = 1$, but there is no immediate criterion for the inclusion $H_K \subset \widetilde{K}$ and we must use for instance the usual numerical computations of § 3.2. We note that in [1, Table 1, § 7] one has many examples of non-p-rational fields K whose p-class group is of order p ($2 \leq p \leq 97$), in complete agreement with our PARI Program I.

The Conjecture 7.1 of [1] may be translated into the similar one :

There are infinitely many imaginary quadratic fields, p-rational for all p.

Indeed, for K fixed, the class group is in general cyclic (from Cohen–Lenstra–Martinet heuristics) and for each p-class group the inclusion of the p-Hilbert class field in $\widetilde{K}$ depends on p-adic values of logarithms of ideals generating the p-classes (see Section 1 and Remark 6.1 below), whose probabilities are

without mystery; but no direction of proof is known and the fact that the set of prime divisors of the class numbers increases as $d \to \infty$ implies some rarefaction of such fields:

In $[b, B] = [1, 10^6]$ the proportion is 0.0766146 (46576 solutions for 607926 fields), in $[b, B] = [10^6, 10^6 + 10^5]$ it is 0.0697357, in $[10^9; 10^9 + 10^5]$ it is 0.0454172, and in $[10^{11}, 10^{11} + 10^5]$ it is 0.0379389 (2306 solutions for 60782 fields).

The following program gives very quickly, in any interval $[b, B]$, the set of imaginary quadratic fields which are p-rational for all $p \geq 2$; for this, it verifies the p-rationalities for $p = 2$, 3, and when p divides the class number:

```
{b=1;B=150;Lrat=listcreate;m=0;for(d=b,B,if(core(d)!=d,next);P=x^2+d;
K=bnfinit(P,1);h=component(component(component(K,8),1),1);
hh=component(factor(6*h),1);t=component(matsize(hh),1);
for(k=1,t,p=component(hh,k);n=2;if(p==2,n=3);Kpn=bnrinit(K,p^n);
Hpn=component(component(Kpn,5),2);e=component(matsize(Hpn),2);
R=0;for(j=1,e,c=component(Hpn,e-j+1);if(Mod(c,p)==0,R=R+1));
T=0;if(R>2,T=1;break));if(T==0,m=m+1;listinsert(Lrat,-d,m)));print(Lrat)}

[-1,-2,-3,-5,-6,-10,-11,-13,-19,-22,-26,-29,-37,-38,-43,-53,-58,-59,-61,
-67,-74,-83,-86,-101,-106,-109,-118,-122,-131,-134,-139,-149] ...
[-1000058,-1000099,-1000117,-1000133,-1000138,-1000166,-1000211,-1000213,
-1000214,-1000253,-1000291,-1000333,-1000357,-1000358,-1000381,-1000394 ...
[-10000000019,-10000000058,-10000000061,-10000000069,-10000000118,
-10000000147,-10000000198,-10000000277,-10000000282,-10000000358 ...
[-123456789062,-123456789094,-123456789133,-123456789194,-123456789322,
-123456789403,-123456789419,-123456789451,-123456789563,-123456789587 ...
```

Remark 6.1. *Let K be an imaginary quadratic field and $p > 3$; the* Log *function is nothing else than the usual logarithm. Then, using the property relying on the characterization "$\mathcal{T}_K = \mathrm{Ker}(\mathrm{Log})$", K is not p-rational as soon as there exists an ideal $\mathfrak{a}$, whose class is of order $p^e \neq 1$, such that $\log(\mathfrak{a}) \in \log(U_K)$ which is equivalent to $\alpha = \xi \cdot u^{p^e}$, where $\mathfrak{a}^{p^e} = (\alpha)$ and $\xi \in \bigoplus_{\mathfrak{p}|p} \mu_{p^f-1}$ where $f \in \{1, 2\}$ is the residue degree of p.*

For instance, in $K = \mathbb{Q}(\sqrt{-383})$, for $p = 17$, the p-class group is generated by the class of $\mathfrak{l} \mid 2$ such that $\mathfrak{l}^{17} = (\alpha)$ where $\alpha = \frac{711+7\sqrt{-383}}{2}$ and $\log(\alpha) \equiv 0 \pmod{17^2}$; since $\log(U_K) = 17\,(\mathbb{Z}_{17} \oplus \mathbb{Z}_{17}\sqrt{-383})$, K is not 17-rational.

So it should be easy to elaborate PARI programs using this point of view.

6.2. **Application to a conjecture of Hajir–Maire.** In [17, Conjecture 0.2] is proposed the following conjecture as a sufficient condition to construct extensions of number fields whose Galois group is a suitable uniform pro-p-group with arbitrary large Iwasawa μ-invariant:

Given a prime p and an integer $m \geq 1$, coprime to p, there exist a totally imaginary field K_0 and a degree m cyclic extension K/K_0 such that K is p-rational.

and it is conjectured that the statement is true taking for K_0 an imaginary p-rational quadratic field [17, Conjecture 4.16].

6.2.1. *Analysis of the conjecture.* Under Leopoldt's conjecture, we have seen that K_0 must be itself p-rational, so the computations in § 6.1 show that, in practice, we may fix K_0 to be any imaginary quadratic field, p-rational for

all p, since they are very numerous. Then one has only to choose p and m and find a degree m cyclic p-rational extension of K_0.

Consider such a field $K_0 = \mathbb{Q}(\sqrt{-d})$; we know that its absolute abelian Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is isomorphic to $\widehat{\mathbb{Z}}^2 \times \prod_{n\geq 1} \mathbb{Z}/n\,\mathbb{Z}$, giving a countable basis of cyclic extensions of degree m, coprime to p, and the defect of the conjecture would say that *any* degree m cyclic extension K of K_0 is non-p-rational (this coming essentialy from the p-class group and/or the normalized p-adic regulator of K, since one can easily realize $\mathcal{W}_K = 1$ for infinitely many K).

The Cohen–Lenstra–Martinet heuristics [4, 5] show that infinitely many cyclic extensions of degree m may have trivial p-class group, so that we must focus on the case of p-adic properties of units of such fields.

For simplicity, we restrict ourselves to extensions K/K_0 unramified at p since in the non-p-part of K^{ab}/K, the inertia groups at the p-places are finite and it remains infinitely many choices.

Let $G = \mathrm{Gal}(K/K_0) \simeq \mathbb{Z}/m\mathbb{Z}$; then E_K/μ_K (still denoted by abuse E_K) is a G-module of $\mathbb{Z}$-rank $m-1$ such that $\mathrm{N}_{K/K_0}(E_K) = 1$, where the norm N_{K/K_0} may also be understood as the "algebraic norm" in $\mathbb{Z}[G]$. It is well known, from generalized Herbrand's theorem (e.g., [10, Lemma I.3.6]), that there exists a "Minkowski unit" $\eta \in E_K$ generating a $\mathbb{Z}[G]$-module E'_K of prime to p index in E_K; so E'_K is a monogenic $\mathbb{Z}[G]/(\mathrm{N}_{K/K_0})$-module.

Thus $\mathcal{R}_K$ is p-adically equivalent to a Frobenius determinant, product of components, indexed with the p-adic characters $\theta \neq 1$ of G, of the form $\mathrm{Reg}_p^{\theta}(\eta) = \prod_{\varphi\,|\,\theta} \mathrm{Reg}_p^{\varphi}(\eta)$ where φ runs trough a set of $\mathbb{Q}_p$-conjugates of some irreducible characters of G.

The field of values of φ only depend on the rational character χ such that $\varphi \mid \theta \mid \chi$ and is denoted C_χ; thus the $\mathrm{Reg}_p^{\varphi}(\eta)$ are C_χ-linear combinations of terms of the form $\frac{1}{p}\log(\eta^\sigma)$, $\sigma \in G$, and the probabilities of $\mathrm{Reg}_p^{\theta}(\eta) \equiv 0 \pmod p$ depend on the residue degree f of p in C_χ and are conjecturaly at most in $\frac{O(1)}{p^f}$ [12, Section 4, § 4.1]. Then each case (p being fixed) strongly depends on the selected value of m, but the probability of non-p-rationality is a constant $\pi(p,m)$ whatever the choice of K.

This gives an incredible probability to have $\mathcal{R}_K \equiv 0 \pmod p$ for all these cyclic fields of degree m; moreover this concerns K_0 fixed among (conjecturally) infinitely many fields.

Remark 6.2. *The above reasoning is valid if we take K as the compositum of K_0 with a real cyclic extension K_1 of degree $m \not\equiv 0 \pmod p$ of $\mathbb{Q}$. If for instance m is odd, K is p-rational if and only if K_1 is p-rational and $\mathcal{C}\ell_K^\infty = 1$ since $\mathcal{R}_K = \mathcal{R}_{K_1} = 1$. So in practice, varying K_1, we are certain to obtain, numerically, a solution. To prove that it is always possible is not yet accessible since we do not have any example, for each m, of a field K_1, p-rational for all p (which is not sufficient to get the p-rationality of K).*

We shall illustrate the two contexts ($K/\mathbb{Q}$ non-abelian and $K = K_0K_1$).

6.2.2. *Numerical examples.* (i) Take $K_0 = \mathbb{Q}(\sqrt{-3})$ and $m = 6$ to define K by means of Kummer theory. Put $K = K_0(\sqrt[6]{u + v\sqrt{-3}})$, $u, v \in \mathbb{Z}$; then K is defined by the polynomial $P = x^{12} - 2u\,x^6 + u^2 + 3v^2$ and $r := r_2 + 1 = 7$.

The program verifies that P is irreducible since u, v are random. The p-rationality is tested for $5 \leq p \leq 100$ and in most cases the p-rationality holds giving many possibilities to illustrate the conjecture.

We give the list of exceptions obtained in an execution of the program (among about one hundred $u + v\sqrt{-3}$):

```
{n=2;r=7;b=5;B=100;for(N=1,100,u=random(10^3);v=random(10^3);
P=x^12-2*u*x^6+u^2+3*v^2;if(polisirreducible(P)!=1,next);K=bnfinit(P,1);
forprime(p=b,B,Kpn=bnrinit(K,p^n);Hpn=component(component(Kpn,5),2);
L=listcreate;e=component(matsize(Hpn),2);R=0;for(k=1,e,
c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;
listinsert(L,p^valuation(c,p),1)));if(R>r,
print(u," ",v," K is not ",p,"-rational")))))}
```

u	v		u	v	
705	960	K is not 7-rational	351	750	K is not 37-rational
705	960	K is not 19-rational	286	682	K is not 7-rational
497	401	K is not 61-rational	60	298	K is not 7-rational
663	465	K is not 7-rational	56	789	K is not 7-rational
593	796	K is not 5-rational	677	538	K is not 7-rational
75	38	K is not 7-rational	884	54	K is not 11-rational
75	38	K is not 11-rational	646	177	K is not 7-rational
351	750	K is not 7-rational	25	130	K is not 13-rational

(ii) For the compositum K of $K_0 = \mathbb{Q}(\sqrt{-1})$ with the cyclic extension of $\mathbb{Q}$ of degree 5 and conductor 11, we obtain that K is not p-rational for $p = 761$, up to 10^4. To have $\mathcal{C}\ell_K \neq 1$, it is necessary that $\mathrm{N}_{K/K_0}(\mathcal{C}\ell_K) = \mathrm{N}_{K/K_1}(\mathcal{C}\ell_K) = 1$, which explains the rarity of the non-p-rationalities:

```
{P=polcompositum(polsubcyclo(11,5),x^2+1);P=component(P,1);K=bnfinit(P,1);
b=2;B=10^4;r=component(component(component(K,7),2),2)+1;
forprime(p=b,B,n=2;if(p==2,n=3); Kpn=bnrinit(K,p^n);
Hpn=component(component(Kpn,5),2);e=component(matsize(Hpn),2);
R=0;for(k=1,e,c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1));
if(R>r,print("rk(T)=",R-r," K is not ",p,"-rational")))}
```

6.3. **Incomplete p-rationality.** As suggested by Hajir and Maire, some cases of "incomplete p-ramification" may be useful for some theoretical aspects when p splits in K in more than one prime ideal; for instance, in the case of imaginary quadratic fields in which p splits, one may consider the ray class field $K(\mathfrak{p})$ of modulus $\mathfrak{p} \mid p$ and class groups formulas in $K(\mathfrak{p})$ with elliptic units, in the framework of the work of [21] for one of the nine principal fields K.

Then one can hope that the groups $\mathcal{T}^{(\ell)}_{K(\mathfrak{p})}$ (for ℓ-ramification theory with any prime ℓ, the base field being $K(\mathfrak{p})$ fixed, insted of K) can be interpreted with these analytic formulas and the corresponding question of the ℓ-rationalities of $K(\mathfrak{p})$ may be of some interest to generalize the classical abelian context.

6.3.1. *General definition of incomplete p-rationality.* Consider the general situation of a number field K with any prime $p \geq 2$. Let P be a subset of the set of p-places of K and let $H_K^{P\mathrm{ram}}$ be the maximal abelian pro-p-extension of K, unramified outside P; the corresponding formula is available in [10, Theorems III.2.5 & III.2.6], stated in the ordinary sense, and gives for the torsion group $\mathcal{T}_K^P$ of $\mathrm{Gal}(H_K^{P\mathrm{ram}}/K)$:

$$\#\mathcal{T}_K^P = \#\mathrm{tor}_{\mathbb{Z}_p}\Big(\bigoplus_{\mathfrak{p}\in P} U_{\mathfrak{p}}\Big/\overline{E_K}^{P}\Big)\cdot\big[H_K : H_K\cap\widetilde{K}^{P}\big], \tag{6.1}$$

where $\overline{E_K}^P$ is the closure of the image of E_K in $\bigoplus_{\mathfrak{p}\in P} U_{\mathfrak{p}}$ (i.e., the projection of $\overline{E}_K$ in this product over P) and $\widetilde{K}^P$ the compositum of the $\mathbb{Z}_p$-extensions contained in $H_K^{\rm Pram}$.

If the second factor $\big[H_K : H_K \cap \widetilde{K}^P\big]$ may be controled, and is trivial in most cases, the first one is more tricky since $\bigoplus_{\mathfrak{p}\in P} U_{\mathfrak{p}}$ is not a Galois module, but the following definition makes sense (under the Leopoldt conjecture):

Let p be a prime number and let P be a subset of the set of p-places of K; the field K is said to be P-rational if $\mathcal{T}_K^P = 1$.

We note that Theorem 2.1 & Corollary 2.2 are still valid for a test of any incomplete p-rationality, using [10, Theorem I.4.5 & Corollary I.4.5.4] for a more general support P: the value of n_0 is the same assuming $e_{\mathfrak{p}} = 1$ for all $\mathfrak{p} \in P$ (otherwise the suitable modulus is $\prod_{\mathfrak{p}\in P} \mathfrak{p}^{e_{\mathfrak{p}} n_0}$).

6.3.2. $\{\mathfrak{p}\}$*-rationality for imaginary quadratic fields.* For an imaginary quadratic fields with p splitted into $\mathfrak{p}\,\mathfrak{p}'$ and $P = \{\mathfrak{p}\}$ one gets:

$$\#\mathcal{T}_K^{\{\mathfrak{p}\}} = \#\big(\mu_{K_{\mathfrak{p}}}/\mu_K\big) \cdot [H_K : H_K \cap \widetilde{K}^{\{\mathfrak{p}\}}],$$

where $\widetilde{K}^{\{\mathfrak{p}\}}$ is a $\mathbb{Z}_p$-extension.

The following program gives the non-$\{\mathfrak{p}\}$-rationality for quadratic fields $\mathbb{Q}(\sqrt{-d})$, $d \in [b, B]$ and $p \in [bp, Bp]$:

```
{b=1;B=10^3;bp=2;Bp=10^4;for(dd=b,B,if(core(dd)!=dd,next);
d=-dd;P=x^2-d;K=bnfinit(P,1);forprime(p=bp,Bp,if(kronecker(d,p)!=1,next);
n=2;if(p==2,n=3);p1=component(component(idealfactor(K,p),1),1);
pn=idealpow(K,p1,n);Kpn=bnrinit(K,pn);Hpn=component(component(Kpn,5),2);
L=listcreate;e=component(matsize(Hpn),2);R=0;for(k=1,e,
c=component(Hpn,e-k+1);if(Mod(c,p)==0,R=R+1;
listinsert(L,p^valuation(c,p),1)));if(R>1,
print("d=",d," rk(T)=",R-1," K is not ",p1,"-rational ",L))))}
```

We get the following non-$\{\mathfrak{p}\}$-rational fields (in the above intervals):

$d \in \{-33, -57, -65, -105, -119, -129, -145, -161, -177, -185, \ldots$
$\ldots, -935, -943, -959, -969, -985, -993\}$, for $\mathfrak{p} \mid 2$,

$d \in \{-107, -302, -362, -419, -503, -509, -533, -602, -617, -713,$
$-863, -974\}$, for $\mathfrak{p} \mid 3$,

$d \in \{-166, -439, -449, -479, -601, -611, -739, -761, -874\}$, for $\mathfrak{p} \mid 5$,

$d \in \{-374, -530, -794, -831, -859, -894\}$, for $\mathfrak{p} \mid 7$,

$d = -758$, for $\mathfrak{p} \mid 11$, $d \in \{-458, -998\}$, for $\mathfrak{p} \mid 13$, $d = -383$, for $\mathfrak{p} \mid 17$.

Remarks 6.3. *(i) In an imaginary quadratic field in which $p \neq 2$ splits, the p-rationality is equivalent to the $\{\mathfrak{p}\}$-rationality: indeed, one verifies that $\widetilde{K}^{\{\mathfrak{p}\}}\widetilde{K}^{\{\mathfrak{p}'\}} = \widetilde{K}$, $\widetilde{K}^{\{\mathfrak{p}\}} \cap \widetilde{K}^{\{\mathfrak{p}'\}} = \widetilde{K} \cap H_K$ (contained in the "anticyclotomic" $\mathbb{Z}_p$-extension of K); thus the p-rationality implies the $\{\mathfrak{p}\}$ and $\{\mathfrak{p}'\}$-rationalities, the converse being obvious.*

For instance, for $p = 3$, $K = \mathbb{Q}(\sqrt{-491})$ where $\mathcal{C}\ell_K \simeq \mathbb{Z}/9\,\mathbb{Z}$, we have $H_K = \widetilde{K}^{\{\mathfrak{p}\}} \cap \widetilde{K}^{\{\mathfrak{p}'\}}$ (whence 3-rationality); the class of $\mathfrak{q} \mid 11$ is of order 9 and $\mathfrak{q}^9 = \big(\frac{1}{2}(95595 + 773\sqrt{-491})\big)$ with $\mathrm{Log}\big(\frac{1}{2}(95595 + 773\sqrt{-491})\big) \equiv 3\sqrt{-491} \pmod{27}$ giving again all the rationalities as expected.

So we may use the above program computing the ray class group modulo $\mathfrak{p}^{n_0}$ which is much faster than the program for the ray class group modulo p^{n_0}.

(ii) In the case $p = 2$ splitted in K, besides the use of Program I (§ 3.2), the computations may be handled with the use of the 2-adic logarithm as in [10, Example III.5.2.2] *for $K = \mathbb{Q}(\sqrt{-15})$, showing its $\{\mathfrak{p}\}$-rationality despite the fact that K is not 2-rational.*

(iii) Let $K/\mathbb{Q}$ be a Galois extension and P as above; let Σ be the set of prime ideals $\mathfrak{p} \mid p$, $\mathfrak{p} \notin P$. The field $H_K^{P\text{ram}}$ is the subfield of H_K^{pr} fixed by the image of $U_K^\Sigma := \bigoplus_{\mathfrak{p}\in\Sigma} U_\mathfrak{p}^1$ (corresponding, by class field theory, to the subgroup generated by the inertia groups) in $U_K/\overline{E}_K$, thus $U_K^\Sigma \cdot \overline{E}_K/\overline{E}_K \simeq U_K^\Sigma/U_K^\Sigma \cap \overline{E}_K$ (then $\mathrm{Gal}(H_K^{P\text{ram}}/H_K) \simeq (U_K/\overline{E}_K)/(U_K^\Sigma \cdot \overline{E}_K/\overline{E}_K)$ is isomorphic to $U_K^P/\overline{E_K}^P$ as expected for the formula (6.1) *giving $\#\mathcal{T}_K^P$).*

We have given generalization of Leopoldt conjecture in this incomplete framework and conjectured a formula for the $\mathbb{Z}_p$-rank of $U_K^\Sigma \cap \overline{E}_K$ which is the set of $\overline{\varepsilon} \in \overline{E}_K$ in U_K whose component on U_K^P is trivial (see [10, Strong p-adic conjecture, III (f), Remarks 4.11.2, 4.11.4, 4.12.1, 4.12.3] *for detailed statements; a* 2013 *arXiv publication, by Dawn C. Nelson, discovers again the same kind of results: https://arxiv.org/abs/1308.4637).*

7. Conclusion

As we have seen, Galois number fields K containing units of infinite order are in general p-rational for "almost all $p \gg 0$" despite the fact that we have no information on $\mathcal{R}_K$ modulo p for $p \to \infty$. More precisely, probabilities are given (in an heuristic approach) by means of p-adic representation theory and the nature of p-adic characters of $\mathrm{Gal}(K/\mathbb{Q})$, via factorization of Frobenius determinants (see [12, Heuristique Principale, § 4.2.2] about the more general question of an algebraic number).

The case of small primes p is different because of local pth roots of unity and p-class groups. For instance, the invariant $\mathcal{W}_K = \big(\bigoplus_{\mathfrak{p}|p} \mu_{K_\mathfrak{p}}\big)/\mu_K$, depending on the splitting of p in $K(\mu_p)/\mathbb{Q}$, may be non-trivial (e.g., the most common case $p = 2$).

For a Galois number field, with sufficiently ramified primes and $p \mid [K : \mathbb{Q}]$, the rank of the p-class group may be an obstruction to the p-rationality because of "genera theory" in $K/\mathbb{Q}$; the case where $p \nmid [K : \mathbb{Q}]$ leads also to an obstruction as soon as $\mathrm{rk}_p(\mathcal{C}\ell_K) > r = r_2 + 1$ and this fact is rather strange in a probabilistic point of view. If K is real and if $p \nmid [K : \mathbb{Q}]$ then $\mathcal{C}\ell_K^\infty = \mathcal{C}\ell_K$ since $\widetilde{K}$ is the cyclotomic $\mathbb{Z}_p$-extension, totally ramified at p.

However, for K fixed and $p \gg 0$, one gets $\mathcal{C}\ell_K^\infty = \mathcal{W}_K = 1$ and the most deep and mysterious invariant about p-rationality is the normalized p-adic regulator which then becomes, for K real and p unramified, $\mathcal{R}_K = \frac{1}{p^{r_1+r_2-1}} R_K$ (R_K being the usual p-adic regulator [29, § 5.5]).

In terms of p-rationality, the real case may be written in a conjectural way as follows (for more information and generalizations to Jaulent's conjecture replacing E_K by the G-module generated by an algebraic number η, see [12, Theorem 1.1 & Heuristic 7.4]):

Conjecture 7.1. *Let $K/\mathbb{Q}$ be a real Galois extension of Galois group G and let $\eta \in E_K$ be a Minkowski unit (i.e., a unit generating a sub-G-module of finite index of E_K). The probability of $\mathcal{R}_K \equiv 0 \pmod p$ is at most:*

$$\frac{1}{p^{\log_2(p)/\log(c_0(\eta))-O(1)}}\ , \text{ for } p \to \infty,$$

where $c_0(\eta) = \max_{\sigma\in G}(|\eta^\sigma|)$, and $\log_2 = \log \circ \log$.

Under the principle of Borel–Cantelli, the number of primes p such that K is non-p-rational is finite.

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