Nonlocal p-Laplacian evolution problems on graphs

Yosra Hafiene, Jalal M. Fadili, Abderrahim Elmoataz

Nonlocal p-Laplacian evolution problems on graphs

Yosra Hafiene[1] Jalal Fadili[2] Abderrahim Elmoataz[3]

[1,2,3] GREYC, Université de Caen

yosra.hafiene@ensicaen.fr

Résumé

L' équation d'évolution du p-Laplacien non-local, gouvernée par un noyau donné, a de très nombreuses applications pour modéliser les phénomènes de diffusion, notamment en traitement du signal et des images sur graphes. En pratique, cette équation d'évolution est implémenté sous une forme discrète (en temps et en espace) comme une approximation numérique du problème continu, où le noyau est remplacé par la matrice d'adjacence d'un graphe. La question naturelle est alors d'étudier la structure des solutions du problème discret et d'en établir la limite continue. C'est l'objectif poursuivi dans ce travail. En combinant des outils issus de la théorie des graphes et des équations d'évolution non-linéaires, nous donnons une interprétation rigoureuse à la limite continue du probléme du p-Laplacien discret sur graphes. Plus spécifiquement, nous considérons une suite de graphes déterministes, pondérés dont l'objet limite est appelé graphon. *L'équation d'évolution du p-Laplacien est alors discrétisée en temps et en espace sur cette suite de graphes. Ainsi, nous prouvons la convergence des solutions de la suite des problèmes discrétisés vers la solution du problème d'évolution continu gouverné par le graphon lorsque le nombre des noeuds du graphe tend vers l'infini. Ce faisant, nous exhibons le vitesse de convergence correspondante.*

Mots Clef

Diffusion nonlocale; p-Laplacien; limites de graphes; approximation numérique.

Abstract

The non-local p-Laplacian evolution equation, governed by given kernel, has various applications to model diffusion phenomena, in particular in signal and image processing. In practice, such an evolution equation is implemented in discrete form (in space and time) as a numerical approximation to a continuous problem, where the kernel is replaced by an adjacency matrix of graph. The natural question that arises is to understand the structure of solutions to the discrete problem, and study their continuous limit. This is the goal pursued in this work. Combining tools from graph theory and non-linear evolution equations, we give a rigorous interpretation to the continuous limit of the discrete p-Laplacian on graphs. More specifically, we consider a sequence of deterministic weighted graphs converging to a so-called graphon. *The continuous p-Laplacian evolution equation is then discretized on this graph sequence both in space and time. We therefore prove that the solutions of the sequence of discrete problems converge to the solution of the continuous evolution problem governed by the graphon, when the number of graph vertices grows to infinity. We exhibit the corresponding convergence rate.*

Keywords

Nonlocal diffusion; p-Laplacian; graphs; graph limits; numerical approximation.

1 Introduction

In its continuous form, the nonlocal p-Laplacian problem with homogeneous Neumann boundary conditions governed by a given kernel K corresponds to the following nonlinear evolution equation

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = -\boldsymbol{\Delta}_p^K u(x,t), & (x,t) \in \Omega \times]0,T], \\ u(x,0) = g(x), & x \in \Omega, \end{cases} \tag{$\mathcal{P}$}$$

where

$$\boldsymbol{\Delta}_p^K = -\int_\Omega K(x,y)\,|u(y,t)-u(x,t)|^{p-2}\,(u(y,t)-u(x,t))dy.$$

$\Omega = [0,1]$ (without loss of generality), $K(\cdot,\cdot)$ is a symmetric, nonnegative and bounded mapping and $p \in [1,+\infty]$. The problem of existence and uniqueness of a solution to ($\mathcal{P}$) is non-trivial. Despite the fact that we will not include the details here, we can show, relying on the theory of nonlinear semi-groups [2], that for $p \in]1,+\infty[$, ($\mathcal{P}$) admits a unique strong solution in $L^p(\Omega)$. There are many applications that integrate equation ($\mathcal{P}$) to model nonlocal diffusion processes. For $p \neq 2$, the discrete p-Laplacien on graphs was studied for the semi-supervised classification, as well as for various image processing applications such as simplification and unsupervised segmentation (see Figure 1 and 2 for some illustrations). These practical considerations naturally lead to a discrete time and space approximation of ($\mathcal{P}$). To do this, we fix $n \in \mathbb{N}$ and consider a partition $\mathcal{Q}_n$ on Ω

$$[(i-1)/n, i/n[, \quad i \in [n], \quad \mathcal{Q}_n = \{\Omega_i^{(n)}, i \in [n]\},$$

Where $[n] = \{1, \cdots, n\}$. Let $\tau_{h-1} := |t_h - t_{h-1}|, h \in [N]$,the time steps corresponding to a division of the interval of time $[0,T]$ of maximum size $\tau = \max \tau_h$. The discrete form in time (explicit) and space of $(\mathcal{P})$ is thus written

$$\begin{cases} \frac{u_i^h - u_i^{h-1}}{\tau_{h-1}} = \frac{1}{n}\sum_{j=1}^{n}(K_n)_{ij}\left|u_j^{h-1} - u_i^{h-1}\right|^{p-2}(u_j^{h-1} - u_i^{h-1}), \\ u_i(0) = g_i^0, \quad i \in [n]. \end{cases}$$

$(K_n)_{ij}$ represents the adjacency matrix of a given convergent graph sequence $\{G_n\}$ converging to a limit object called graphon $K(\cdot,\cdot)$ (see [1] for more details about graph limits). Our goal is to study the continuum limit of the discrete p-Laplacian on graphs and quantify the rate convergence and the error estimates. All the proofs of the results can be found in the long version [3].

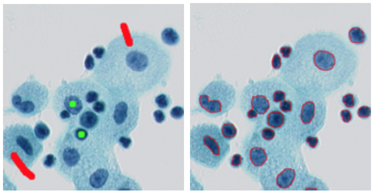

Figure 1 – Semi-supervised segmentation. Left : image with labeled vertices. Right : classified graph.

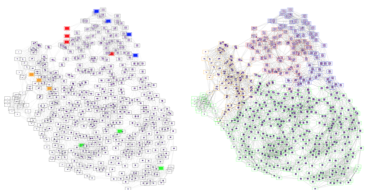

Figure 2 – Semi-supervised classification. Left : graph with initial labels. Right : segmented image.

1.1 Networks on weighted graphs

Let $K : \Omega^2 \to [0,1]$, be a symmetric measurable function which will be used to assign weights to the edges of the graphs considered bellow, we allow only positive weights. We define the quotient of K and $\mathcal{Q}_n$ as a weighted graph with n nodes $K/\mathcal{Q}_n = \left([n], [n]\times[n], \hat{K}_n\right)$. Weights $(\hat{K}_n)_{ij}$ obtained by averaging K over the sets in $\mathcal{Q}_n$

$$(\hat{K}_n)_{ij} = n^2 \int_{\Omega_i^{(n)}\times\Omega_j^{(n)}} K(x,y)dxdy. \tag{1}$$

We consider the totally discrete counterpart of $(\mathcal{P})$ on $K/\mathcal{Q}_n$

$$\begin{cases} \frac{u_i^h - u_i^{h-1}}{\tau_{h-1}} = \frac{1}{n}\sum_{j=1}^{n}(\hat{K}_n)_{ij}\left|u_j^{h-1} - u_i^{h-1}\right|^{p-2}(u_j^{h-1} - u_i^{h-1}), \\ u_i(0) = g_i^0, \quad i \in [n]. \end{cases}$$

Let us recall that our main goal is to compare the solutions of the discrete and continuous models and establish some consistency results. Since the solutions do not live in the same spaces, it is convenient to represent some intermediate model that is the continuous extension of the discrete problem, using the vector $U^h = (u_1^h, u_2^h, \cdots, u_n^h)^T$ whose components solve the previous system to obtain the following linear interpolation on Ω, for $x \in \Omega_i^{(n)}$, $t \in]t_{h-1}, t_h]$

$$\check{u}_n(x,t) = \frac{t_h - t}{\tau_{h-1}} u_i^{h-1} + \frac{t - t_{h-1}}{\tau_{h-1}} u_i^h \tag{2}$$

and $\bar{u}_n(x,t) = \sum_{h=1}^{N} u_i^{h-1}\chi_{]t_{h-1},t_h]}(t)\chi_{\Omega_i^{(n)}}(x)$. So that $\check{u}_n(x,t)$ satisfies the following problem :

$$\begin{cases} \frac{\partial}{\partial t}\check{u}_n(x,t) = -\boldsymbol{\Delta}_p^{K_n^w}(\bar{u}_n(x,t)), \\ \check{u}_n^0(x) = g_n^0(x), \end{cases} \tag{$\mathcal{P}_n^w$}$$

where K_n^w and g_n^0 are constant piecewise interpolations of $(\hat{K}_n)_{ij}$ and g_i.

Theorem 1.1. *Suppose that $p \in]1,+\infty[$, $K : \Omega^2 \to [0,1]$ is a symmetric measurable function, and $g \in L^\infty(\Omega)$. Let u and $\check{u}_n$ be the solutions of $(\mathcal{P})$ and $(\mathcal{P}_n^w)$, respectively. Then*

$$\|u - \check{u}_n\|_{C(0,T;L^p(\Omega))} \underset{n\to\infty,\Delta\to 0}{\longrightarrow} 0. \tag{3}$$

To quantify the rate of convergence in (3), we need to add some supplementary assumptions on the kernel K and the initial data g.

Definition 1.1. *The total variation of a function K is defined by duality : For $K \in L^1_{loc}(\Omega^2)$ it is given by*

$$J(K) = \sup\left\{-\int_{\Omega^2} K \operatorname{div}(\phi)\, dxdy\right\}, \tag{TV}$$

where

$$\phi \in S := \{\phi \in C_c^\infty(\Omega^2;\mathbb{R}^N), |\phi(x,y)| \le 1 \forall (x,y) \in \Omega^2\}.$$

A function is said to have bounded variation whenever $J(K) < +\infty$. We call $\mathrm{BV}(\Omega^2)$ the set of functions with bounded variation $K \in L^1(\Omega^2)$ such that $J(K) < +\infty$.

Theorem 1.2. *Suppose that $p \in]1,+\infty[$, $K : \Omega^2 \to [0,1]$ is a symmetric and measurable function in $\mathrm{BV}(\Omega^2)$, and $g \in L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$. Let u and $\check{u}_n$ be the solutions of $(\mathcal{P})$ and $(\mathcal{P}_n^w)$ respectively. Then*

$$\|u - \check{u}_n\|_{C(0,T;L^p(\Omega))} \le O(n^{-\frac{1}{p}}) + O(\tau).$$

References

[1] S L.Lovasz and B.Sezgedy, Limits of dense graph sequences, J.Combin Theory ser.B.96 2006.

[2] F. Andreu-Vaillo, J. M. Mazon, J. D. Rossi, and J. J. Toledo-Melero. Nonlocal Diffusion Problems, v.165 of Mathematical Surveys and Monographs. American Mathematical Society, 2010.

[3] S Y. Hafiene, M. J. Fadili, and A. Elmoataz. Nonlocal p-laplacian evolution problems on graphs. Technical Report arXiv :1612.07156, 2016.