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Rate of convergence to equilibrium for discrete-time stochastic dynamics with memory

Maylis Varvenne*

September 21, 2017

Abstract

The main objective of the paper is to study the long-time behavior of general discrete dynamics driven by ergodic stationary Gaussian noise. To this end, we first explain how is it possible to define invariant distributions in this generally non-Markovian setting and to get existence results under appropriate conditions. Then, we get a uniqueness result and a rate of convergence to the invariant distribution in total variation thanks to a coupling procedure (with a step specific to non-Markovian framework).

Keywords: Discrete stochastic dynamics; Stationary Gaussian noise; Rate of convergence to equilibrium; Total variation distance; Lyapunov function; Toeplitz operator.

1 Introduction

Convergence to equilibrium for Stochastic dynamics is one of the most natural and most studied problems in probability theory. Regarding Markov processes, this topic has been deeply undertaken through various approaches: spectral analysis, functional inequalities or coupling methods. However, in many applications (Physics, Biology, Finance...) the future evolution of a quantity may depend on its own history, and thus, noise with independent increments does not accurately reflect reality. A classical way to overcome this problem is to consider dynamical systems driven by processes with stationary increments like fractional Brownian motion for instance. In a continuous time framework, Stochastic Differential Equations (SDEs) driven by Gaussian processes with stationary increments have been introduced to model random evolution phenomena with long range dependence properties. Consider SDEs of the following form

$$dX_t = b(X_t) + \sigma(X_t)dZ_t \quad (1.1)$$

where $(Z_t)_{t \geq 0}$ is a Gaussian process with ergodic stationary increments and $\sigma : \mathbb{R}^d \rightarrow \mathcal{M}_d(\mathbb{R})$, $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are functions defined in a such a way that existence of a solution holds. As concerns long-time behavior, different properties have been studied like approximation of stationary solution in [8] or the rate of convergence to an equilibrium distribution. For this last property, the case when $(Z_t)_{t \geq 0}$ is fractional Brownian motion (fBm) has received significant attention from Hairer [14], Fontbona and Panloup [11], Deya, Panloup and Tindel [9] over the last decade. They used coalescent coupling strategy to compute the rate of convergence. In the additive noise setting, Hairer proved that the process converges in total variation to the stationary regime with a rate upper-bounded by $C_\varepsilon t^{-(\alpha_H - \varepsilon)}$ for any $\varepsilon > 0$, with

$$\alpha_H = \begin{cases} \frac{1}{8} & \text{if } H \in (\frac{1}{4}, 1) \setminus \{\frac{1}{2}\} \\ H(1 - 2H) & \text{if } H \in (0, \frac{1}{4}). \end{cases} \quad (1.2)$$

In the multiplicative noise setting, Fontbona and Panloup extended those results under selected assumptions on σ to the case where $H \in (\frac{1}{2}, 1)$ and finally Deya, Panloup and Tindel obtained this type of results

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in the rough setting $H \in (\frac{1}{3}, 1/2)$. Here, we focus on a general class of recursive discrete dynamics driven by a stationary sequence which includes in particular discretization of (1.1), that is

$$X_{n+1} = F(X_n, \Delta_{n+1}) \quad (1.3)$$

where $(\Delta_n)_{n \in \mathbb{Z}}$ is an ergodic stationary Gaussian sequence. This type of discrete stochastic dynamics, which is not Markovian in general, has not been much discussed except in the linear case like Autoregressive Moving-average (ARMA) models [5] whose main objective is the prediction of stationary processes. When F is linear, dynamics like (1.3) are truly related to ARMA processes through the so-called Wold's decomposition theorem which implies that we can see $(\Delta_n)_{n \in \mathbb{Z}}$ as a moving-average of infinite order (see [5] to get more details).

Here, we investigate the problem of the long-time behavior of (1.3) for a general class of functions F . To this end, we first explain how is it possible to define invariant distributions in this non-Markovian setting and to obtain existence results, and then we use a coalescent coupling strategy to get the rate of convergence to equilibrium of processes following (1.3) under selected assumptions. This discrete time framework has several advantages. First, it allows us to better target the impact of the memory thanks to the moving average (MA) representation of the noise process (see (2.2)). The deterministic sequence defined by the coefficients involved in this representation measures, in a sense, the weight of the past since the covariance function of $(\Delta_n)_{n \in \mathbb{Z}}$ is entirely determined by those coefficients (see Remark 2.1). Then, one of our motivations to work in this discrete context is to see if the speed of convergence to equilibrium is affected by the difficulty of the coupling strategy which is a priori greater in a continuous time setting.

Now, let us briefly recall how this coupling method is organized. First, one takes two processes $(X_n^1, (\Delta_{n+k}^1)_{k \leq 0})_{n \geq 0}$ and $(X_n^2, (\Delta_{n+k}^2)_{k \leq 0})_{n \geq 0}$ following (1.3) starting respectively from μ_0 and μ_* (the invariant distribution). As a preliminary step, one waits that the two paths get close. Then, at each trial, the coupling attempt is divided in two steps. First, one tries in Step 1 to stick the positions together at a given time. Then, in Step 2, one tries to ensure that the paths stay clustered until $+\infty$. Actually, oppositely to the Markovian setting where the paths remain naturally fastened together (by putting the same innovation on each path), the main difficulty here is that, staying together has a cost. In other words, this property can be ensured only with a non trivial coupling of the noises. Finally, if one of the two previous steps fails, one begins Step 3 by putting the same noise on each coordinate until the "cost" to attempt Step 1 is not too big. In other words, during this step one waits again for the paths to get close but also for the memory of the coupling cost to decrease sufficiently.

Thanks to this strategy, we are able to prove that the law of the process $(X_{n+k})_{k \geq 0}$ following (1.3) converges in total variation to the stationary regime with a rate upper-bounded by Cn^{-v} where v is a quantity which is directly linked to the asymptotic behavior of the sequence of coefficients appearing in the MA representation of the noise process. In particular, we focus on Gaussian noise with exponential and polynomial memory (see Section 2 for more details). For the polynomial case, a more precise example is also studied, namely noise with fractional memory (see Subsection 2.6). This example coupled with the fact that we apply our result to the discretization of (1.1) (see Subsection 2.4) allows us to contrast with the continuous time results [14, 11, 9].

The following section gives more details on the studied dynamics and describes the assumptions required to get the main result, namely Theorem 2.1. Then, the proof of Theorem 2.1 is achieved in Sections 3, 4, 5, 6 and 7, which are outlined at the end of Section 2.

2 Setting and main results

2.1 Setting

Let $X := (X_n)_{n \geq 0}$ denote an \mathbb{R}^d -valued sequence defined by: X_0 is a random variable with distribution denoted by μ_0 and

$$\forall n \geq 0, \quad X_{n+1} = F(X_n, \Delta_{n+1}), \quad (2.1)$$

where $F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is continuous and $(\Delta_n)_{n \in \mathbb{Z}}$ is a stationary and purely non-deterministic Gaussian sequence. Hence, by Wold's decomposition theorem [5] it has a moving average representation

$$\forall n \in \mathbb{Z}, \quad \Delta_n = \sum_{k=0}^{+\infty} a_k \xi_{n-k} \quad (2.2)$$

with

$$\begin{cases} (a_k)_{k \geq 0} \in \mathbb{R}^{\mathbb{N}} \text{ such that } a_0 \neq 0 \text{ and } \sum_{k=0}^{+\infty} a_k^2 < +\infty \\ (\xi_k)_{k \in \mathbb{Z}} \text{ an i.i.d sequence such that } \xi_1 \sim \mathcal{N}(0, I_d). \end{cases} \quad (2.3)$$

Without loss of generality, we assume that $a_0 = 1$. Actually, if $a_0 \neq 1$, we can come back to this case by setting

$$\tilde{\Delta}_n = \sum_{k=0}^{+\infty} \tilde{a}_k \xi_{n-k}$$

with $\tilde{a}_k := \frac{a_k}{a_0}$.

Remark 2.1. *The asymptotic behavior of the sequence $(a_k)_{k \geq 0}$ certainly plays a key role to compute the rate of convergence to equilibrium of the process $(X_n)_{n \geq 0}$. Actually, the memory induced by the noise process is quantified by the sequence $(a_k)_{k \geq 0}$ through the identity*

$$\forall n \in \mathbb{Z}, \forall k \geq 0, \quad c(k) := \mathbb{E}[\Delta_n \Delta_{n+k}] = \sum_{i=0}^{+\infty} a_i a_{k+i}.$$

In the sequel, the *state space* of the process X and the *noise space* associated to $((\Delta_{n+k})_{k \leq 0})_{n \geq 0}$ will be respectively denoted by $\mathcal{X} := \mathbb{R}^d$ and $\mathcal{W} := (\mathbb{R}^d)^{\mathbb{Z}^-}$. These notations will be clarified in Subsection 3.1.

2.2 Preliminary tool: a Toeplitz type operator

The moving-average representation of the Gaussian sequence $(\Delta_n)_{n \in \mathbb{Z}}$ naturally leads us to define an operator related to the coefficients $(a_k)_{k \geq 0}$. First, set

$$\ell_a(\mathbb{Z}^-, \mathbb{R}^d) := \left\{ w \in (\mathbb{R}^d)^{\mathbb{Z}^-} \mid \forall k \geq 0, \sum_{l=0}^{+\infty} a_l w_{-k-l} < +\infty \right\}$$

and define \mathbf{T}_a on $\ell_a(\mathbb{Z}^-, \mathbb{R}^d)$ by

$$\mathbf{T}_a(w) = \left(\sum_{l=0}^{+\infty} a_l w_{-k-l} \right)_{k \geq 0}. \quad (2.4)$$

Due to the Cauchy-Schwarz inequality, we can check that for instance $\ell^2(\mathbb{Z}^-, \mathbb{R}^d)$ is included in $\ell_a(\mathbb{Z}^-, \mathbb{R}^d)$ due to the assumption $\sum_{k \geq 0} a_k^2 < +\infty$. This Toeplitz type operator \mathbf{T}_a links $(\Delta_n)_{n \in \mathbb{Z}}$ to $(\xi_n)_{n \in \mathbb{Z}}$. The following proposition spells out the reverse operator.

Proposition 2.1. *Let \mathbf{T}_b be the operator defined on $\ell_b(\mathbb{Z}^-, \mathbb{R}^d)$ in the same way as \mathbf{T}_a but with the following sequence $(b_k)_{k \geq 0}$*

$$b_0 = \frac{1}{a_0} \quad \text{and} \quad \forall k \geq 1, \quad b_k = -\frac{1}{a_0} \sum_{l=1}^k a_l b_{k-l}. \quad (2.5)$$

Then,

$$\forall w \in \ell_a(\mathbb{Z}^-, \mathbb{R}^d), \quad \mathbf{T}_b(\mathbf{T}_a(w)) = w \quad \text{and} \quad \forall w \in \ell_b(\mathbb{Z}^-, \mathbb{R}^d), \quad \mathbf{T}_a(\mathbf{T}_b(w)) = w$$

that is $\mathbf{T}_b = \mathbf{T}_a^{-1}$ and $\ell_b(\mathbb{Z}^-, \mathbb{R}^d) = \mathbf{T}_a(\ell_a(\mathbb{Z}^-, \mathbb{R}^d))$.

Proof. Let $w \in \ell_a(\mathbb{Z}^-, \mathbb{R}^d)$. Then let $n \geq 0$,

$$\begin{aligned}
(\mathbf{T}_b(\mathbf{T}_a(w)))_{-n} &= \sum_{k=0}^{+\infty} b_k (\mathbf{T}_a(w))_{-n-k} \\
&= \sum_{k=0}^{+\infty} b_k \sum_{l=0}^{+\infty} a_l w_{-n-k-l} \\
&= \sum_{k=0}^{+\infty} \sum_{i=k}^{+\infty} b_k a_{i-k} w_{-n-i} \quad (\text{by setting } i = k + l) \\
(\mathbf{T}_b(\mathbf{T}_a(w)))_{-n} &= \sum_{i=0}^{+\infty} \underbrace{\left(\sum_{k=0}^i b_k a_{i-k} \right)}_{=0 \text{ except for } i=0} w_{-n-i} = w_{-n}
\end{aligned}$$

We show in the same way that for $w \in \ell_b(\mathbb{Z}^-, \mathbb{R}^d)$, we have $\mathbf{T}_a(\mathbf{T}_b(w)) = w$. \square

Remark 2.2. *The sequence $(b_k)_{k \geq 0}$ is of first importance in the sequel. The sketch of the proof of Theorem 2.1 will use an important property linked to the sequence $(b_k)_{k \geq 0}$: if two sequences u and v are such that*

$$\forall n \geq 1, \quad u_n = \sum_{k=0}^{n-1} a_k v_{n-k}$$

then

$$\forall n \geq 1, \quad v_n = \sum_{k=0}^{n-1} b_k u_{n-k}.$$

This reverse identity and the asymptotic behavior of $(b_k)_{k \geq 0}$ play a significant role in the computation of the rate of convergence.

The following section is devoted to outline assumptions on $(a_k)_{k \geq 0}$ and $(b_k)_{k \geq 0}$ and then on F to get the main result.

2.3 Assumptions and general theorem

First of all, let us introduce assumptions on $(a_k)_{k \geq 0}$ and $(b_k)_{k \geq 0}$. All along the paper, we will switch between two types of assumptions called respectively the *polynomial case* and the *exponential case*.

Hypothesis (\mathbf{H}_{poly}): The following conditions hold,

- there exist $\rho, \beta > 0$ and $C_\rho, C_\beta > 0$ such that

$$\forall k \geq 0, \quad |a_k| \leq C_\rho (k+1)^{-\rho} \quad \text{and} \quad \forall k \geq 0, \quad |b_k| \leq C_\beta (k+1)^{-\beta}.$$

- there exist $\kappa \geq \rho + 1$ and $C_\kappa > 0$ such that

$$\forall k \geq 0, \quad |a_k - a_{k+1}| \leq C_\kappa (k+1)^{-\kappa}.$$

Hypothesis (\mathbf{H}_{exp}): There exist $\lambda, \mu > 0$ and $C_\lambda, C_\mu > 0$ such that,

$$\forall k \geq 0, \quad |a_k| \leq C_\lambda e^{-\lambda k} \quad \text{and} \quad \forall k \geq 0, \quad |b_k| \leq C_\mu e^{-\mu k}.$$

Remark 2.3. \triangleright (\mathbf{H}_{poly}) and (\mathbf{H}_{exp}) are general parametric hypothesis which apply to a large class of Gaussian driven dynamics. These assumptions involve the memory of the noise process through the sequence $(a_k)_{k \geq 0}$ but also through the coefficients appearing in the reverse Toeplitz operator \mathbf{T}_a^{-1} (see

Proposition 2.1. Due to the strategy of the proof (coalescent coupling in a non Markovian setting) we also need a bound on the discrete derivative of $(a_k)_{k \geq 0}$.

▷ Even though $(a_k)_{k \geq 0}$ and $(b_k)_{k \geq 0}$ are related by (2.5), there is no general rule which connects ρ and β . This fact will be highlighted in Subsection 2.6.

Let us now introduce some assumptions on the function F which defines the dynamics (2.1). Throughout this paper $F : \mathcal{X} \times \mathbb{R}^d \rightarrow \mathcal{X}$ is a continuous function and the following hypothesis (\mathbf{H}_1) and (\mathbf{H}_2) are satisfied.

Hypothesis (\mathbf{H}_1) : There exists a continuous function $V : \mathcal{X} \rightarrow \mathbb{R}_+^*$ satisfying $\lim_{|x| \rightarrow +\infty} V(x) = +\infty$ and $\exists \gamma \in (0, 1)$ and $C > 0$ such that for all $(x, w) \in \mathcal{X} \times \mathbb{R}^d$,

$$V(F(x, w)) \leq \gamma V(x) + C(1 + |w|).$$

Remark 2.4. We will see in Subsection 3.2 that this condition on F ensures the existence of an invariant distribution (in a sense made precise below).

We define $\tilde{F} : \mathcal{X} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{X}$ by $\tilde{F}(x, u, y) = F(x, u + y)$. We assume that \tilde{F} satisfies the following conditions:

Hypothesis (\mathbf{H}_2) : Let $K > 0$. We assume that there exists $\tilde{K} > 0$ such that for every $\mathbf{x} := (x, x', y, y')$ in $B(0, K)^4$, there exist $\Lambda_{\mathbf{x}} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $M_K > 0$ and $C_{\tilde{K}}$ such that the following holds

- $\Lambda_{\mathbf{x}}$ is a measurable, invertible and almost everywhere differentiable function such that $\Lambda_{\mathbf{x}}^{-1}$ is also measurable.
- for all $u \in B(0, \tilde{K})$,

$$\tilde{F}(x, u, y) = \tilde{F}(x', \Lambda_{\mathbf{x}}(u), y') \tag{2.6}$$

$$|\det(J_{\Lambda_{\mathbf{x}}}(u))| \geq C_{\tilde{K}} \tag{2.7}$$

- for all $u \in \mathbb{R}^d$,

$$|\Lambda_{\mathbf{x}}(u) - u| \leq M_K \tag{2.8}$$

Remark 2.5. Let us make a few precisions on the arguments of \tilde{F} : x is the position of the process, u the increment of the innovation process and y is related to the past of the process (see (4.7) for more details). The boundary $C_{\tilde{K}}$ and M_K are independent from x, x', y and y' . This assumption can be viewed as a kind of controlability assumption in the following sense: the existence of $\Lambda_{\mathbf{x}}$ leads to the coalescence of the positions by (2.6). This is of first importance to achieve the first step of the coupling procedure (see Subsection 4.2).

We are now in position to state our main result. Let $\mathcal{L}((X_n^{\mu_0})_{n \geq 0})$ denote the distribution of the process X starting from an initial condition μ_0 (see Subsection 3.1 below for detailed definitions of initial condition and invariant distribution) and for an invariant distribution μ_{\star} denote by $S\mu_{\star}$ the law of the stationary solution. Finally, we denote by $\|\cdot\|_{TV}$ the classical total variation norm.

Theorem 2.1. Assume (\mathbf{H}_1) and (\mathbf{H}_2) . Then,

(i) There exists an invariant distribution μ_{\star} associated to (2.1).

(ii) Assume that $(\mathbf{H}_{\text{poly}})$ is true with $\rho, \beta > 1/2$ and $\rho + \beta > 3/2$. Then, uniqueness holds for the invariant distribution μ_{\star} . Furthermore, for every initial distribution μ_0 for which $\int_{\mathcal{X}} V(x) \Pi_{\mathcal{X}}^* \mu_0(dx) < +\infty$ and for all $\varepsilon > 0$, there exists $C_{\varepsilon} > 0$ such that

$$\|\mathcal{L}((X_{n+k}^{\mu_0})_{k \geq 0}) - S\mu_{\star}\|_{TV} \leq C_{\varepsilon} n^{-(v(\beta, \rho) - \varepsilon)}.$$

where the function v is defined by

$$v(\beta, \rho) = \sup_{\alpha \in (\frac{1}{2}v(\frac{3}{2} - \beta), \rho)} \min\{1, 2(\rho - \alpha)\} (\min\{\alpha, \beta, \alpha + \beta - 1\} - 1/2).$$

(iii) Assume that $(\mathbf{H}_{\text{exp}})$ is true, then uniqueness holds for the invariant distribution μ_\star . Furthermore, for every initial distribution μ_0 for which $\int_{\mathcal{X}} V(x) \Pi_{\mathcal{X}}^* \mu_0(dx) < +\infty$ and for all $p > 0$, there exists $C_p > 0$ such that

$$\|\mathcal{L}((X_{n+k}^{\mu_0})_{k \geq 0}) - \mathcal{S}\mu_\star\|_{TV} \leq C_p n^{-p}.$$

Remark 2.6. Assumption (\mathbf{H}_2) is only required to perform the first step of the coupling strategy to get (ii) and (iii) (see Section 4 for more details).

In the following subsection, we test the assumptions of our main result Theorem 2.1 (especially (\mathbf{H}_1) and (\mathbf{H}_2)) on the Euler scheme of SDEs like (1.1).

2.4 The Euler Scheme

Recall that $\mathcal{X} = \mathbb{R}^d$. In this subsection, set

$$\begin{aligned} F_h : \quad \mathcal{X} \times \mathbb{R}^d &\rightarrow \mathcal{X} \\ (x, w) &\mapsto x + hb(x) + \sigma(x)w. \end{aligned} \quad (2.9)$$

where $h > 0$, $b : \mathcal{X} \rightarrow \mathcal{X}$ is continuous and $\sigma : \mathcal{X} \rightarrow \mathcal{M}_d(\mathbb{R})$ is a continuous and bounded function on \mathcal{X} . For all $x \in \mathcal{X}$ we suppose that $\sigma(x)$ is invertible and we denote by $\sigma^{-1}(x)$ the inverse. Moreover, we assume that σ^{-1} is a continuous function and that b satisfies a Lyapunov type assumption that is:

(L1) $\exists C > 0$ such that

$$\forall x \in \mathcal{X}, \quad |b(x)| \leq C(1 + |x|) \quad (2.10)$$

(L2) $\exists \tilde{\beta} \in \mathbb{R}$ and $\tilde{\alpha} > 0$ such that

$$\forall x \in \mathcal{X}, \quad \langle x, b(x) \rangle \leq \tilde{\beta} - \tilde{\alpha}|x|^2. \quad (2.11)$$

Remark 2.7. This function F_h corresponds to the Euler scheme associated to SDEs like (1.1). The conditions on the function b are classical to get existence of invariant distribution.

In this setting the function \tilde{F}_h (introduced in Hypothesis (\mathbf{H}_2)) is given by

$$\begin{aligned} \tilde{F}_h : \quad \mathcal{X} \times \mathbb{R}^d \times \mathbb{R}^d &\rightarrow \mathcal{X} \\ (x, u, y) &\mapsto x + hb(x) + \sigma(x)(u + y). \end{aligned}$$

Theorem 2.2. Let $h > 0$. Let F_h be the function defined above. Assume that $b : \mathcal{X} \rightarrow \mathcal{X}$ is a continuous function satisfying **(L1)** and **(L2)** and $\sigma : \mathcal{X} \rightarrow \mathcal{M}_d(\mathbb{R})$ is a continuous and bounded function such that for all $x \in \mathcal{X}$, $\sigma(x)$ is invertible and $x \mapsto \sigma^{-1}(x)$ is a continuous function. Then, (\mathbf{H}_1) and (\mathbf{H}_2) hold for F_h with $h > 0$ small enough.

Proof. Set $V(x) = |x|$. Let us begin by proving that (\mathbf{H}_1) holds with V for F_h with $h > 0$ small enough. We have:

$$|F_h(x, w)|^2 = |x|^2 + h^2|b(x)|^2 + 2h\langle x, b(x) \rangle + 2\langle x, \sigma(x)w \rangle + 2h\langle b(x), \sigma(x)w \rangle + |\sigma(x)w|^2.$$

Then, using the inequality $|\langle a, b \rangle| \leq \frac{1}{2}(\varepsilon|a|^2 + \frac{1}{\varepsilon}|b|^2)$ for all $\varepsilon > 0$, we get

$$|\langle x, \sigma(x)w \rangle| \leq \frac{1}{2}(\varepsilon|x|^2 + \frac{1}{\varepsilon}|\sigma(x)w|^2) \quad \text{et} \quad |\langle b(x), \sigma(x)w \rangle| \leq \frac{1}{2}(\varepsilon|b(x)|^2 + \frac{1}{\varepsilon}|\sigma(x)w|^2).$$

Moreover, assumptions **(L1)** and **(L2)** on b give

$$|\langle b(x), x \rangle| \leq \tilde{\beta} - \tilde{\alpha}|x|^2 \quad \text{et} \quad |b(x)|^2 \leq \tilde{C}(1 + |x|^2).$$

Hence, we finally have

$$\begin{aligned}
|F_h(x, w)|^2 &\leq |x|^2 + \tilde{C}h^2(1 + |x|^2) + 2h(\tilde{\beta} - \tilde{\alpha}|x|^2) + \varepsilon|x|^2 + \frac{1}{\varepsilon}|\sigma(x)w|^2 \\
&\quad + \tilde{C}h\varepsilon(1 + |x|^2) + \frac{h}{\varepsilon}|\sigma(x)w|^2 + |\sigma(x)w|^2 \\
&\leq |x|^2 + 2h(\tilde{\beta} - \tilde{\alpha}|x|^2) + \tilde{C}(\varepsilon + h\varepsilon + h^2)(1 + |x|^2) + \left(1 + \frac{h+1}{\varepsilon}\right)|\sigma(x)w|^2.
\end{aligned}$$

Now, set $\varepsilon = h^2$ and choose h small enough such that $\tilde{C}(\varepsilon + h\varepsilon + h^2) \leq \tilde{\alpha}h$. Therefore,

$$|F_h(x, w)|^2 \leq |x|^2 + h(\tilde{\gamma} - \tilde{\alpha}|x|^2) + \left(1 + \frac{h+1}{\varepsilon}\right)|\sigma(x)w|^2$$

where $\tilde{\gamma} = 2\tilde{\beta} + \tilde{\alpha}$. Then

$$|F_h(x, w)|^2 \leq (1 - \tilde{\alpha}h)|x|^2 + h\tilde{\gamma} + \left(1 + \frac{h+1}{\varepsilon}\right)|\sigma(x)w|^2.$$

By assumption σ is a bounded function on \mathbb{R}^d . Then, there exists $\tilde{C} > 0$ depending on h and σ such that

$$|F_h(x, w)|^2 \leq (1 - \tilde{\alpha}h)|x|^2 + \tilde{C}(1 + |w|^2).$$

Using the classical inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we finally get the existence of $\gamma \in (0, 1)$ and $C > 0$ such that for all $(x, w) \in \mathbb{R}^d \times \mathbb{R}^d$

$$|F_h(x, w)| \leq \gamma|x| + C(1 + |w|) \quad (2.12)$$

which achieves the proof of **(H₁)**.

We now turn to the proof of **(H₂)**. Let $K > 0$ and take $\mathbf{x} = (x, x', y, y') \in B(0, K)^4$. Here we take $\tilde{K} = K$. Hence, let us now define $\Lambda_{\mathbf{x}}$. For all $u \in B(0, K)$, we set

$$\Lambda_{\mathbf{x}}(u) = \sigma^{-1}(x')\sigma(x)u + \sigma^{-1}(x')(x - x' + h(b(x) - b(x'))) + \sigma^{-1}(x')\sigma(x)y - y' \quad (2.13)$$

which is equivalent to $\tilde{F}_h(x, u, y) = \tilde{F}_h(x', \Lambda_{\mathbf{x}}(u), y')$ for all $u \in B(0, K)$.

Hence, for all $u \in B(0, K)$,

$$J_{\Lambda_{\mathbf{x}}}(u) = \sigma^{-1}(x')\sigma(x). \quad (2.14)$$

Since σ, σ^{-1} and b are continuous, there exist $C_K > 0$ and $m_K > 0$ independent from \mathbf{x} such that for all $u \in B(0, K)$,

$$\begin{aligned}
|\det(J_{\Lambda_{\mathbf{x}}}(u))| &\geq C_K \\
|\Lambda_{\mathbf{x}}(u) - u| &\leq m_K.
\end{aligned}$$

Then, we extend $\Lambda_{\mathbf{x}}$ to a continuous and invertible function in such a way that there exists $M_K \geq m_K$ such that for all $u \in \mathbb{R}^d$,

$$|\Lambda_{\mathbf{x}}(u) - u| \leq M_K.$$

Finally, the function \tilde{F}_h satisfies **(H₂)**. □

The two following subsections are devoted to outline examples of sequences which satisfy hypothesis **(H_{exp})** or **(H_{poly})**. In particular, Subsection 2.6 includes the case where the process $(\Delta_n)_{n \in \mathbb{Z}}$ corresponds to fractional Brownian motion increments.

2.5 An explicit case which satisfies $(\mathbf{H}_{\text{exp}})$

In this subsection, we investigate an explicit exponential case with the following definition for the sequence $(a_k)_{k \geq 0}$

$$a_0 = 1 \quad \text{and} \quad \forall k \in \mathbb{N}^*, \quad a_k = C_a e^{-\lambda k} \quad (2.15)$$

with $C_a \in \mathbb{R}$. Let us recall that $b_0 = 1$ (since $a_0 = 1$) and for all $k \geq 1$, we can get the following general expression of b_k (see Appendix A):

$$b_k = \sum_{p=1}^k \frac{(-1)^p}{a_0^{p+1}} \left(\sum_{\substack{k_1, \dots, k_p \geq 1 \\ k_1 + \dots + k_p = k}} \prod_{i=1}^p a_{k_i} \right). \quad (2.16)$$

A classical combinatorial argument shows that $\#\{(k_1, \dots, k_p) \in \mathbb{N}^* \mid k_1 + \dots + k_p = k\} = \binom{k-1}{p-1}$. As a consequence, when the sequence $(a_k)_{k \geq 0}$ is defined by (2.15), we can easily compute the coefficients b_k for $k \geq 1$,

$$\begin{aligned} b_k &= \sum_{p=1}^k (-C_a)^p e^{-\lambda k} \#\{(k_1, \dots, k_p) \in \mathbb{N}^* \mid k_1 + \dots + k_p = k\} \\ &= \sum_{p=1}^k \binom{k-1}{p-1} (-C_a)^p e^{-\lambda k} \\ b_k &= -C_a (1 - C_a)^{k-1} e^{-\lambda k}. \end{aligned} \quad (2.17)$$

Hence, to satisfy $(\mathbf{H}_{\text{exp}})$, we only need C_a to be such that $\mu := \lambda - \ln |1 - C_a| > 0$ and then for all $k \in \mathbb{N}^*$, we get

$$|b_k| \leq C_b e^{-\mu k} \quad (2.18)$$

with $C_b > 0$ a constant depending on C_a .

Remark 2.8. \triangleright In this setting where everything is computable, it's interesting to see that the asymptotic decrease of the sequence $(|b_k|)$ is not only related to the one of the sequence $(|a_k|)$. For instance, if we take $C_a < 0$, the simple fact that $a_0 > 0$ and $a_k < 0$ for all $k > 0$ makes (b_k) diverge to $+\infty$ and nevertheless, $(|a_k|)$ decreases to 0 at an exponential rate.

\triangleright If we take $C_a = 1$, we can reduce $(\Delta_n)_{n \in \mathbb{Z}}$ to the following induction

$$\forall n \in \mathbb{Z}, \quad \Delta_{n+1} = \xi_{n+1-k} + e^{-\lambda} \Delta_n. \quad (2.19)$$

2.6 Stationary Gaussian sequence of fractional type

Let $H \in (0, 1)$. In the sequel, we will speak about stationary Gaussian sequence of *fractional type* if the sequence (a_k) is such that

$$\forall k \geq 0, \quad |a_k| \leq C_\rho (k+1)^{-\rho} \quad \text{and} \quad |a_k - a_{k+1}| \leq \tilde{C}_\rho (k+1)^{-(\rho+1)} \quad (2.20)$$

with $\rho := 3/2 - H$.

As we will see below, this condition includes the case where $(\Delta_n)_{n \in \mathbb{Z}}$ corresponds to the fractional Brownian motion (fBm) increments. Unfortunately, computing the rate of convergence of the corresponding sequence $(b_k)_{k \geq 0}$ is a hard task and strongly depends on the variations of $(a_k)_{k \geq 0}$. Actually, in Proposition 2.2 and 2.3, dealing with the same order of memory, we will see that the orders of rate of convergence are really different. Note that the first corresponds to the case where $a_k := (k+1)^{-(3/2-H)}$ for all $k \in \mathbb{N}$ whereas the second deals with fBm increments.

Proposition 2.2. *Assume (\mathbf{H}_1) and (\mathbf{H}_2) . Let $H \in (0, 1)$ and set $\rho := 3/2 - H \in (1, 3/2)$. If for all $k \geq 0$, $a_k = (k+1)^{-\rho}$, then (a_k) is of fractional type and we have $|b_k| \leq (k+1)^{-\rho}$. Moreover, if $\rho > 3/4$ Theorem 2.1 (ii) holds with the rate*

$$v(\rho, \rho) = v(3/2 - H, 3/2 - H) = \frac{1}{2} \begin{cases} (1-H)^2 & \text{if } H \in (0, 1/2] \\ (3-4H)^2 & \text{if } H \in (1/2, 3/4). \end{cases}$$

Remark 2.9. *This result follows from the proof of the inequality $|b_k| \leq (k+1)^{-\rho}$ for all $k \geq 0$ which is outlined in Appendix B. The key argument in this proof is the property of log-convexity of the sequence $(a_k)_{k \in \mathbb{N}}$, which means that for all $k \in \mathbb{N}$, $a_k \geq 0$ and for $k \geq 1$, $a_k^2 - a_{k-1}a_{k+1} \leq 0$.*

As mentioned before, the terminology “fractional type” refers to the fractional Brownian motion. Indeed, in a continuous-time setting, a famous and classical example of non-Markovian dynamics is SDE driven by fBm

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \quad (2.21)$$

Recall that a d -dimensional fBm with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $(B_t)_{t \geq 0}$ with stationary increments satisfying

$$\forall t, s \geq 0, \forall i, j \in \{1, \dots, d\}, \quad \mathbb{E} \left[(B_t^i - B_s^i)(B_t^j - B_s^j) \right] = \delta_{ij} |t - s|^{2H}.$$

The study by a coupling argument of the rate of convergence to equilibrium for this kind of dynamics has been undertaken by Hairer [14], Fontbona and Panloup [11], Deya, Panloup and Tindel [9], respectively in the additive noise, multiplicative noise with $H > 1/2$ and multiplicative noise with $H \in (1/3, 1/2)$. Here in our discrete-time setting, we are thus concerned by the long time behavior of (2.1) if we take for $h > 0$

$$(\Delta_n)_{n \in \mathbb{Z}} = (B_{nh} - B_{(n-1)h})_{n \in \mathbb{Z}} \quad (2.22)$$

which is a stationary Gaussian sequence. It can be realized through a moving average representation with coefficients $(a_k^H)_{k \geq 0}$ defined by (see [19]):

$$a_0^H = h^H \kappa(H) 2^{1/2-H} \quad \text{and for } k \geq 1, \quad a_k^H = h^H \kappa(H) \left(\left(k + \frac{1}{2} \right)^{H-1/2} - \left(k - \frac{1}{2} \right)^{H-1/2} \right) \quad (2.23)$$

where

$$\kappa(H) = \frac{\sqrt{\sin(\pi H) \Gamma(2H+1)}}{\Gamma(H+1/2)}.$$

One can easily check that $a_k^H \underset{k \rightarrow +\infty}{\sim} C_{h,H} (k+1)^{-(3/2-H)}$ and $|a_k^H - a_{k+1}^H| \leq C'_{h,H} (k+1)^{-(5/2-H)}$.

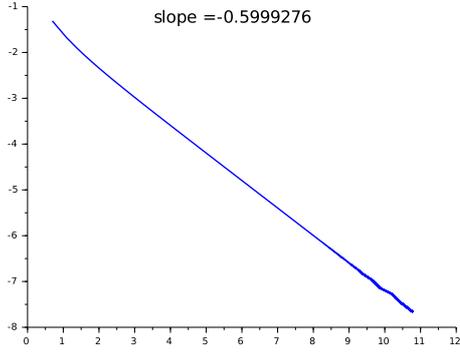
Hence $(a_k^H)_{k \geq 0}$ is of fractional type in the sense of (2.20). Now, the question is: how does the corresponding (b_k^H) behave? When H belongs to $(0, 1/2)$, only a_0^H is non-negative and then (a_k^H) is not log-convex. Therefore, we cannot use this property to get the asymptotic behavior of (b_k^H) as we did in Proposition 2.2. However, thanks to simulations (see Figure 1a and 1b), we conjectured and we proved (see Appendix E) the following proposition.

Proposition 2.3. *There exists $C''_{h,H} > 0$ such that for all $H \in (0, 1/2)$*

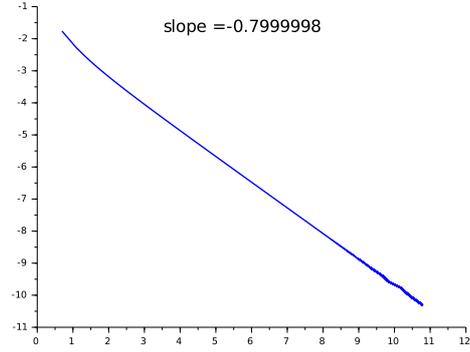
$$\forall k \geq 0, \quad |b_k^H| \leq C''_{h,H} (k+1)^{-(H+1/2)}. \quad (2.24)$$

Then, if we assume (\mathbf{H}_1) and (\mathbf{H}_2) , Theorem 2.1 (ii) holds with the rate

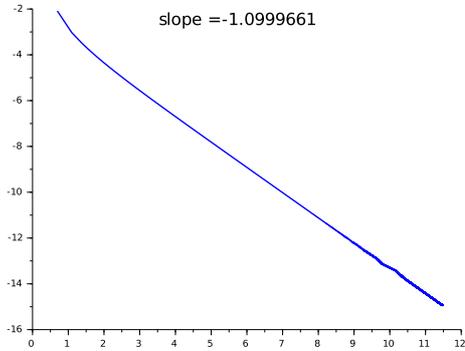
$$v(\rho, 2-\rho) = v(3/2 - H, H + 1/2) = \frac{1}{2} \begin{cases} H(1-2H) & \text{if } H \in (0, 1/4] \\ \frac{1}{8} & \text{if } H \in (1/4, 1/2). \end{cases}$$



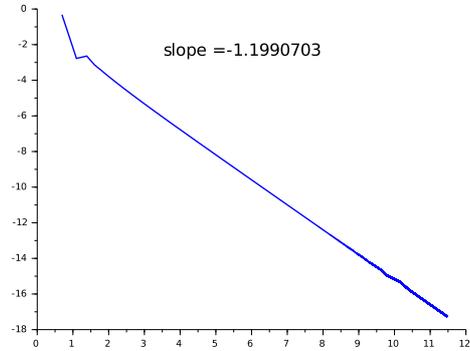
(a) parameter: $H = 0.1$



(b) parameter: $H = 0.3$



(c) parameter: $H = 0.6$



(d) parameter: $H = 0.9$

Figure 1: $(\log |b_k^H|)$ according to $(\log(k+1))$ with different Hurst parameters H .

As concerns the case where H belongs to $(1/2, 1)$, we conjecture that (see Figure 1c and 1d)

Conjecture: There exists $C''_{h,H} > 0$ and $\beta_H > 1$ such that

$$\forall k \geq 0, \quad |b_k^H| \leq C''_{h,H}(k+1)^{-\beta_H}. \quad (2.25)$$

Remark 2.10. We do not have a precise idea of the expression of β_H with respect to H . But, we can note that if $\rho < 1$ and $\beta > 1$ in $(\mathbf{H}_{\text{poly}})$, then the rate of convergence in Theorem 2.1 is $v(\rho, \beta) = \frac{(2\rho-1)^2}{8}$ and does not depend on β . Hence, if $H \in (1/2, 1)$, $\rho = 3/2 - H$ and $\beta_H > 1$, we fall into this case and then the dependence of β_H in terms of H does not matter.

If the previous conjecture is true we get the following rate of convergence for $H \in (1/2, 1)$ in Theorem 2.1:

$$v(\rho, \beta_H) = v(3/2 - H, \beta_H) = \frac{(1-H)^2}{2}.$$

Now, let us briefly discuss about those results. First, Proposition 2.2 and 2.3 indeed confirm that the fluctuations of the sequence $(a_k)_{k \in \mathbb{N}}$ have an impact on $(b_k)_{k \in \mathbb{N}}$ and on the corresponding rate of convergence. Then, let us focus on the specific statements on fBm increments (especially Proposition 2.3 and the

conjecture below) and compare with the continuous time setting. When H belongs to $(0, 1/2)$ Proposition 2.3 gives exactly the same rate of convergence obtained in [14, 11, 9]. However, when $H > 1/2$ if the conjecture is true which seems to be confirmed by simulations, we get a smaller rate than in a continuous time setting. The reason for this may be that Theorem 2.1 is a result with quite general hypothesis on the Gaussian noise process $(\Delta_n)_{n \in \mathbb{Z}}$. In the case of fBm increments, the moving average representation is explicit. Hence, we may use a more specific approach and significantly closer to Hairer's, especially with regard to Step 2 in the coupling method (see Subsection 5.2.2) by not exploiting the technical lemma 5.3 for instance. This seems to be a right track in order to improve our results in this precise example.

We are now ready to begin the proof of Theorem 2.1. In Section 3, we establish a Markovian structure to define invariant distribution and to get the first part of the theorem, i.e. (i). Then, in Section 4 we explain the scheme of coupling before implementing this strategy in Sections 5 and 6. Finally, in Section 7, we achieve the proof of (ii) and (iii) of Theorem 2.1.

3 Existence of invariant distribution

The stochastic dynamics described in (2.1) is clearly non-Markovian. We will see in this section how is it possible to introduce a Markovian structure and then the notion of invariant distribution. This method is inspired by [15].

3.1 Markovian structure

The first idea is now to look at $(X_n, (\Delta_{n+k})_{k \leq 0})_{n \geq 0}$ instead of $(X_n)_{n \geq 0}$. Let us introduce the following concatenation operator

$$\begin{aligned} \sqcup : \mathcal{W} \times \mathbb{R}^d &\rightarrow \mathcal{W} \\ (w, w') &\mapsto w \sqcup w' \end{aligned} \tag{3.1}$$

where $(w \sqcup w')_0 = w'$ and $\forall k < 0, (w \sqcup w')_k = w_{k+1}$. Then (2.1) is equivalent to the system

$$(X_{n+1}, (\Delta_{n+1+k})_{k \leq 0}) = \varphi((X_n, (\Delta_{n+k})_{k \leq 0}), \Delta_{n+1}) \tag{3.2}$$

where

$$\begin{aligned} \varphi : (\mathcal{X} \times \mathcal{W}) \times \mathbb{R}^d &\rightarrow \mathcal{X} \times \mathcal{W} \\ ((x, w), w') &\mapsto (F(x, w'), w \sqcup w'). \end{aligned}$$

Therefore, $(X_n, (\Delta_{n+k})_{k \leq 0})_{n \geq 0}$ can be realized through the Feller Markov transition kernel \mathcal{Q} defined by

$$\int_{\mathcal{W}} g(x', w') \mathcal{Q}((x, w), (dx', dw')) = \int_{\mathbb{R}^d} g(\varphi((x, w), \delta)) \mathcal{P}(w, d\delta). \tag{3.3}$$

where $\mathcal{P}(w, d\delta) := \mathcal{L}(\Delta_{n+1} | (\Delta_{n+k})_{k \leq 0} = w)$ does not depend on n since (Δ_n) is a stationary sequence, and $g : \mathcal{X} \times \mathcal{W} \rightarrow \mathbb{R}$ is a measurable function.

Definition 3.1. A measure $\mu \in \mathcal{M}_1(\mathcal{X} \times \mathcal{W})$ is said to be an invariant distribution for (3.2) and then for (2.1) if it is invariant by \mathcal{Q} , i.e.

$$\mathcal{Q}\mu = \mu.$$

However, the concept of uniqueness will be slightly different from the classical setting. Indeed, denote by $\mathcal{S}\mu$ the distribution of $(X_n^\mu)_{n \geq 0}$ when we realize $(X_n^\mu, (\Delta_{n+k})_{k \leq 0})_{n \geq 0}$ through the transition \mathcal{Q} and with initial distribution μ . Then, we will speak of uniqueness of the invariant distribution up to the equivalence relation: $\mu \sim \nu \iff \mathcal{S}\mu = \mathcal{S}\nu$.

Moreover, here uniqueness will be deduced from the coupling procedure. There exist some results about uniqueness using ergodic theory, like in [15], but they will be not outlined here.

3.2 Lyapunov condition

Denote by \mathbb{P}_w the law of $(\Delta_k)_{k \leq 0}$. Since $(\Delta_n)_{n \in \mathbb{Z}}$ is stationary we immediately get the following property:

Property 3.1. *If a measure $\mu \in \mathcal{M}_1(\mathcal{X} \times \mathcal{W})$ is such that $\Pi_{\mathcal{W}}^* \mu = \mathbb{P}_w$ then $\Pi_{\mathcal{W}}^* \mathcal{Q} \mu = \mathbb{P}_w$.*

We can now define the notion of Lyapunov function.

Definition 3.2. *A function $\psi : \mathcal{X} \rightarrow [0, +\infty)$ is called a Lyapunov function for \mathcal{Q} if ψ is continuous and if the following holds:*

(i) $\psi^{-1}([0, a])$ is compact for all $a \in [0, +\infty)$.

(ii) $\exists \beta > 0$ and $\alpha \in (0, 1)$ such that:

$$\int_{\mathcal{X} \times \mathcal{W}} \psi(x) \mathcal{Q} \mu(dx, dw) \leq \beta + \alpha \int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* \mu)(dx)$$

for all $\mu \in \mathcal{M}_1(\mathcal{X} \times \mathcal{W})$ such that $\Pi_{\mathcal{W}}^* \mu = \mathbb{P}_w$ and $\int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* \mu)(dx) < +\infty$.

The following result ensures the existence of invariant distribution for \mathcal{Q} .

Theorem 3.1. *If there exists a Lyapunov function ψ for \mathcal{Q} , then \mathcal{Q} has at least one invariant distribution μ_* , in other words $\mathcal{Q} \mu_* = \mu_*$.*

A detailed proof of this result is given in Appendix C. Finally, we get the first part (i) of Theorem 2.1 about the existence of an invariant distribution by setting $\psi := V$ (with V the function appearing in **(H₁)**) and by saying that ψ is a Lyapunov function for \mathcal{Q} .

4 General coupling procedure

We now turn to the proof of the main result of the paper, i.e. Theorem 2.1 (ii) and (iii) about the convergence in total variation. This result is based on a coupling method first introduced in [14], but also used in [11] and [9], in a continuous time framework. The coupling strategy is slightly different in our discrete context, the following part is devoted to explain this procedure.

4.1 Scheme of coupling

Let $(\Delta_n^1)_{n \in \mathbb{Z}}$ and $(\Delta_n^2)_{n \in \mathbb{Z}}$ be two stationary and purely non-deterministic Gaussian sequences with the following moving average representations

$$\begin{cases} \Delta_n^1 = \sum_{k=0}^{+\infty} a_k \xi_{n-k}^1 \\ \Delta_n^2 = \sum_{k=0}^{+\infty} a_k \xi_{n-k}^2 \end{cases}$$

with

$$\begin{cases} (a_k)_{k \geq 0} \in \mathbb{R}^{\mathbb{N}} \text{ such that } a_0 = 1 \text{ and } \sum_{k=0}^{+\infty} a_k^2 < +\infty \\ \xi^i := (\xi_k^i)_{k \in \mathbb{Z}} \text{ an i.i.d sequence such that } \xi_1^i \sim \mathcal{N}(0, I_d) \text{ for } i = 1, 2. \end{cases} \quad (4.1)$$

We denote by (X^1, X^2) the solution of the system:

$$\begin{cases} X_{n+1}^1 = F(X_n^1, \Delta_{n+1}^1) \\ X_{n+1}^2 = F(X_n^2, \Delta_{n+1}^2) \end{cases} \quad (4.2)$$

with initial conditions $(X_0^1, (\Delta_k^1)_{k \leq 0})$ and $(X_0^2, (\Delta_k^2)_{k \leq 0})$. We assume that $(X_0^2, (\Delta_k^2)_{k \leq 0}) \sim \mu_*$ where μ_* denotes a fixed invariant distribution associated to (2.1). The previous section ensures that such a measure exists. We define the natural filtration associated to (4.2) by

$$(\mathcal{F}_n)_{n \in \mathbb{N}} = (\sigma((\xi_k^1)_{k \leq n}, (\xi_k^2)_{k \leq n}, X_0^1, X_0^2))_{n \geq 0}.$$

To lower the “weight of the past” at the beginning of the coupling procedure, we assume that a.s.,

$$(\Delta_k^1)_{k \leq 0} = (\Delta_k^2)_{k \leq 0}$$

which is actually equivalent to assume that a.s. $(\xi_k^1)_{k \leq 0} = (\xi_k^2)_{k \leq 0}$ since the invertible Toeplitz operator defined in Subsection 2.2 links $(\Delta_k^i)_{k \leq 0}$ to $(\xi_k^i)_{k \leq 0}$ for $i = 1, 2$. Lastly, we denote by $(g_n)_{n \in \mathbb{Z}}$ and $(f_n)_{n \in \mathbb{Z}}$ the random variable sequences defined by

$$\xi_{n+1}^1 = \xi_{n+1}^2 + g_n \quad \text{and} \quad \Delta_{n+1}^1 = \Delta_{n+1}^2 + f_n. \quad (4.3)$$

They respectively represent the “drift” between the underlying noises (ξ_k^i) and the real noises (Δ_k^i) . By assumption, we have $g_n = f_n = 0$ for $n < 0$.

Remark 4.1. *From the moving average representations, we deduce immediately the following relation for all $n \geq 0$,*

$$f_n = \sum_{k=0}^{+\infty} a_k g_{n-k} = \sum_{k=0}^n a_k g_{n-k} \quad (4.4)$$

The aim is now to build $(g_n)_{n \geq 0}$ and $(f_n)_{n \geq 0}$ in order to stick X^1 and X^2 . We set

$$\tau_\infty = \inf\{n \geq 0 \mid X_k^1 = X_k^2, \forall k \geq n\}.$$

In a purely Markovian setting, when the paths coincide at time n then they remain stuck for all $k \geq n$ by putting the same innovation into both processes. Due to the memory this phenomenon cannot happen here. Hence, this involves a new step in the coupling scheme: try to keep the paths fastened together (see below).

Recall that $\mathcal{L}((X_k^2)_{k \geq n}) = \mathcal{S}\mu_\star$. The purpose of the coupling procedure is to bound the quantity $\mathbb{P}(\tau_\infty > n)$ since by a classical result we have

$$\|\mathcal{L}((X_k^1)_{k \geq n}) - \mathcal{S}\mu_\star\|_{TV} \leq \mathbb{P}(\tau_\infty > n). \quad (4.5)$$

Hence, we realize the coupling after a series of trials which follows three steps:

- * **Step 1:** Try to stick the positions at a given time with a “controlled cost”.
- * **Step 2:** (specific to non-Markov processes) Try to keep the paths fastened together.
- * **Step 3:** If Step 2 fails, we wait long enough so as to allow Step 1 to be realized with a “controlled cost” and with a non-negative probability. During this step, we assume that $g_n = 0$.

More precisely, let us introduce some notations,

- Let $\tau_0 \geq 0$. We begin the first trial at time $\tau_0 + 1$, in other words we try to stick $X_{\tau_0+1}^1$ and $X_{\tau_0+1}^2$. Hence, we assume that

$$\forall n < \tau_0, \quad g_n = f_n = 0. \quad (4.6)$$

- For $j \geq 1$, let τ_j denote the end of trial j . More specifically,
 - ▷ If $\tau_j = +\infty$ for some $j \geq 1$, it means that the coupling tentative has been successful.
 - ▷ Else, τ_j corresponds to the end of Step 3, that is $\tau_j + 1$ is the beginning of Step 1 of trial $j + 1$.

The real meaning of “controlled cost” will be clarified on Subsection 5.1. But the main idea is that at Step 1 of trial j , the “cost” is represented by the quantity $g_{\tau_{j-1}}$ that we need to build to get $X_{\tau_{j-1}+1}^1 = X_{\tau_{j-1}+1}^2$ with non-negative probability. Here the cost does not only depend on the positions at time τ_{j-1} but also on all the past of the underlying noises ξ^1 and ξ^2 . Hence, we must have a control on $g_{\tau_{j-1}}$ in case of failure and to this end we have to wait enough during Step 3 before beginning a new attempt of coupling.

4.2 Coupling lemmas to achieve Step 1 and 2

This section is devoted to establish coupling lemmas in order to build (ξ^1, ξ^2) during Step 1 and Step 2.

4.2.1 Hitting step

If we want to stick X^1 and X^2 at time $n+1$, we need to build $(\xi_{n+1}^1, \xi_{n+1}^2)$ in order to get $F(X_n^1, \Delta_{n+1}^1) = F(X_n^2, \Delta_{n+1}^2)$ with non-negative probability, that is to get

$$\begin{aligned} F\left(X_n^1, a_0 \xi_{n+1}^1 + \sum_{k=1}^{+\infty} a_k \xi_{n+1-k}^1\right) &= F\left(X_n^2, a_0 \xi_{n+1}^2 + \sum_{k=1}^{+\infty} a_k \xi_{n+1-k}^2\right) \\ \iff \tilde{F}\left(X_n^1, \xi_{n+1}^1, \sum_{k=1}^{+\infty} a_k \xi_{n+1-k}^1\right) &= \tilde{F}\left(X_n^2, \xi_{n+1}^2, \sum_{k=1}^{+\infty} a_k \xi_{n+1-k}^2\right) \end{aligned} \quad (4.7)$$

since $a_0 = 1$. The following lemma will be the main tool to achieve this goal.

Lemma 4.1. *Let $K > 0$ and $\mu := \mathcal{N}(0, I_d)$. Under the controllability assumption **(H₂)**, there exists $\tilde{K} > 0$ (given by **(H₂)**), such that for every $\mathbf{x} := (x, x', y, y')$ in $B(0, K)^4$, we can build a random variable (Z_1, Z_2) with values in $(\mathbb{R}^d)^2$ such that*

(i) $\mathcal{L}(Z_1) = \mathcal{L}(Z_2) = \mu$,

(ii) there exists $\delta_{\tilde{K}} > 0$ depending only on \tilde{K} such that

$$\mathbb{P}(\tilde{F}(x, Z_1, y) = \tilde{F}(x', Z_2, y')) \geq \mathbb{P}(Z_2 = \Lambda_{\mathbf{x}}(Z_1), |Z_1| \leq \tilde{K}) \geq \delta_{\tilde{K}} > 0 \quad (4.8)$$

where $\Lambda_{\mathbf{x}}$ is the function given by hypothesis **(H₂)**,

(iii) there exists $M_K > 0$ depending only on K such that

$$\mathbb{P}(|Z_2 - Z_1| \leq M_K) = 1. \quad (4.9)$$

Proof. Let $\mathbf{x} := (x, x', y, y') \in B(0, K)^4$. First, let us denote by π_1 (resp. π_2) the projection from $\mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R}^d of the first (resp. the second) coordinate. Introduce the two following functions defined on \mathbb{R}^d

$$\begin{aligned} \Lambda_1 &: u_1 \mapsto (u_1, \Lambda_{\mathbf{x}}(u_1)) \\ \Lambda_2 &: u_2 \mapsto (\Lambda_{\mathbf{x}}^{-1}(u_2), u_2) \end{aligned}$$

where $\Lambda_{\mathbf{x}}$ is the function given by **(H₂)**. Now, we set

$$\mathbf{P}_1 = \frac{1}{2}(\Lambda_1^* \mu \wedge \Lambda_2^* \mu).$$

Let us find a simplest expression for \mathbf{P}_1 . For every measurable function $f : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$, we have

$$\Lambda_1^* \mu(f) = \int_{\mathbb{R}^d} f(u_1, \Lambda_{\mathbf{x}}(u_1)) \mu(du_1) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(u_1, u_2) \delta_{\Lambda_{\mathbf{x}}(u_1)}(du_2) \mu(du_1)$$

and

$$\begin{aligned}
\Lambda_2^* \mu(f) &= \int_{\mathbb{R}^d} f(\Lambda_{\mathbf{x}}^{-1}(u_2), u_2) \mu(du_2) \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\Lambda_{\mathbf{x}}^{-1}(u_2), u_2) \exp\left(-\frac{|u_2|^2}{2}\right) du_2 \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(u_1, \Lambda_{\mathbf{x}}(u_1)) \exp\left(-\frac{|\Lambda_{\mathbf{x}}(u_1)|^2}{2}\right) |\det(J_{\Lambda_{\mathbf{x}}}(u_1))| du_1 \quad (\text{by setting } u_1 = \Lambda_{\mathbf{x}}^{-1}(u_2)) \\
&= \int_{\mathbb{R}^d} f(u_1, \Lambda_{\mathbf{x}}(u_1)) \underbrace{\exp\left(\frac{|u_1|^2}{2} - \frac{|\Lambda_{\mathbf{x}}(u_1)|^2}{2}\right) |\det(J_{\Lambda_{\mathbf{x}}}(u_1))|}_{=: D_{\Lambda_{\mathbf{x}}}(u_1)} \mu(du_1) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} f(u_1, u_2) \delta_{\Lambda_{\mathbf{x}}(u_1)}(du_2) D_{\Lambda_{\mathbf{x}}}(u_1) \mu(du_1).
\end{aligned}$$

By construction, we then have

$$\mathbf{P}_1(du_1, du_2) = \frac{1}{2} \delta_{\Lambda_{\mathbf{x}}(u_1)}(du_2) (D_{\Lambda_{\mathbf{x}}}(u_1) \wedge 1) \mu(du_1). \quad (4.10)$$

Write $S(u_1, u_2) = (u_2, u_1)$ and denote by $\tilde{\mathbf{P}}_1$ the ‘‘symmetrized’’ non-negative measure induced by \mathbf{P}_1 ,

$$\tilde{\mathbf{P}}_1 = \mathbf{P}_1 + S^* \mathbf{P}_1. \quad (4.11)$$

We then define (Z_1, Z_2) as follows:

$$\mathcal{L}(Z_1, Z_2) = \tilde{\mathbf{P}}_1 + \Delta^*(\mu - \pi_1^* \tilde{\mathbf{P}}_1) = \mathbf{P}_1 + \mathbf{P}_2 \quad (4.12)$$

with $\Delta(u) = (u, u)$ and $\mathbf{P}_2 = S^* \mathbf{P}_1 + \Delta^*(\mu - \pi_1^* \tilde{\mathbf{P}}_1)$. It remains to prove that $\mathcal{L}(Z_1, Z_2)$ is well defined and satisfies all the properties required by the lemma.

First step: Prove that \mathbf{P}_2 is the sum of two positive measures.

Using (4.10), we can check that for all non-negative function f ,

$$\pi_1^* \mathbf{P}_1(f) \leq \frac{1}{2} \mu(f)$$

and

$$\pi_2^* \mathbf{P}_1(f) = \pi_1^*(S^* \mathbf{P}_1)(f) \leq \frac{1}{2} \mu(f).$$

By adding the two previous inequalities, we deduce that the measure $\mu - \pi_1^* \tilde{\mathbf{P}}_1$ is positive. This concludes the first step.

Second step: Prove that $\pi_1^*(\mathbf{P}_1 + \mathbf{P}_2) = \pi_2^*(\mathbf{P}_1 + \mathbf{P}_2) = \mu$.

This fact is almost obvious. We just need to use the fact that

$$\pi_1 \circ \Delta = \pi_2 \circ \Delta = \text{Id}$$

and the symmetry property of $\tilde{\mathbf{P}}_1$,

$$\text{i.e. } \pi_1^* \tilde{\mathbf{P}}_1 = \pi_2^* \tilde{\mathbf{P}}_1.$$

Third step: Prove (4.8) and (4.9).

Let us first remark that the support of $\mathbf{P}_1 + \mathbf{P}_2$ is included in

$$\{(u, v) \in \mathbb{R}^d \times \mathbb{R}^d \mid v = \Lambda_{\mathbf{x}}(u)\} \cup \{(u, v) \in \mathbb{R}^d \times \mathbb{R}^d \mid v = \Lambda_{\mathbf{x}}^{-1}(u)\} \cup \{(u, v) \in \mathbb{R}^d \times \mathbb{R}^d \mid v = u\}.$$

Therefore, by (2.8) in (\mathbf{H}_2) and the fact that

$$(\forall u \in \mathbb{R}^d, |\Lambda_{\mathbf{x}}(u) - u| \leq M_K) \iff (\forall u \in \mathbb{R}^d, |\Lambda_{\mathbf{x}}^{-1}(u) - u| \leq M_K)$$

since $\Lambda_{\mathbf{x}}$ is invertible on \mathbb{R}^d , we finally get (4.9).

Then, using again (\mathbf{H}_2) where \tilde{K} is defined and the definition of the subprobability \mathbf{P}_1 we get

$$\mathbb{P}(\tilde{F}(x, Z_1, y) = \tilde{F}(x', Z_2, y')) \geq \underbrace{\mathbf{P}_1(B(0, \tilde{K}) \times \Lambda(B(0, \tilde{K})))}_{=\mathbb{P}(Z_2 = \Lambda_{\mathbf{x}}(Z_1), |Z_1| \leq \tilde{K})} \quad (4.13)$$

and

$$\mathbb{P}(Z_2 = \Lambda_{\mathbf{x}}(Z_1), |Z_1| \leq \tilde{K}) = \frac{1}{2} \int_{B(0, \tilde{K})} (D_{\Lambda_{\mathbf{x}}}(u) \wedge 1) \mu(du).$$

It just remains to use (2.7) and (2.8) in (\mathbf{H}_2) to conclude. Indeed,

$$\begin{aligned} \mathbb{P}(Z_2 = \Lambda_{\mathbf{x}}(Z_1), |Z_1| \leq \tilde{K}) &= \frac{1}{2} \int_{B(0, \tilde{K})} \left(\exp\left(\frac{|u|^2}{2} - \frac{|\Lambda_{\mathbf{x}}(u)|^2}{2}\right) |\det(J_{\Lambda_{\mathbf{x}}}(u))| \right) \wedge 1 \mu(du) \\ &\geq \frac{1}{2} \mu(B(0, \tilde{K})) \left[\left(\exp\left(-\frac{(M_{\tilde{K}} + \tilde{K})^2}{2}\right) C_{\tilde{K}} \right) \wedge 1 \right] =: \delta_{\tilde{K}} > 0 \end{aligned}$$

which concludes the proof. \square

4.2.2 Sticking step

Now, if the positions X_{n+1}^1 and X_{n+1}^2 are stuck together, we want that they remain fastened together for all $k > n + 1$ which means that:

$$\begin{aligned} \forall k \geq n + 1, \quad F(X_k^1, \Delta_{k+1}^1) &= F(X_k^2, \Delta_{k+1}^2) \\ \iff \forall k \geq n + 1, \quad F(X_k^1, \xi_{k+1}^1 + \sum_{l=1}^{+\infty} a_l \xi_{k+1-l}^1) &= F(X_k^1, \xi_{k+1}^2 + \sum_{l=1}^{+\infty} a_l \xi_{k+1-l}^2) \end{aligned} \quad (4.14)$$

since $X_k^1 = X_k^2$ and $a_0 = 1$. Recall that for all $k \in \mathbb{Z}$, $g_k = \xi_{k+1}^1 - \xi_{k+1}^2$ is the drift between the underlying noises. Then, if we have

$$\begin{aligned} \forall k \geq n + 1, \quad \xi_{k+1}^1 + \sum_{l=1}^{+\infty} a_l \xi_{k+1-l}^1 &= \xi_{k+1}^2 + \sum_{l=1}^{+\infty} a_l \xi_{k+1-l}^2 \\ \iff \forall k \geq n + 1, \quad g_k &= - \sum_{l=1}^{+\infty} a_l g_{k-l} \end{aligned} \quad (4.15)$$

the identity (4.14) is automatically satisfied.

Remark 4.2. *The successful g_k defined by relation (4.15) is \mathcal{F}_k -measurable. This explains why we chose to index it by k even if it represents the drift between ξ_{k+1}^1 and ξ_{k+1}^2 .*

Hence, we will try to get (4.15) on successive finite intervals to finally get a bound on the successful-coupling probability. The size choice of those intervals will be important according to the hypothesis $(\mathbf{H}_{\text{poly}})$ or $(\mathbf{H}_{\text{exp}})$ that we made. The two next results will be our tools to get (4.15) on Subsection 5.2. For the sake of simplicity we set out these results on \mathbb{R} . On \mathbb{R}^d we just have to apply them on every marginal. Lemma 4.2 is almost the statement of Lemma 5.13 of [14] or Lemma 3.2 of [11].

Lemma 4.2. Let $\mu := \mathcal{N}(0, 1)$. Let $a \in \mathbb{R}$, $b \geq |a|$ and $M_b := \max(4b, -2 \log(b/8))$.

(i) For all $b \geq |a|$, there exist δ_b^1 and $\delta_b^2 \in (0, 1)$, such that we can build a probability measure $\mathcal{N}_{a,b}^2$ on \mathbb{R}^2 with every marginal equal to μ and such that:

$$\mathcal{N}_{a,b}^2(\{(x, y) \mid y = x + a\}) \geq \delta_b^1 \quad \text{and} \quad \mathcal{N}_{a,b}^2(\{(x, y) \mid |y - x| \leq M_b\}) = 1.$$

(ii) Moreover, if $b \in (0, 1)$, the previous statement holds with $\delta_b^1 = 1 - b$.

The following corollary is an adapted version of Lemma 3.3 of [11] to our discrete context.

Corollary 4.1. Let $T > 0$ be an integer, $b > 0$, $g = (g_0, g_1, \dots, g_T) \in \mathbb{R}^{T+1}$ such that $\|g\| \leq b$ where $\|\cdot\|$ is the euclidian norm on \mathbb{R}^{T+1} and set $M_b := \max(4b, -2 \log(b/8))$.

(i) Then, there exists $\delta_b^1 \in (0, 1)$, for which we can build a random variable $((\xi_{k+1}^1)_{k \in \llbracket 0, T \rrbracket}, (\xi_{k+1}^2)_{k \in \llbracket 0, T \rrbracket})$ with values in $(\mathbb{R}^{T+1})^2$, with distribution $\mathcal{N}(0, I_{T+1})$ and satisfying:

$$\mathbb{P}(\xi_{k+1}^1 = \xi_{k+1}^2 + g_k \quad \forall k \in \llbracket 0, T \rrbracket) \geq \delta_b^1$$

and

$$\mathbb{P}(\|\xi^1 - \xi^2\| \leq M_b) = 1.$$

(ii) Moreover, if $b \in (0, 1)$, the previous statement holds with $\delta_b^1 = 1 - b$.

Proof. Let $(u_k)_{k \in \llbracket 0, T \rrbracket}$ be an orthonormal basis of \mathbb{R}^{T+1} with $u_0 = \frac{g}{\|g\|}$. We denote by (U_1, U_2) a random variable which has distribution $\mathcal{N}_{a,b}^2$ (with $a = \|g\|$) given in the lemma 4.2. Let $(\varepsilon_k)_{k \in \llbracket 1, T \rrbracket}$ be an iid random variable sequence with $\varepsilon_1 \sim \mathcal{N}(0, 1)$ and independent from (U_1, U_2) . Then, for $i = 1, 2$ we define the isometry:

$$\begin{aligned} \mathcal{W}^i : \mathbb{R}^{T+1} &\rightarrow \mathcal{W}^i(\mathbb{R}^{T+1}) \subset L^2(\Omega, \mathcal{F}, \mathbb{P}) \\ u_0 &\mapsto U_i \\ u_k &\mapsto \varepsilon_k \quad \text{for } k \in \llbracket 1, T \rrbracket. \end{aligned} \tag{4.16}$$

And we set for all $n \in \llbracket 0, T \rrbracket$, $\xi_{n+1}^i := \mathcal{W}^i(e_n)$ where e_n is the vector of \mathbb{R}^{T+1} for which every coordinate is 0 except the n^{th} which is 1. Since $(u_k)_{k \in \llbracket 0, T \rrbracket}$ is an orthonormal basis of \mathbb{R}^{T+1} , we then have:

$$e_n = \sum_{k=0}^T \langle e_n, u_k \rangle u_k.$$

Hence,

$$\xi_{n+1}^i = \mathcal{W}^i \left(\sum_{k=0}^T \langle e_n, u_k \rangle u_k \right) = U_i \frac{g_n}{\|g\|} + \sum_{k=1}^T \langle e_n, u_k \rangle \varepsilon_k.$$

ξ_{n+1}^i is clearly centered and Gaussian as a linear combination of independent centered Gaussian random variables and using that \mathcal{W}^i is an isometry, we get that $(\xi_{k+1}^i)_{k \in \llbracket 0, T \rrbracket}$ has distribution $\mathcal{N}(0, I_{T+1})$ for $i = 1, 2$. Therefore, we built ξ^1 and ξ^2 as announced. Indeed, by Lemma 4.2

$$\mathbb{P}(\xi_{n+1}^1 = \xi_{n+1}^2 + g_n \quad \forall n \in \llbracket 0, T \rrbracket) = \mathbb{P}(U_1 = U_2 + \|g\|) \geq \delta_b^1$$

and

$$\mathbb{P}(\|\xi^1 - \xi^2\| \leq M_b) = \mathbb{P}(|U_1 - U_2| \leq M_b) = 1.$$

(ii) also follows immediately from Lemma 4.2. □

5 Coupling under $(\mathbf{H}_{\text{poly}})$ or $(\mathbf{H}_{\text{exp}})$

We can now move on the real coupling procedure to finally get a lower-bound for the successful-coupling probability. In a first subsection, we explain exactly what we called ‘‘controlled cost’’ and in a second subsection we spell out our bound.

5.1 Admissibility condition

The “controlled cost” is called “admissibility” in [14]. Here, we will talk about (K, α) -admissibility, as in [11], but in the following sense:

Definition 5.1. *Let $K > 0$ and $\alpha > 0$ be two constants and τ a random variable with values in \mathbb{N} . We say that the system is (K, α) -admissible at time τ if $\tau(\omega) < +\infty$ and if $(X_\tau^1(\omega), X_\tau^2(\omega), (\xi_n^1(\omega), \xi_n^2(\omega))_{n \leq \tau})$ satisfies*

$$\forall n \geq 0, \left| \sum_{k=n+1}^{+\infty} a_k g_{\tau+n-k}(\omega) \right| \leq v_n \quad (5.1)$$

and

$$|X_\tau^i(\omega)| \leq K, \quad \left| \sum_{k=1}^{+\infty} a_k \xi_{\tau+1-k}^i(\omega) \right| \leq K \quad \text{for } i = 1, 2 \quad (5.2)$$

with

$$v_n = (n+1)^{-\alpha} \text{ under } (\mathbf{H}_{\text{poly}}) \quad \text{and} \quad v_n = e^{-\alpha n} \text{ under } (\mathbf{H}_{\text{exp}}). \quad (5.3)$$

Remark 5.1. *On the one hand, condition (5.1) measures the distance between the past of the noises (before time τ). On the other hand, condition (5.2) has two parts: the first one ensures that at time τ both processes are not far from each other and the second part is a constraint on the memory part of the Gaussian noise $\Delta_{\tau+1}^i$.*

The aim is to prove that under those two conditions, the coupling will be successful with a probability lower-bounded by a non-negative constant. To this end, we will need to ensure that at every time τ_j , the system will be (K, α) -admissible with non-negative probability. We set:

$$\Omega_{\alpha, \tau}^1 := \left\{ \omega, \tau(\omega) < +\infty, \left| \sum_{k=n+1}^{+\infty} a_k g_{\tau+n-k}(\omega) \right| \leq v_n \quad \forall n \in \mathbb{N} \right\} \quad (5.4)$$

and

$$\Omega_{K, \tau}^2 := \left\{ \omega, \tau(\omega) < +\infty, |X_\tau^i(\omega)| \leq K \text{ and } \left| \sum_{k=1}^{+\infty} a_k \xi_{\tau+1-k}^i(\omega) \right| \leq K \quad \text{for } i = 1, 2 \right\}. \quad (5.5)$$

We define

$$\Omega_{K, \alpha, \tau} = \Omega_{\alpha, \tau}^1 \cap \Omega_{K, \tau}^2. \quad (5.6)$$

If $\omega \in \Omega_{K, \alpha, \tau}$, we will try to couple at time $\tau+1$. Otherwise, we say that Step 1 fails and one begins Step 3. Hence, Step 1 of trial j has two ways to fail: either ω belongs to $\Omega_{K, \alpha, \tau_{j-1}}^c$ and one moves directly to Step 3 or ω belongs to $\Omega_{K, \alpha, \tau_{j-1}}$, one tries to couple and it fails.

5.2 Lower-bound for the successful-coupling probability

The main purpose of this subsection is to get a non-negative lower-bound for the successful-coupling probability which will be independent of j (the number of the tentative), in other words we want to prove the following proposition

Proposition 5.1. *Assume (\mathbf{H}_1) and (\mathbf{H}_2) . Let $K > 0$, $\alpha > \frac{1}{2} \vee (\frac{3}{2} - \beta)$ if we are under $(\mathbf{H}_{\text{poly}})$ and $\alpha > 0$ different from μ if we are under $(\mathbf{H}_{\text{exp}})$. In both cases, there exists δ_0 in $(0, 1)$ such that for all $j \geq 1$,*

$$\delta_0 \leq \mathbb{P}(\Delta\tau_j = +\infty | \Omega_{K, \alpha, \tau_{j-1}}) \quad (5.7)$$

where $\Delta\tau_j := \tau_j - \tau_{j-1}$ and τ_j is defined in Subsection 4.1 as the end of trial j .

Another result will appear in this subsection: a lower-bound (independent from j) of the failure-coupling probability, that is we can choose $\delta_1 \in (0, 1)$ such that

$$\forall j \geq 1, \quad \delta_1 \leq \mathbb{P}(\tau_j < \infty | \tau_{j-1} < \infty). \quad (5.8)$$

This result may appear of weak interest but will be of first importance in Subsection 6.2 and we will get this bound thanks to Step 1, that is why we talk about this here.

5.2.1 Step 1 (hitting step)

Lemma 5.1. *Let $K > 0$ and $\alpha > 0$. Assume **(H₁)** and **(H₂)**. Let $\tilde{K} > 0$ be the constant appearing in **(H₂)**, $\delta_1 \in (0, 1)$ and τ be a stopping time with respect to $(\mathcal{F}_n)_{n \in \mathbb{Z}}$ such that $\mathbb{P}(\Omega_{K, \alpha, \tau}) > 0$. We can build $(\xi_{\tau+1}^1, \xi_{\tau+1}^2)$ with $\xi_{\tau+1}^1 \sim \mathcal{N}(0, I_d)$ and $\xi_{\tau+1}^2 \sim \mathcal{N}(0, I_d)$ such that*

(i) *There exist $K_1 \in (0, \tilde{K}]$ and $\delta_{K_1} \in (0, 1)$ such that*

$$\mathbb{P}(X_{\tau+1}^1 = X_{\tau+1}^2 | \Omega_{K, \alpha, \tau}) \geq \mathbb{P}(\xi_{\tau+1}^2 = \Lambda_{\mathbf{x}}(\xi_{\tau+1}^1), |\xi_{\tau+1}^1| \leq K_1 | \Omega_{K, \alpha, \tau}) \geq \delta_{K_1} > 0 \quad (5.9)$$

and

$$\mathbb{P}(\Omega_{K, \alpha, \tau}^c \cup (\{\xi_{\tau+1}^2 \neq \Lambda_{\mathbf{x}}(\xi_{\tau+1}^1) \text{ or } |\xi_{\tau+1}^1| > K_1\} \cap \Omega_{K, \alpha, \tau})) \geq \delta_1 \quad (5.10)$$

where $\mathbf{x} := (X_{\tau}^1, X_{\tau}^2, \sum_{k=0}^{+\infty} a_k \xi_{\tau+1-k}^1, \sum_{k=0}^{+\infty} a_k \xi_{\tau+1-k}^2)$ and $\Lambda_{\mathbf{x}}$ comes from **(H₂)**.

(ii) *There exists $M_K > 0$ such that*

$$|g_{\tau}| = |\xi_{\tau+1}^1 - \xi_{\tau+1}^2| \leq M_K \quad \text{a.s.}$$

Remark 5.2. *The constant δ_1 is chosen independently from K and α .*

Before proving this result, let us explain a bit why we add the lower-bound (5.10). As we already said, we will see further (in Subsection 6.2) that we need a (uniform) bound on the failure-coupling probability $\mathbb{P}(\tau_j < \infty | \tau_{j-1} < \infty)$ for every $j \geq 1$. Therefore, for every $j \geq 1$, we will consider that Step 1 of trial j fails if and only if $\omega \in \Omega_{K, \alpha, \tau_{j-1}}^c \cup (\{\xi_{\tau_{j-1}+1}^2 \neq \Lambda_{\mathbf{x}}(\xi_{\tau_{j-1}+1}^1) \text{ or } |\xi_{\tau_{j-1}+1}^1| > K_1\} \cap \Omega_{K, \alpha, \tau_{j-1}})$ and in this case one immediately begins Step 3. Hence, for all $j \geq 1$, thanks to Lemma 5.1 we get the existence of K_1 such that:

$$\begin{aligned} & \mathbb{P}(\tau_j < \infty | \tau_{j-1} < \infty) \\ & \geq \mathbb{P}\left(\Omega_{K, \alpha, \tau_{j-1}}^c \cup \left(\{\xi_{\tau_{j-1}+1}^2 \neq \Lambda_{\mathbf{x}}(\xi_{\tau_{j-1}+1}^1) \text{ or } |\xi_{\tau_{j-1}+1}^1| > K_1\} \cap \Omega_{K, \alpha, \tau_{j-1}}\right)\right) \geq \delta_1. \end{aligned}$$

This construction may seem artificial but it is necessary to prove Proposition 6.2. Moreover, this has no impact on the computation of the rate of convergence to equilibrium since it only affects Step 1. We can now move on the proof of Lemma 5.1.

Proof. (i) Set $\mathbf{x} := (X_{\tau}^1, X_{\tau}^2, \sum_{k=1}^{+\infty} a_k \xi_{\tau+1-k}^1, \sum_{k=1}^{+\infty} a_k \xi_{\tau+1-k}^2)$. Conditionally to $\Omega_{K, \alpha, \tau}$ we have $\mathbf{x} \in B(0, K)^4$ and we can build (Z_1, Z_2) as in Lemma 4.1. Let $\xi \sim \mathcal{N}(0, 1)$ be independent from (Z_1, Z_2) and set

$$(\xi_{\tau+1}^1, \xi_{\tau+1}^2) = (\mathbb{1}_{\Omega_{K, \alpha, \tau}} Z_1 + \mathbb{1}_{\Omega_{K, \alpha, \tau}^c} \xi, \mathbb{1}_{\Omega_{K, \alpha, \tau}} Z_2 + \mathbb{1}_{\Omega_{K, \alpha, \tau}^c} \xi). \quad (5.11)$$

Therefore, we deduce by Lemma 4.1 and its proof that for all $K_1 \in (0, \tilde{K}]$,

$$\mathbb{P}(X_{\tau+1}^1 = X_{\tau+1}^2 | \Omega_{K, \alpha, \tau}) \geq \underbrace{\mathbb{P}(Z_2 = \Lambda_{\mathbf{x}}(Z_1), |Z_1| \leq K_1 | \Omega_{K, \alpha, \tau})}_{=\mathbb{P}(\xi_{\tau+1}^2 = \Lambda_{\mathbf{x}}(\xi_{\tau+1}^1), |\xi_{\tau+1}^1| \leq K_1 | \Omega_{K, \alpha, \tau})} \geq \delta_{K_1} > 0. \quad (5.12)$$

And the first part of (i) is proven. It remains to choose the good $K_1 \in (0, \tilde{K}]$ to get the second part. Set $p_K := \mathbb{P}(\Omega_{K, \alpha, \tau})$ and $\mu := \mathcal{N}(0, I_d)$, then

$$\begin{aligned} & \mathbb{P}(\Omega_{K, \alpha, \tau}^c \cup (\{\xi_{\tau+1}^2 \neq \Lambda_{\mathbf{x}}(\xi_{\tau+1}^1) \text{ or } |\xi_{\tau+1}^1| > K_1\} \cap \Omega_{K, \alpha, \tau})) \\ & = 1 - p_K + p_K \mathbb{P}(\{\xi_{\tau+1}^2 \neq \Lambda_{\mathbf{x}}(\xi_{\tau+1}^1) \text{ or } |\xi_{\tau+1}^1| > K_1\} | \Omega_{K, \alpha, \tau}) \\ & \geq 1 - p_K + p_K \mathbb{P}(|\xi_{\tau+1}^1| > K_1 | \Omega_{K, \alpha, \tau}) \\ & \geq 1 - p_K + p_K \mu(B(0, K_1)^c) = 1 - p_K + p_K(1 - \mu(B(0, K_1))) \end{aligned}$$

where the last inequality is due to Lemma 4.1 one more time. Finally, it remains to choose $K_1 \in (0, \tilde{K}]$ small enough in order to get $1 - p_K + p_K(1 - \mu(B(0, K_1))) \geq \delta_1$.

(ii) If $\omega \in \Omega_{K, \alpha, \tau}$, by the previous construction and Lemma 4.1, we have $|g_{\tau}(\omega)| = |Z_1(\omega) - Z_2(\omega)| \leq M_K$. And if $\omega \in \Omega_{K, \alpha, \tau}^c$ then $|g_{\tau}(\omega)| = |\xi(\omega) - \xi(\omega)| = 0$ which concludes the proof of (ii). \square

To fix the ideas let us recall what we mean by “success of Step 1” and “failure of Step 1” of trial j ($j \geq 1$) :

$$\{\text{success of Step 1}\} = \Omega_{K,\alpha,\tau_{j-1}} \cap \{\xi_{\tau_{j-1}+1}^2 = \Lambda_{\mathbf{x}}(\xi_{\tau_{j-1}+1}^1), |\xi_{\tau_{j-1}+1}^1| \leq K_1\} \quad (5.13)$$

$$\{\text{failure of Step 1}\} = \Omega_{K,\alpha,\tau_{j-1}}^c \cup \left(\{\xi_{\tau_{j-1}+1}^2 \neq \Lambda_{\mathbf{x}}(\xi_{\tau_{j-1}+1}^1) \text{ or } |\xi_{\tau_{j-1}+1}^1| > K_1\} \cap \Omega_{K,\alpha,\tau_{j-1}} \right) \quad (5.14)$$

where $\mathbf{x} := \left(X_{\tau_{j-1}}^1, X_{\tau_{j-1}}^2, \sum_{k=0}^{+\infty} a_k \xi_{\tau_{j-1}+1-k}^1, \sum_{k=0}^{+\infty} a_k \xi_{\tau_{j-1}+1-k}^2 \right)$.

5.2.2 Step 2 (sticking step)

Step 2 of trial j consists in trying to keep the paths fastened together on successive intervals $I_{j,\ell}$. More precisely, during trial j , we set

$$\begin{aligned} I_{j,0} &:= \{\tau_{j-1} + 1\}, & I_{j,1} &:= \llbracket \tau_{j-1} + 2, \tau_{j-1} + 2c_2 - 1 \rrbracket \\ \text{and } \forall \ell \geq 2, & I_{j,\ell} &:= \llbracket \tau_{j-1} + c_2 s_\ell, \tau_{j-1} + c_2 s_{\ell+1} - 1 \rrbracket \end{aligned} \quad (5.15)$$

where $c_2 \geq 2$ will be chosen further and with

$$\forall \ell \geq 2, \quad s_\ell = \begin{cases} 2^\ell & \text{under } (\mathbf{H}_{\text{poly}}) \\ \ell & \text{under } (\mathbf{H}_{\text{exp}}). \end{cases} \quad (5.16)$$

We denote

$$\ell_j^* := \sup\{\ell \geq 1 \mid \forall n \in I_{j,\ell-1}, g_{n-1} = g_{n-1}^{(s)}\} \quad (5.17)$$

where $g_{n-1}^{(s)}$ is the successful-coupling drift defined by (4.15), i.e. $g_{n-1}^{(s)} = -\sum_{l=1}^{+\infty} a_l g_{n-1-l}$. In other words, I_{j,ℓ_j^*} is the interval where the failure occurs. If $\{\ell \geq 1 \mid \forall n \in I_{j,\ell-1}, g_{n-1} = g_{n-1}^{(s)}\} = \emptyset$, we adopt the convention $\ell_j^* = 0$, it corresponds to the case where the failure occurs at Step 1. When $\ell_j^* = +\infty$, trial j is successful. For a given positive α and $K > 0$, we set

$$\mathcal{B}_{j,\ell} := \Omega_{K,\alpha,\tau_{j-1}} \cap \{\ell_j^* > \ell\} \quad \forall j \geq 1, \ell \geq 0. \quad (5.18)$$

which means that failure of Step 2 may occur at most after ℓ trials. With this notations we get

$$\mathbb{P}(\Delta\tau_j = +\infty \mid \Omega_{K,\alpha,\tau_{j-1}}) = \mathbb{P}(\text{success of Step 1} \mid \Omega_{K,\alpha,\tau_{j-1}}) \prod_{\ell=1}^{+\infty} \mathbb{P}(\mathcal{B}_{j,\ell} \mid \mathcal{B}_{j,\ell-1}) \quad (5.19)$$

where the event {success of Step 1} is defined by (5.13).

Remark 5.3. *There is an infinite product in this expression of the successful-coupling probability. Hence, the size choice of the intervals $I_{j,\ell}$ defined in (5.16) will play a significant role in the convergence of the product to a non-negative limit.*

In the following lemma, similarly to the above definitions, we consider for a stopping time τ the intervals $(I_{\tau,\ell})_{\ell \geq 1}$, the integer ℓ_τ^* and the events $\mathcal{B}_{\tau,\ell}$, replacing τ_{j-1} by τ .

Lemma 5.2. *Let $K > 0$, assume (\mathbf{H}_1) and (\mathbf{H}_2) . Let $\alpha > \frac{1}{2} \vee \left(\frac{3}{2} - \beta\right)$ under $(\mathbf{H}_{\text{poly}})$ or $\alpha > 0$ different from μ under $(\mathbf{H}_{\text{exp}})$. Let τ be a stopping time with respect to $(\mathcal{F}_n)_{n \in \mathbb{Z}}$ (defined in Subsection 4.1) and assume that the system is (K, α) -admissible at time τ , then there exists $C_K > 0$ such that for $c_2 \geq 2$ large enough the successful drift $g^{(s)}$ satisfies*

$$\text{for } \ell = 1, \quad \|g^{(s)}\|_{I_{\tau,1}} \leq C_K$$

and

$$\forall \ell \geq 2, \quad \|g^{(s)}\|_{I_{\tau,\ell}} = \left(\sum_{k=\tau+c_2 s_\ell}^{\tau+c_2 s_{\ell+1}-1} |g_{k-1}^{(s)}|^2 \right)^{1/2} \leq 2^{-\tilde{\alpha}\ell}$$

where $\tilde{\alpha} := \begin{cases} \min\{\alpha, \beta, \alpha + \beta - 1\} - 1/2 - \varepsilon & \text{for all } \varepsilon > 0 \quad \text{under } (\mathbf{H}_{\text{poly}}) \\ \min(\alpha, \mu) & \text{under } (\mathbf{H}_{\text{exp}}) \end{cases}$.

Therefore, for all $\ell \geq 1$, we can build thanks to Corollary 4.1 $((\xi_k^1)_{k \in I_{\tau, \ell}}, (\xi_k^2)_{k \in I_{\tau, \ell}})$ during Step 2 in such a way that

$$\mathbb{P}(\mathcal{B}_{\tau, 1} | \mathcal{B}_{\tau, 0}) \geq \delta_K^1 \quad \text{and} \quad \forall \ell \geq 2, \quad \mathbb{P}(\mathcal{B}_{\tau, \ell} | \mathcal{B}_{\tau, \ell-1}) \geq 1 - 2^{-\tilde{\alpha}\ell}$$

where $\delta_K^1 \in (0, 1)$.

Moreover, if $2 \leq \ell_\tau^* < +\infty$, there exists $C_\alpha > 0$ independent from K such that

$$\left(\sum_{k=\tau+c_2 s_{\ell_\tau^*}}^{\tau+c_2 s_{\ell_\tau^*+1}-1} |g_{k-1}|^2 \right)^{1/2} \leq C_\alpha (\ell_\tau^* + 1)$$

and if $\ell_\tau^* = 1$, $\|g\|_{I_{\tau, 1}} \leq C'_K$ for some constant $C'_K > 0$.

Remark 5.4. \triangleright Under hypothesis $(\mathbf{H}_{\text{poly}})$ the condition $\alpha > \frac{1}{2} \vee (\frac{3}{2} - \beta)$ will ensure that $\min\{\alpha, \beta, \alpha + \beta - 1\} - 1/2 > 0$.

\triangleright In the polynomial case, for technical reasons $\tilde{\alpha}$ depends on $\varepsilon > 0$. This expression allows us to put together different cases and simplify the lemma. Indeed, if $(\alpha, \beta) \notin \{1\} \times (0, 1] \cup (0, 1] \times \{1\}$, we can take $\varepsilon = 0$.

To prove this lemma we will use in the *polynomial case* the following technical result which a more precise statement and a proof are given in Appendix D.

Lemma 5.3 (Technical lemma). *Let $\alpha > 0$ and $\beta > 0$ such that $\alpha + \beta > 1$. Then, there exists $C(\alpha, \beta) > 0$ such that for every $\varepsilon > 0$,*

$$\forall n \geq 0, \quad \sum_{k=0}^n (k+1)^{-\beta} (n+1-k)^{-\alpha} \leq C(\alpha, \beta) (n+1)^{-\min\{\alpha, \beta, \alpha+\beta-1\}+\varepsilon}.$$

When $(\alpha, \beta) \notin \{1\} \times (0, 1] \cup (0, 1] \times \{1\}$, we can take $\varepsilon = 0$ in the previous inequality.

We can now move on the proof of Lemma 5.2.

Proof. Let us prove the first part of the lemma, namely the upper-bound of the ℓ^2 norm for the successful-coupling drift term on the intervals I_ℓ^r . Indeed, the second part is just an application of corollary 4.1. Since the system is (K, α) -admissible at time τ , we get by (5.1)

$$\forall n \geq 0, \quad \left| \sum_{k=n+1}^{+\infty} a_k g_{\tau+n-k} \right| \leq v_n.$$

But, if Step 2 is successful, we recall that by (4.15) the successful drift satisfies

$$\forall n \geq 1, \quad g_{\tau+n}^{(s)} = - \sum_{k=1}^{+\infty} a_k g_{\tau+n-k}$$

and then

$$\forall n \geq 1, \quad \sum_{k=0}^n a_k g_{\tau+n-k}^{(s)} = - \underbrace{\sum_{k=n+1}^{+\infty} a_k g_{\tau+n-k}}_{=: u_n} \quad \text{and we set} \quad u_0 := a_0 g_\tau^{(s)} = g_\tau^{(s)}.$$

Therefore thanks to remark 2.2 this is equivalent to

$$\forall n \geq 0, \quad g_{\tau+n}^{(s)} = \sum_{k=0}^n b_k u_{n-k}.$$

By the admissibility assumption, we have $\forall k \in \{0, \dots, n-1\}$, $|u_{n-k}| \leq v_{n-k}$ and by Lemma 5.1 (ii), $|u_0| \leq M_K$. Hence, we get

$$\forall n \geq 0, |g_{\tau+n}^{(s)}| \leq M_K \sum_{k=0}^n |b_k| v_{n-k}. \quad (5.20)$$

• Polynomial case: Assume $(\mathbf{H}_{\text{poly}})$. Then for all $n \geq 0$, $v_n = (n+1)^{-\alpha}$ with $\alpha > \frac{1}{2} \vee (\frac{3}{2} - \beta)$. Here, (5.20) is equivalent to

$$\begin{aligned} \forall n \geq 0, |g_{\tau+n}^{(s)}| &\leq M_K C_\beta \sum_{k=0}^n (k+1)^{-\beta} (n+1-k)^{-\alpha} \\ &\leq M'_K (n+1)^{-\min\{\alpha, \beta, \alpha+\beta-1\} + \varepsilon}. \end{aligned}$$

for all $\varepsilon > 0$ by applying the technical lemma 5.3 and setting $M'_K := C(\alpha, \beta) M_K C_\beta$. We then set $\tilde{\alpha} := \min\{\alpha, \beta, \alpha + \beta - 1\} - 1/2 - \varepsilon$.

Hence, for all $\ell \geq 2$,

$$\begin{aligned} \|g^{(s)}\|_{I_{\tau, \ell}} &= \left(\sum_{k=\tau+c_2 2^\ell}^{\tau+c_2 2^{\ell+1}-1} \left(g_{k-1}^{(s)} \right)^2 \right)^{1/2} \\ &\leq M'_K \left(\sum_{k=c_2 2^\ell}^{c_2 2^{\ell+1}-1} k^{-2\tilde{\alpha}-1} \right)^{1/2} \\ &\leq M'_K \times (c_2 2^\ell \times (c_2 2^\ell)^{-2\tilde{\alpha}-1})^{1/2} = M'_K c_2^{-\tilde{\alpha}} 2^{-\tilde{\alpha}\ell}. \end{aligned}$$

It remains to choose $c_2 \geq 2 \vee (M'_K)^{1/\tilde{\alpha}}$ to get the desired bound.

• Exponential case: Assume $(\mathbf{H}_{\text{exp}})$. Then for all $n \geq 0$, $v_n = e^{-\alpha n}$ with $\alpha > 0$ and $\alpha \neq \mu$. Here, (5.20) is equivalent to

$$\begin{aligned} \forall n \geq 0, |g_{\tau+n}^{(s)}| &\leq M_K C_\mu \sum_{k=0}^n e^{-\mu k} e^{-\alpha(n-k)} \\ &\leq M'_K e^{-\min\{\alpha, \mu\}n}. \end{aligned}$$

by setting $M'_K := M_K C_\mu$. We then set $\tilde{\alpha} := \min(\alpha, \mu)$.

Hence, for all $\ell \geq 2$,

$$\begin{aligned} \|g^{(s)}\|_{I_\ell^{(\tau)}} &= \left(\sum_{k=\tau+c_2 \ell}^{\tau+c_2(\ell+1)-1} \left(g_{k-1}^{(s)} \right)^2 \right)^{1/2} \\ &\leq M'_K \left(\sum_{k=c_2 \ell}^{c_2(\ell+1)-1} e^{-2\tilde{\alpha}(k-1)} \right)^{1/2} \\ &\leq M'_K \times \left(\frac{e^{-2\tilde{\alpha}(c_2 \ell - 2)}}{2\tilde{\alpha}} [1 - e^{-2\tilde{\alpha}c_2}] \right)^{1/2} \quad (\text{by integral upper-bound}) \\ &\leq e^{-\tilde{\alpha}\ell} \quad (\text{by choosing } c_2 \geq 2 \text{ large enough}) \\ &\leq 2^{-\tilde{\alpha}\ell} \end{aligned}$$

For $\ell = 1$, in both polynomial and exponential cases, the same approach gives us the existence of $C_K > 0$ such that

$$\|g^{(s)}\|_{I_{\tau, 1}} \leq C_K.$$

□

By combining Lemma 5.1, Lemma 5.2 and the expression (5.19) we finally get Proposition 5.1.

6 (K, α) -admissibility

6.1 On condition (5.1)

Let $\Delta t_3^{(j)}$ be the duration of Step 3 of trial j for $j \geq 1$. The purpose of the next proposition is to prove that thanks to a calibration of $\Delta t_3^{(j)}$, one satisfies almost surely condition (5.1) at time τ_j .

Proposition 6.1. *Assume (\mathbf{H}_1) and (\mathbf{H}_2) . Let $\alpha \in (\frac{1}{2} \vee (\frac{3}{2} - \beta), \rho)$ if we are under $(\mathbf{H}_{\text{poly}})$ and $\alpha \in (0, \lambda)$ different from μ if we are under $(\mathbf{H}_{\text{exp}})$. Assume that for all $j \geq 1$,*

$$\Delta t_3^{(j)} = \begin{cases} t_* \varsigma^j 2^{\theta \ell_j^*} & \text{with } \theta > (2(\rho - \alpha))^{-1} & \text{under } (\mathbf{H}_{\text{poly}}) \\ t_* + \varsigma^j + \theta \ell_j^* & \text{with } \theta > 0 & \text{under } (\mathbf{H}_{\text{exp}}) \end{cases}$$

where ℓ_j^* is defined in (5.17) and with $\varsigma > 1$ arbitrary. Then for every $K > 0$, there exists a choice of t_* such that, for all $j \geq 0$, condition (5.1) is a.s. true at time τ_j on the event $\{\tau_j < +\infty\}$. In other words, for all $j \geq 0$,

$$\mathbb{P}(\Omega_{\alpha, \tau_j}^1 | \{\tau_j < +\infty\}) = 1.$$

Proof. Let us begin by the first coupling trial, in other words for $j = 0$. We recall that $g_k = 0$ for all $k < \tau_0$ (see (4.6)), therefore

$$\forall n \geq 0, \left| \sum_{k=n+1}^{+\infty} a_k g_{\tau_0+n-k} \right| = 0 \leq v_n$$

and then condition (5.1) is a.s. true at time τ_0 . Now, we assume $j \geq 1$ and we work on the event

$$\{\tau_j < +\infty\} \quad (\supset \{\tau_m < +\infty\} \text{ for all } 0 \leq m \leq j-1).$$

Let us prove that on this event we have

$$\forall n \geq 0, \left| \frac{1}{a_0} \sum_{k=n+1}^{+\infty} a_k g_{\tau_j+n-k} \right| \leq v_n.$$

Set $u_n := \sum_{k=n+1}^{+\infty} a_k g_{\tau_j+n-k}$. Since $g_k = 0$ for all $k < \tau_0$, we get

$$u_n = \sum_{k=n+1}^{n+\tau_j-\tau_0} a_k g_{\tau_j+n-k}.$$

Let us now separate the right term into the contributions of the different coupling trials. We get

$$u_n = \sum_{m=1}^j \left(\sum_{k=n+\tau_j-\tau_m+1}^{n+\tau_j-\tau_{m-1}} a_k g_{\tau_j+n-k} \right) = \sum_{m=1}^j \underbrace{\left(\sum_{k=\tau_{m-1}}^{\tau_m-1} a_{n+\tau_j-k} g_k \right)}_{(\star)_m}$$

and $(\star)_m$ corresponds to the contribution of trial m , divided into two parts: success and failure. We have now to distinguish two cases:

First case: $\ell_m^* \geq 1$, in other words the failure occurs during Step 2. We recall that in this case the system was automatically (K, α) -admissible at time τ_{m-1} , which will allow us to use Lemma 5.2 on τ_{m-1} .

Then, since $g_k = 0$ on $[\tau_{m-1} + c_2 s_{\ell_m^*+1}, \tau_m - 1]$ by definition of Step 3 of the coupling procedure,

$$\begin{aligned}
(\star)_m &= \sum_{k=\tau_{j_{m-1}}}^{\tau_{m-1}+c_2 s_{\ell_m^*+1}-1} a_{n+\tau_j-k} g_k = \sum_{k=0}^{c_2 s_{\ell_m^*+1}-1} a_{n+\tau_j-\tau_{m-1}-k} g_{\tau_{m-1}+k} \\
&= \underbrace{\sum_{k=0}^{c_2 s_{\ell_m^*}-1} a_{n+\tau_j-\tau_{m-1}-k} g_{\tau_{m-1}+k}}_{\text{success}} \\
&\quad + \underbrace{\sum_{k=c_2 s_{\ell_m^*}}^{c_2 s_{\ell_m^*+1}-1} a_{n+\tau_j-\tau_{m-1}-k} g_{\tau_{m-1}+k}}_{\text{failure}}.
\end{aligned}$$

We have now to make the distinction between the polynomial and the exponential case.

▷ Under $(\mathbf{H}_{\text{poly}})$: $s_\ell = 2^\ell$, $|a_k| \leq C_\rho(k+1)^{-\rho}$ and then $a_k^2 \leq C_\rho^2(k+1)^{-2\rho}$

Using Cauchy-Schwarz inequality, the domination assumption on (a_k) , and the fact that

$$n + \tau_j - \tau_{m-1} - k + 1 \geq n + \tau_m - \tau_{m-1} - k + 1 = \Delta t_3^{(m)} + c_2 2^{\ell_m^*+1} + n - k$$

we get,

$$\begin{aligned}
|\text{success}| &\leq \left(\sum_{k=0}^{c_2 2^{\ell_m^*}-1} a_{n+\tau_j-\tau_{m-1}-k} \right)^{1/2} \left(\sum_{k=0}^{c_2 2^{\ell_m^*}-1} |g_{\tau_{m-1}+k}|^2 \right)^{1/2} \\
&\leq C_\rho \left(\sum_{k=0}^{c_2 2^{\ell_m^*}-1} (n + \tau_j - \tau_{m-1} - k + 1)^{-2\rho} \right)^{1/2} \left(\sum_{k=0}^{c_2 2^{\ell_m^*}-1} |g_{\tau_{m-1}+k}|^2 \right)^{1/2} \\
&\leq C_\rho \left(\sum_{k=0}^{c_2 2^{\ell_m^*}-1} (\Delta t_3^{(m)} + c_2 2^{\ell_m^*+1} + n - k)^{-2\rho} \right)^{1/2} \left(\sum_{k=0}^{c_2 2^{\ell_m^*}-1} |g_{\tau_{m-1}+k}|^2 \right)^{1/2} \\
&= C_\rho \left(\sum_{k=\Delta t_3^{(m)}+c_2 2^{\ell_m^*+n+1}}^{\Delta t_3^{(m)}+c_2 2^{\ell_m^*+1}+n} k^{-2\rho} \right)^{1/2} \left(\sum_{k=0}^{c_2 2^{\ell_m^*}-1} |g_{\tau_{m-1}+k}|^2 \right)^{1/2} \\
&\leq C_\rho \left(c_2 2^{\ell_m^*} (n + \Delta t_3^{(m)})^{-2\rho} \right)^{1/2} \left(\sum_{k=0}^{c_2 2^{\ell_m^*}-1} |g_{\tau_{m-1}+k}|^2 \right)^{1/2}.
\end{aligned}$$

By the same arguments, we obtain

$$|\text{failure}| \leq C_\rho \left(c_2 2^{\ell_m^*} (n + \Delta t_3^{(m)})^{-2\rho} \right)^{1/2} \left(\sum_{k=c_2 2^{\ell_m^*}}^{c_2 2^{\ell_m^*+1}-1} |g_{\tau_{m-1}+k}|^2 \right)^{1/2}.$$

Hence, by the triangular inequality, we have

$$|(\star)_m| \leq C_\rho \sqrt{c_2} 2^{\ell_m^*/2} (n + \Delta t_3^{(m)})^{-\rho} \left[\underbrace{\left(\sum_{k=0}^{c_2 2^{\ell_m^*}-1} (g_{\tau_{m-1}+k})^2 \right)^{1/2}}_{(1)} + \underbrace{\left(\sum_{k=c_2 2^{\ell_m^*}}^{c_2 2^{\ell_m^*+1}-1} (g_{\tau_{m-1}+k})^2 \right)^{1/2}}_{(2)} \right].$$

Since $\ell_j^* \geq 1$, by Lemma 5.1 (ii) and Lemma 5.2, we have

$$(1) \leq M_K + \sum_{\ell=1}^{\ell_m^*-1} \|g^{(s)}\|_{I_\ell} \leq M_K + C_K + \sum_{\ell=2}^{+\infty} 2^{-\tilde{\alpha}\ell} =: \tilde{C}_K$$

and

$$(2) \leq \max(C_\alpha, C'_K)(\ell_m^* + 1).$$

Therefore

$$|(\star)_m| \leq C_K^{(2)} 2^{\ell_m^*/2} (\ell_m^* + 1) (n + \Delta t_3^{(m)})^{-\rho} \quad (6.1)$$

where $C_K^{(2)} := \max(C_\alpha, C'_K, \tilde{C}_K)$.

Moreover, recall that under $(\mathbf{H}_{\text{poly}})$

$$\Delta t_3^{(m)} = t_* \varsigma^m 2^{\theta \ell_m^*} \quad \text{with } \theta > (2(\rho - \alpha))^{-1} \text{ and } \varsigma > 1. \quad (6.2)$$

Plugging the definition of $\Delta t_3^{(m)}$ into (6.1) and using that for all $x, y > 0$, $(x + y)^{-\rho} \leq x^{-(\rho - \alpha)} y^{-\alpha}$

$$\begin{aligned} |(\star)_m| &\leq C_K^{(2)} 2^{\ell_m^*/2} (\ell_m^* + 1) (n + \Delta t_3^{(m)})^{-\rho} \\ &\leq C_K^{(2)} (\ell_m^* + 1) 2^{(1/2 - \theta(\rho - \alpha))\ell_m^*} (t_* \varsigma^m)^{-(\rho - \alpha)} (n + 1)^{-\alpha}. \end{aligned} \quad (6.3)$$

Since $\theta > (2(\rho - \alpha))^{-1}$ we have

$$C_{\alpha, K} = \sup_{\ell^* > 0} C_K^{(2)} (\ell^* + 1) 2^{(1/2 - \theta(\rho - \alpha))\ell^*} < +\infty,$$

and (6.3) yields

$$|(\star)_m| \leq C_{\alpha, K} (t_* \varsigma^m)^{-(\rho - \alpha)} (n + 1)^{-\alpha} \quad \text{under } (\mathbf{H}_{\text{poly}}). \quad (6.4)$$

▷ Under $(\mathbf{H}_{\text{exp}})$: $s_\ell = \ell$, $|a_k| \leq C_\lambda e^{-\lambda k}$ and then $a_k^2 \leq C_\lambda^2 e^{-2\lambda k}$

Since the proof is almost the same in the exponential case, we will go faster and skip some details. Using again Cauchy-Schwarz inequality, the domination assumption on (a_k) , and the fact that

$$n + \tau_j - \tau_{m-1} - k \geq n + \tau_m - \tau_{m-1} - k = \Delta t_3^{(m)} + c_2(\ell_m^* + 1) + n - k - 1$$

we get

$$\begin{aligned} |\text{success}| &\leq \left(\sum_{k=0}^{c_2 \ell_m^* - 1} a_{n + \tau_j - \tau_{m-1} - k}^2 \right)^{1/2} \left(\sum_{k=0}^{c_2 \ell_m^* - 1} |g_{\tau_{m-1} + k}|^2 \right)^{1/2} \\ &\leq C_\lambda \left(\sum_{k = \Delta t_3^{(m)} + c_2 + n}^{\Delta t_3^{(m)} + c_2(\ell_m^* + 1) + n - 1} e^{-2\lambda k} \right)^{1/2} \left(\sum_{k=0}^{c_2 \ell_m^* - 1} |g_{\tau_{m-1} + k}|^2 \right)^{1/2} \\ &\leq \frac{C_\lambda}{\sqrt{2\lambda}} e^{-\lambda(n + \Delta t_3^{(m)} + c_2 - 1)} \left(\sum_{k=0}^{c_2 2^{\ell_m^*} - 1} |g_{\tau_{m-1} + k}|^2 \right)^{1/2} \quad (\text{by integral bound}) \\ &\leq \frac{C_\lambda e^\lambda}{\sqrt{2\lambda}} e^{-\lambda(n + \Delta t_3^{(m)})} \left(\sum_{k=0}^{c_2 2^{\ell_m^*} - 1} |g_{\tau_{m-1} + k}|^2 \right)^{1/2} \end{aligned}$$

and by the same arguments,

$$|\text{failure}| \leq \frac{C_\lambda e^\lambda}{\sqrt{2\lambda}} e^{-\lambda(n+\Delta t_3^{(m)})} \left(\sum_{k=c_2 \ell_m^*}^{c_2(\ell_m^*+1)-1} |g_{\tau_{m-1}+k}|^2 \right)^{1/2}.$$

As in the polynomial case, by using Lemma 5.1 and 5.2 we get the existence of $C_K^{(3)} > 0$ such that

$$|(\star)_m| \leq C_K^{(3)} (\ell_m^* + 1) e^{-\lambda(n+\Delta t_3^{(m)})}. \quad (6.5)$$

Moreover, recall that under $(\mathbf{H}_{\text{exp}})$

$$\Delta t_3^{(m)} = t_* + \varsigma^m + \theta \ell_m^* \quad \text{with } \theta > 0 \text{ and } \varsigma > 1. \quad (6.6)$$

Plugging the definition of $\Delta t_3^{(m)}$ into (6.5), we get

$$\begin{aligned} |(\star)_m| &\leq C_K^{(3)} (\ell_m^* + 1) e^{-\lambda(n+\Delta t_3^{(m)})} \\ &\leq C_K^{(3)} (\ell_m^* + 1) e^{-(\lambda-\alpha)\Delta t_3^{(m)}} e^{-\alpha n} \\ &= C_K^{(3)} (\ell_m^* + 1) e^{-(\lambda-\alpha)(t_*+\varsigma^m+\theta\ell_m^*)} e^{-\alpha n}. \end{aligned}$$

We set

$$C'_{\alpha,K} = \sup_{\ell^* > 0} C_K^{(3)} (\ell^* + 1) e^{-\theta(\lambda-\alpha)\ell^*} < +\infty.$$

And this gives us

$$|(\star)_m| \leq C'_{\alpha,K} e^{-(\lambda-\alpha)(t_*+\varsigma^m)} e^{-\alpha n} \quad \text{under } (\mathbf{H}_{\text{exp}}). \quad (6.7)$$

Second case: $\ell_m^* = 0$, in other words, failure occurs during Step 1. This includes the case when the system is not (K, α) -admissible at time τ_{m-1} .

We have

$$(\star)_m = a_{n+\tau_j-\tau_{m-1}} g_{\tau_{m-1}}.$$

By Lemma 5.1 (ii), $|g_{\tau_{m-1}}| \leq M_K$.

Moreover, since $n + \tau_j - \tau_{m-1} \geq n + \tau_m - \tau_{m-1} = n + \Delta t_3^{(m)}$, we obtain by using the same method as in the first case,

$$|(\star)_m| \leq \begin{cases} M_K (t_* \varsigma^m)^{-(\rho-\alpha)} (n+1)^{-\alpha} & \text{under } (\mathbf{H}_{\text{poly}}) \\ M_K e^{-(\lambda-\alpha)(t_*+\varsigma^m)} e^{-\alpha n} & \text{under } (\mathbf{H}_{\text{exp}}) \end{cases}. \quad (6.8)$$

By putting (6.4), (6.7) and (6.8) together, we finally get

$$|(\star)_m| \leq \begin{cases} \max(M_K, C_{\alpha,K}) (t_* \varsigma^m)^{-(\rho-\alpha)} (n+1)^{-\alpha} & \text{under } (\mathbf{H}_{\text{poly}}) \\ \max(M_K, C'_{\alpha,K}) e^{-(\lambda-\alpha)(t_*+\varsigma^m)} e^{-\alpha n} & \text{under } (\mathbf{H}_{\text{exp}}) \end{cases}.$$

Set $S_1 = \sum_{m=1}^{+\infty} \varsigma^{-(\rho-\alpha)m}$ and $S_2 = \sum_{m=1}^{+\infty} e^{-(\lambda-\alpha)\varsigma^m}$. By choosing t_* large enough, we obtain for all $1 \leq m \leq j$:

$$|(\star)_m| \leq \begin{cases} \frac{1}{S_1} \varsigma^{-(\rho-\alpha)m} (n+1)^{-\alpha} & \text{under } (\mathbf{H}_{\text{poly}}) \\ \frac{1}{S_2} e^{-(\lambda-\alpha)\varsigma^m} e^{-\alpha n} & \text{under } (\mathbf{H}_{\text{exp}}) \end{cases}. \quad (6.9)$$

Finally, by adding (6.9) for $m = 1, \dots, j$ we have

$$\forall n \geq 0, \quad |u_n| \leq \begin{cases} (n+1)^{-\alpha} & \text{under } (\mathbf{H}_{\text{poly}}) \\ e^{-\alpha n} & \text{under } (\mathbf{H}_{\text{exp}}) \end{cases}$$

which concludes the proof. \square

6.2 Compact return condition (5.2)

In the sequel, we set

$$\mathcal{E}_j := \{\tau_j < \infty\} (= \{\tau_1 < \infty, \dots, \tau_j < \infty\}). \quad (6.10)$$

The aim of this subsection is to prove the following proposition:

Proposition 6.2. *Assume (\mathbf{H}_1) and (\mathbf{H}_2) . For all $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that*

$$\mathbb{P}(\Omega_{K_\varepsilon, \tau_j}^2 | \mathcal{E}_j) \geq 1 - \varepsilon. \quad (6.11)$$

At this stage, we assume that $(\mathbf{H}_{\text{poly}})$ is true. Indeed, the exponential case will immediately follow from the polynomial one since $(\mathbf{H}_{\text{exp}})$ implies $(\mathbf{H}_{\text{poly}})$.

Since for every events A_1, A_2, A_3 and A_4 , we have $\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4) \geq \sum_{i=1}^4 \mathbb{P}(A_i) - 3$, it is enough to prove that for all $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that

$$\mathbb{P}(|X_{\tau_j}^i| \leq K_\varepsilon | \mathcal{E}_j) \geq 1 - \varepsilon \quad \text{and} \quad \mathbb{P}\left(\left|\sum_{k=1}^{+\infty} a_k \xi_{\tau_j+1-k}^i\right| \leq K_\varepsilon \middle| \mathcal{E}_j\right) \geq 1 - \varepsilon \quad \text{for } i = 1, 2 \quad (6.12)$$

to get (6.11). Let us first focus on the first part of (6.12) concerning $|X_{\tau_j}^i|$ for $i = 1, 2$. Recall that the function $V : \mathbb{R}^d \rightarrow \mathbb{R}_+^*$ appearing in (\mathbf{H}_1) is such that $\lim_{|x| \rightarrow +\infty} V(x) = +\infty$. For $K > 0$ large enough we then have: $|x| \geq K \Rightarrow V(x) \geq K$. Therefore, for $i = 1, 2$ and K_ε large enough, using Markov inequality we get

$$\mathbb{P}(|X_{\tau_j}^i| \geq K_\varepsilon | \mathcal{E}_j) \leq \mathbb{P}(V(X_{\tau_j}^i) \geq K_\varepsilon | \mathcal{E}_j) \leq \frac{\mathbb{E}(V(X_{\tau_j}^i) | \mathcal{E}_j)}{K_\varepsilon} \quad (6.13)$$

Hence, the first part of (6.12) is true if there exists a constant C such that for every $j \in \mathbb{N}$ and for every $K > 0$,

$$\mathbb{E}(V(X_{\tau_j}^i) | \mathcal{E}_j) \leq C \quad \text{for } i = 1, 2. \quad (6.14)$$

Indeed, plugging (6.14) into (6.13) and taking $K_\varepsilon \geq \frac{C}{\varepsilon}$ yield the desired inequality. We see here that the independence of C with respect to K is essential.

For the sake of simplicity, we will first use the following hypothesis to prove (6.14):

(\mathbf{H}'_1) : Let $\gamma \in (0, 1)$. There exists $C_\gamma > 0$ such that for all $j \in \mathbb{N}$, for every $K > 0$ and for $i = 1, 2$,

$$\mathbb{E}\left[\sum_{l=1}^{\Delta\tau_j} \gamma^{\Delta\tau_j-l} |\Delta_{\tau_{j-1}+l}^i| \middle| \mathcal{E}_j\right] < C_\gamma$$

where $\Delta\tau_j := \tau_j - \tau_{j-1}$ and Δ^i is the stationary Gaussian sequence defined by (2.2).

Proposition 6.3. *Assume (\mathbf{H}_1) , (\mathbf{H}_2) and (\mathbf{H}'_1) . Let $(X_n^1, X_n^2)_{n \in \mathbb{N}}$ be a solution of (4.2) with initial condition (X_0^1, X_0^2) satisfying $\mathbb{E}(V(X_0^i)) < \infty$ for $i = 1, 2$. Moreover, assume that $\tau_0 = 0$ and that $(\tau_j)_{j \geq 1}$ is built in such a way that for all $j \geq 1$, $\mathbb{P}(\mathcal{E}_j | \mathcal{E}_{j-1}) \geq \delta_1 > 0$ (where δ_1 is not depending on j) and $\Delta\tau_j \geq \frac{\log(\delta_1/2)}{\log(\gamma)}$. Then, there exists a constant C such that for all $j \in \mathbb{N}$ and for every $K > 0$,*

$$\mathbb{E}(V(X_{\tau_j}^i) | \mathcal{E}_j) \leq C \quad \text{for } i = 1, 2.$$

Remark 6.1. \triangleright Actually, hypothesis (\mathbf{H}'_1) is true under $(\mathbf{H}_{\text{poly}})$ and will be proven in Subsection 6.3.

\triangleright The existence of $\delta_1 > 0$ independent from j is proven in Subsection 5.2.

\triangleright To get $\Delta\tau_j \geq \frac{\log(\delta_1/2)}{\log(\gamma)}$, it is sufficient to choose t_* large enough in the expression of $\Delta t_3^{(j)}$ (see Proposition 6.1).

\triangleright Since in Theorem 2.1 we made the assumption $\int_{\mathcal{X}} V(x) \Pi_{\mathcal{X}}^* \mu_0(dx) < +\infty$ and since an invariant distribution (extracted thanks to Theorem 3.1) also satisfies $\int_{\mathcal{X}} V(x) \Pi_{\mathcal{X}}^* \mu_*(dx) < +\infty$, we get that $\mathbb{E}(V(X_0^i)) < \infty$ for $i = 1, 2$. Hence, we can set $\tau_0 = 0$.

Proof. By (\mathbf{H}_1) , there exist $\gamma \in (0, 1)$ and $C > 0$ such that for all $n \geq 0$ and for $i = 1, 2$ we have

$$V(X_{n+1}^i) \leq \gamma V(X_n^i) + C(1 + |\Delta_{n+1}^i|).$$

By applying this inequality at time $n = \tau_j - 1$, and by induction, we immediately get:

$$V(X_{\tau_j}^i) \leq \gamma^{\Delta\tau_j} V(X_{\tau_j-1}^i) + C \sum_{l=1}^{\Delta\tau_j} \gamma^{\Delta\tau_j-l} (1 + |\Delta_{\tau_j-1+l}^i|). \quad (6.15)$$

By assumption $\Delta\tau_j \geq \frac{\log(\delta_1/2)}{\log(\gamma)}$ then $\gamma^{\Delta\tau_j} \leq \frac{\delta_1}{2}$. Moreover, since $\mathcal{E}_j \subset \mathcal{E}_{j-1}$ and $\mathbb{P}(\mathcal{E}_j | \mathcal{E}_{j-1}) \geq \delta_1$, we get

$$\begin{aligned} \mathbb{E}[V(X_{\tau_j-1}^i) | \mathcal{E}_j] &= \frac{1}{\mathbb{P}(\mathcal{E}_j)} \mathbb{E}[V(X_{\tau_j-1}^i) \mathbf{1}_{\mathcal{E}_j}] \leq \underbrace{\frac{\mathbb{P}(\mathcal{E}_{j-1})}{\mathbb{P}(\mathcal{E}_j)}}_{=\mathbb{P}(\mathcal{E}_j | \mathcal{E}_{j-1})^{-1}} \mathbb{E}[V(X_{\tau_j-1}^i) | \mathcal{E}_{j-1}] \\ &\leq \delta_1^{-1} \mathbb{E}[V(X_{\tau_j-1}^i) | \mathcal{E}_{j-1}] \end{aligned}$$

therefore

$$\mathbb{E}[\gamma^{\Delta\tau_j} V(X_{\tau_j-1}^i) | \mathcal{E}_j] \leq \frac{\delta_1}{2} \delta_1^{-1} \mathbb{E}[V(X_{\tau_j-1}^i) | \mathcal{E}_{j-1}] = \frac{1}{2} \mathbb{E}[V(X_{\tau_j-1}^i) | \mathcal{E}_{j-1}].$$

Hence, by taking (6.15), we have

$$\begin{aligned} \mathbb{E}[V(X_{\tau_j}^i) | \mathcal{E}_j] &\leq \frac{1}{2} \mathbb{E}[V(X_{\tau_j-1}^i) | \mathcal{E}_{j-1}] + C \sum_{l=0}^{+\infty} \gamma^l + C \mathbb{E} \left[\sum_{l=1}^{\Delta\tau_j} \gamma^{\Delta\tau_j-l} |\Delta_{\tau_j-1+l}^i| \middle| \mathcal{E}_j \right] \\ &= \frac{1}{2} \mathbb{E}[V(X_{\tau_j-1}^i) | \mathcal{E}_{j-1}] + \frac{C}{1-\gamma} + C \mathbb{E} \left[\sum_{l=1}^{\Delta\tau_j} \gamma^{\Delta\tau_j-l} |\Delta_{\tau_j-1+l}^i| \middle| \mathcal{E}_j \right]. \end{aligned}$$

Hypothesis (\mathbf{H}'_1) allows us to say

$$\mathbb{E}[V(X_{\tau_j}^i) | \mathcal{E}_j] \leq \frac{1}{2} \mathbb{E}[V(X_{\tau_j-1}^i) | \mathcal{E}_{j-1}] + C \left(\frac{1}{1-\gamma} + C_\gamma \right).$$

By induction, we get the existence of a constant $\tilde{C}_\gamma > 0$ such that

$$\mathbb{E}[V(X_{\tau_j}^i) | \mathcal{E}_j] \leq \left(\frac{1}{2} \right)^j \mathbb{E}[V(X_{\tau_0}^i) | \mathcal{E}_0] + \tilde{C}_\gamma.$$

since $\mathbb{P}(\mathcal{E}_0) = 1$, we get

$$\mathbb{E}[V(X_{\tau_j}^i) | \mathcal{E}_j] \leq \left(\frac{1}{2} \right)^j \mathbb{E}[V(X_{\tau_0}^i)] + \tilde{C}_\gamma.$$

Since $\tau_0 = 0$ and we assumed that $\mathbb{E}[V(X_0^i)] < \infty$, the proof is over. \square

It remains now to prove (\mathbf{H}'_1) to get the first part of (6.12). The second part will be deduced from the proof of (\mathbf{H}'_1) thanks to Remark 6.2.

6.3 Proof of hypothesis (\mathbf{H}'_1)

We recall that we want to prove that under (\mathbf{H}_{poly}), the following hypothesis is true:

(\mathbf{H}'_1): Let $\gamma \in (0, 1)$. There exists $C_\gamma > 0$ such that for all $j \in \mathbb{N}$, for every $K > 0$ and for $i = 1, 2$,

$$\mathbb{E} \left[\sum_{l=1}^{\Delta\tau_j} \gamma^{\Delta\tau_j-l} |\Delta_{\tau_{j-1}+l}^i| \middle| \mathcal{E}_j \right] < C_\gamma$$

where $\Delta\tau_j := \tau_j - \tau_{j-1}$ and Δ^i is the stationary Gaussian sequence defined by (2.2).

Remark 6.2. *Since the proof of this assumption will exclusively use the domination assumption on $(a_k)_{k \geq 0}$ and since $(\tilde{a}_k)_{k \geq 0} := (a_{k+1})_{k \geq 0}$ satisfies the same domination assumption, we will also get that for $i = 1, 2$,*

$$\mathbb{E} \left[\sum_{l=1}^{\Delta\tau_j} \gamma^{\Delta\tau_j-l} |\tilde{\Delta}_{\tau_{j-1}+l}^i| \middle| \mathcal{E}_j \right] < C_\gamma$$

where $\tilde{\Delta}_{\tau_{j-1}+l}^i = \sum_{k=0}^{+\infty} a_{k+1} \xi_{\tau_{j-1}+l-k}^i$. Hence, we will get that for $i = 1, 2$

$$\mathbb{E} \left[|\tilde{\Delta}_{\tau_j}^i| \middle| \mathcal{E}_j \right] = \mathbb{E} \left[\left| \sum_{k=0}^{+\infty} a_{k+1} \xi_{\tau_j-k}^i \right| \middle| \mathcal{E}_j \right] < C_\gamma.$$

Then, by the Markov inequality we finally get the second part of (6.12).

We now turn to the proof of (\mathbf{H}'_1). We work on the set $\mathcal{E}_j = \{\tau_j < +\infty\}$. We have

$$\sum_{l=1}^{\Delta\tau_j} \gamma^{\Delta\tau_j-l} |\Delta_{\tau_{j-1}+l}^i| = \sum_{u=\tau_{j-1}+1}^{\tau_j} \gamma^{\tau_j-u} |\Delta_u^i|.$$

But,

$$|\Delta_u^i| = \left| \sum_{k=-\infty}^{k=u} a_{u-k} \xi_k^i \right| = \left| \sum_{m=0}^j \Lambda_m^i(u) \right|.$$

where

$$\Lambda_m^i(u) = \sum_{k=\tau_{m-1}+1}^{\tau_m} a_{u-k} \xi_k^i \quad \text{pour } m \in \{1, \dots, j-1\}, \quad (6.16)$$

$$\Lambda_0^i(u) = \sum_{k=-\infty}^{k=\tau_0} a_{u-k} \xi_k^i \quad \text{et} \quad \Lambda_j^i(u) = \sum_{k=\tau_{j-1}+1}^u a_{u-k} \xi_k^i. \quad (6.17)$$

With these notations, we get the following upper-bound

$$\sum_{l=1}^{\Delta\tau_j} \gamma^{\Delta\tau_j-l} |\Delta_{\tau_{j-1}+l}^i| \leq \sum_{m=0}^j \sum_{u=\tau_{j-1}+1}^{\tau_j} \gamma^{\tau_j-u} |\Lambda_m^i(u)|. \quad (6.18)$$

The goal of the following lemmas is to get an upper-bound of the quantity $\mathbb{E} \left[\sup_{u \in [\tau_{j-1}+1, \tau_j]} |\Lambda_m^i(u)| \middle| \mathcal{E}_j \right]$

when $m \in \{0, \dots, j-1\}$.

Lemma 6.1. *Assume (\mathbf{H}_{poly}). Let $t_0, t_1 \in \mathbb{Z}$ and $u \in \mathbb{N}$ such that $t_0 < t_1 < u$. Let $(\xi_k)_{k \in \mathbb{Z}}$ be a sequence with values in \mathbb{R}^d . Then,*

$$\begin{aligned} \left| \sum_{k=t_0}^{t_1} a_{u-k} \xi_k \right| &\leq C_\rho (u+1-t_0)^{-\rho} \left| \sum_{k=t_0}^{t_1} \xi_k \right| + C_\kappa \sum_{k=1}^{t_1-t_0} \left| \sum_{l=k+t_0}^{t_1} \xi_l \right| (u+1-t_0-k)^{-\kappa} \\ &\leq C_\rho (u+1-t_0)^{-\rho} \left| \sum_{k=t_0}^{t_1} \xi_k \right| + C_\kappa \sum_{k=1}^{t_1-t_0} \left| \sum_{l=k+t_0}^{t_1} \xi_l \right| (u+1-t_0-k)^{-(\rho+1)}. \end{aligned}$$

Remark 6.3. *The last inequality just follows from the fact that $\kappa \geq \rho + 1$ by assumption.*

Proof. The proof is essentially based on a summation by parts argument. We set

$$\sum_{k=t_0}^{t_1} a_{u-k} \xi_k = \sum_{k=0}^{t_1-t_0} \underbrace{a_{u-t_0-k}}_{=: a'_k} \underbrace{\xi_{t_0+k}}_{=: \xi'_k}$$

and

$$\tilde{B}_k := \sum_{l=k}^{t_1-t_0} \xi'_l \quad \text{for } k \in \llbracket 0, t_1 - t_0 \rrbracket.$$

We then have

$$\begin{aligned} \sum_{k=0}^{t_1-t_0} a'_k \xi'_k &= \sum_{k=0}^{t_1-t_0-1} a'_k (\tilde{B}_k - \tilde{B}_{k+1}) + a'_{t_1-t_0} \xi'_{t_1-t_0} \\ &= \sum_{k=0}^{t_1-t_0} a'_k \tilde{B}_k - \sum_{k=1}^{t_1-t_0} a'_{k-1} \tilde{B}_k \\ &= a'_0 \tilde{B}_0 + \sum_{k=1}^{t_1-t_0} (a'_k - a'_{k-1}) \tilde{B}_k \\ \sum_{k=0}^{t_1-t_0} a'_k \xi'_k &= a_{u-t_0} \left(\sum_{k=t_0}^{t_1} \xi_k \right) + \sum_{k=1}^{t_1-t_0} \left(\sum_{l=k+t_0}^{t_1} \xi_l \right) [a_{u-t_0-k} - a_{u-t_0-(k-1)}]. \end{aligned}$$

Finally, by using triangular inequality and $(\mathbf{H}_{\text{poly}})$ we deduce that

$$\left| \sum_{k=t_0}^{t_1} a_{u-k} \xi_k \right| \leq C_\rho (u+1-t_0)^{-\rho} \left| \sum_{k=t_0}^{t_1} \xi_k \right| + C_\kappa \sum_{k=1}^{t_1-t_0} \left| \sum_{l=k+t_0}^{t_1} \xi_l \right| (u+1-t_0-k)^{-\kappa}.$$

□

In the next lemma we adopt the convention $\sum_{\emptyset} = 1$. Moreover, recall that by Proposition 6.1, we have for every $j \in \mathbb{N}^*$, $\Delta \tau_j \geq \varsigma^j$ for an arbitrary $\varsigma > 1$.

Lemma 6.2. *Assume $(\mathbf{H}_{\text{poly}})$. We suppose that $\tau_0 = 0$ and that there exists $\delta_1 \in (0, 1)$ such that for all $m \geq 1$ and $K > 0$, $\mathbb{P}(\mathcal{E}_m | \mathcal{E}_{m-1}) \geq \delta_1$. Then, for $i = 1, 2$, for all $p > 1$ and for every $\varepsilon \in (0, \rho - 1/2)$, there exists $C > 0$ such that for all $j \geq 1$, $m \in \{0, \dots, j-1\}$ and $K > 0$,*

$$\mathbb{E} \left[\sup_{u \in \llbracket \tau_{j-1}+1, \tau_j \rrbracket} |\Lambda_m^i(u)| | \mathcal{E}_j \right] \leq C \frac{\left(\sum_{l=m+1}^{j-1} \varsigma^l \right)^{1/2-\rho+\varepsilon}}{\delta_1^{\frac{j-m}{p}}}. \quad (6.19)$$

Consequently, there exist $\eta \in (0, 1)$ and $C > 0$ such that for all $j \geq 1$ and $m \in \{0, \dots, j-1\}$,

$$\mathbb{E} \left[\sup_{u \in \llbracket \tau_{j-1}+1, \tau_j \rrbracket} |\Lambda_m^i(u)| | \mathcal{E}_j \right] \leq C \eta^{j-m}. \quad (6.20)$$

Proof. First of all, let us prove that (6.19) induces (6.20). Let $\alpha_1 \in (0, +\infty)$ such that $\varsigma = \delta_1^{-\alpha_1}$. One just have to remark that for $j \geq 2$ and $m \in \{1, \dots, j-2\}$,

$$\left(\sum_{l=m+1}^{j-1} \varsigma^l \right)^{1/2-\rho+\varepsilon} \delta_1^{\frac{j-m}{p}} \leq \delta_1^{(\alpha_1(\rho-1/2-\varepsilon)-1/p)(j-m)}$$

We choose for instance $\varepsilon = \frac{1}{2}(\rho - 1/2)$ and $p > \frac{2}{\alpha_1(\rho-1/2)}$ in such a way that

$$\alpha_1(\rho - 1/2 - \varepsilon) - 1/p > 0.$$

We then deduce (6.20).

Now, it remains to show (6.19). For clarity, we set

$$E_j := \llbracket \tau_{j-1} + 1, \tau_j \rrbracket.$$

Using that for $m \geq 1$, $\mathcal{E}_m \subset \mathcal{E}_{m-1}$ and $\mathbb{P}(\mathcal{E}_m | \mathcal{E}_{m-1}) \geq \delta_1 \in (0, 1)$ and Hölder inequality we deduce the following inequalities,

$$\begin{aligned} \mathbb{E}[\sup_{u \in E_j} |\Lambda_m^i(u)| | \mathcal{E}_j] &\leq \mathbb{E}[\sup_{u \in E_j} |\Lambda_m^i(u)|^p | \mathcal{E}_j]^{1/p} = \left(\mathbb{E}[\sup_{u \in E_j} |\Lambda_m^i(u)|^p \mathbf{1}_{\mathcal{E}_j}] \frac{1}{\mathbb{P}(\mathcal{E}_j)} \right)^{1/p} \\ &\leq \left(\mathbb{E}[\sup_{u \in E_j} |\Lambda_m^i(u)|^p \mathbf{1}_{\mathcal{E}_{j-1}}] \frac{1}{\mathbb{P}(\mathcal{E}_j)} \right)^{1/p} = \left(\mathbb{E}[\sup_{u \in E_j} |\Lambda_m^i(u)|^p | \mathcal{E}_{j-1}] \frac{\mathbb{P}(\mathcal{E}_{j-1})}{\mathbb{P}(\mathcal{E}_j)} \right)^{1/p} \\ &\leq \delta_1^{-1/p} \mathbb{E}[\sup_{u \in E_j} |\Lambda_m^i(u)|^p | \mathcal{E}_{j-1}]^{1/p} \\ &\leq (\delta_1^{-1})^{\frac{j-m}{p}} \mathbb{E}[\sup_{u \in E_j} |\Lambda_m^i(u)|^p | \mathcal{E}_m]^{1/p} \quad (\text{by induction}). \end{aligned}$$

It remains to prove the existence of C such that for all $j \geq 1$, $m \in \{0, \dots, j-1\}$ and $K > 0$,

$$\mathbb{E}[\sup_{u \in E_j} |\Lambda_m^i(u)|^p | \mathcal{E}_m]^{1/p} \leq C \left(\sum_{l=m+1}^{j-1} \varsigma^l \right)^{1/2-\rho+\varepsilon}$$

with again the convention $\sum_{\emptyset} = 1$. We separate the end of the proof into three cases.

Case 1: $j \geq 3$ and $m \in \{1, \dots, j-2\}$.

By Lemma 6.1, applied with $t_0 = \tau_{m-1} + 1$ and $t_1 = \tau_m$

$$|\Lambda_m^i(u)| \leq C_\rho (u - \tau_{m-1})^{-\rho} \left| \sum_{k=\tau_{m-1}+1}^{\tau_m} \xi_k^i \right| + C_\kappa \sum_{k=1}^{\tau_m - \tau_{m-1} - 1} \left| \sum_{l=k+\tau_{m-1}+1}^{\tau_m} \xi_l^i \right| (u - \tau_{m-1} - k)^{-(\rho+1)}.$$

But, $u - \tau_{m-1} \geq \tau_{j-1} - \tau_{m-1} \geq \sum_{l=m+1}^{j-1} \varsigma^l$ and $u - \tau_{m-1} - k \geq \tau_{j-1} - \tau_m + 1 \geq \sum_{l=m+1}^{j-1} \varsigma^l$. Let $\varepsilon \in (0, \rho - 1/2)$, we then have

$$\begin{aligned} |\Lambda_m^i(u)| &\leq \left(\sum_{l=m+1}^{j-1} \varsigma^l \right)^{1/2-\rho+\varepsilon} \left[C_\rho (u - \tau_{m-1})^{-(1/2+\varepsilon)} \left| \sum_{k=\tau_{m-1}+1}^{\tau_m} \xi_k^i \right| \right. \\ &\quad \left. + C_\kappa \sum_{k=1}^{\tau_m - \tau_{m-1} - 1} \left| \sum_{l=k+\tau_{m-1}+1}^{\tau_m} \xi_l^i \right| (u - \tau_{m-1} - k)^{-(3/2+\varepsilon)} \right]. \end{aligned}$$

We denote by $\tilde{\Lambda}_m^i(u)$ the above quantity between brackets. Hence

$$\mathbb{E}[\sup_{u \in E_j} |\Lambda_m^i(u)|^p | \mathcal{E}_m]^{1/p} \leq \left(\sum_{l=m+1}^{j-1} \varsigma^l \right)^{1/2-\rho+\varepsilon} \mathbb{E}[\sup_{u \in E_j} |\tilde{\Lambda}_m^i(u)|^p | \mathcal{E}_m]^{1/p}.$$

We now have to prove the existence of C such that

$$\mathbb{E}[\sup_{u \in E_j} |\tilde{\Lambda}_m^i(u)|^p | \mathcal{E}_m]^{1/p} \leq C \quad \text{for all } p \in (1, +\infty).$$

We write $\mathcal{E}_m = \cup_{\ell \geq 0} \mathcal{A}_{m,\ell}$ with

$$\mathcal{A}_{m,\ell} = \mathcal{B}_{m,\ell}^c \cap \mathcal{B}_{m,\ell-1} \quad (6.21)$$

where $\mathcal{B}_{m,\ell}$ is defined in (5.18). In other words, $\mathcal{A}_{m,0}$ is the failure of Step 1 of tentative m and for $\ell \geq 1$, $\mathcal{A}_{m,\ell}$ is the event ‘‘Step 2 of trial m fails after exactly ℓ attempts’’.

Let $\ell \in \mathbb{N}$, we begin by studying $\mathbb{E}[\sup_{u \in E_j} |\tilde{\Lambda}_m^i(u)|^p | \mathcal{A}_{m,\ell}]^{1/p}$. Since $u > \tau_m$,

$$|\tilde{\Lambda}_m^i(u)| \leq C_\rho (\Delta \tau_m)^{-(1/2+\varepsilon)} \left| \sum_{k=\tau_{m-1}+1}^{\tau_{m-1}+\Delta \tau_m} \xi_k^i \right| + C_\kappa \sum_{k=1}^{\Delta \tau_{m-1}} \left| \sum_{l=k+\tau_{m-1}+1}^{\tau_{m-1}+\Delta \tau_m} \xi_l^i \right| (\Delta \tau_m - k)^{-(3/2+\varepsilon)}.$$

By Minkowski inequality and the fact that $\Delta \tau_m =: \Delta(m, \ell)$ is constant on $\mathcal{A}_{m,\ell}$, we get

$$\begin{aligned} \mathbb{E}[\sup_{u \in E_j} |\tilde{\Lambda}_m^i(u)|^p | \mathcal{A}_{m,\ell}]^{1/p} &\leq C_\rho (\Delta(m, \ell))^{-(1/2+\varepsilon)} \mathbb{E} \left[\left| \sum_{k=\tau_{m-1}+1}^{\tau_{m-1}+\Delta(m, \ell)} \xi_k^i \right|^p | \mathcal{A}_{m,\ell} \right]^{1/p} \\ &\quad + C_\kappa \sum_{k=1}^{\Delta(m, \ell)-1} (\Delta(m, \ell) - k)^{-(3/2+\varepsilon)} \mathbb{E} \left[\left| \sum_{l=k+\tau_{m-1}+1}^{\tau_{m-1}+\Delta(m, \ell)} \xi_l^i \right|^p | \mathcal{A}_{m,\ell} \right]^{1/p}. \end{aligned} \quad (6.22)$$

Moreover, using Cauchy-Schwarz inequality,

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{k=\tau_{m-1}+1}^{\tau_{m-1}+\Delta(m, \ell)} \xi_k^i \right|^p | \mathcal{A}_{m,\ell} \right]^{1/p} &= \mathbb{E} \left[\left| \sum_{k=\tau_{m-1}+1}^{\tau_{m-1}+\Delta(m, \ell)} \xi_k^i \right|^p \mathbf{1}_{\mathcal{A}_{m,\ell}} | \mathcal{E}_{m-1} \right]^{1/p} \mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_{m-1})^{-1/p} \\ &\leq \mathbb{E} \left[\left(\sum_{k=\tau_{m-1}+1}^{\tau_{m-1}+\Delta(m, \ell)} \xi_k^i \right)^{2p} | \mathcal{E}_{m-1} \right]^{1/2p} \mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_{m-1})^{-1/2p} \\ &\leq c_p \sqrt{\Delta(m, \ell)} \mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_{m-1})^{-1/2p}. \end{aligned} \quad (6.23)$$

In the last inequality we use the fact that $\sum_{k=\tau_{m-1}+1}^{\tau_{m-1}+\Delta(m, \ell)} \xi_k^i$ is independent from \mathcal{E}_{m-1} and that its law is $\mathcal{N}(0, \Delta(m, \ell))$. In the same way, we obtain

$$\mathbb{E} \left[\left| \sum_{l=k+\tau_{m-1}+1}^{\tau_{m-1}+\Delta(m, \ell)} \xi_l^i \right|^p | \mathcal{A}_{m,\ell} \right]^{1/p} \leq c_p \sqrt{\Delta(m, \ell) - k} \mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_{m-1})^{-1/2p}. \quad (6.24)$$

We deduce from (6.23) and (6.24) that in (6.22)

$$\begin{aligned} \mathbb{E}[\sup_{u \in E_j} |\tilde{\Lambda}_m^i(u)|^p | \mathcal{A}_{m,\ell}]^{1/p} &\leq \left(c_p C_\rho (\Delta(m, \ell))^{-\varepsilon} + c_p C_\kappa \sum_{k=1}^{\Delta(m, \ell)-1} (\Delta(m, \ell) - k)^{-(1+\varepsilon)} \right) \mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_{m-1})^{-1/2p} \\ &\leq C_{p,\varepsilon} \mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_{m-1})^{-1/2p}. \end{aligned} \quad (6.25)$$

Then by using the inequality $(a + b)^{1/p} \leq a^{1/p} + b^{1/p}$ for $p > 1$ and (6.25) for ℓ from 0 to $+\infty$, we get

$$\begin{aligned} \mathbb{E}[\sup_{u \in E_j} |\tilde{\Lambda}_m^i(u)|^p | \mathcal{E}_m]^{1/p} &= \left(\sum_{\ell \geq 0} \mathbb{E}[\sup_{u \in E_j} |\tilde{\Lambda}_m^i(u)|^p | \mathcal{A}_{m,\ell}] \mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_m) \right)^{1/p} \\ &\leq \sum_{\ell \geq 0} \mathbb{E}[\sup_{u \in E_j} |\tilde{\Lambda}_m^i(u)|^p | \mathcal{A}_{m,\ell}]^{1/p} \mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_m)^{1/p} \\ &\leq C_{p,\varepsilon} \sum_{\ell \geq 0} \mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_{m-1})^{-1/2p} \left(\frac{\mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_{m-1})}{\mathbb{P}(\mathcal{E}_m | \mathcal{E}_{m-1})} \right)^{1/p} \\ &\leq C_{p,\varepsilon} \delta_1^{-1/p} \sum_{\ell \geq 0} \mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_{m-1})^{1/2p}. \end{aligned}$$

But $\mathcal{A}_{m,\ell} \subset \mathcal{E}_m \subset \mathcal{E}_{m-1}$ hence for $\ell > 0$,

$$\mathbb{P}(\mathcal{A}_{m,\ell} | \mathcal{E}_{m-1}) = \mathbb{P}(\mathcal{B}_{m,\ell-1} | \mathcal{E}_{m-1}) \mathbb{P}(\mathcal{B}_{m,\ell}^c | \mathcal{B}_{m,\ell-1}) \leq \mathbb{P}(\mathcal{B}_{m,\ell-1}^c | \mathcal{B}_{m,\ell-1}),$$

Therefore, for all $\varepsilon \in (0, \rho - 1/2)$ and $p \in (1, +\infty)$, by applying Lemma 5.2, this gives us the existence of C such that

$$\mathbb{E}[\sup_{u \in E_j} |\tilde{\Lambda}_m^i(u)|^p | \mathcal{E}_m]^{1/p} \leq C_{p,\varepsilon} \delta_1^{-1/p} (2 + \sum_{\ell \geq 2} 2^{-\alpha \ell / 2p}) < C.$$

The first case is now achieved.

Case 2: Let $j \geq 2$ and $m = j - 1$.

The proof is almost exactly the same as in case 1. We simply use the following controls

$$u - \tau_{j-2} > \Delta \tau_{j-1} \quad \text{and} \quad u - \tau_{j-2} - k > \Delta \tau_{j-1} - k$$

and we do not introduce ε which is useless here since $\sum_{l=m+1}^{j-1} = \sum_{\emptyset}$.

Case 3: Let $j \geq 1$ and $m = 0$. By assumption $\tau_0 = 0$, then

$$\Lambda_0^i(u) = \sum_{k=-\infty}^{k=0} a_{u-k} \xi_k^i.$$

By Lemma 6.1, for all $M > 0$,

$$\begin{aligned} \left| \sum_{k=-M}^{k=0} a_{u-k} \xi_k^i \right| &\leq C_\rho (u + 1 + M)^{-\rho} \left| \sum_{k=-M}^0 \xi_k^i \right| + C_\kappa \sum_{k=1}^M \left| \sum_{l=k-M}^0 \xi_l^i \right| (u + 1 + M - k)^{-(\rho+1)} \\ &\leq C_\rho (1 + M)^{-\rho} \left| \sum_{k=-M}^0 \xi_k^i \right| + C_\kappa \sum_{k=1}^M \left| \sum_{l=k-M}^0 \xi_l^i \right| (M + 1 - k)^{-(\rho+1)} \\ &= C_\rho (1 + M)^{-\rho} \left| \sum_{k=-M}^0 \xi_k^i \right| + C_\kappa \sum_{k=0}^{M-1} \left| \sum_{l=-k}^0 \xi_l^i \right| (k + 1)^{-(\rho+1)} \\ &= C_\rho (1 + M)^{-\rho} \left| \sum_{k=0}^M \xi_{-k}^i \right| + C_\kappa \sum_{k=0}^{M-1} \left| \sum_{l=0}^k \xi_{-l}^i \right| (k + 1)^{-(\rho+1)}. \end{aligned}$$

Since $\rho > 1/2$, by means of Borel-Cantelli Lemma and the fact that $\sum_{k=0}^M \xi_{-k}^i \sim \mathcal{N}(0, M + 1)$, we can show that $\lim_{M \rightarrow +\infty} (1 + M)^{-\rho} \left(\sum_{k=0}^M \xi_{-k}^i \right) = 0$ a.s.

We then get

$$|\Lambda_0^i(u)| \leq C_\kappa \sum_{k=0}^{+\infty} \left| \sum_{l=0}^k \xi_{-l}^i \right| (k + 1)^{-(\rho+1)}.$$

Set $W_k^i = \sum_{l=0}^{k-1} \xi_{-l}^i$ for $k > 0$ and $W_0^i = \xi_0^i$. Using Minkowski inequality, we have for all $p \in (1, +\infty)$ and for all $\varepsilon \in (0, \rho - 1/2)$

$$\begin{aligned} \mathbb{E}[\sup_{u \in E_j} |\Lambda_0^i(u)|^p | \mathcal{E}_0]^{1/p} &= \mathbb{E}[\sup_{u \in E_j} |\Lambda_0^i(u)|^p]^{1/p} \leq C_\kappa \sum_{k=0}^{+\infty} (k+1)^{-(\rho+1/2-\varepsilon)} \mathbb{E} \left[\left(\frac{|W_{k+1}^i|}{(k+1)^{1/2+\varepsilon}} \right)^p \right]^{1/p} \\ &\leq C_\kappa \mathbb{E} \left[\left(\sup_{k \geq 0} \frac{|W_{k+1}^i|}{(k+1)^{1/2+\varepsilon}} \right)^p \right]^{\frac{1}{p}} \sum_{k=0}^{+\infty} (k+1)^{-(\rho+1/2-\varepsilon)} \\ &\leq C \mathbb{E} \left[\left(\sup_{k \geq 0} \frac{|W_{k+1}^i|}{(k+1)^{1/2+\varepsilon}} \right)^p \right]^{\frac{1}{p}} \end{aligned}$$

because $\rho + 1/2 - \varepsilon > 1$. It remains to prove that for $\varepsilon \in (0, \rho - 1/2)$ and for $p \in (1, +\infty)$

$$\mathbb{E} \left[\left(\sup_{k \geq 0} \frac{|W_{k+1}^i|}{(k+1)^{1/2+\varepsilon}} \right)^p \right]^{\frac{1}{p}} < +\infty.$$

Thanks to a summation by parts, we can show that

$$\frac{W_{k+1}^i}{(k+1)^{1/2+\varepsilon}} = \sum_{l=0}^k \frac{\xi_{-l}^i}{(l+1)^{1/2+\varepsilon}} + \sum_{l=1}^k W_l^i \left(\frac{1}{(l+1)^{1/2+\varepsilon}} - \frac{1}{l^{1/2+\varepsilon}} \right).$$

Hence, using that $\left| \frac{1}{(l+1)^{1/2+\varepsilon}} - \frac{1}{l^{1/2+\varepsilon}} \right| \leq (1/2 + \varepsilon) \frac{1}{l^{3/2+\varepsilon}}$, we get

$$\frac{|W_{k+1}^i|}{(k+1)^{1/2+\varepsilon}} \leq \left| \sum_{l=0}^k \frac{\xi_{-l}^i}{(l+1)^{1/2+\varepsilon}} \right| + (1/2 + \varepsilon) \sum_{l=1}^k \frac{|W_l^i|}{l^{3/2+\varepsilon}}.$$

Therefore

$$\sup_{k \geq 0} \frac{|W_{k+1}^i|}{(k+1)^{1/2+\varepsilon}} \leq \sup_{k \geq 0} \left| \sum_{l=0}^k \frac{\xi_{-l}^i}{(l+1)^{1/2+\varepsilon}} \right| + (1/2 + \varepsilon) \sum_{l=1}^{+\infty} \frac{|W_l^i|}{l^{3/2+\varepsilon}}.$$

We use again Minkowski inequality, which gives

$$\left\| \sup_{k \geq 0} \frac{|W_{k+1}^i|}{(k+1)^{1/2+\varepsilon}} \right\|_p \leq \left\| \sup_{k \geq 0} \left| \sum_{l=0}^k \frac{\xi_{-l}^i}{(l+1)^{1/2+\varepsilon}} \right| \right\|_p + (1/2 + \varepsilon) \sum_{l=1}^{+\infty} \frac{\|W_l^i\|_p}{l^{3/2+\varepsilon}}.$$

On the one hand, we have $\|W_l^i\|_p \leq c_p \sqrt{l}$ because $W_l^i \sim \mathcal{N}(0, l)$. On the other hand, set $N_k := \sum_{l=0}^k \frac{\xi_{-l}^i}{(l+1)^{1/2+\varepsilon}}$. This is a martingale with distribution $\mathcal{N}\left(0, \sum_{l=0}^k (l+1)^{-(1+2\varepsilon)}\right)$ therefore $\|N_k\|_p \leq c_p$ with c_p independent from k . Hence, $(N_k)_{k \in \mathbb{N}}$ converges a.s. and in L^p into $N_\infty \in L^p$. We then deduce by Doob's inequality that

$$\left\| \sup_{k \geq 0} |N_k| \right\|_p \leq \left(\frac{p-1}{p} \right) \|N_\infty\|_p.$$

Finalement,

$$\left\| \sup_{k \geq 0} \frac{|W_{k+1}^i|}{(k+1)^{1/2+\varepsilon}} \right\|_p < +\infty.$$

which achieves the third case. \square

Proposition 6.4. *Assume $(\mathbf{H}_{\text{poly}})$. We suppose that $\tau_0 = 0$ and that for all $m \geq 1$ and $K > 0$, $\mathbb{P}(\mathcal{E}_m | \mathcal{E}_{m-1}) \geq \delta_1 \in (0, 1)$. Then (\mathbf{H}'_1) holds true.*

Proof. First, thanks to (6.18), we have

$$\sum_{l=1}^{\Delta\tau_j} \gamma^{\Delta\tau_j-l} |\Delta_{\tau_{j-1}+l}^i| \leq \sum_{m=0}^j \sum_{u=\tau_{j-1}+1}^{\tau_j} \gamma^{\tau_j-u} |\Lambda_m^i(u)|.$$

The aim is to bound every term in the right-hand side. For $m \in \{0, \dots, j-1\}$ and for all $u \in E_j := \llbracket \tau_{j-1} + 1, \tau_j \rrbracket$,

$$|\Lambda_m^i(u)| \leq \sup_{u \in E_j} |\Lambda_m^i(u)|.$$

Since the right-hand side does not depend on u anymore, we deduce that for all $m \in \{0, \dots, j-1\}$

$$\sum_{u=\tau_{j-1}+1}^{\tau_j} \gamma^{\tau_j-u} |\Lambda_m^i(u)| \leq \sup_{u \in E_j} |\Lambda_m^i(u)| \sum_{w=0}^{+\infty} \gamma^w = \sup_{u \in E_j} |\Lambda_m^i(u)| \frac{1}{1-\gamma}.$$

Hence, Lemma 6.2, gives that for all $m \in \{0, \dots, j-1\}$

$$\mathbb{E} \left[\sum_{u=\tau_{j-1}+1}^{\tau_j} \gamma^{\tau_j-u} |\Lambda_m^i(u)| \middle| \mathcal{E}_j \right] \leq \frac{C}{1-\gamma} \eta^{j-m}$$

where $\eta \in (0, 1)$. Consequently,

$$\mathbb{E} \left[\sum_{m=0}^{j-1} \sum_{u=\tau_{j-1}+1}^{\tau_j} \gamma^{\tau_j-u} |\Lambda_m^i(u)| \middle| \mathcal{E}_j \right] \leq \frac{C}{(1-\gamma)(1-\eta)}. \quad (6.26)$$

In inequality (6.18), it then remains to bound the term with $\Lambda_j^i(u)$. By substitution, we obtain for $m = j$

$$\sum_{u=\tau_{j-1}+1}^{\tau_j} \gamma^{\tau_j-u} |\Lambda_j^i(u)| = \sum_{v=1}^{\Delta\tau_j} \gamma^{\Delta\tau_j-v} |\Lambda_j^i(v + \tau_{j-1})|.$$

As in the proof of Lemma 6.2, we use the decomposition of \mathcal{E}_j through the events $\mathcal{A}_{j,\ell}$ and that $\Delta\tau_j =: \Delta(j, \ell)$ is constant on $\mathcal{A}_{j,\ell}$:

$$\mathbb{E} \left[\sum_{u=\tau_{j-1}+1}^{\tau_j} \gamma^{\tau_j-u} |\Lambda_j^i(u)| \middle| \mathcal{E}_j \right] = \sum_{\ell \geq 0} \sum_{v=1}^{\Delta(j,\ell)} \gamma^{\Delta(j,\ell)-v} \mathbb{E} [|\Lambda_j^i(v + \tau_{j-1})| \middle| \mathcal{A}_{j,\ell}] \mathbb{P}(\mathcal{A}_{j,\ell} | \mathcal{E}_j). \quad (6.27)$$

Using that $\mathcal{A}_{j,\ell} \subset \mathcal{E}_j \subset \mathcal{E}_{j-1}$ and Cauchy-Schwarz inequality, one notes that

$$\begin{aligned} \mathbb{E} [|\Lambda_j^i(v + \tau_{j-1})| \middle| \mathcal{A}_{j,\ell}] \mathbb{P}(\mathcal{A}_{j,\ell} | \mathcal{E}_j) &\leq \mathbb{E} [|\Lambda_j^i(v + \tau_{j-1})|^2 \middle| \mathcal{E}_j]^{1/2} \mathbb{P}(\mathcal{A}_{j,\ell} | \mathcal{E}_j)^{1/2} \\ &\leq \frac{\mathbb{E} [|\Lambda_j^i(v + \tau_{j-1})|^4 \middle| \mathcal{E}_{j-1}]^{1/4}}{\mathbb{P}(\mathcal{E}_j | \mathcal{E}_{j-1})^{1/4}} \mathbb{P}(\mathcal{A}_{j,\ell} | \mathcal{E}_j)^{1/2} \\ &\leq \frac{\sup_{v \in \mathbb{N}^*} \mathbb{E} [|\Lambda_j^i(v + \tau_{j-1})|^4 \middle| \mathcal{E}_{j-1}]^{1/4}}{\mathbb{P}(\mathcal{E}_j | \mathcal{E}_{j-1})^{1/4}} \mathbb{P}(\mathcal{A}_{j,\ell} | \mathcal{E}_j)^{1/2}. \end{aligned} \quad (6.28)$$

But $\mathbb{P}(\mathcal{E}_j | \mathcal{E}_{j-1}) \geq \delta_1 > 0$ and by Lemma 5.2, we have for all $\ell \geq 2$,

$$\mathbb{P}(\mathcal{A}_{j,\ell} | \mathcal{E}_j) = \frac{\mathbb{P}(\mathcal{A}_{j,\ell} | \mathcal{E}_{j-1})}{\mathbb{P}(\mathcal{E}_j | \mathcal{E}_{j-1})} \leq \delta_1^{-1} \mathbb{P}(\mathcal{B}_{j,\ell-1} | \mathcal{E}_{j-1}) \mathbb{P}(\mathcal{B}_{j,\ell}^c | \mathcal{B}_{j,\ell-1}) \leq \delta_1^{-1} 2^{-\bar{\alpha}\ell}. \quad (6.29)$$

We now use (6.28) and (6.29) into (6.27) and this gives the existence of $C_{\delta_1, \gamma}$ such that

$$\mathbb{E} \left[\sum_{u=\tau_{j-1}+1}^{\tau_j} \gamma^{\tau_j-u} |\Lambda_j^i(u)| \middle| \mathcal{E}_j \right] \leq C_{\delta_1, \gamma} \sup_{v \in \mathbb{N}^*} \mathbb{E} [|\Lambda_j^i(v + \tau_{j-1})|^4 \middle| \mathcal{E}_{j-1}]^{1/4}. \quad (6.30)$$

It only remains to show that

$$\sup_{v \in \mathbb{N}^*} \mathbb{E}[|\Lambda_j^i(v + \tau_{j-1})|^4 | \mathcal{E}_{j-1}]^{1/4} < +\infty.$$

By Lemma 6.1 and the definition of Λ_j^i in (6.17),

$$\begin{aligned} |\Lambda_j^i(v + \tau_{j-1})| &= \left| \sum_{k=\tau_{j-1}+1}^{v+\tau_{j-1}} a_{v+\tau_{j-1}-k} \xi_k^i \right| \\ &\leq C_\rho v^{-\rho} \left| \sum_{k=\tau_{j-1}+1}^{v+\tau_{j-1}} \xi_k^i \right| + C_\kappa \sum_{k=1}^{v-1} \left| \sum_{l=k+\tau_{j-1}+1}^{v+\tau_{j-1}} \xi_l^i \right| (v-k)^{-(\rho+1)}. \end{aligned}$$

We again apply Minkowski inequality

$$\begin{aligned} &\mathbb{E}[|\Lambda_j^i(v + \tau_{j-1})|^4 | \mathcal{E}_{j-1}]^{1/4} \\ &\leq C_\rho v^{-\rho} \mathbb{E} \left[\left| \sum_{k=\tau_{j-1}+1}^{v+\tau_{j-1}} \xi_k^i \right|^4 \middle| \mathcal{E}_{j-1} \right]^{1/4} + C_\kappa \sum_{k=1}^{v-1} (v-k)^{-(\rho+1)} \mathbb{E} \left[\left| \sum_{l=k+\tau_{j-1}+1}^{v+\tau_{j-1}} \xi_l^i \right|^4 \middle| \mathcal{E}_{j-1} \right]^{1/4} \\ &= C_\rho v^{-\rho} \mathbb{E} \left[\left| \sum_{k=1}^v \xi_k^i \right|^4 \right]^{1/4} + C_\kappa \sum_{k=1}^{v-1} (v-k)^{-(\rho+1)} \mathbb{E} \left[\left| \sum_{l=k+1}^v \xi_l^i \right|^4 \right]^{1/4} \\ &\leq c_4 \left(C_\rho v^{-\rho+1/2} + C_\kappa \sum_{k=1}^{v-1} (v-k)^{-(\rho+1/2)} \right) \end{aligned}$$

where c_4 is related to the 4th moment of a centered and reduced Gaussian random variable. Since $\rho + 1/2 > 1$, we immediately deduce that

$$\sup_{v \in \mathbb{N}^*} \mathbb{E}[|\Lambda_j^i(v + \tau_{j-1})|^4 | \mathcal{E}_{j-1}]^{1/4} < +\infty. \quad (6.31)$$

We put together (6.26), (6.30) and (6.31) to conclude the proof of (\mathbf{H}'_1) . \square

7 Proof of Theorem 2.1

Now we have all the necessary elements to prove the second part of the main theorem 2.1 concerning the convergence in total variation to the unique invariant distribution (where the uniqueness will immediately follow from this convergence).

We recall that $\Delta\tau_j$ denotes the duration of coupling trial j and we set

$$j^{(s)} := \inf\{j > 0, \Delta\tau_j = +\infty\}. \quad (7.1)$$

$j^{(s)}$ corresponds to the trial where the coupling procedure is successful. The aim of this section is to bound $\mathbb{P}(\tau_\infty > n)$, where

$$\tau_\infty = \inf\{n \geq 0 \mid X_k^1 = X_k^2, \forall k \geq n\},$$

since

$$\|\mathcal{L}((X_k^1)_{k \geq n}) - \mathcal{S}\mu_\star\|_{TV} \leq \mathbb{P}(\tau_\infty > n).$$

But, we have

$$\mathbb{P}(\tau_\infty > n) = \mathbb{P} \left(\sum_{k=1}^{+\infty} \Delta\tau_k \mathbb{1}_{\{j^{(s)} > k\}} > n \right)$$

where $j^{(s)}$ is defined in (7.1). It remains to bound the right term. Let $p \in (0, +\infty)$.

If $p \in (0, 1)$, then by the Markov inequality and the simple inequality $|a + b|^p \leq |a|^p + |b|^p$, we get

$$\begin{aligned} \mathbb{P}\left(\sum_{k=1}^{+\infty} \Delta\tau_k \mathbf{1}_{\{j^{(s)} > k\}} > n\right) &\leq \frac{1}{n^p} \sum_{k=1}^{+\infty} \mathbb{E}[|\Delta\tau_k|^p \mathbf{1}_{\{j^{(s)} > k\}}] \\ &\leq \frac{1}{n^p} \sum_{k=1}^{+\infty} \mathbb{E}[\mathbb{E}[|\Delta\tau_k|^p \mathbf{1}_{\{\Delta\tau_k < +\infty\}} \mid \{\tau_{k-1} < +\infty\}}] \mathbf{1}_{\{\tau_{k-1} < +\infty\}}]. \end{aligned} \quad (7.2)$$

Else, if $p \geq 1$, by Markov inequality and Minkowski inequality, we have

$$\begin{aligned} \mathbb{P}\left(\sum_{k=1}^{+\infty} \Delta\tau_k \mathbf{1}_{\{j^{(s)} > k\}} > n\right) &\leq \frac{1}{n^p} \mathbb{E}\left[\left(\sum_{k=1}^{+\infty} \Delta\tau_k \mathbf{1}_{\{j^{(s)} > k\}}\right)^p\right] \\ &\leq \frac{1}{n^p} \left(\sum_{k=1}^{+\infty} \mathbb{E}[|\Delta\tau_k|^p \mathbf{1}_{\{j^{(s)} > k\}}]^{1/p}\right)^p \\ &\leq \frac{1}{n^p} \left(\sum_{k=1}^{+\infty} \mathbb{E}[\mathbb{E}[|\Delta\tau_k|^p \mathbf{1}_{\{\Delta\tau_k < +\infty\}} \mid \{\tau_{k-1} < +\infty\}}] \mathbf{1}_{\{\tau_{k-1} < +\infty\}}]^{1/p}\right)^p. \end{aligned} \quad (7.3)$$

We recall that the event $\mathcal{A}_{k,\ell}$ defined in (6.21) corresponds to the failure of Step 2 after ℓ attempts at trial k . Both in (7.2) and (7.3), we separate the term $\mathbb{E}[|\Delta\tau_k|^p \mathbf{1}_{\{\Delta\tau_k < +\infty\}} \mid \{\tau_{k-1} < +\infty\}]$ through the events $\mathcal{A}_{k,\ell}$ which gives

$$\mathbb{E}[|\Delta\tau_k|^p \mathbf{1}_{\{\Delta\tau_k < +\infty\}} \mid \{\tau_{k-1} < +\infty\}] = \sum_{\ell=1}^{+\infty} \mathbb{E}[\mathbf{1}_{\mathcal{A}_{k,\ell}} |\Delta\tau_k|^p \mathbf{1}_{\{\Delta\tau_k < +\infty\}} \mid \{\tau_{k-1} < +\infty\}]. \quad (7.4)$$

Moreover, thanks to Lemma 5.2 and the definition of the events $\mathcal{A}_{k,\ell}$, we deduce that for $\ell \geq 2$,

$$\mathbb{P}(\mathcal{A}_{k,\ell} \mid \{\tau_{k-1} < +\infty\}) \leq 2^{-\tilde{\alpha}\ell} \quad (7.5)$$

where $\tilde{\alpha} := \begin{cases} \min\{\alpha, \beta, \alpha + \beta - 1\} - 1/2 - \varepsilon & \text{for all } \varepsilon > 0 \quad \text{under } (\mathbf{H}_{\text{poly}}) \\ \min(\alpha, \mu) & \text{under } (\mathbf{H}_{\text{exp}}). \end{cases}$

We have now to distinguish the polynomial case from the exponential one.

▷ Under $(\mathbf{H}_{\text{poly}})$:

We have a bound of type $\Delta\tau_k \leq C_1 \varsigma^k 2^{\theta\ell}$ (due to the value of $\Delta t_3^{(k)}$, see Proposition 6.1) on the event $\mathcal{A}_{k,\ell}$ where $\varsigma > 1$ is arbitrary. Indeed, on $\mathcal{A}_{k,\ell}$, we have

$$\begin{aligned} \Delta\tau_k = \tau_k - \tau_{k-1} &\leq c_2 2^{\ell+1} + \Delta t_3^{(k)} \\ &= c_2 2^{\ell+1} + t_* \varsigma^k 2^{\theta\ell} \\ &\leq C_1 \varsigma^k 2^{(\theta \vee 1)\ell} \quad (\text{for } C_1 \text{ large enough.}) \end{aligned}$$

Hence, in (7.4) we get

$$\begin{aligned} \mathbb{E}[|\Delta\tau_k|^p \mathbf{1}_{\{\Delta\tau_k < +\infty\}} \mid \{\tau_{k-1} < +\infty\}] &\leq C_1^p \varsigma^{kp} \left(\sum_{\ell=1}^{+\infty} 2^{(\theta \vee 1)p\ell} \mathbb{P}(\mathcal{A}_{k,\ell} \mid \{\tau_{k-1} < +\infty\})\right) \\ &\leq C_1^p \varsigma^{kp} \left(2^{(\theta \vee 1)p} + \sum_{\ell=2}^{+\infty} 2^{((\theta \vee 1)p - \tilde{\alpha})\ell}\right) \quad \text{using (7.5)} \\ &\leq C \varsigma^{kp} \quad \text{if } p \in \left(0, \frac{\tilde{\alpha}}{\theta \vee 1}\right). \end{aligned}$$

Then for $p \in (0, \frac{\tilde{\alpha}}{\theta\sqrt{1}})$,

$$\mathbb{E}[\mathbb{E}[|\Delta\tau_k|^p \mathbf{1}_{\{\Delta\tau_k < +\infty\}} \mid \{\tau_{k-1} < +\infty\}] \mathbf{1}_{\{\tau_{k-1} < +\infty\}}] \leq C\zeta^{kp} \mathbb{P}(j^{(s)} > k-1) \quad (7.6)$$

and it remains to control $\mathbb{P}(j^{(s)} > k-1)$. We have

$$\mathbb{P}(j^{(s)} > k-1) = \prod_{j=1}^{k-1} \mathbb{P}(\mathcal{E}_j \mid \mathcal{E}_{j-1}) = \prod_{j=1}^{k-1} (1 - \mathbb{P}(\mathcal{E}_j^c \mid \mathcal{E}_{j-1}))$$

where \mathcal{E}_j is defined in (6.10). By Proposition 5.1 and 6.2 applied for $\varepsilon = 1/2$, we get for every $j \geq 2$,

$$\mathbb{P}(\mathcal{E}_j^c \mid \mathcal{E}_{j-1}) \geq \mathbb{P}(\Delta\tau_j = +\infty \mid \Omega_{K_{1/2}, \alpha, \tau_{j-1}}) \mathbb{P}(\Omega_{K_{1/2}, \alpha, \tau_{j-1}} \mid \mathcal{E}_{j-1}) \geq \frac{\delta_0}{2}$$

where $\delta_0 > 0$ depends on $K_{1/2}$. Therefore,

$$\mathbb{P}(j^{(s)} > k-1) \leq \left(1 - \frac{\delta_0}{2}\right)^{k-1}$$

and by (7.6)

$$\mathbb{E}[\mathbb{E}[|\Delta\tau_k|^p \mathbf{1}_{\{\Delta\tau_k < +\infty\}} \mid \{\tau_{k-1} < +\infty\}] \mathbf{1}_{\{\tau_{k-1} < +\infty\}}] \leq C\zeta^{kp} \left(1 - \frac{\delta_0}{2}\right)^{k-1}. \quad (7.7)$$

Finally, by choosing $1 < \varsigma < (1 - \frac{\delta_0}{2})^{-1/p}$, we get using (7.2) or (7.3) that for all $p \in (0, \frac{\tilde{\alpha}}{\theta\sqrt{1}})$, there exists $C_p > 0$ such that

$$\mathbb{P}(\tau_\infty > n) \leq \mathbb{P}\left(\sum_{k=1}^{+\infty} \Delta\tau_k \mathbf{1}_{\{j^{(s)} > k\}} > n\right) \leq C_p n^{-p}. \quad (7.8)$$

It remains to optimize the upper-bound $\frac{\tilde{\alpha}}{\theta\sqrt{1}}$ for p . Since $\tilde{\alpha} := \min\{\alpha, \beta, \alpha + \beta - 1\} - 1/2 - \varepsilon$ with $\varepsilon > 0$ as small as necessary and since by Proposition 6.1

$$\theta > (2(\rho - \alpha))^{-1} \text{ and } \alpha \in \left(\frac{1}{2} \vee \left(\frac{3}{2} - \beta\right), \rho\right),$$

we finally get (7.8) for all $p \in (0, v(\beta, \rho))$ where

$$v(\beta, \rho) = \sup_{\alpha \in (\frac{1}{2} \vee (\frac{3}{2} - \beta), \rho)} \min\{1, 2(\rho - \alpha)\} (\min\{\alpha, \beta, \alpha + \beta - 1\} - 1/2)$$

which concludes the proof of Theorem 2.1 in the polynomial case.

▷ Under (**H_{exp}**):

The proof is almost the same. The only differences are that we use the following bound

$$\begin{aligned} \Delta\tau_k = \tau_k - \tau_{k-1} &\leq c_2(\ell + 1) + \Delta t_3^{(k)} \\ &= c_2(\ell + 1) + t_* + \varsigma^k + \theta\ell \\ &\leq C_1 \varsigma^k \theta\ell \quad (\text{for } C_1 \text{ large enough}) \end{aligned}$$

on the events $\mathcal{A}_{k, \ell}$ and the upperbound $\mathbb{P}(\mathcal{A}_{k, \ell} \mid \{\tau_{k-1} < +\infty\}) \leq 2^{-\tilde{\alpha}\ell}$ given in (7.5). And then we get for all $p > 0$ the existence of $C_p > 0$ such that

$$\mathbb{P}(\tau_\infty > n) \leq \mathbb{P}\left(\sum_{k=1}^{+\infty} \Delta\tau_k \mathbf{1}_{\{j^{(s)} > k\}} > n\right) \leq C_p n^{-p}. \quad (7.9)$$

by choosing $1 < \varsigma < (1 - \frac{\delta_0}{2})^{-1/p}$ and the proof of Theorem 2.1 is over.

A Explicit formula for the sequence $(b_k)_{k \geq 0}$

Theorem A.1. Let $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ be two sequences such that for $n \in \mathbb{N}$,

$$u_n = \sum_{k=0}^n a_k v_{n-k} \quad (\text{A.1})$$

then we have:

$$v_n = \sum_{k=0}^n b_k u_{n-k} \quad (\text{A.2})$$

where

$$b_0 := \frac{1}{a_0} \quad \text{and} \quad \forall k \geq 1, \quad b_k := \sum_{p=1}^k \frac{(-1)^p}{a_0^{p+1}} \left(\sum_{\substack{k_1, \dots, k_p \geq 1 \\ k_1 + \dots + k_p = k}} \prod_{i=1}^p a_{k_i} \right).$$

Proof. It suffices to reverse a triangular Toeplitz matrix. Indeed, equation (A.1) is equivalent to:

$$\forall n \in \mathbb{N}, \quad \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} a_0 & 0 & \dots & 0 \\ a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n-1} & \dots & a_1 & a_0 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_n \end{pmatrix}. \quad (\text{A.3})$$

Denote by A the matrix associated to the system. Denote by N the following nilpotent matrix:

$$N = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ (0) & & 1 & 0 \end{pmatrix}.$$

Then, $A = a_0 I_n + a_1 N + \dots + a_{n-1} N^{n-1}$ and we are looking for B such that $B = b_0 I_n + b_1 N + \dots + b_{n-1} N^{n-1}$ and $AB = I_n$. Denote by

$$S(z) = \sum_{k \geq 0} a_k z^k \quad \text{and} \quad S^{-1}(z) = \sum_{k \geq 0} b_k z^k,$$

we are interested in the $(n-1)$ first coefficients of $S^{-1}(z)$.

And formally,

$$\begin{aligned} S^{-1}(z) &= \frac{1}{S(z)} = \frac{1}{a_0} \left(\frac{1}{1 + \sum_{k \geq 1} \frac{a_k}{a_0} z^k} \right) \\ &= \frac{1}{a_0} \sum_{p \geq 0} \frac{(-1)^p}{a_0^p} \left(\sum_{k \geq 1} a_k z^k \right)^p \\ &= \frac{1}{a_0} + \sum_{p \geq 1} \frac{(-1)^p}{a_0^{p+1}} \sum_{k \geq p} \left(\sum_{\substack{k_1, \dots, k_p \geq 1 \\ k_1 + \dots + k_p = k}} a_{k_1} a_{k_2} \dots a_{k_p} \right) z^k \\ &= \frac{1}{a_0} + \sum_{k \geq 1} \sum_{p=1}^k \left(\frac{(-1)^p}{a_0^{p+1}} \sum_{\substack{k_1, \dots, k_p \geq 1 \\ k_1 + \dots + k_p = k}} a_{k_1} a_{k_2} \dots a_{k_p} \right) z^k. \end{aligned}$$

Finally, we identify the desired coefficients. □

B Particular case: when $(a_k)_{k \geq 0}$ is log-convex

This section is based on a work made by N.Ford, D.V.Savostyanov et N.L.Zamarashkin in [12].

Lemma B.1. *Let $(a_n)_{n \in \mathbb{N}}$ be a log-convex sequence in the following sense*

$$a_n \geq 0 \quad \text{for } n \geq 0 \quad \text{and} \quad a_n^2 \leq a_{n-1}a_{n+1} \quad \text{for } n \geq 1.$$

If $a_0 > 0$, then the sequence $(b_n)_{n \in \mathbb{N}}$ defined by

$$b_0 = \frac{1}{a_0} \quad \text{et} \quad \forall n \geq 1, \quad b_n = -\frac{1}{a_0} \sum_{k=1}^n a_k b_{n-k}$$

satisfies

$$\forall n \geq 1, \quad b_n \leq 0 \quad \text{et} \quad |b_n| \leq b_0 a_n \tag{B.1}$$

Remark B.1. *The sequence $a_n = (n+1)^{-\rho}$ is log-convex for all $\rho > 0$, then the corresponding $(b_n)_{n \in \mathbb{N}}$ is such that $\forall n \in \mathbb{N}, \quad |b_n| \leq (n+1)^{-\rho}$.*

Proof. Without loss of generality, we assume that $a_0 = 1$.

• We first prove by strong induction on $n \geq 1$ the following property:

$$(P_n) : b_n \leq 0$$

- For $n = 1$: We have

$$b_1 = -\frac{a_1}{a_0} \leq 0$$

- Heredity: Let $n \geq 1$ and assume that P_k is true for $k \in \llbracket 1, n \rrbracket$.

For all $n \geq 1$:

$$\begin{aligned} b_n &= -\sum_{k=1}^n a_k b_{n-k} \implies \sum_{k=0}^n a_k b_{n-k} = 0 \\ &\implies \sum_{k=0}^n a_{n-k} b_k = 0 \\ &\implies a_n = -\sum_{k=1}^n a_{n-k} b_k \end{aligned}$$

We divide the last equality by a_{n-1} and we deduce the two following equalities:

$$-\frac{a_n}{a_{n-1}} = \sum_{k=1}^n \frac{a_{n-k}}{a_{n-1}} b_k \tag{B.2}$$

$$-\frac{a_{n+1}}{a_n} = \sum_{k=1}^{n+1} \frac{a_{n+1-k}}{a_n} b_k \tag{B.3}$$

We make (B.3) – (B.2) and we get

$$\left(\frac{a_n}{a_{n-1}} - \frac{a_{n+1}}{a_n} \right) = \sum_{k=1}^n \left(\frac{a_{n+1-k}}{a_n} - \frac{a_{n-k}}{a_{n-1}} \right) b_k + \frac{b_{n+1}}{a_n}$$

then

$$\frac{b_{n+1}}{a_n} = \underbrace{\left(\frac{a_n}{a_{n-1}} - \frac{a_{n+1}}{a_n} \right)}_{\leq 0 \text{ by log-convexity of } (a_n)} - \sum_{k=1}^n \underbrace{\left(\frac{a_{n+1-k}}{a_n} - \frac{a_{n-k}}{a_{n-1}} \right)}_{(*)} \underbrace{b_k}_{\leq 0}$$

But, (*) have the same sign as

$$\frac{a_{n+1-k}}{a_{n-k}} - \frac{a_n}{a_{n-1}} = \left(\frac{a_{n+1-k}}{a_{n-k}} - \frac{a_{n+2-k}}{a_{n+1-k}} \right) + \left(\frac{a_{n+2-k}}{a_{n+1-k}} - \frac{a_{n+3-k}}{a_{n+2-k}} \right) + \dots + \left(\frac{a_{n-1}}{a_{n-2}} - \frac{a_n}{a_{n-1}} \right) \leq 0$$

where every term is negative by log-convexity of (a_n) .

We finally deduce that

$$b_{n+1} \leq 0.$$

- Conclusion: Property P_n is shown for all $n \geq 1$.

• The second property satisfied by (b_n) directly follows from the first one. Let $n \geq 1$, as we just saw $b_n \leq 0$ therefore $|b_n| = -b_n$. But,

$$\begin{aligned} -b_n &= \sum_{k=1}^{n-1} a_k \underbrace{(b_{n-k})}_{\leq 0} + a_n b_0 \\ &\leq b_0 a_n \end{aligned}$$

and the lemma is proven. \square

C Proof of Theorem 3.1

Let $x_0 \in \mathcal{X}$ and $\mu = \delta_{x_0} \times \mathbb{P}_w$. We have $\Pi_{\mathcal{W}}^* \mu = \mathbb{P}_w$ therefore by property 3.1 we get $\forall k \in \mathbb{N}$, $\Pi_{\mathcal{W}}^*(\mathcal{Q}^k \mu) = \mathbb{P}_w$. Moreover, we clearly have $\int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* \mu)(dx) = \psi(x_0) < +\infty$.

We now set for all $n \in \mathbb{N}^*$,

$$R_n \mu = \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{Q}^k \mu.$$

The aim is to prove that the sequence $(R_n \mu)_{n \in \mathbb{N}^*}$ is tight. First, let us prove that $(\Pi_{\mathcal{X}}^* R_n \mu)_{n \in \mathbb{N}^*}$ is tight. By (ii) of Definition 3.2, we have $\forall k \geq 0$:

$$\int_{\mathcal{X} \times \mathcal{W}} \psi(x) \mathcal{Q}^{k+1} \mu(dx, dw) - \alpha \int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* \mathcal{Q}^k \mu)(dx) \leq \beta.$$

By adding for k from 0 to $n-1$, dividing by n and reordering the terms, we get:

$$\begin{aligned} (1 - \alpha) \int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* R_n \mu)(dx) \\ + \frac{1}{n} \sum_{k=0}^{n-1} \left(\int_{\mathcal{X} \times \mathcal{W}} \psi(x) \mathcal{Q}^k \mu(dx, dw) - \int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* \mathcal{Q}^k \mu)(dx) \right) \\ + \frac{1}{n} \int_{\mathcal{X} \times \mathcal{W}} \psi(x) \mathcal{Q}^{n+1} \mu(dx, dw) - \frac{1}{n} \int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* \mu)(dx) \leq \beta. \end{aligned} \quad (\text{C.1})$$

Since we are in a Polish space (here $\mathcal{X} \times \mathcal{W}$) we can “disintegrate” $\mathcal{Q}^k \mu$ for all $k \in \{0, \dots, n-1\}$ (see [1] for background):

$$\mathcal{Q}^k \mu(dx, dw) = (\mathcal{Q}^k \mu)^x(dw) (\Pi_{\mathcal{X}}^* \mathcal{Q}^k \mu)(dx).$$

By integrating first with respect to w and then with respect to x , we get:

$$\int_{\mathcal{X} \times \mathcal{W}} \psi(x) \mathcal{Q}^k \mu(dx, dw) - \int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* \mathcal{Q}^k \mu)(dx) = 0.$$

Let us return to (C.1),

$$\begin{aligned} (1 - \alpha) \int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* R_n \mu)(dx) &\leq \beta + \frac{1}{n} \left(\int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* \mu)(dx) - \int_{\mathcal{X} \times \mathcal{W}} \psi(x) \mathcal{Q}^{n+1} \mu(dx, dw) \right) \\ &= \beta + \frac{1}{n} \int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* \mu - \Pi_{\mathcal{X}}^* (\mathcal{Q}^{n+1} \mu))(dx). \end{aligned}$$

Set $A_{n+1} = \int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* \mathcal{Q}^{n+1} \mu)(dx)$. By (ii) of Definition 3.2 and by induction, we have

$$0 \leq \frac{A_{n+1}}{n} \leq \frac{\beta}{n} \sum_{k=0}^n \alpha^k + \frac{\alpha^{n+1}}{n} A_0 = \frac{\beta}{n} \frac{1 - \alpha^{n+1}}{1 - \alpha} + \frac{\alpha^{n+1}}{n} \psi(x_0).$$

Hence we deduce that $\lim_{n \rightarrow +\infty} \frac{A_{n+1}}{n} = 0$. Then, there exists $C > 0$ such that $\forall n \in \mathbb{N}^*$:

$$(1 - \alpha) \int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* R_n \mu)(dx) \leq C$$

and then

$$\sup_{n \geq 1} \int_{\mathcal{X}} \psi(x) (\Pi_{\mathcal{X}}^* R_n \mu)(dx) \leq \frac{C}{1 - \alpha}.$$

Let $\delta > 0$ and $K_\delta := \{x \in \mathcal{X} \mid \psi(x) \leq \delta\} = \psi^{-1}([0, \delta])$. By (ii) of Definition 3.2, K_δ is a compact set. For all $x \in \mathcal{X}$, we have $\mathbb{1}_{K_\delta^c}(x) \leq \frac{\psi(x)}{\delta}$, so

$$\forall n \in \mathbb{N}^*, \quad (\Pi_{\mathcal{X}}^* R_n \mu)(K_\delta^c) \leq \frac{C}{\delta(1 - \alpha)}.$$

By setting $\frac{\varepsilon}{2} = \frac{C}{\delta(1 - \alpha)}$, we deduce that $(\Pi_{\mathcal{X}}^* R_n \mu)_{n \in \mathbb{N}^*}$ is tight.

Let us now go back to the tightness of $(R_n \mu)_{n \in \mathbb{N}^*}$. Let K be a compact set of \mathcal{W} such that $\mathbb{P}_w(K^c) < \frac{\varepsilon}{2}$, this is possible since \mathcal{W} is Polish. We then get

$$\begin{aligned} R_n \mu((K_\delta \times K)^c) &\leq R_n \mu(K_\delta^c \times \mathcal{W}) + R_n \mu(\mathcal{X} \times K^c) \\ &= (\Pi_{\mathcal{X}}^* R_n \mu)(K_\delta^c) + (\Pi_{\mathcal{W}}^* R_n \mu)(K^c) \\ &= (\Pi_{\mathcal{X}}^* R_n \mu)(K_\delta^c) + (\mathbb{P}_w)(K^c) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Finally, $(R_n \mu)_{n \in \mathbb{N}^*}$ is tight. Let μ_\star be one of its accumulation points. By the Krylov-Bogoliubov criterium we deduce that μ_\star is an invariant distribution for \mathcal{Q} .

D Proof of Lemma 5.3

Recall that we want to prove that for all $\alpha, \beta > 0$ such that $\alpha + \beta > 1$, there exists $C(\alpha, \beta) > 0$ such that for all $n \geq 0$,

$$\sum_{k=0}^n (k+1)^{-\beta} (n+1-k)^{-\alpha} \leq C(\alpha, \beta) \begin{cases} (n+1)^{-\beta} \ln(n) & \text{if } \alpha = 1 \text{ and } \beta \leq 1 \\ (n+1)^{-\alpha} \ln(n) & \text{if } \beta = 1 \text{ and } \alpha \leq 1 \\ (n+1)^{-\min\{\alpha, \beta, \alpha+\beta-1\}} & \text{else} \end{cases} \quad (\text{D.1})$$

For the sake of simplicity, we will prove this result when n is odd. If n is even, the proof is almost the same. Set $N := \frac{n+1}{2}$. Then, we get

$$\begin{aligned}
\sum_{k=0}^n (k+1)^{-\beta} (n+1-k)^{-\alpha} &= \sum_{k=1}^{n+1} k^{-\beta} (n+2-k)^{-\alpha} \\
&= \sum_{k=1}^N k^{-\beta} (n+2-k)^{-\alpha} + \sum_{k=N+1}^{n+1} k^{-\beta} (n+2-k)^{-\alpha} \\
&= \sum_{k=1}^N k^{-\beta} (n+2-k)^{-\alpha} + \sum_{k=1}^N k^{-\alpha} (n+2-k)^{-\beta} \\
\sum_{k=0}^n (k+1)^{-\beta} (n+1-k)^{-\alpha} &= S_N(\beta, \alpha) + S_N(\alpha, \beta)
\end{aligned} \tag{D.2}$$

by setting $S_N(\alpha, \beta) := \sum_{k=1}^N k^{-\alpha} (n+2-k)^{-\beta} = \sum_{k=1}^N k^{-\alpha} (2N - (k-1))^{-\beta}$.

▷ If $\alpha \in (0, 1)$, we have $\alpha + \beta - 1 \leq \beta$ and $S_N(\alpha, \beta) \leq \tilde{C}(\alpha, \beta) (n+1)^{-(\alpha+\beta-1)}$.
Indeed,

$$\begin{aligned}
S_N(\alpha, \beta) &= \sum_{k=1}^N k^{-\alpha} (2N - (k-1))^{-\beta} \\
&= N^{-(\alpha+\beta-1)} \times \frac{1}{N} \sum_{k=1}^N \left(\frac{k}{N}\right)^{-\alpha} \left(2 - \frac{k-1}{N}\right)^{-\beta} \\
&\leq N^{-(\alpha+\beta-1)} \times \frac{1}{N} \sum_{k=1}^N \left(\frac{k}{N}\right)^{-\alpha} \left(2 - \frac{k}{N}\right)^{-\beta}
\end{aligned}$$

and

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{k=1}^N \left(\frac{k}{N}\right)^{-\alpha} \left(2 - \frac{k}{N}\right)^{-\beta} = \int_0^1 x^{-\alpha} (2-x)^{-\beta} dx$$

where the integral is well defined since $\alpha \in (0, 1)$. Therefore, since $N = \frac{n+1}{2}$, we deduce that there exists $\tilde{C}(\alpha, \beta) > 0$ such that $S_N(\alpha, \beta) \leq \tilde{C}(\alpha, \beta) (n+1)^{-(\alpha+\beta-1)}$.

▷ If $\alpha > 1$, we have $\alpha + \beta - 1 > \beta$ and $S_N(\alpha, \beta) \leq \tilde{C}(\alpha, \beta) (n+1)^{-\beta}$.
Indeed,

$$\begin{aligned}
S_N(\alpha, \beta) &= \sum_{k=1}^N k^{-\alpha} (2N - (k-1))^{-\beta} \\
&\leq (2N - (N-1))^{-\beta} \sum_{k=1}^N k^{-\alpha} \\
&\leq (N+1)^{-\beta} \sum_{k=1}^{+\infty} k^{-\alpha}.
\end{aligned}$$

Therefore, as before we deduce that there exists $\tilde{C}(\alpha, \beta) > 0$ such that $S_N(\alpha, \beta) \leq \tilde{C}(\alpha, \beta) (n+1)^{-\beta}$.

▷ If $\alpha = 1$, in the same way as in the case $\alpha > 1$, we get

$$\begin{aligned} S_N(\alpha, \beta) &\leq (N+1)^{-\beta} \sum_{k=1}^N \frac{1}{k} \\ &\leq \tilde{C}(N+1)^{-\beta} \ln(N) \end{aligned}$$

Therefore, there exists $\tilde{C}(\alpha, \beta) > 0$ such that $S_N(\alpha, \beta) \leq \tilde{C}(\alpha, \beta)(n+1)^{-\beta} \ln(n)$.

Finally, we get that for all $\alpha > 0$ and $\beta > 0$ such that $\alpha + \beta > 1$,

$$S_N(\alpha, \beta) \leq \tilde{C}(\alpha, \beta) \begin{cases} (n+1)^{-\min\{\alpha, \beta, \alpha+\beta-1\}} & \text{if } \alpha \neq 1 \\ (n+1)^{-\beta} \ln(n) & \text{if } \alpha = 1 \end{cases} \quad (\text{D.3})$$

Putting this inequality into (D.2) we finally get the desired inequality and the proof is finished.

E Proof of Proposition 2.3

We recall that $\rho = 3/2 - H$ where $H \in (0, 1/2)$ is the Hurst parameter and $(b_n)_{n \in \mathbb{N}}$ is defined by

$$b_0 = \frac{1}{a_0^H} \quad \text{et pour } n \geq 1, \quad b_n = -\frac{1}{a_0^H} \sum_{k=1}^n a_k^H b_{n-k}. \quad (\text{E.1})$$

and for all $k \geq 1$,

$$\frac{a_k^H}{a_0^H} = \left((2k+1)^{1-\rho} - (2k-1)^{1-\rho} \right).$$

We want to show that $|b_n| \leq C_b(n+1)^{-(2-\rho)}$ by induction. To this end we only need to prove that for n large enough,

$$S_n := \sum_{k=1}^n \left((2k-1)^{1-\rho} - (2k+1)^{1-\rho} \right) (n+1-k)^{-(2-\rho)} \leq (n+1)^{-(2-\rho)}. \quad (\text{E.2})$$

For the sake of simplicity we assume that n is even.

$$\begin{aligned} S_n &= \sum_{k=1}^{n/2} \left((2k-1)^{1-\rho} - (2k+1)^{1-\rho} \right) (n+1-k)^{-(2-\rho)} + \sum_{k=n/2+1}^n \left((2k-1)^{1-\rho} - (2k+1)^{1-\rho} \right) (n+1-k)^{-(2-\rho)} \\ S_n &=: S_n^{(1)} + S_n^{(2)}. \end{aligned} \quad (\text{E.3})$$

▷ We begin with $S_n^{(1)}$. Summation by parts:

$$\begin{aligned} S_n^{(1)} &= \left(\frac{n}{2} + 1 \right)^{-(2-\rho)} \left(1 - (n+1)^{1-\rho} \right) - \sum_{k=1}^{n/2-1} \left(1 - (2k+1)^{1-\rho} \right) \left((n-k)^{-(2-\rho)} - (n+1-k)^{-(2-\rho)} \right) \\ &= \left(\frac{n}{2} + 1 \right)^{-(2-\rho)} \left(1 - (n+1)^{1-\rho} \right) - \left(\left(\frac{n}{2} + 1 \right)^{-(2-\rho)} - n^{-(2-\rho)} \right) \\ &\quad + \sum_{k=1}^{n/2-1} (2k+1)^{1-\rho} \left((n-k)^{-(2-\rho)} - (n+1-k)^{-(2-\rho)} \right) \\ &= n^{-(2-\rho)} - \left(\frac{n}{2} + 1 \right)^{-(2-\rho)} (n+1)^{1-\rho} + \sum_{k=1}^{n/2-1} (2k+1)^{1-\rho} \left((n-k)^{-(2-\rho)} - (n+1-k)^{-(2-\rho)} \right) \\ S_n^{(1)} &= n^{-(2-\rho)} - \left(\frac{n}{2} \right)^{-(2-\rho)} (n+1)^{1-\rho} + \sum_{k=1}^{n/2} (2k+1)^{1-\rho} \left((n-k)^{-(2-\rho)} - (n+1-k)^{-(2-\rho)} \right). \end{aligned} \quad (\text{E.4})$$

We set $\tilde{S}_n := \sum_{k=1}^{n/2} (2k+1)^{1-\rho} \left((n-k)^{-(2-\rho)} - (n+1-k)^{-(2-\rho)} \right)$. Then,

$$\begin{aligned}\tilde{S}_n &= \frac{1}{n} \sum_{k=1}^{n/2} \left(\frac{2k+1}{n} \right)^{1-\rho} \left(\left(1 - \frac{k}{n} \right)^{-(2-\rho)} - \left(1 - \frac{k-1}{n} \right)^{-(2-\rho)} \right) \\ &= \frac{2^{1-\rho}}{n} \sum_{k=1}^{n/2} \left(\frac{k+1/2}{n} \right)^{1-\rho} \left(\left(1 - \frac{k}{n} \right)^{-(2-\rho)} - \left(1 - \frac{k-1}{n} \right)^{-(2-\rho)} \right).\end{aligned}$$

Moreover,

$$\begin{aligned}& \sum_{k=1}^{n/2} \left(\frac{k+1/2}{n} \right)^{1-\rho} \left(\left(1 - \frac{k}{n} \right)^{-(2-\rho)} - \left(1 - \frac{k-1}{n} \right)^{-(2-\rho)} \right) \\ &= (2-\rho) \left(\int_0^{1/2} \left(x + \frac{1}{2n} \right)^{1-\rho} (1-x)^{-(3-\rho)} dx - \sum_{k=1}^{n/2} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\left(x + \frac{1}{2n} \right)^{1-\rho} - \left(\frac{k+1/2}{n} \right)^{1-\rho} \right) (1-x)^{-(3-\rho)} dx \right)\end{aligned}\tag{E.5}$$

and

$$\begin{aligned}\int_0^{1/2} \left(x + \frac{1}{2n} \right)^{1-\rho} (1-x)^{-(3-\rho)} dx &= \left[\frac{(1-x)^{\rho-2} \left(x + \frac{1}{2n} \right)^{2-\rho}}{(2-\rho) \left(1 + \frac{1}{2n} \right)} \right]_0^{1/2} \\ &= \frac{1}{2-\rho} \left(1 + \frac{1}{2n} \right)^{-1} \left[\left(1 + \frac{1}{n} \right)^{2-\rho} - \left(\frac{1}{2n} \right)^{2-\rho} \right].\end{aligned}\tag{E.6}$$

Hence by putting together (E.5) and (E.6) we get

$$\begin{aligned}\tilde{S}_n &= \frac{2^{1-\rho}}{n} \left(\left(1 + \frac{1}{2n} \right)^{-1} \left[\left(1 + \frac{1}{n} \right)^{2-\rho} - \left(\frac{1}{2n} \right)^{2-\rho} \right] \right. \\ &\quad \left. - (2-\rho) \sum_{k=1}^{n/2} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\left(x + \frac{1}{2n} \right)^{1-\rho} - \left(\frac{k+1/2}{n} \right)^{1-\rho} \right) (1-x)^{-(3-\rho)} dx \right).\end{aligned}\tag{E.7}$$

We deduce from (E.4) and (E.7) that

$$S_n^{(1)} \leq n^{-(2-\rho)} - \left(\frac{n}{2} \right)^{-(2-\rho)} (n+1)^{1-\rho} + \frac{2^{1-\rho}}{n} \left(1 + \frac{1}{2n} \right)^{-1} \left[\left(1 + \frac{1}{n} \right)^{2-\rho} - \left(\frac{1}{2n} \right)^{2-\rho} \right].\tag{E.8}$$

▷ Now, we look after $S_n^{(2)}$:

As before, using the fact that

$$\int_0^{1/2} \left(1 - x + \frac{1}{2n} \right)^{-\rho} x^{-(2-\rho)} dx = \left[\frac{\left(1 - x + \frac{1}{2n} \right)^{1-\rho} x^{\rho-1}}{(\rho-1) \left(1 + \frac{1}{2n} \right)} \right]_0^{1/2} = \frac{1}{\rho-1} \left(1 + \frac{1}{2n} \right)^{-1} \left(1 + \frac{1}{n} \right)^{1-\rho}\tag{E.9}$$

we get

$$S_n^{(2)} = \frac{2^{1-\rho}}{n} \left(\left(1 + \frac{1}{2n} \right)^{-1} \left(1 + \frac{1}{n} \right)^{1-\rho} - (\rho-1) \sum_{k=1}^{n/2} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(x^{-(2-\rho)} - \left(\frac{k}{n} \right)^{-(2-\rho)} \right) \left(1 - x + \frac{1}{2n} \right)^{-\rho} dx \right).\tag{E.10}$$

Now, for all $k \in \{1, \dots, n/2\}$, we set

$$I_k := \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(x^{-(2-\rho)} - \left(\frac{k}{n}\right)^{-(2-\rho)} \right) \left(1 - x + \frac{1}{2n} \right)^{-\rho} dx.$$

Thanks to the substitution $t = x - \frac{k-1}{n}$, we have

$$I_k = \int_0^{1/n} \left(\left(t + \frac{k-1}{n} \right)^{-(2-\rho)} - \left(\frac{k}{n}\right)^{-(2-\rho)} \right) \left(1 + \frac{1}{2n} - t - \frac{k-1}{n} \right)^{-\rho} dt$$

Taylor-Lagrange expansion:

- $\left(t + \frac{k-1}{n} \right)^{-(2-\rho)} - \left(\frac{k}{n}\right)^{-(2-\rho)} = \left(\frac{1}{n} - t\right) (2-\rho) \left(\frac{k}{n}\right)^{-(3-\rho)} + \frac{1}{2} \left(\frac{1}{n} - t\right)^2 (2-\rho)(3-\rho) \xi^{-(4-\rho)}$
with $\xi \in]t + (k-1)/n, k/n[$.
- $\left(1 + \frac{1}{2n} - t - \frac{k-1}{n} \right)^{-\rho} = \left(1 + \frac{1}{2n} - \frac{k-1}{n} \right)^{-\rho} + t\rho \left(1 + \frac{1}{2n} - c \right)^{-\rho-1}$
with $c \in](k-1)/n, t + (k-1)/n[$.

Therefore, we deduce that

$$I_k \geq \int_0^{1/n} \left(\frac{1}{n} - t\right) (2-\rho) \left(\frac{k}{n}\right)^{-(3-\rho)} \left(1 + \frac{1}{2n} - \frac{k-1}{n} \right)^{-\rho} dt = \frac{2-\rho}{2n^2} \left(\frac{k}{n}\right)^{-(3-\rho)} \left(1 + \frac{1}{2n} - \frac{k-1}{n} \right)^{-\rho}.$$

Then we add the inequality for k from 1 to $n/2$,

$$\sum_{k=1}^{n/2} I_k \geq \frac{2-\rho}{2n} \times \underbrace{\frac{1}{n} \sum_{k=1}^{n/2} \left(\frac{k}{n}\right)^{-(3-\rho)} \left(1 + \frac{1}{2n} - \frac{k-1}{n} \right)^{-\rho}}_{U_n}. \quad (\text{E.11})$$

We easily show that

$$U_n \geq \int_0^{1/2} \left(y + \frac{1}{n} \right)^{-(3-\rho)} \left(1 + \frac{3}{2n} - y \right)^{-\rho} dy =: J_n. \quad (\text{E.12})$$

By integration by parts on J_n we get:

$$\begin{aligned} J_n &= \left[\frac{-(y + \frac{1}{n})^{-(2-\rho)}}{2-\rho} \left(1 + \frac{3}{2n} - y \right)^{-\rho} \right]_0^{1/2} + \frac{\rho}{2-\rho} \int_0^{1/2} \left(y + \frac{1}{n} \right)^{-(2-\rho)} \left(1 + \frac{3}{2n} - y \right)^{-\rho-1} dy \\ &= \frac{1}{2-\rho} \underbrace{\left[\left(\frac{1}{n}\right)^{-(2-\rho)} \left(1 + \frac{3}{2n} \right)^{-\rho} - \left(\frac{1}{2} + \frac{1}{n}\right)^{-(2-\rho)} \left(\frac{1}{2} + \frac{3}{2n} \right)^{-\rho} \right]}_{\underset{n \rightarrow +\infty}{\sim} n^{2-\rho}} \\ &\quad + \frac{\rho}{2-\rho} \underbrace{\int_0^{1/2} \left(y + \frac{1}{n} \right)^{-(2-\rho)} \left(1 + \frac{3}{2n} - y \right)^{-\rho-1} dy}_{\underset{n \rightarrow +\infty}{\rightarrow} \int_0^{1/2} y^{-(2-\rho)} (1-y)^{-\rho-1} dy}. \end{aligned}$$

Hence, for n large enough, we have

$$U_n \geq J_n \geq \frac{1}{2(2-\rho)} n^{2-\rho} \quad (\text{E.13})$$

By combining (E.11) and (E.13) we get for n large enough

$$\sum_{k=1}^{n/2} I_k \geq \frac{1}{4} n^{1-\rho}.$$

Finally we get for $S_n^{(2)}$ the following upper-bound for n large enough,

$$S_n^{(2)} \leq \frac{2^{1-\rho}}{n} \left(\left(1 + \frac{1}{2n}\right)^{-1} \left(1 + \frac{1}{n}\right)^{1-\rho} - \frac{\rho-1}{4} n^{1-\rho} \right). \quad (\text{E.14})$$

By putting together (E.8) and (E.14) and when factoring by $(n+1)^{-(2-\rho)}$, we get for S_n

$$S_n \leq (n+1)^{-(2-\rho)} \left(1 - \frac{1}{n}\right)^{-(2-\rho)} \times u_n \quad (\text{E.15})$$

with

$$u_n = 1 - 2^{2-\rho} (n+1)^{1-\rho} + 2^{1-\rho} n^{1-\rho} \left[\left(1 + \frac{1}{2n}\right)^{-1} \left(\left(1 + \frac{1}{n}\right)^{2-\rho} - \left(\frac{1}{2n}\right)^{2-\rho} \right) + \left(1 + \frac{1}{2n}\right)^{-1} \left(1 + \frac{1}{n}\right)^{1-\rho} - \frac{\rho-1}{4} n^{1-\rho} \right]$$

Lastly, we have the following asymptotic expansion:

$$\left(1 - \frac{1}{n}\right)^{-(2-\rho)} \times u_n = 1 - \frac{2^{1-\rho}(\rho-1)}{4} n^{2-2\rho} + o\left(\frac{1}{n}\right)$$

Since $\rho \in (1, 3/2)$ we have $2 - 2\rho \in (-1, 0)$ therefore for n large enough we conclude that

$$S_n \leq (n+1)^{-(2-\rho)}.$$

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