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ABSTRACT. We investigate the flat holomorphic vector bundles over compact complex parallelizable manifolds G/Γ, where G is a complex connected Lie group and Γ is a cocompact lattice in it. The main result proved here is a structure theorem for flat holomorphic vector bundles E_ρ associated to any irreducible representation $\rho : \Gamma \longrightarrow \mathrm{GL}(r, \mathbb{C})$. More precisely, we prove that E_ρ is holomorphically isomorphic to a vector bundle of the form $E^{\oplus n}$, where E is a stable vector bundle. All the rational Chern classes of E vanish, in particular, its degree is zero.

We deduce a stability result for flat holomorphic vector bundles E_ρ of rank 2 over G/Γ. If an irreducible representation $\rho : \Gamma \longrightarrow \mathrm{GL}(2, \mathbb{C})$ satisfies the condition that the induced homomorphism $\Gamma \longrightarrow \mathrm{PGL}(2, \mathbb{C})$ does not extend to a homomorphism from G, then E_ρ is proved to be stable.

Résumé. Nous étudions les fibrés holomorphes plats sur les variétés parallélisables compactes G/Γ (avec G groupe de Lie connexe complexe et Γ réseau cocompact). Notre résultat principal décrit les fibrés holomorphes plats E_ρ associés à des représentations irréductibles $\rho : \Gamma \longrightarrow \mathrm{GL}(r, \mathbb{C})$. Nous démontrons que ces fibrés E_ρ sont isomorphes à une somme directe $E^{\oplus n}$, avec E fibré vectoriel stable de degré zéro.

Nous en déduisons un résultat de stabilité concernant les fibrés holomorphes plats E_ρ de rang 2 sur les quotients G/Γ. Si $\rho : \Gamma \longrightarrow \mathrm{GL}(2, \mathbb{C})$ est une représentation irréductible telle que le morphisme induit $\rho' : \Gamma \longrightarrow \mathrm{PGL}(2, \mathbb{C})$ ne s'étend pas à G, alors E_ρ est stable.

Version française abrégée

Nous étudions les fibrés plats holomorphes sur les variétés parallélisables compactes G/Γ, avec G groupe de Lie complexe connexe et Γ réseau cocompact. Ces fibrés plats de rang r sont donnés par des représentations $\rho : \Gamma \longrightarrow \mathrm{GL}(r, \mathbb{C})$. Un tel fibré est holomorphiquement trivial si et seulement si le morphisme ρ s'étend en un morphisme de groupes de Lie $G \longrightarrow \mathrm{GL}(r, \mathbb{C})$ (voir, par exemple, [16, p. 801, Proposition 3.1]).

Nous nous intéressons à la notion de stabilité de ces fibrés. Même si pour G non abélien, les quotients G/Γ ne sont pas kähleriens, ces variétés portent des métriques *balancées* (i.e. qui ont la propriété $d\omega^{m-1} = 0$, avec m la dimension complexe de la variété) [1, 3]. Par rapport à ces métriques les notions classiques de degré, pente (=degré divisé par le rang) et (semi-)stabilité et polystabilité au sens des pentes se définissent comme dans le cas classique (projective ou kählerien) [10].

Le résultat principal de cette note détermine la structure des fibrés holomorphes plats (de rang r) E_ρ au-dessus de G/Γ qui sont construits à partir de représentations irréductibles $\rho : \Gamma \longrightarrow \mathrm{GL}(r, \mathbb{C})$. Nous démontrons que ces fibrés sont nécesairement isomorphes à un fibré de la forme $E^{\oplus n}$, avec E fibré vectoriel stable de degré zéro.

Dans la preuve nous utilisons un résultat de [3] qui donne la semistabilité de E_ρ. Ensuite nous démontrons la polystabilité qui est une étape importante de la preuve.

Une conséquence du théorème précédent est un résultat de stabilité pour les fibrés plats holomorphes de rang 2 au-dessus de G/Γ. Plus précisément, si $\rho : \Gamma \longrightarrow \mathrm{GL}(2, \mathbb{C})$ est une représentation irréductible telle que la représentation induite $\rho' : \Gamma \longrightarrow \mathrm{PGL}(2, \mathbb{C})$ ne s'étend pas à G, alors E_ρ est stable.

1. Introduction

An interesting class of compact non-Kähler manifolds which generalizes compact complex tori consists of those manifolds whose holomorphic tangent bundle is holomorphically trivial. By a result of Wang [15], those so-called *parallelizable* manifolds are known to be biholomorphic to a quotient of a complex connected Lie group G by a cocompact lattice Γ in G. Those quotients G/Γ are Kähler exactly when G is abelian (and, consequently, the quotient is a complex torus).

In particular, for G nonabelian, the above quotients are not algebraic. Moreover, for G semi-simple, the corresponding parallelizable manifolds are known to have algebraic dimension zero (meaning that the only meromorphic functions on G/Γ are the constant ones).

For those parallelizable manifolds of algebraic dimension zero, Ghys's arguments in [8] prove that all foliations and all holomorphic distributions on G/Γ are *homogeneous* (i.e., they all descend from G-right invariant foliations (respectively, distributions) on G). In particular, any complex subbundle of the holomorphic tangent bundle is isomorphic to a trivial vector bundle. It was also proved in [5] that all holomorphic geometric structures in Gromov's sense [9] (constructed from higher order frame bundles) on parallelizable manifolds G/Γ of algebraic dimension zero are also necessarily homogeneous (e.g. their pull-back on G are G-right invariant).

The simplest examples of parallelizable manifolds of algebraic dimension zero are compact quotients of $\mathrm{SL}(2, \mathbb{C})$. Those manifolds are closely related to the 3-hyperbolic manifolds in a natural way. Indeed, $\mathrm{PSL}(2, \mathbb{C})$ being the group of direct isometries of the hyperbolic 3-space, the direct orthonormal frame bundle of a compact oriented hyperbolic 3-manifold V is diffeomorphic to a quotient $\mathrm{PSL}(2, \mathbb{C})/\Gamma$ (with the lattice Γ being isomorphic to the fundamental group of V). Those quotients are both geometrically interesting and very abundant. In particular, they can have arbitrarily large first Betti number (see [14], [7, Section 6.2], or [12]), meaning that the rank of the abelianization of Γ (modulo torsion) can be arbitrarily large. This remark was used by Ghys [7] in order to prove that the quotients of $\mathrm{SL}(2, \mathbb{C})$ with first Betti number ≥ 1 are not rigid (as

complex manifolds). Ghys constructed the corresponding deformation space using group homomorphisms of Γ into $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ which are close to the natural embedding $\Gamma \subset \mathrm{SL}(2,\mathbb{C}) \times \{I_2\} \subset \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$, where I_2 is the identity element. The images of those homomorphisms are (up to conjugacy) graphs $\gamma \longmapsto (\gamma, \rho(\gamma))$, $\gamma \in \Gamma$, of homomorphisms $\rho : \Gamma \longrightarrow \mathrm{SL}(2,\mathbb{C})$ that are close to the map of Γ to the identity element (the trivial homomorphism).

As soon as the first Betti number of Γ is positive, there exists nontrivial group homomorphisms $u : \Gamma \longrightarrow \mathbb{C}$ and one can consider $\rho : \Gamma \longrightarrow \mathrm{SL}(2,\mathbb{C})$ defined by $\gamma \longmapsto \exp(u(\gamma)\xi)$, with exp being the exponential map and ξ some fixed element of $\mathrm{sl}(2,\mathbb{C})$. Notice that those homomorphisms do not extend to homomorphisms from $\mathrm{SL}(2,\mathbb{C})$; indeed, $\mathrm{SL}(2,\mathbb{C})$ is a *perfect* group (meaning generated by its commutators) and hence $\mathrm{SL}(2,\mathbb{C})$ does not admit nontrivial homomorphisms into abelian groups. This implies that the associated flat holomorphic line bundle E_ρ over $\mathrm{SL}(2,\mathbb{C})/\Gamma$ is nontrivial [16, p. 801, Proposition 3.1].

For any $r \geq 2$, any any element ξ of $\mathrm{Lie}(\mathrm{SL}(r,\mathbb{C}))$, the previous homomorphisms $u : \Gamma \longrightarrow \mathbb{C}$ produce group homomorphisms $\Gamma \longrightarrow \mathrm{SL}(r,\mathbb{C})$, $\gamma \longmapsto \exp(u(\gamma)\xi)$, taking values in a one parameter subgroup.

For cocompact lattices Γ with first Betti number ≥ 2, Ghys constructed in [7] group homomorphisms from Γ to $\mathrm{SL}(2,\mathbb{C})$ which are close to the identity and such that the image is Zariski dense. One can also see [12] in which the author constructs many cocompact lattices Γ in $\mathrm{SL}(2,\mathbb{C})$ admitting a surjective homomorphism onto a nonabelian free group (this also implies that the lattices can have arbitrarily large first Betti number). Since nonabelian free groups admit many linear irreducible representations, we get many linear irreducible representations of those Γ furnishing nontrivial flat holomorphic vector bundles over the corresponding quotients $\mathrm{SL}(2,\mathbb{C})/\Gamma$.

In this note we deal with flat holomorphic vector bundles over parallelizable manifolds G/Γ. Our main result (see Theorem 2.1 in Section 2) is a structure theorem for flat holomorphic vector bundles E_ρ given by irreducible representations $\rho : \Gamma \longrightarrow \mathrm{GL}(r,\mathbb{C})$. We prove that E_ρ is isomorphic to a direct sum $E^{\oplus n}$, where E is a stable vector bundle. All the rational Chern classes of E vanish, in particular, the degree of E is zero (see Remark 2.3).

As a consequence, we deduce in Section 4 a stability result for flat holomorphic vector bundles E_ρ of rank two over G/Γ. If an irreducible homomorphism $\rho : \Gamma \longrightarrow \mathrm{GL}(2,\mathbb{C})$ has the property that the homomorphism $\Gamma \longrightarrow \mathrm{PGL}(2,\mathbb{C})$ obtained by composing ρ with the natural projection of $\mathrm{GL}(2,\mathbb{C})$ to $\mathrm{PGL}(2,\mathbb{C})$ does not extend to a homomorphism from G, then E_ρ is stable.

It should be mentioned that the stability of vector bundles in the previous results is slope-stability with respect to a *balanced* metric on the parallelizable manifold (see definition and explanations in Section 2).

In a further work the authors will address the question of (semi)stability of flat holomorphic vector bundles over Ghys's deformations of parallelizable manifolds $SL(2, \mathbb{C})/\Gamma$ constructed in [7]. It should be mentioned that generic small deformations of $SL(2, \mathbb{C})/\Gamma$ do not admit nontrivial holomorphic 2-forms (see Lemma 3.3 in [4]) and, consequently, Corollary 8 in [6] implies that those generic small deformations admit balanced metrics.

2. IRREDUCIBLE REPRESENTATIONS AND POLYSTABLE BUNDLES

Let G be a complex connected Lie group and $\Gamma \subset G$ a discrete subgroup such that the quotient

$$M := G/\Gamma \tag{2.1}$$

is a compact (complex) manifold. Such a Γ is called a cocompact lattice in G. The left–translation action of G on itself produces a holomorphic action of G on the complex manifold M. The Lie algebra of G will be denoted by $\mathfrak{g}$. Take a maximal compact subgroup $K \subset G$. Fix a Hermitian form $h \in \mathfrak{g}^* \otimes \overline{\mathfrak{g}}^*$ on $\mathfrak{g}$ such that the action of K on $\mathfrak{g}^* \otimes \overline{\mathfrak{g}}^*$, given by the adjoint action of K on $\mathfrak{g}$, fixes h. Consider the translations of h by the right–multiplication action of G on itself. The resulting Hermitian structure on G descends to a Hermitian structure on the quotient M in (2.1). The $(1, 1)$–form on M corresponding to this Hermitian structure will be denoted by ω. We know that

$$d\omega^{m-1} = 0, \tag{2.2}$$

where $m = \dim_{\mathbb{C}} M$ [3, p. 277, Theorem 1.1(1)]. Such a Hermitian metric structure is called balanced (one can also see [1]).

For a torsion-free coherent analytic sheaf F on M, define

$$\text{degree}(F) := \int_M c_1(\det F) \wedge \omega^{m-1} \in \mathbb{R},$$

where $\det F$ is the determinant line bundle for F [11, Ch. V, § 6]. From (2.1) it follows that the degree is well-defined. Indeed, any two first Chern forms for F differ by an exact 2–form on M, and

$$\int_M (d\alpha) \wedge \omega^{m-1} = -\int_M \alpha \wedge d\omega^{m-1} = 0.$$

In fact, this show that degree is a topological invariant. Define

$$\mu(F) := \frac{\text{degree}(F)}{\text{rank}(F)} \in \mathbb{R}$$

which is called the slope of F. A torsion-free coherent analytic sheaf F on M is called *stable* (respectively, *semistable*) if for every coherent analytic subsheaf $V \subset F$ such the $\text{rank}(V) \in [1, \text{rank}(F) - 1]$, the inequality $\mu(V) < \mu(F)$ (respectively, $\mu(V) \leq \mu(F)$) holds (see [11, Ch. V, § 7]). Hence, *throughout the paper, (semi)stability means slope-(semi)stability.*

A torsion-free coherent analytic sheaf F is called *polystable* if the following two conditions hold:

(1) F is semistable, and

(2) F is a direct sum of stable sheaves.

Consider any homomorphism

$$\rho : \Gamma \longrightarrow \mathrm{GL}(r, \mathbb{C}). \tag{2.3}$$

Let (E_ρ, ∇^ρ) be the flat holomorphic vector bundle or rank r over M associated to ρ. We recall that the total space of E_ρ is the quotient of $G \times \mathbb{C}^r$ where two points $(z_1, v_1), (z_2, v_2) \in G \times \mathbb{C}^r$ are identified if there is an element $\gamma \in \Gamma$ such that $z_2 = z_1 \gamma$ and $v_2 = \rho(\gamma^{-1})(v_1)$. We note that the fiber of E_ρ over the point $e\Gamma \in M$, where e is the identity element of G, is identified with $\mathbb{C}^r$ by sending $w \in \mathbb{C}^r$ to the equivalence class of (e, w). The trivial connection on the trivial vector bundle $G \times \mathbb{C}^r \longrightarrow G$ descends to a connection on E_ρ which is denoted by ∇^ρ. The left–translation action of G on G and the trivial action of G on $\mathbb{C}^r$ together define an action of G on $G \times \mathbb{C}^r$. It descends to an action of G on E_ρ. This action of G on E_ρ is a lift of the left–translation action of G on M. In particular, the holomorphic vector bundle E_ρ is equivariant.

A homomorphism ρ as in (2.3) is called *reducible* if there is no nonzero proper subspace of $\mathbb{C}^r$ preserved by $\rho(\Gamma)$ for the standard action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$.

Theorem 2.1. *Assume that the homomorphism ρ is irreducible. Then E_ρ is isomorphic to a vector bundle of the form $E^{\oplus n}$, where E is a stable vector bundle.*

Proof. Since E_ρ admits a flat connections, namely ∇^ρ, the rational Chern class $c_1(E_\rho) \in H^2(M, \mathbb{Q})$ vanishes. Hence from the definition of degree it follows that $\mathrm{degree}(E_\rho) = 0$. Since E_ρ is equivariant, it follows that E_ρ is semistable [3, p. 279, Lemma 3.2].

We will now prove that E_ρ is polystable.

Let

$$W \subset E_\rho \tag{2.4}$$

be the unique maximal polystable subsheaf of degree zero in E_ρ given by Proposition 3.1 in Section 3. Consider the action of G on the equivariant bundle E_ρ. From the uniqueness of W it follows that the action of G on E_ρ preserves the subsheaf W in (2.4); indeed, the subsheaf $g \cdot W \subset E_\rho$ given by the action on W of any fixed $g \in G$ is again a maximal polystable subsheaf of degree zero, and hence $g \cdot W$ coincides with W. This implies that W is a subbundle of E_ρ (the subset of M over which W is subbundle is preserved by the action of G on M, and hence it must be entire M as the action is transitive). As noted earlier, the fiber of E_ρ over the point $e\Gamma \in M$ is identified with $\mathbb{C}^r$. The isotropy subgroup for $e\Gamma$ is Γ itself. Since W is preserved by the action of G on E_ρ, we conclude that the subspace $W_{e\Gamma} \subset (E_\rho)_{e\Gamma} = \mathbb{C}^r$ is preserved by the action of Γ on $\mathbb{C}^r$ given by ρ. Since ρ is irreducible, it follows that $W_{e\Gamma} = (E_\rho)_{e\Gamma}$. This implies that $W = E_\rho$. Hence E_ρ is polystable.

Let

$$E_\rho = \bigoplus_{i=1}^{n} F_i$$

be a decomposition of the polystable bundle E_ρ into a direct sum of stable bundles. Since $\mathrm{degree}(E_\rho) = 0$, and E_ρ is semistable, it follows that $\mathrm{degree}(F_i) = 0$ for all $1 \leq i \leq n$.

Order the above vector bundles F_i in such a way that

(1) F_i is isomorphic to F_1 for all $1 \leq i \leq \ell$, and
(2) F_i is not isomorphic to F_1 for every $\ell + 1 \leq i \leq n$.

Note that ℓ may be 1. Let

$$F = \bigoplus_{i=1}^{\ell} F_i \subset \bigoplus_{i=1}^{n} F_i = E_\rho$$

be the subbundle given by the direct sum of the first ℓ summands. We will show that the flat connection ∇^ρ on E_ρ preserves F.

Let Ω_M denote the holomorphic cotangent bundle of M. Consider the composition

$$F \hookrightarrow E_\rho \xrightarrow{\nabla^\rho} E_\rho \otimes \Omega_M \longrightarrow (E_\rho/F) \otimes \Omega_M \, ;$$

it is an $\mathcal{O}_M$–linear homomorphism known as the second fundamental form of F for ∇^ρ. We will denote this second fundamental form by $S(\nabla^\rho, F)$. Now, Ω_M is trivial (a trivialization is given by a right translation invariant trivialization of the holomorphic cotangent bundle of G). Also, we have

$$E_\rho/F = \bigoplus_{i=\ell+1}^{n} F_i \, .$$

For any $\ell + 1 \leq i \leq n$, since F_i is a stable bundle of degree zero not isomorphic to the stable vector bundle F_1 of degree zero, it follows that

$$H^0(M, \mathrm{Hom}(F_1, F_i)) = H^0(M, F_i \otimes F_1^*) = 0 \, . \tag{2.5}$$

Since F is a direct sum copies of F_1, and Ω_M is trivial, from (2.5) we conclude that $S(\nabla^\rho, F) = 0$. This implies that the connection ∇^ρ on E_ρ preserves F.

Since ∇^ρ preserves F, and ρ is irreducible, it follows that $F = E_\rho$. This completes the proof of the theorem. $\qquad\square$

Remark 2.2. If the homomorphism ρ extends to a homomorphism $\widetilde{\rho} : G \longrightarrow \mathrm{GL}(r, \mathbb{C})$, then the vector bundle E_ρ is holomorphically trivial. Indeed, the map $G \times \mathbb{C}^r \longrightarrow G \times \mathbb{C}^r$ that sends any (z, v) to $(z, \widetilde{\rho}(z)(v))$ descends to a holomorphic isomorphism of E_ρ with the trivial vector bundle $M \times \mathbb{C}^r$ (recall that E_ρ is a quotient of $G \times \mathbb{C}^r$). Therefore, in that case the integer n in Theorem 2.1 is the rank r.

Remark 2.3. Since E_ρ admits a flat connection, all the rational Chern classes of E_ρ of positive degree vanish [2]. Therefore, the condition $E^{\oplus n} = E_\rho$ in Theorem 2.1 implies that all the rational Chern classes of E of positive degree vanish.

3. UNIQUENESS OF SOCLE OF SEMISTABLE REFLEXIVE SHEAVES

The aim of this section is to prove the following Proposition 3.1 about semi-stable reflexive sheaves (see Definition 1.1.9 on page 6 in [10]). The analogous (weaker) statement for bundles is needed in the proof of Theorem 2.1.

Proposition 3.1. *Consider a compact complex manifold X equipped with a Gauduchon metric ω. Let E be a semistable reflexive sheaf on X of slope μ (with respect to ω). Then there is a unique polystable sheaf $E' \subset E$ with $\mu(E') = \mu$ that is maximal among all such subsheaves.*

Proof. Let $F \subset E$ be any subsheaf of slope $\mu(F) = \mu$. Taking the double dual of the inclusion we see that F^{**} embeds into $E^{**} = E$. Moreover, since F^{**}/F is a torsion sheaf it follows that $\mu(F) \leq \mu(F^{**})$. Indeed, by Proposition V. 6.14 (p. 166) in [11] the determinant line bundle $\mathcal{L}$ of the torsion sheaf F^{**}/F admits a nontrivial holomorphic section. Then we make use of the following Lemma (stated as Proposition 1.3.5 on page 35 in [13]) which is a consequence of Poincaré-Lelong formula:

Lemma 3.2. *If the line bundle $\mathcal{L}$ admits a nontrivial holomorphic section t with vanishing divisor D_t, then $degree(\mathcal{L}) = c \cdot Vol_\omega(D_t)$, where c is a positive constant and the volume $Vol_\omega(D_t)$ of D_t is computed with the fixed Gauduchon metric ω. In particular, if $\mathcal{L}$ is nontrivial and admits a nontrivial holomorphic section, then $degree(\mathcal{L}) > 0$.*

Since E is semistable, it follows that

$$\mu(F) = \mu(F^{**}) = \mu,$$

so that F^{**} is semistable. Assume that F is stable and that $V \subset F^{**}$ is a non-trivial subsheaf of slope $\mu(V) = \mu$. Then $V \cap F$ is a subsheaf of the same slope and rank as V. Since F is stable, one has

$$\operatorname{rank}(V) = \operatorname{rank}(F) = \operatorname{rank}(F^{**}).$$

This shows that F^{**} is stable as well. Therefore, every stable subsheaf of E of slope μ is contained in a reflexive stable subsheaf of the same rank and slope.

Let $E' \subset E$ be the sum of all stable reflexive subsheaves of slope μ. By the argument of the first paragraph, E' contains all stable subsheaves of slope μ in E. It suffices to show that E' is polystable. Let

$$E'' := E_1 \oplus \cdots \oplus E_s \subset E'$$

be a polystable subsheaf of E' with a maximal number s of stable reflexive subsheaves E_i of slope μ. If $E'' = E'$ there is nothing to show. Otherwise, there is a reflexive stable subsheaf $F \subset E'$ of slope μ not contained in E''. If $F \cap E'' = 0$, then $E'' \oplus F \subset E'$, contradicting the maximality of s. Assume therefore that $F \cap E'' \neq 0$.

If $\mu(F \cap E'') < \mu$, then $\mu(E'' + F) = \mu(E'' \oplus F/(E'' \cap F)) > \mu$, contradicting the semistability of E. Hence $\mu(F \cap E'') = \mu$. Since F is stable, $F \cap E'' \subset F$ is a subsheaf of

the same rank, so that $F/(F \cap E'')$ is a torsion module. Since E'' is reflexive, the inclusion $F \cap E'' \subset E''$ extends to $F \subset E''$, contradicting the assumptions on F.

This shows that $E' = E''$ is polystable. By construction, E' is unique. $\qquad\square$

4. Rank two flat bundles on quotients of $\mathrm{SL}(2, \mathbb{C})$

We set $r = 2$ in (2.3). Take any irreducible ρ as in Theorem 2.1. Let $\rho' : \Gamma \longrightarrow \mathrm{PGL}(2, \mathbb{C})$ be the composition of ρ with the canonical projection of $\mathrm{GL}(2, \mathbb{C})$ to $\mathrm{PGL}(2, \mathbb{C})$.

Proposition 4.1. *Assume that the homomorphism ρ' does not extend to a homomorphism from G. Then the vector bundle E_ρ is stable.*

Proof. Assume that E_ρ is not stable. From Theorem 2.1 it follows that $E_\rho = L \oplus L$, where L is a holomorphic line bundle. Hence the projective bundle $\mathbb{P}(E_\rho)$ is holomorphically trivial. Now [16, p. 801, Proposition 3.1] contradicts the given condition that ρ' does not extend to a homomorphism from G. $\qquad\square$

Remark 4.2. In an earlier proof of Proposition 4.1, a longer argument was given after proving that $\mathbb{P}(E_\rho)$ is holomorphically trivial. The referee pointed out that [16, p. 801, Proposition 3.1] completes the proof at this point. This also enabled us to remove the assumption $G = \mathrm{SL}(2, \mathbb{C})$ in the earlier version.

Let $\rho, \eta : \Gamma \longrightarrow \mathrm{GL}(2, \mathbb{C})$ be two irreducible homomorphisms such that

- neither of the two corresponding homomorphisms $\rho', \eta' : \Gamma \longrightarrow \mathrm{PGL}(2, \mathbb{C})$ extends to a homomorphism from G, and
- for the action $\rho \otimes \eta^*$ of Γ on $\mathrm{End}(\mathbb{C}^2) = \mathbb{C}^2 \otimes (\mathbb{C}^2)^*$, where η^* is the action of Γ on $(\mathbb{C}^2)^*$ corresponding to its action on $\mathbb{C}^2$ given by η, the vector space $\mathrm{End}(\mathbb{C}^2)$ decomposes into a direct sum of irreducible Γ–modules such that all the direct summands are nontrivial and the action of Γ on none of them extends to an action of G.

From Proposition 4.1 we know that the associated vector bundles E_ρ and E_η are stable.

Lemma 4.3. *The two stable vector bundle E_ρ and E_η are not isomorphic.*

Proof. It suffices to show that

$$H^0(M, \mathrm{Hom}(E_\eta, E_\rho)) = 0. \qquad (4.1)$$

Since $\mathrm{End}(\mathbb{C}^2)$ decomposes into a direct sum of irreducible Γ–modules, from Theorem 2.1 we know that

$$\mathrm{Hom}(E_\eta, E_\rho) = \bigoplus_{i=1}^{b}(F_i)^{\oplus d_i}, \qquad (4.2)$$

where each F_i is a stable vector bundle of degree zero (here b is the number of irreducible Γ–modules in $\mathrm{End}(\mathbb{C}^2)$). To prove (4.1) it is enough to show that each F_i is nontrivial,

because a nontrivial stable vector bundle of degree zero does not admit any nonzero section.

Let V be an irreducible Γ module and E_V the corresponding vector bundle on M. If E_V is trivial, then the action of Γ on V extends to an action of G [16, p. 801, Proposition 3.1]. Therefore, each F_i in (4.2) is nontrivial. $\qquad\square$

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: `indranil@math.tifr.res.in`

Université Côte d'Azur, CNRS, LJAD, France

E-mail address: `dumitres@unice.fr`

Institut für Mathematik, Johannes Gutenberg Universität Mainz, 55099 Mainz, Germany

E-mail address: `lehn@mathematik.uni-mainz.de`