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Abstract

Integral inequalities for quadratic functions play an important role in the derivation of delay-dependent stability criteria for linear time-delay systems. Based on the Jensen inequality, a reciprocally convex combination approach was introduced in [17] for deriving delay-dependent stability criterion, which achieves the same upper bound of the time-varying delay as the one on the use of the Moon et al.'s inequality. Recently, a new inequality called Wirtinger-based inequality that encompasses the Jensen inequality was proposed in [20] for the stability analysis of time-delay systems. In this paper, *it is revealed that the reciprocally convex combination approach is effective only with the use of Jensen inequality. When the Jensen inequality is replaced by Wirtinger-based inequality, the Moon et al.'s inequality together with convex analysis can lead to less conservative stability conditions than the reciprocally convex combination inequality.* Moreover, we prove that *the feasibility of an LMI condition derived by the Moon et al.'s inequality as well as convex analysis implies the feasibility of an LMI condition induced by the reciprocally convex combination inequality.* Finally, the efficiency of the methods is demonstrated by some numerical examples, even though the corresponding system with zero-delay as well as the system without the delayed term are not stable.

Keywords: Systems with time-varying delays, Jensen inequality, Wirtinger-based inequality, convex method, Lyapunov-Krasovskii functionals.

1 Introduction

During the last two decades, a considerable amount of attention has been paid to stability and control of linear systems with time-varying delays (see e.g., [2], [19] and the references therein).

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One of the popular methods is the use of Lyapunov-Krasovskii functionals (LKF) to derive so-called delay-dependent sufficient conditions in terms of linear matrix inequalities (LMIs) (see e.g., [4], [17], [23], [12], [26], [27]). There are some degrees of freedom: 1) the selection of the functional; 2) the application of one of the existing integral inequalities; 3) the application of matrix inequalities. These three steps are the important ways to derive sufficient stability conditions and to reduce their conservatism. Many bounding techniques have been developed for delay-dependent stability analysis, for example, the Park's inequality [15], the Moon et al.'s inequality [14], the Jensen inequality [6], [13], the free-weighting matrix approach [8], [9], the convex approach [16], [23] and the combinations of some techniques above.

It is noted that the utilization of Jensen inequality to estimate the upper bound of the derivative of the LKF usually yields the following quadratic terms (see e.g., [11], [17]):

$$-\tfrac{1}{\alpha(t)}\xi_1^T(t)R\xi_1(t) - \tfrac{1}{1-\alpha(t)}\xi_2^T(t)R\xi_2(t), \tag{1}$$

where $0 < \alpha(t) < 1$ is a time-varying continuous function, $\xi_1(t)$ and $\xi_2(t)$ are two real column vectors with appropriate dimension, and R is a positive symmetric matrix with the same dimension as ξ_1 and ξ_2. The main difficulty relies on the fact that this term is not convex with respect to $\alpha(t)$ and, consequently, yields some difficulties when one wants to implement and to test the resulting LMI conditions. Therefore, to obtain stability criteria via LMI setup, the upper bound of (1) can be further estimated by virtue of the Park's inequality or the Moon et al.'s inequality together with the convex analysis [16]. Among the recent results, [17] introduced a reciprocally convex approach, which not only achieves the same upper bound of the time-varying delay as the one provided by [16] but also decreases the number of decision variables dramatically.

On the other hand, many different techniques have been introduced to reduce the bound on the gap of the Jensen inequality, see e.g., [1], [10], [30], [32], [33] and [35]. Among them, an alternative inequality called Wirtinger-based inequality, which encompasses Jensen inequality as a particular case, was developed in [20] and [22]. By the reciprocally convex combination inequality introduced in [17], the resulting stability conditions in [20] and [22] are less conservative than those of [16] and [17] that are based on the Jensen inequality.

In this paper, we present a comparison of bounding methods for cross terms in deriving the delay-dependent stability criteria for linear systems with time-varying delays. The main contributions are as follows:

1. We reveal that the reciprocally convex combination approach is effective only with the use of Jensen inequality. *When the Jensen inequality is replaced by Wirtinger-based inequality, the Moon et al.'s inequality together with convex analysis can lead to less conservative stability conditions than the reciprocally convex combination inequality.* This is different from the utilization of Jensen inequality, where the reciprocally convex combination inequality and the Moon et al.'s

inequality with convex analysis lead to identical admissible upper bound of the time-varying delay.

2. Moreover, we prove that the feasibility of an LMI condition derived by the Moon et al.'s inequality as well as convex analysis implies the feasibility of an LMI condition induced by the reciprocally convex combination inequality.

The structure of this paper is as follows. The system and some preliminaries are described in Section 2. Section 3 recalls some bounding techniques for cross terms and provides a theoretical comparison. In Section 4 we present several delay-dependent stability conditions, the comparison of conservatism and complexity of which is given. Examples of numerical simulation are illustrated in Section 5.

Notations: Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The notation $P \succ 0$, for $P \in \mathbb{R}^{n\times n}$, means that P is symmetric and positive definite. The set $\mathbb{S}_+^n$ represents the set of symmetric positive definite matrices of $\mathbb{R}^{n\times n}$. Moreover, for any square matrix $A \in \mathbb{R}^{n\times n}$, we define $\mathrm{He}(A) = A + A^T$. The matrix I represents the identity matrix of appropriate dimension. The notation $0_{n,m}$ stands for the matrix in $\mathbb{R}^{n\times m}$ whose entries are zero and, when no confusion is possible, the subscript will be omitted. For any function $x :\ [-h,\ +\infty) \to \mathbb{R}^n$, the notation $x_t(\theta)$ stands for $x(t+\theta)$, for all $t \geq 0$ and all $\theta \in [-h,\ 0]$.

2 Problem formulation

In order to illustrate the comparison of stability criteria for system with time-varying delays, we will consider a linear time-delay system of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h(t)), & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h_2, 0], \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, ϕ is the initial condition, A and A_d are constant matrices with appropriate dimensions. The delay is assumed to be time-varying and satisfies the following constraint

$$h(t) \in [h_1, h_2], \tag{3}$$

where $0 \leq h_1 \leq h_2$. We also assume that the derivative of the delay is not constrained. For simplicity, the time argument is omitted when there is no possible confusion, meaning, more especially, that in the sequel h stands for $h(t)$.

Among the LKFs that applied to the delay-dependent stability analysis of such time-delay systems, one of the most relevant terms, which was introduced in [3] is a double integral quadratic term given by

$$V(\dot{x}_t) = h_{12} \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{x}^T(s) R_2 \dot{x}(s) \mathrm{d}s \mathrm{d}\theta,$$

where $h_{12} = h_2 - h_1$, $R_2 \in \mathbb{S}_+^n$, and $\dot{x}_t(\theta) = \dot{x}(t+\theta)$ represents the time-derivative state of the time-delay system. This class of Lyapunov-Krasovskii terms has been widely used in the literature mainly because the computation of its time-derivative leads to conditions which depend on the explicit value of the delay h_1, h_2. Indeed when differentiating this term with respect to the time variable t, we obtain

$$\dot{V}(\dot{x}_t) = h_{12}^2 \dot{x}^T(t) R_2 \dot{x}(t) - h_{12} \int_{-h_2}^{-h_1} \dot{x}_t^T(s) R_2 \dot{x}_t(s) \mathrm{d}s. \tag{4}$$

This term is relevant to ensure the negativity of $\dot{V}(x_t)$ because of the negative contribution of the second term. In order to transform (4) into a suitable LMI setup, this integral term should be expressed appropriately in terms of $x_t(-h_1)$, $x_t(-h)$ and $x_t(-h_2)$. Therefore, in order to obtain a more accurate bound for this integral term, and thus, to reduce the conservatism of the resulting stability conditions, various bounding techniques have been employed in the literature. Among them, we are concentrated on the use of the Jensen inequality [6], Wirtinger-based inequality [20] together with the use of convex approach.

The objective of this paper is to compare the conservatism of several bounding methods in the derivation of stability criteria for systems with time-varying delays.

3 Bounding techniques

In this section, we introduce several efficient bounding techniques to be used to deal with the quadratic terms that arise in the derivation of the LKF.

3.1 Integral inequalities

Integral inequalities for quadratic functions play an important role in deriving the delay-dependent stability criteria for linear time-delay systems. In the following, a brief recall of three integral inequalities are proposed.

3.1.1 Jensen inequality

The first method to analyze the stability of time-delay systems is based on the Jensen inequality formulated in the next lemma.

Lemma 1 *[6] For any matrix R in $\mathbb{S}_+^n$ and any differentiable function $x : [a, b] \to \mathbb{R}^n$, the following inequality holds:*

$$\int_a^b \dot{x}^T(u) R \dot{x}(u) \mathrm{d}u \geq \tfrac{1}{b-a} \xi_0^T R \xi_0, \tag{5}$$

where

$$\xi_0 = x(b) - x(a). \tag{6}$$

By applying Lemma 1 to the second term of (4) after splitting the integral into two parts, we arrive at

$$\begin{array}{l} -h_{12} \int_{-h_2}^{-h_1} \dot{x}_t^T(s) R_2 \dot{x}_t(s) \mathrm{d}s \\ = -h_{12} \Big[\int_{-h}^{-h_1} \dot{x}_t^T(s) R_2 \dot{x}_t(s) \mathrm{d}s + \int_{-h_2}^{-h} \dot{x}_t^T(s) R_2 \dot{x}_t(s) \mathrm{d}s \Big] \\ \leq -\frac{h_{12}}{h-h_1} \eta_0^T(t) R_2 \eta_0(t) - \frac{h_{12}}{h_2-h} \eta_1^T(t) R_2 \eta_1(t) \\ = - \begin{bmatrix} \eta_0(t) \\ \eta_1(t) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\alpha} R_2 & 0 \\ * & \frac{1}{1-\alpha} R_2 \end{bmatrix} \begin{bmatrix} \eta_0(t) \\ \eta_1(t) \end{bmatrix}, \end{array} \tag{7}$$

where

$$\begin{array}{rcl} \alpha & = & \frac{h-h_1}{h_{12}}, \\ \eta_0(t) & = & x(t-h_1) - x(t-h), \\ \eta_1(t) & = & x(t-h) - x(t-h_2). \end{array} \tag{8}$$

3.1.2 Wirtinger-based integral inequality

The following lemma provides an inequality called Wirtinger-based inequality, which encompasses Jensen inequality as a particular case, and was recently proposed in [20].

Lemma 2 *[20] For any matrix $R \in \mathbb{S}_n^+$ and any differentiable function $x : [a, b] \to \mathbb{R}^n$, the following inequality holds:*

$$\int_a^b \dot{x}^T(u) R \dot{x}(u) \mathrm{d}u \geq \frac{1}{b-a} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix}^T \tilde{R} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix}, \tag{9}$$

where ξ_0 is given by (6) and

$$\begin{array}{rcl} \xi_1 & = & x(b) + x(a) - \frac{2}{b-a} \int_a^b x(u) \mathrm{d}u, \\ \tilde{R} & = & \mathrm{diag}(R, 3R). \end{array} \tag{10}$$

The application of Lemma 2 to the second term of (4) yields

$$\begin{array}{l} -h_{12} \int_{-h_2}^{-h_1} \dot{x}_t^T(s) R_2 \dot{x}_t(s) \mathrm{d}s \\ \leq -\frac{h_{12}}{h-h_1} \zeta_0^T(t) \tilde{R}_2 \zeta_0(t) - \frac{h_{12}}{h_2-h} \zeta_1^T(t) \tilde{R}_2 \zeta_1(t) \\ = - \begin{bmatrix} \zeta_0(t) \\ \zeta_1(t) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\alpha} \tilde{R}_2 & 0 \\ * & \frac{1}{1-\alpha} \tilde{R}_2 \end{bmatrix} \begin{bmatrix} \zeta_0(t) \\ \zeta_1(t) \end{bmatrix}, \end{array} \tag{11}$$

where α is given in (8) and

$$\begin{aligned}
&\tilde{R}_2 = \mathrm{diag}(R_2, 3R_2),\\
&\zeta_0(t)=\begin{bmatrix} x(t-h_1)-x(t-h) \\ x(t-h_1)+x(t-h)-\frac{2}{h-h_1}\int_{t-h}^{t-h_1} x(s)\mathrm{d}s \end{bmatrix},\\
&\zeta_1(t)=\begin{bmatrix} x(t-h)-x(t-h_2) \\ x(t-h)+x(t-h_2)-\frac{2}{h_2-h}\int_{t-h_2}^{t-h} x(s)\mathrm{d}s \end{bmatrix}.
\end{aligned} \tag{12}$$

3.1.3 Free-matrix-based integral inequality

Recently, a free-matrix-based integral inequality was provided in [28] to estimate the bound of the second term of (4).

Lemma 3 *[28] Let x be a differentiable function $[a,b] \to \mathbb{R}^n$. For any matrices $R \in \mathbb{S}_n^+$ and $Z_1, Z_3, \in \mathbb{S}_{3n}^+$, and matrices $Z_2 \in \mathbb{R}^{3n\times 3n}$, $N_1, N_2 \in \mathbb{R}^{3n\times n}$, satisfying*

$$\begin{bmatrix} Z_1 & Z_2 & N_1 \\ * & Z_3 & N_2 \\ * & * & R \end{bmatrix} \succeq 0, \tag{13}$$

the following inequality holds

$$\int_a^b \dot{x}^T(u)R\dot{x}(u)\mathrm{d}u \geq \rho^T\Sigma\rho, \tag{14}$$

where

$$\begin{aligned}
\rho &= \begin{bmatrix} x^T(b) & x^T(a) & \frac{1}{b-a}\int_a^b x^T(u)\mathrm{d}u \end{bmatrix}^T,\\
\Sigma &= (b-a)(-Z_1-\tfrac{Z_3}{3}) - \mathrm{He}(N_1\Pi_1 + N_2\Pi_2),\\
\Pi_1 &= \begin{bmatrix} I & -I & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} I & I & -2I \end{bmatrix}.
\end{aligned} \tag{15}$$

The utilization of Lemma 3 to the second term of (4) leads to

$$\begin{aligned}
&-h_{12}\int_{-h_2}^{-h_1} \dot{x}_t^T(s)R_2\dot{x}_t(s)\mathrm{d}s\\
&= -h_{12}\Big[\int_{-h}^{-h_1} \dot{x}_t^T(s)R_2\dot{x}_t(s)\mathrm{d}s + \int_{-h_2}^{-h} \dot{x}_t^T(s)R_2\dot{x}_t(s)\mathrm{d}s\Big]\\
&\leq -\rho_1^T\Sigma_1\rho_1 - \rho_2^T\Sigma_2\rho_2,
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\rho_1 &= \begin{bmatrix} x_t^T(-h_1) & x_t^T(-h) & \frac{1}{h-h_1}\int_{-h}^{-h_1} x_t^T(u)\mathrm{d}u \end{bmatrix}^T,\\
\rho_2 &= \begin{bmatrix} x_t^T(-h) & x_t^T(-h_2) & \frac{1}{h_2-h}\int_{-h_2}^{-h} x_t^T(u)\mathrm{d}u \end{bmatrix}^T,\\
\Sigma_1 &= (h-h_1)(X_1^1 + \tfrac{X_3^1}{3}) + \mathrm{He}(N_1^1\Pi_1 + N_2^1\Pi_2),\\
\Sigma_2 &= (h_2-h)(X_1^2 + \tfrac{X_3^2}{3}) + \mathrm{He}(N_1^2\Pi_1 + N_2^2\Pi_2)
\end{aligned} \tag{17}$$

with

$$\begin{bmatrix} X_1^i & X_2^i & N_1^i \\ * & X_3^i & N_2^i \\ * & * & R_2 \end{bmatrix} \succeq 0,\ i = 1, 2. \tag{18}$$

The right-hand side of the inequality (16) can be transformed into a suitable LMI setup by convex optimization approach with $h = h_1$ and $h = h_2$ [5].

3.1.4 Comparison of Wirtinger-based and free-matrix-based integral inequalities

In this section, we illustrate a theoretical comparison of Wirtinger-based and free-matrix-based integral inequalities. By defining in Lemma 3

$$N = \begin{bmatrix} N_1^T \\ N_2^T \end{bmatrix},\quad \Pi = \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix},$$

it is easy to see that

$$\mathrm{He}(N_1\Pi_1 + N_2\Pi_2) = \mathrm{He}(N^T\Pi) = \tilde{\Theta} - \tfrac{1}{b-a}\Pi^T\tilde{R}\Pi - (b-a)N^T\tilde{R}^{-1}N, \tag{19}$$

where $\tilde{R}$ is defined in (10) and

$$\tilde{\Theta} = (b-a)(\tfrac{1}{b-a}\tilde{R}\Pi + N)^T\tilde{R}^{-1}(\tfrac{1}{b-a}\tilde{R}\Pi + N). \tag{20}$$

Then, it follows that Σ given in (17) can be rewritten as

$$\begin{aligned} \Sigma &= (b-a)(-Z_1 - \tfrac{Z_3}{3}) - \tilde{\Theta} + \tfrac{1}{b-a}\Pi^T\tilde{R}\Pi + (b-a)N^T\tilde{R}^{-1}N \\ &= \tfrac{1}{b-a}\Pi^T\tilde{R}\Pi - \tilde{\Theta} - (b-a)(Z_1 - N_1R^{-1}N_1^T) - \tfrac{b-a}{3}(Z_3 - N_2R^{-1}N_2^T) \\ &\preceq \tfrac{1}{b-a}\Pi^T\tilde{R}\Pi. \end{aligned} \tag{21}$$

The latter inequality holds because of $\tilde{\Theta} \succeq 0$ and the fact that application of Schur complement to (13) implies $Z_1 - N_1R^{-1}N_1^T \succeq 0$ and $Z_3 - N_2R^{-1}N_2^T \succeq 0$. Hence, from (21) and $\rho^T\Pi^T = \begin{bmatrix} \xi_0^T & \xi_1^T \end{bmatrix}$, it is verified that the free-matrix-based integral inequality (14) with (13) cannot deliver a more tight lower bound of $\int_a^b \dot{x}^T(u)R\dot{x}(u)\mathrm{d}u$ than (9) although more free matrices are involved in (14).

3.2 Convex approaches

It is noted that in (7) and (11), the positive definite matrix

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ * & \frac{1}{1-\alpha}R \end{bmatrix} \tag{22}$$

with $R = R_2$ or $\tilde{R}_2$ is time-varying due to time-varying $\frac{1}{\alpha}$ and $\frac{1}{1-\alpha}$ with $0 < \alpha < 1$. To derive stability conditions in terms of LMIs, the time-varying matrix (22) needs to be estimated by some bounding techniques. For example, in [4] and [25], the matrix (22) was estimated as follows:

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ * & \frac{1}{1-\alpha}R \end{bmatrix} \succeq \begin{bmatrix} R & 0 \\ * & 0 \end{bmatrix}. \tag{23}$$

By including a useful term, the matrix (22) was further estimated in [8] as

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ * & \frac{1}{1-\alpha}R \end{bmatrix} \succeq \begin{bmatrix} R & 0 \\ * & R \end{bmatrix}. \tag{24}$$

Compared to (23), inequality (24) possesses a more tight lower bound of (22) and thus, can derive less conservative LMI stability conditions than (23). However, (24) still leaves some room for a more tight lower bound of (22).

3.2.1 Moon et al.'s inequality

Lemma 4 *[14] Suppose that there exists matrices Q, Z in $\mathbb{S}_+^n$, matrix $X \in \mathbb{R}^{n\times n}$ satisfying $\begin{bmatrix} Q & X \\ * & Z \end{bmatrix} \succeq 0$. Then the following inequality holds for any $x, y \in \mathbb{R}^n$, any matrix $N \in \mathbb{R}^{n\times n}$*

$$-2x^T Ny \leq \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} Q & X - N \\ * & Z \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{25}$$

For any scalar $\varepsilon > 0$, any matrix R in $\mathbb{S}_+^n$, it follows from (25) with the choice of $Q = \varepsilon^{-1}R$, $Z = \varepsilon R^{-1}$, $X = N = -I$ that

Lemma 5 *For any $x, y \in \mathbb{R}^n$, any scalar $\varepsilon > 0$, any matrix R in $\mathbb{S}_+^n$, the following inequality holds*

$$2x^T y \leq \varepsilon^{-1} x^T Rx + \varepsilon y^T R^{-1} y. \tag{26}$$

Then for any matrices M_i in $\mathbb{R}^{2n\times n}$, $i = 1, 2$, we have

$$\frac{1}{\alpha}\begin{bmatrix} I \\ 0 \end{bmatrix} R \begin{bmatrix} I & 0 \end{bmatrix} + \alpha M_1 R^{-1} M_1^T \succeq \mathrm{He}(M_1[I \quad 0]) \quad \textit{and}$$

$$\frac{1}{1-\alpha}\begin{bmatrix} 0 \\ I \end{bmatrix} R \begin{bmatrix} 0 & I \end{bmatrix} + (1-\alpha) M_2 R^{-1} M_2^T \succeq \mathrm{He}(M_2[0 \quad I]).$$

Therefore, the equality

$$\begin{aligned} \begin{bmatrix} \frac{1}{\alpha}R & 0 \\ * & \frac{1}{1-\alpha}R \end{bmatrix} &= \mathrm{He}(M_1[I \quad 0] + M_2[0 \quad I]) - \alpha M_1 R^{-1} M_1^T \\ &\quad - (1-\alpha) M_2 R^{-1} M_2^T + \Theta_1(\alpha) + \Theta_2(\alpha) \end{aligned} \tag{27}$$

holds for all scalar $\alpha \in (0,1)$, *where*

$$\begin{array}{rcl} \Theta_1(\alpha) & = & \alpha(\frac{1}{\alpha}R[I \;\; 0] - M_1^T)^T R^{-1}(\frac{1}{\alpha}R[I \;\; 0] - M_1^T) \succeq 0, \\ \Theta_2(\alpha) & = & (1-\alpha)(\frac{1}{1-\alpha}R[0 \;\; I] - M_2^T)^T R^{-1}(\frac{1}{1-\alpha}R[0 \;\; I] - M_2^T) \succeq 0. \end{array} \tag{28}$$

3.2.2 Reciprocally convex combination inequality

Recall the reciprocally convex combination lemma (RCCL) provided in [17]:

Lemma 6 (RCCL) *For a given matrix* $R \in \mathbb{S}_+^n$, *assume that there exists a matrix* $X \in \mathbb{R}^{n\times n}$ *such that*

$$\begin{bmatrix} R & X \\ * & R \end{bmatrix} \succeq 0. \tag{29}$$

Then the equality

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ * & \frac{1}{1-\alpha}R \end{bmatrix} = \begin{bmatrix} R & X \\ * & R \end{bmatrix} + \Theta(\alpha) \tag{30}$$

holds for all scalar $\alpha \in (0,1)$, *where*

$$\begin{array}{rcl} \Theta(\alpha) & = & \begin{bmatrix} \frac{1-\alpha}{\alpha}R & -X \\ * & \frac{\alpha}{1-\alpha}R \end{bmatrix} \\ & = & \begin{bmatrix} \sqrt{\frac{1-\alpha}{\alpha}}\, I & 0 \\ * & -\sqrt{\frac{\alpha}{1-\alpha}}\, I \end{bmatrix} \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1-\alpha}{\alpha}}\, I & 0 \\ * & -\sqrt{\frac{\alpha}{1-\alpha}}\, I \end{bmatrix} \succeq 0. \end{array}$$

3.2.3 Comparison of (27) and (30) with (29)

In order to apply (27) or (30) to derive LMI stability conditions of system (2), the lower bound of $\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ * & \frac{1}{1-\alpha}R \end{bmatrix}$ needs to be estimated since both $\Theta_i(\alpha)$, $i = 1, 2$, in (27) and $\Theta(\alpha)$ in (30) are not affine in the convex parameter α. Then we have the following result:

Lemma 7 *The equality (27) provides a tighter lower bound of (22) that is affine in* α *than equality (30) with (29).*

Proof 1 *Suppose now that there exist matrices* $R \in \mathbb{S}_+^n$ *and* $X \in \mathbb{R}^{n\times n}$ *such that (29) and (30) are satisfied. In (27), choose*

$$\begin{array}{rcl} M_1^T & = & R[I \;\; 0] + X[0 \;\; I], \\ M_2^T & = & R[0 \;\; I] + X^T[I \;\; 0]. \end{array}$$

Then, the sum of $\Theta_1(\alpha)$ and $\Theta_2(\alpha)$ in (28) can be presented as

$$\begin{aligned}
&\Theta_1(\alpha)+\Theta_2(\alpha)\\
&=\begin{bmatrix} \frac{(1-\alpha)^2}{\alpha}R & 0 \\ * & \frac{\alpha^2}{1-\alpha}R \end{bmatrix}+\begin{bmatrix} 0 & -X \\ * & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ * & \alpha X^T R^{-1} X \end{bmatrix}+\begin{bmatrix} (1-\alpha)XR^{-1}X^T & 0 \\ * & 0 \end{bmatrix}\\
&=\Theta(\alpha)-\begin{bmatrix} (1-\alpha)(R-XR^{-1}X^T) & 0 \\ * & \alpha(R-X^T R^{-1}X) \end{bmatrix}\\
&\preceq \Theta(\alpha).
\end{aligned}$$

The latter inequality is guaranteed by the fact that application of Schur complement to (29) yields $R-XR^{-1}X^T\succeq 0$ and $R-X^TR^{-1}X\succeq 0$. Then, by comparing (30) and (27) we arrive at

$$\begin{bmatrix} R & X \\ * & R \end{bmatrix}\preceq \mathrm{He}(M_1[I\ \ 0]+M_2[0\ \ I])-\alpha M_1R^{-1}M_1^T-(1-\alpha)M_2R^{-1}M_2^T,$$

which shows that the RCCL is a particular case of the Moon et al.'s inequality.

An alternative proof of Lemma 7 can be found in [31]. From Lemma 7, it is then expected that the stability conditions resulting from the application of the RCCL is more conservative than the ones obtained from the application of Moon et al.'s inequality.

In Section 3.1.4, although it is verified that the free-matrix-based inequality (14) with (13) cannot deliver a more tight lower bound of $\int_a^b \dot{x}^T(u)R\dot{x}(u)\mathrm{d}u$ than Wirtinger-based inequality (9), this formulation still has some interest with respect to the original Wirtinger-based inequality, as noticed in [7]. Indeed, when one has to test stability of time-varying delay systems, Lemma 3 can deliver tighter lower bounds than the one based on the Wirtinger-based integral inequality together with the RCCL. In light of [7] and on the previous considerations, this reduction of the conservatism is mainly due to the application of the convex optimization approach. Indeed when one uses the Wirtinger-based inequality together with the Moon et al.'s inequality and convex analysis, one can derive less conservative stability conditions than the one based on inequality (14) and also with a lower number of decision variables.

4 Delay-dependent stability conditions and comparison of conservatism and complexity

This section first presents several delay-dependent stability conditions obtained by means of the Jensen inequality or Wirtinger-based integral inequality together with the convex approaches provided in Section 3.2 and then provides a comparison of the conservatism and numerical complexity of different methods. For the simplicity of presentation, the following notations will be used in this section.

$$
\begin{aligned}
e_i &= [0_{n\times(i-1)n}\ I_n\ 0_{n\times(7-i)n}],\ i=1,\dots,7,\\
G_0 &= Ae_1 + A_1e_3, \quad \Gamma = [G_3^T\ \ G_4^T]^T,\ G_1(h) = [e_1^T\ \ h_1e_5^T\ \ (h-h_1)e_6^T + (h_2-h)e_7^T]^T,\\
G_2 &= [e_1^T - e_2^T\ \ \ e_1^T + e_2^T - 2e_5^T]^T,\ \ G_3 = [e_2^T - e_3^T\ \ \ e_2^T + e_3^T - 2e_6^T]^T,\\
G_4 &= [e_3^T - e_4^T\ \ \ e_3^T + e_4^T - 2e_7^T]^T,\ G_5 = [e_2^T\ \ \ e_3^T\ \ \ e_6^T]^T,\ \ G_6 = [e_3^T\ \ \ e_4^T\ \ \ e_7^T]^T.
\end{aligned} \tag{31}
$$

4.1 Stability conditions

Consider the standard LKF for the stability analysis of systems with time-varying delay from the interval $[h_1, h_2]$ (see e.g., [17]). The application of Moon et al.'s inequality in Lemma 5 to (7) and the Jensen inequality leads to the following condition for stability of system (2).

Lemma 8 (Jensen-Moon-Convex) *Assume that there exist two scalars $h_2 > h_1 \geq 0$, matrices P, S_i, and R_i in $\mathbb{S}_+^n$, and two matrices Y_i in $\mathbb{R}^{4n\times n}$, $i = 1, 2$, such that the following LMIs are satisfied*

$$
\begin{bmatrix} \Phi - h_{12}\mathrm{He}(Y_1F_{23} + Y_2F_{34}) & h_{12}Y_i \\ * & -R_2 \end{bmatrix} \prec 0,\ i = 1, 2, \tag{32}
$$

where

$$
\begin{aligned}
\Phi &= \mathrm{He}(F_1^TPF_0) + S + F_0^T(h_1^2R_1 + h_{12}^2R_2)F_0 - F_{12}^TR_1F_{12},\\
F_0 &= \begin{bmatrix} A & 0 & A_d & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix},\\
F_{12} &= \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix},\ F_{23} = \begin{bmatrix} 0 & I & -I & 0 \end{bmatrix},\\
F_{34} &= \begin{bmatrix} 0 & 0 & I & -I \end{bmatrix}, \quad S = \mathrm{diag}(S_1, S_2 - S_1, 0, -S_2).
\end{aligned} \tag{33}
$$

Then the system (2) *is asymptotically stable for the time-varying delay h satisfying* (3).

Recently, Lemma 6 together with the Jensen inequality is widely employed to derive stability condition for systems with time-varying delay. The stability condition is summarized in the following theorem, taken from [17].

Lemma 9 (Jensen-RCCL) *Assume that there exist two scalars $h_2 > h_1 \geq 0$, matrices P, S_i, and R_i, $i = 1, 2$, in $\mathbb{S}_+^n$, and a matrix X in $\mathbb{R}^{n\times n}$ such that the following LMIs are satisfied*

$$
\Psi = \begin{bmatrix} R_2 & X \\ * & R_2 \end{bmatrix} \succeq 0,\ \Phi - \begin{bmatrix} F_{23}^T & F_{34}^T \end{bmatrix} \Psi \begin{bmatrix} F_{23}^T & F_{34}^T \end{bmatrix}^T \prec 0, \tag{34}
$$

where Φ, F_{23} and F_{34} are given in (33). Then the system (2) *is asymptotically stable for the time-varying delay h satisfying* (3).

To employ Lemma 2 for delay-dependent analysis of systems with time-varying delays, an augmented LKF was suggested in [22]. The stability condition that derived in [22] by means of Lemma 6 as well as Wirtinger-based integral inequality in Lemma 2 is summarized in the following theorem.

Lemma 10 (Wirtinger-RCCL) *Assume that there exist matrices P in $\mathbb{S}_+^{3n}$, S_i, and R_i, $i = 1, 2$, in $\mathbb{S}_+^n$, and a matrix X in $\mathbb{R}^{2n\times 2n}$ such that the following matrix inequalities are satisfied for h in $\{h_1,\ h_2\}$*

$$\tilde{\Psi} = \begin{bmatrix} \tilde{R}_2 & X \\ * & \tilde{R}_2 \end{bmatrix} \succeq 0, \quad \Phi(h) = \Phi_0(h) - \Gamma^T \tilde{\Psi} \Gamma \prec 0, \tag{35}$$

where

$$\begin{aligned} \Phi_0(h) &= \mathrm{He}(G_1^T(h) P G_0) + \hat{S} + G_0^T(h_1^2 R_1 + h_{12}^2 R_2) G_0 - G_2^T \tilde{R}_1 G_2, \\ \hat{S} &= \mathrm{diag}(S_1, S_2 - S_1, -S_2, 0_{3n}), \\ \tilde{R}_1 &= \mathrm{diag}(R_1, 3R_1), \end{aligned} \tag{36}$$

and the notations are given in (12) and (31). Then the system (2) *is asymptotically stable for all time-varying delay functions h satisfying* (3).

Moreover, following [28] and applying Lemma 3 to the second term of (4) after splitting the integral into two parts, we derive the following result:

Lemma 11 (Free-matrix and Convex) *Assume that there exist matrices P, X_1^i, X_3^i in $\mathbb{S}_+^{3n}$, S_i, R_i in $\mathbb{S}_+^n$, and matrices X_2^i in $\mathbb{R}^{3n\times 3n}$, N_1^i, N_2^i in $\mathbb{R}^{3n\times n}$, $i = 1, 2$, such that (18) and the following matrix inequalities are satisfied for h in $\{h_1,\ h_2\}$*

$$\begin{aligned} &\Phi_0(h) + h_{12}(h - h_1) G_5^T (X_1^1 + \tfrac{1}{3} X_3^1) G_5 + h_{12}(h_2 - h) G_6^T (X_1^2 + \tfrac{1}{3} X_3^2) G_6 \\ &\quad + h_{12} \mathrm{He}(G_5^T N_1^1 [I \quad 0] G_3 + G_5^T N_2^1 [0 \quad I] G_3 + G_6^T N_1^2 [I \quad 0] G_4 + G_6^T N_2^2 [0 \quad I] G_4) \prec 0, \end{aligned} \tag{37}$$

where the notations $\Phi_0(h)$ and G_5, G_6 are given in (36) and (31), respectively. Then, the system (2) *is asymptotically stable for all time-varying delay functions h satisfying* (3).

The application of Lemma 5 to (11) and Wirtinger-based integral inequality also allows us to derive stability criterion via LMI setup. In such situation, the following theorem is provided.

Theorem 1 (Wirtinger-Moon-Convex) *Assume that there exist matrices P in $\mathbb{S}_+^{3n}$, S_i, and R_i in $\mathbb{S}_+^n$, and two matrices Y_i in $\mathbb{R}^{7n\times 2n}$, $i = 1, 2$, such that the following LMIs are satisfied for h in $\{h_1,\ h_2\}$*

$$\Omega(h) = \begin{bmatrix} \Phi_0(h) - h_{12}\mathrm{He}(Y_1 G_3 + Y_2 G_4) & h_{12} Y_i \\ * & -\tilde{R}_2 \end{bmatrix} \prec 0, \ i = 1, 2, \tag{38}$$

where the notations $\tilde{R}_2$, $\Phi_0(h)$ and G_3, G_4 are given in (12), (36) and (31), respectively. Then the system (2) *is asymptotically stable for all time-varying delay functions h satisfying* (3).

Proof 2 *The proof follows from the standard arguments for the delay-dependent stability analysis with the use of Wirtinger-based integral inequality (9), Moon et al.'s inequality (27) and convex analysis. Consider the augmented Lyapunov functional $V(x_t, \dot{x}_t)$ given in [22] and define $\zeta(t) =$*

col$\{\eta(t), \frac{1}{h_1}\int_{-h_1}^{0} x_t(s)\mathrm{d}s, \frac{1}{h-h_1}\int_{-h}^{-h_1} x_t(s)\mathrm{d}s, \frac{1}{h_2-h}\int_{-h_2}^{-h} x_t(s)\mathrm{d}s\}$ *with* $\eta(t) =$ *col*$\{x(t), x(t-h_1), x(t-h), x(t-h_2)\}$.

By Lemma 2 the differentiation of $V(x_t, \dot{x}_t)$ *along the trajectories of (2) leads to*

$$\dot{V}(x_t, \dot{x}_t) \leq \zeta^T(t)\Phi_0(h)\zeta(t) - h_{12}\int_{-h_2}^{-h_1} \dot{x}_t^T(s)R_2\dot{x}_t(s)\mathrm{d}s, \tag{39}$$

where $\Phi_0(h)$ *is given in (36). Furthermore, from (11) the following inequality*

$$\begin{aligned}
&-h_{12}\int_{-h_2}^{-h_1} \dot{x}_t^T(s)R_2\dot{x}_t(s)\mathrm{d}s \\
&\leq -\zeta^T(t)\left[\begin{array}{cc} G_3^T & G_4^T \end{array}\right]\left[\begin{array}{cc} \frac{1}{\alpha}\tilde{R}_2 & 0 \\ * & \frac{1}{1-\alpha}\tilde{R}_2 \end{array}\right]\left[\begin{array}{c} G_3 \\ G_4 \end{array}\right]\zeta(t) \\
&\leq -\zeta^T(t)\left[\begin{array}{cc} G_3^T & G_4^T \end{array}\right]\Big[\mathrm{He}\Big(\left[\begin{array}{c} h_{12}\hat{Y}_1 \\ 0 \end{array}\right][I \quad 0] + \left[\begin{array}{c} 0 \\ h_{12}\hat{Y}_2 \end{array}\right][0 \quad I]\Big) - h_{12}(h-h_1)\left[\begin{array}{c} \hat{Y}_1 \\ 0 \end{array}\right]\tilde{R}_2^{-1}[\hat{Y}_1^T \quad 0] \\
&\qquad\qquad - h_{12}(h_2-h)\left[\begin{array}{c} 0 \\ \hat{Y}_2 \end{array}\right]\tilde{R}_2^{-1}[0 \quad \hat{Y}_2^T]\Big]\left[\begin{array}{c} G_3 \\ G_4 \end{array}\right]\zeta(t)
\end{aligned} \tag{40}$$

holds for any matrices $\hat{Y}_1, \hat{Y}_2 \in \mathbb{R}^{2n\times 2n}$, *where* $\alpha = (h-h_1)/h_{12}$. *The latter inequality is guaranteed by Lemma 5, with* $M_1 = h_{12}[\hat{Y}_1^T \quad 0]^T$ *and* $M_2 = h_{12}[0 \quad \hat{Y}_2^T]^T$. *Thus, by letting* $Y_1 = G_3^T\hat{Y}_1$ *and* $Y_2 = G_4^T\hat{Y}_2$, *we obtain from (39) and (40) that*

$$\dot{V}(x_t, \dot{x}_t) \leq \zeta^T(t)\Phi(h)\zeta(t),$$

where $\Phi(h) = \Phi_0(h) - h_{12}\mathrm{He}(Y_1G_3 + Y_2G_4) + h_{12}(h-h_1)Y_1\tilde{R}_2^{-1}Y_1^T + h_{12}(h_2-h)Y_2\tilde{R}_2^{-1}Y_2^T$. *Since* $\Phi(h)$ *is affine with respect to* h, *the two matrix inequalities* $\Phi(h_1) \prec 0$ *and* $\Phi(h_2) \prec 0$ *imply* $\Phi(h) \prec 0$ *for all* $h \in [h_1,\ h_2]$. *This means that by Schur complement if the two LMIs* $\Omega(h)_{|h=h_i} \prec 0, i = 1, 2$, *then* $\dot{V}(x_t, \dot{x}_t) < 0$, *implying asymptotic stability of system (2) for all time-varying delay in the interval* $[h_1,\ h_2]$.

Remark 1 *Lemmas 10, 11 and Theorem 1 are also applicable to the stability analysis of systems with interval delays, which may be unstable for small delays (or without delays). It is worth noting that Lemmas 8 and 9 that correspond to classical Lyapunov-Krasovskii approaches based on Jensen inequality cannot assess stability of such systems.*

Remark 2 *More recently, generalized integral inequalities were developed in [21] based on Bessel's inequality and Legendre polynomials, which includes Jensen and Wirtinger-based inequalities and the recent inequalities based on auxiliary functions ([18], [29]) as particular cases. Therefore, the stability criteria of Lemmas 8-11 and Theorem 1 could be further improved by employing generalized integral inequalities together with Lemma 5 or 6.*

Table 1: The comparison of admissible upper bound h_2 for different methods.

Methods	Lemma 1 (Jensen)		Lemma 2 (Wirtinger)		Lemma 3 (free-matrix)
Lemma 6 (reciprocally convex)	$h_2\{\text{Lem. }9\}$	$\Longrightarrow (\leq)$	$h_2\{\text{Lem.}10\}$		——
	$\Updownarrow (=)$		$\Downarrow (\leq)$		
Lemma 5 (Moon et al.)	$h_2\{\text{Lem.}8\}$		$h_2\{\text{Th.}1\}$	$(\geq) \Longleftarrow$	$h_2\{\text{Lem.}11\}$

Table 2: The comparison of the numerical complexity of different methods.

Decision variables No. and order of LMIs	Lemma 1 (Jensen)	Lemma 2 (Wirtinger)	Lemma 3 (free-matrix)
Lemma 6 (reciprocally convex)	$3.5n^2 + 2.5n$ 1 of $2n \times 2n$ 1 of $4n \times 4n$ (Lem.9)	$10.5n^2 + 3.5n$ 1 of $4n \times 4n$ 1 of $7n \times 7n$ (Lem.10)	——
Lemma 5 (Moon et al.)	$10.5n^2 + 2.5n$ 2 of $5n \times 5n$ (Lem.8)	$20.5n^2 + 3.5n$ 2 of $8n \times 8n$ (Th.1)	$54.5n^2 + 9.5n$ 2 of $3n \times 3n$, 1 of $7n \times 7n$ (Lem.11)

4.2 Comparison of numerical complexity of different conditions

Based on the discussions in Sections 3.1.4 and 3.2.3, and the fact of Wirtinger-based inequality encompassing the Jensen inequality, Table 1 shows the comparison of the maximum values of h_2 that preserve the stability by applying Lemmas 8-11 and Theorem 1 with given lower bound h_1. The numerical complexity of the resulting LMIs under different bounding techniques is illustrated in Table 2.

From Tables 1 and 2, it is seen that Lemma 9, which is derived by Jensen inequality and the reciprocally convex combination approach, possesses the least number of scalar decision variables while *Theorem 1, which is obtained by Wirtinger-based inequality and the Moon et al.'s inequality together with convex analysis, leads to the least conservative results regardless of the complexity.*

5 Numerical Examples

To demonstrate the effectiveness and the comparison of the stability criteria Lemmas 8-11 and Theorem 1, we consider two numerical examples as follows.

Table 3: Admissible upper bound h_2 for various h_1 for the system described in Example (41).

Methods \ h_1	0.0	0.4	0.7	1.0	2.0	3.0
Lemma 8	1.86	1.88	1.95	2.06	2.61	3.31
Lemma 9	1.86	1.88	1.95	2.06	2.61	3.31
Lemma 10	2.11	2.17	2.23	2.31	2.79	3.49
Lemma 11	2.18	2.21	2.25	2.32	2.79	3.49
Theorem 1	2.24	2.27	2.29	2.34	2.80	3.49

5.1 Example 1

Consider the following linear time-delay system (2) with:

$$A = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix}. \tag{41}$$

It is well-known that this system is stable for constant delay $h \leq 6.1725$. The results for time-varying delays by applying Lemmas 8-11 and Theorem 1 are summarized in Table 3. In Lemmas 8 and 9, the stability conditions are restricted by the use of the Jensen inequality. The results obtained by solving Lemma 10 show a clear reduction of the conservatism. This is due to the use of both reciprocally convex combination Lemma 6 and Wirtinger-based integral inequality provided in Lemma 2. The conservatism can be further reduced by substitution Lemma 5 for reciprocally convex combination Lemma 6.

Moreover, less conservative criteria can be obtained by including other techniques, e.g., additional triple integral term in LKF [24], delay-partitioning approach [5], developed integral inequalities [30], [32], [33], [34]. This is not our focus in the present paper.

5.2 Example 2

Consider the system (2) with:

$$A = \begin{bmatrix} 0.0 & 1.0 \\ -2.0 & 0.1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0.0 & 0.0 \\ 1.0 & 0.0 \end{bmatrix}. \tag{42}$$

Notice that $\mathrm{Re}(eig(A + A_d)) = 0.05 > 0$, the delay free system is unstable, therefore, in this case, Lemmas 8 and 9 that correspond to classical Lyapunov-Krasovskii approaches based on Jensen inequality are not applicable any more. For the constant delay case, a frequency approach shows that the solutions of this system are stable if the delay belongs to the interval $[0.10017,\ 1.7178]$ [6]. The stability conditions of Lemma 10 and Theorem 1 can be applied to assess the stability of such systems due to the use of Wirtinger-based integral inequality provided in Lemma 2. Table 4 shows that Theorem 1 leads to a larger delay interval that preserve the stability of systems than Lemma 10.

Table 4: Admissible upper bound h_2 for various h_1 for the system described in Example (42).

Methods \ h_1	0.11	0.3	0.5	0.8	1.0	1.3
Lemma 11	0.40	1.09	1.34	1.49	1.53	1.54
Lemma 10	0.42	1.09	1.36	1.52	1.56	1.57
Theorem 1	0.42	1.10	1.38	1.54	1.57	1.57

6 Conclusions

In this paper, we have revealed that the reciprocally convex combination approach is effective only with the use of Jensen inequality. When the Jensen inequality is replaced by Wirtinger-based inequality, the Moon et al.'s inequality instead of the reciprocally convex combination approach is suggested for delay-dependent stability analysis of linear time-delay systems. Polytopic uncertainties in the system model can be easily included in the analysis.

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