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Review of explicit approximations to the Colebrook relation for flow friction

Dejan Brkić

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Abstract: Because of Moody's chart as demonstrated applicability of the Colebrook equation over a very wide range of Reynolds number and relative roughness values, this equation become the accepted standard of accuracy for calculated hydraulic friction factor. Colebrook equation suffers from being implicit in unknown friction factor and thus requires an iterative solution where convergence to 0.01% typically requires less than 7 iterations. Implicit Colebrook equation cannot be rearranged to derive friction factor directly in one step. Iterative calculus can causes problem in simulation of flow in a pipe system in which it may be necessary to evaluate friction factor hundreds or thousands of times. This is main reason for attempting to develop a relationship that is a reasonable approximation for the Colebrook equation but which is explicit in friction factor. Review of existing explicit approximation of the implicit Colebrook equation with estimated accuracy is shown in this paper. Estimated accuracy compared with iterative solution of implicit Colebrook equation is shown for entire range of turbulence where Moody diagram should be used as the reference. Finally, it can be concluded that most of available approximations of the Colebrook equation, with few exceptions, are very accurate with deviation of no more than few percentages.

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Because of Moody's chart as demonstrated applicability of the Colebrook equation over a very wide range of Reynolds number and relative roughness values, this equation become the accepted standard of accuracy for calculated hydraulic friction factor. Colebrook equation suffers from being implicit in unknown friction factor and thus requires an iterative solution where convergence to 0.01% typically requires less than 7 iterations. Implicit Colebrook equation cannot be rearranged to derive friction factor directly in one step. Iterative calculus can causes problem in simulation of flow in a pipe system in which it may be necessary to evaluate friction factor hundreds or thousands of times. This is main reason for attempting to develop a relationship that is a reasonable approximation for the Colebrook equation but which is explicit in friction factor. Review of existing explicit approximation of the implicit Colebrook equation with estimated accuracy is shown in this paper. Estimated accuracy compared with iterative solution of implicit Colebrook equation is shown for entire range of turbulence where Moody diagram should be used as the reference. Finally, it can be concluded that most of available approximations of the Colebrook equation, with few exceptions, are very accurate with deviation of no more than few percentages.

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I wish to thank reviewers and editors for valuable comments.

Reviewer 1:

1. I hope that my expression of English is now better.

Reviewer 2:

Missing

Reviewer 3:

Missing

Reviewer 4:

1. Abstract was rearranged now as you suggested (last two sentences). Please, see also Research highlights. My text has now graphical abstract also....

2. I disagree with your suggestion to reduce number of approximation shown in my review. Honestly, I have considered your suggestion seriously, but in the meantime I have found new approximations such as Papaevangelou et al (2010) and Vatankhah and Kouchakzadeh (2008, 2009). They are very accurate. Therefore, I have added these two approximations...

3. I agree with you that problem of range of applicability of certain approximation is very important. This problem is very complex. So to avoid any further speculations I have added diagram of accuracy of each approximation over the entire practical range of Reynolds number and relative roughness. According to these diagrams, one can choose will he/she use in certain case this particular approximation or not.

4. Detailed discussion with results is now presented in more appropriate way.

5. Conclusion is now rearranged after your suggestions.

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1 **Review of explicit approximations to the Colebrook relation for flow friction**

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6 Abstract: Because of Moody's chart as demonstrated applicability of the Colebrook equation
7 over a very wide range of Reynolds number and relative roughness values, this equation become
8 the accepted standard of accuracy for calculated hydraulic friction factor. Colebrook equation
9 suffers from being implicit in unknown friction factor and thus requires an iterative solution
10 where convergence to 0.01% typically requires less than 7 iterations. Implicit Colebrook
11 equation cannot be rearranged to derive friction factor directly in one step. Iterative calculus can
12 causes problem in simulation of flow in a pipe system in which it may be necessary to evaluate
13 friction factor hundreds or thousands of times. This is main reason for attempting to develop a
14 relationship that is a reasonable approximation for the Colebrook equation but which is explicit
15 in friction factor. Review of existing explicit approximation of the implicit Colebrook equation
16 with estimated accuracy is shown in this paper. Estimated accuracy compared with iterative
17 solution of implicit Colebrook equation is shown for entire range of turbulence where Moody
18 diagram should be used as the reference. Finally, it can be concluded that most of available
19 approximations of the Colebrook equation, with few exceptions, are very accurate with deviation
20 of no more than few percentages.

21 **Keywords:** Colebrook equation; Approximations; Fluid mechanics; Turbulent flow; Friction
22 coefficient; Darcy friction factor; Hydraulic resistances; Excel spreadsheet

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1 **1. Introduction**

2 Difficulty of solving turbulent flow problems in pipes lies in the fact that hydraulic friction factor
3 is a complex function of relative surface roughness and Reynolds number. Precisely, hydraulic
4 resistance depends on flow rate. Similar situation is with electrical resistance when diode is in a
5 circuit. Furthermore, being more complex, widely used empirical Colebrook equation is
6 transcendental which means that it cannot be solved by using only elementary functions and
7 basic arithmetic operations in definitive form. Problem is that, since the Colebrook equation is
8 implicit (i.e. both the right and left-hand terms contain friction factor), containing the unknown
9 friction factor in implicit form, the Reynolds number and the pipe roughness, it has to be solved
10 iteratively. Even today in the era of advance computer technology, explicit approximations of the
11 implicit Colebrook relation is very often used for calculation of friction factor in pipes. The
12 reason that so many researchers propose approximate solutions to Colebrook equation is that
13 these correlations are necessary to calculate the pressure drop and average velocity in conduits in
14 one step. Friction factor can be derived using logarithmic or power law formulation (Zagarola et
15 al 1997). Colebrook equation belongs to the logarithmic law. The laws of resistances to fluid
16 flow through rough pipe are of great significance. Colebrook equation is valid for turbulent
17 regime in rough pipes including so called rough and smooth turbulent regime with special accent
18 on transient regime between them. There is no perfectly smooth inner surface of pipe. All pipe
19 walls have physically rough surfaces (Figure 1). Degree of roughness varies depending on the
20 manufacturing process, surface finish, type of pipe material (Hammad 1999), age, conditions of
21 exploitation, etc. In turbulent flow, thin layer of fluid very close to inner pipe surface in which
22 flow is laminar is called “laminar sub-layer”. If the pipe roughness (protrusions of inner pipe
23 surface) is completely covered by the sub-layer, the surface is smooth from the hydraulic point of

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1 view. With increasing of Reynolds number, thickness of laminar sub-layer decreases baring
2 protrusions and fluid flow through pipe become consequently rough from the hydraulic point of
3 view. At very low Reynolds number, relative roughness does not have influence on friction
4 factor and it depends only on value of Reynolds number. But, on the contrary, at very high
5 Reynolds number, Reynolds number does not have influence on friction factor and it depends
6 only on relative roughness. Between these two opposite regime friction factor depends on both,
7 Reynolds number and relative roughness. Colebrook equation was developed to cover this
8 transient zone of turbulence, but it also covers completely smooth and completely rough regime.
9 In laminar flow, all pipes behave as smooth but Colebrook equation is not valid for this regime.
10 One of the presented approximations in this paper includes laminar regime (Churchill 1977).

11
12 Figure 1. Hydraulic regimes; A) Hydraulically “smooth”, B) Partially turbulent, and C)
13 Turbulent (rough)

14
15 The Colebrook equation is widely used in the petroleum industry for calculations of oil and gas
16 pipelines, in civil engineering for calculation of water distribution systems, for drainage systems,
17 ventilation systems, in chemical engineering, and in all fields of engineering where fluid flow
18 can be occurred.

19
20 **2. On Colebrook equation for flow friction**

21 Before 1939. when Colebrook equation was published, for turbulent regime in smooth pipes
22 widely was used Prandtl equation also implicit in friction factor (Colebrook 1939). Prandtl

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4 1 derived a formula from the logarithmic velocity profile and experimental data on smooth pipes
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6 2 (1):
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10 3
$$\frac{1}{\sqrt{\lambda}} = 2 \cdot \log_{10} \left(\frac{\text{Re} \cdot \sqrt{\lambda}}{2.51} \right) = 2 \cdot \log_{10} (\text{Re} \cdot \sqrt{\lambda}) - 0.8 \quad (1)$$

11
12

13 4 The development of approximate equations for the calculation of friction factor in rough pipes
14
15 5 began with Nikuradse's turbulent pipe flow investigations in 1932 and 1933 (Hager and Liiv
16
17 6 2008). For turbulent regime in rough pipes widely was used von Karman's relation (2):
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21 7
$$\frac{1}{\sqrt{\lambda}} = 2 \cdot \log_{10} \left(\frac{3.71 \cdot D}{\varepsilon} \right) = 1.74 - 2 \cdot \log_{10} \left(\frac{2 \cdot \varepsilon}{D} \right) = 1.14 - 2 \cdot \log_{10} \left(\frac{\varepsilon}{D} \right) \quad (2)$$

22
23

24 8 Greek ε is the equivalent Nikuradse's sand-grain roughness value for the inner surface of pipe (or
25
26 9 so called uniform roughness). Prandtl's and von Karman's relations are also known as NPK
27
28 10 (Nikuradse-Prandtl-Karman) equations. Colebrook later performed experiments on sixteen spun
29
30 11 concrete-lined pipes and six spun bitumastic-lined pipes ranging in diameter from 101.6 mm to
31
32 12 1524 mm with average surface roughness values between 0.04318 mm and 0.254 mm (Taylor et
33
34
35 13 al 2006).
36
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41 15 In an attempt to classify the data available at the time and those from experiment conducted by
42
43 16 himself and his colleague White (Colebrook and White 1937), Colebrook (1939) developed a
44
45 17 curve fit to describe transitional roughness (3):
46
47
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49 18
$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10} \left(\frac{2.51}{\text{Re} \cdot \sqrt{\lambda}} + \frac{\varepsilon}{3.71 \cdot D} \right) \quad (3)$$

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53 19 Colebrook equation also can be noted as (3a):
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$$\frac{1}{\sqrt{\lambda}} = 1.14 - 2 \cdot \log_{10} \left(\frac{\varepsilon}{D} + \frac{9.35}{\text{Re} \sqrt{\lambda}} \right) \quad (3a)$$

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1 Colebrook equation describes a monotonic change in the friction factor from smooth to fully
2 rough (Figure 2). It is valid especially for commercial steel pipes. Strictly mathematically is
3 incorrect what Colebrook had done, i.e. $\log(A+B) \neq \log(A) + \log(B)$, but physically this relation
4 gives good results. Problem can be treated as inverse; according to logarithmic rules equally is
5 incorrect to split the Colebrook relation into two pieces.

6
7 Figure 2. Colebrook relation make transitional curve between hydraulically “smooth” regime
8 described by Prandtl (1), and turbulent (rough) regime described von Karman (2)

9
10 Colebrook equation is also basis for Rouse (1943) and widely used Moody (1944) chart. Many
11 people seem to believe that the Moody diagram has surprisingly good properties. In fact, all it is
12 a plot of solutions of the nonlinear transcendental Colebrook equation. In principle, Moody
13 diagram is used for solution of three types of problems, i.e. problem in which head loss is
14 unknown, in which volume flow rate is unknown and in which diameter is unknown. Solving for
15 unknown head loss with Moody diagram is relative straightforward but use of implicit Colebrook
16 formula complicate solving all three types of problems.

17
18 Many researchers adopt a modification of the Colebrook equation (4) recommended by the
19 American Gas Association (AGA) in case of natural gas pipelines calculations, using 2.825
20 constant instead of 2.51 (Haaland 1983, Coelho and Pinho 2007). This procedure produces
21 maximal deviation up to 3.2% (Figure 3).

22
$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10} \left(\frac{2.825}{\text{Re} \cdot \sqrt{\lambda}} + \frac{\epsilon}{3.71 \cdot D} \right) \quad (4)$$

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1 Figure 3. Distribution of estimated deviation of implicit Colebrook equation modified by AGA
2 (4) compared with standard implicit Colebrook equation (3)

4 Figure 3 is three-dimensional. Similar figures in further text are two-dimensional.

6 Also, some researchers use Fanning factor which is not the same as the Darcy friction factor
7 (here noted as λ). Darcy friction factor is 4 times greater than the Fanning friction factor, but
8 physical meaning is equal. Darcy, Darcy-Weisbach and Moody friction factors are synonyms.

10 Colebrook equation is somewhere known as Colebrook-White equation (CW equation). White
11 was not actually a co-author of the paper in which this equation was presented (Colebrook 1939).
12 But, Colebrook made a special point of acknowledging important contribution of White to the
13 development of the equation (Colebrook and White 1937, Colebrook 1939). Letter W has
14 additional symbolic value because alternate explicit reformulations of Colebrook equation with
15 Lambert W-function involved exist (Brkić 2011a, Clamond 2009, Goudar and Sonnad 2003,
16 Keady 1998, More 2006, Nandakumar 2007, Sonnad and Goudar 2004, 2005, 2006, 2007).
17 Approximations proposed by Brkić (2011a) were also developed using Lambert W-function and
18 its solution proposed by Barry et al (2000). Further about Lambert-W function, readers can see in
19 paper of Hayes (2005).

21 **3. Available explicit approximations of the Colebrook equation with analysis of their**
22 **estimated accuracy**

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4 1 How well Colebrook equation fits the experimental data is beyond the scope of presented
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6 2 approximations. Perhaps one of these equations even fits the available data better than the
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9 3 Colebrook equation. Until the comparison is made with real, measured values, however, this will
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11 4 not be known. According to Cipra (1996), some of the key formulas of turbulence are off by as
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14 5 much as 65%. Yoo and Singh (2004, 2010) found that the Colebrook equation produced an
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16 6 average error of more than 11% while the roughness height of commercial pipes varied quite
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19 7 significantly, depending on the pipe size and type.
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24 9 As it will be shown, there were some early expressions of Colebrook equation in explicit form
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26 10 which were not particularly accurate, but in the years 1973-1984 there was a flurry of activity
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29 11 obtaining more accurate approximations that appeared mainly in the chemical engineering
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31 12 literature. Note that some of presented approximations exist in several versions. Here has to be
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33 13 very careful because typographical errors are always possible (Concha 2008, Brkić 2009a).
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36 14 Approximations will be presented starting from the oldest. Estimated accuracy compared with
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38 15 iterative solution of implicit Colebrook equation will be shown in figures 4-23 for entire range of
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41 16 turbulence where Moody diagram should be used as the reference.
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45 18 **3.1 Moody approximation**

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48 19 Approximation proposed by Moody (1947) is the oldest approximation of implicit Colebrook
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51 20 relation (5):
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$$53 \lambda \approx 0.0055 \cdot \left(1 + \left(2 \cdot 10^4 \cdot \frac{\varepsilon}{D} + \frac{10^6}{\text{Re}} \right)^{\frac{1}{3}} \right) \quad (5)$$

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1 With error up to 21.49% compared with implicit Colebrook equation (Figure 4), it has today only
2 historical value.

3
4 Figure 4. Distribution of estimated error of Moody approximation compared with implicit
5 Colebrook equation

3.2 Wood approximation

8 Approximation proposed by Wood (1966) is, equal as those by Moody (1947), power-law type
9 equation (6). Its accuracy is not improved compared with Moody approximation.

$$\lambda \approx 0.094 \cdot \left(\frac{\varepsilon}{D}\right)^{0.225} + 0.53 \cdot \left(\frac{\varepsilon}{D}\right) + 88 \cdot \left(\frac{\varepsilon}{D}\right)^{0.44} \cdot \text{Re}^{-V} \quad (6)$$

11 Where V is (7):

$$V = 1.62 \cdot \left(\frac{\varepsilon}{D}\right)^{0.134} \quad (7)$$

13 Estimated error of Wood approximation is up to 23.79% compared with implicit Colebrook
14 equation (Figure 5).

16 Figure 5. Distribution of estimated error of Wood approximation compared with implicit
17 Colebrook equation

3.3 Eck approximation

20 Approximation proposed by Eck (1973) is most simple but not very accurate (8), but better than
21 those by Moody (1947) and Wood (1966).

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.715 \cdot D} + \frac{15}{\text{Re}} \right) \quad (8)$$

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1 Estimated error of Eck approximation is up to 8.2% compared with implicit Colebrook equation
2 (Figure 6).

4 Figure 6. Distribution of estimated error of Eck approximation compared with implicit
5 Colebrook equation

7 **3.4 Churchill approximation (only for turbulent regime)**

8 Approximation proposed by Churchill (1973) is very similar with approximations proposed by
9 Swamee and Jain (1976) and Jain (1976). It is first approximation with improved accuracy (9):

10
$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.71 \cdot D} + \left(\frac{7}{\text{Re}} \right)^{0.9} \right) \quad (9)$$

11 Estimated error of Churchill approximation (valid only for turbulent regime) is up to 2.18%
12 compared with implicit Colebrook equation (Figure 7).

14 Figure 7. Distribution of estimated error of approximation by Churchill (valid only for turbulent
15 regime), Swamee and Jain, approximation by Jain and Churchill approximation (valid for full
16 range of flow) compared with implicit Colebrook equation

18 **3.5 Swamee and Jain approximation**

19 Approximation proposed by Swamee and Jain (1976) (10) with error up to 2.04% is almost
20 identical as those proposed by Churchill (1973). Distribution of estimated error of Swamee and
21 Jain approximation over turbulent part of Moody's chart is shown in figure 7.

22
$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.7 \cdot D} + \frac{5.74}{\text{Re}^{0.9}} \right) \quad (10)$$

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1 Some further details on this approximation readers can see in paper of Swamee and Rathie
2 (2007).

3
4 **3.6 Jain approximation**

5 Approximation proposed by Jain (1976) (11) with error up to 2.05% is comparable with those
6 proposed by Churchill (1973) and Swamee and Jain (1976):

$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.715 \cdot D} + \left(\frac{6.943}{\text{Re}} \right)^{0.9} \right) \tag{11}$$

8 Distribution of estimated error of Jain approximation over turbulent part of Moody’s chart is
9 shown in figure 7.

10
11 **3.7 Churchill approximation (full range of turbulence including laminar regime)**

12 Approximation proposed by Churchill (1977) covers entire laminar and turbulent regime
13 (including unstable zone between them) with error up to 2.19% (12):

$$\lambda \approx 8 \cdot \left(\left(\frac{8}{\text{Re}} \right)^{12} + \frac{1}{(C_1 + C_2)^{1.5}} \right)^{\frac{1}{12}} \tag{12}$$

15 Where C_1 is (13):

$$C_1 = \left(2.457 \cdot \ln \frac{1}{\left(\frac{7}{\text{Re}} \right)^{0.9} + 0.27 \cdot \frac{\varepsilon}{D}} \right)^{16} \tag{13}$$

17 And C_2 is (14):

$$C_2 = \left(\frac{37530}{\text{Re}} \right)^{16} \tag{14}$$

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5 Distribution of estimated error of Churchill approximation (full range of turbulence including
6 2
7 laminar regime and unstable zone between them) over turbulent part of Moody's chart is shown
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10 in figure 7.
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15 5 **3.8 Chen approximation**

16 6
17
18 First, really accurate approximation (15) was developed by Chen (1979).
19

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21 7
$$\frac{1}{\sqrt{\lambda}} \approx -2.0 \cdot \log_{10} \left(\frac{\varepsilon}{3.7065 \cdot D} - \frac{5.0452}{\text{Re}} \cdot \log_{10} \left(\frac{1}{2.8257} \cdot \left(\frac{\varepsilon}{D} \right)^{1.1098} + \frac{5.8506}{\text{Re}^{0.8981}} \right) \right) \quad (15)$$

22
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24
25 8 Estimated error of Chen approximation is up to 0.35% compared with implicit Colebrook
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27 9 equation (Figure 8).
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30 10
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32 11 Figure 8. Distribution of estimated error of Chen approximation compared with implicit
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34 12 Colebrook equation
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39 14 For some details on Chen (1979) approximation readers can see discussion by Schorle et al
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41 (1980) and closure of Chen (1980).
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44 16 45 17 **3.9 Round approximation**

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49 18 Approximation proposed by Round (1980) is relative simple but not very accurate (16):
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53 19
$$\frac{1}{\sqrt{\lambda}} \approx 1.8 \cdot \log_{10} \left(\frac{\text{Re}}{0.135 \cdot \text{Re} \cdot \left(\frac{\varepsilon}{D} \right) + 6.5} \right) \quad (16)$$

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1 Estimated error of Round approximation is up to 10.92% compared with implicit Colebrook
2 equation (Figure 9).

3
4 Figure 9. Distribution of estimated error of Round approximation compared with implicit
5 Colebrook equation

6
7 **3.10 Barr approximation**

8 Approximation proposed by Barr (1981) was very accurate for the time when it was developed
9 (17). It does not require internal iterative calculus.

10
$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.7 \cdot D} + \frac{4.518 \cdot \log_{10} \left(\frac{Re}{7} \right)}{Re \cdot \left(1 + \frac{Re^{0.52}}{29} \cdot \left(\frac{\varepsilon}{D} \right)^{0.7} \right)} \right) \quad (17)$$

11 Estimated error of Barr approximation is up to 0.27% compared with implicit Colebrook
12 equation (Figure 10).

13
14 Figure 10. Distribution of estimated error of Barr approximation compared with implicit
15 Colebrook equation

16
17 **3.11 Zigrang and Sylvester approximations**

18 Approximations proposed by Zigrang and Sylvester (1982) use internal iterative procedure to
19 achieve high accuracy (18, 19):

20
$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.7 \cdot D} - \frac{5.02}{Re} \cdot \log_{10} \left(\frac{\varepsilon}{3.7 \cdot D} + \frac{13}{Re} \right) \right) \quad (18)$$

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$$\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.7 \cdot D} - \frac{5.02}{\text{Re}} \cdot \log_{10} \left(\frac{\varepsilon}{3.7 \cdot D} - \frac{5.02}{\text{Re}} \cdot \log_{10} \left(\frac{\varepsilon}{3.7 \cdot D} + \frac{13}{\text{Re}} \right) \right) \right) \quad (19)$$

Form of approximation by Zigrang and Sylvester (1982) (18) is less accurate than (19), since the first one is based on two internal iterations while the second one uses three internal iterations. Estimated error of more complex but also more accurate approximation by Zigrang and Sylvester (1982) is up to 0.13% compared with implicit Colebrook equation (Figure 11). For simpler form of Zigrang and Sylvester approximation error is up to 1% (Figure 11).

Figure 11. Distribution of estimated error of Zigrang and Sylvester approximations compared with implicit Colebrook equation

3.12 Haaland approximation

Approximation proposed by Haaland (1983) is very accurate and simple. It was first one, designed equally for calculation of friction factor for liquid and gaseous flow (20):

$$\frac{1}{\sqrt{\lambda}} \approx -\frac{1.8}{n} \cdot \log_{10} \left(\left(\frac{\varepsilon}{3.7 \cdot D} \right)^{1.11n} + \left(\frac{6.9}{\text{Re}} \right)^n \right) \quad (20)$$

For n=1, Haaland equation (20) is valid for flow of liquid. Haaland (1983) suggested that n=3 yields friction factors in consonance with those recommended for use in gas transmission lines.

Estimated error of approximation by Haaland (1983) valid for liquid flow is up to 1.4% compared with implicit Colebrook equation (Figure 12).

Figure 12. Distribution of estimated error of Haaland approximation for liquid flow compared with implicit Colebrook equation

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2 Same comparison is done for Haaland approximation for gaseous flow (Figure 13). This was
3 done in comparisons with standard implicit Colebrook equation and with modified implicit
4 Colebrook equation rearranged by AGA.

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6 Figure 13. Distribution of estimated error of Haaland approximation for gaseous flow compared
7 with implicit Colebrook equation (upper) and with modified implicit Colebrook equation
8 rearranged by AGA (lower)

9

10 **3.13 Serghides approximations**

11 Approximations proposed by Serghides (1984)¹. are accurate but with internal iterative calculus
12 (21, 22):

13
$$\lambda \approx \left(S_1 - \frac{(S_2 - S_1)^2}{S_3 - 2 \cdot S_2 + S_1} \right)^{-2} \tag{21}$$

14
$$\lambda \approx \left(4.781 - \frac{(S_1 - 4.781)^2}{S_2 - 2 \cdot S_1 + 4.781} \right)^{-2} \tag{22}$$

15 Where S_1 is (23):

16
$$S_1 = -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.7 \cdot D} + \frac{12}{Re} \right) \tag{23}$$

17 Where S_2 is (24):

¹ Cronologically, after approximation by Serghides (1984) are approximations proposed by Chen (1984, 1985). But these approximations are similar with Altshul, Russian power-law equation from Soviet practice and therefore they will be presented with this equation later in the text. Note also that author of Chen approximation from 1979 is Chen N.H. (Chen 1979), while the author of Chen approximations from 1984 is Chen J.J.J. (Chen 1984).

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1 $S_2 = -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.7 \cdot D} + \frac{2.51 \cdot S_1}{\text{Re}} \right)$ (24)

2 Where S_3 is (25):

3 $S_3 = -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.7 \cdot D} + \frac{2.51 \cdot S_2}{\text{Re}} \right)$ (25)

4 Form of approximation by Serghides (1984) (21) is more accurate than (22), since the first one is
5 based on three internal steps while the second one use two internal steps. More complex version
6 is with accuracy up to 0.13% while less complex one is up to 0.35% (Figure 14).

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8 Figure 14. Distribution of estimated error of Serghides approximations compared with implicit
9 Colebrook equation

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11 **3.14 Manadilli approximation**

12 Approximation by Manadilli (1997) contains signomial terms (26).

13 $\frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.7 \cdot D} + \frac{95}{\text{Re}^{0.983}} - \frac{96.82}{\text{Re}} \right)$ (26)

14 A special group of functions appearing in mathematical models of many processes is the
15 signomial functions. A signomial function is defined as the sum of signomial terms, which in
16 turn are products of power functions multiplied with a real constant. Estimated error of
17 approximation by Manadilli (1997) is up to 2.06% compared with implicit Colebrook equation
18 (Figure 15).

19
20 Figure 15. Distribution of estimated error of Manadilli approximation compared with implicit
21 Colebrook equation

1 3.15 Romeo, Royo and Monzón approximation

2 Approximation proposed by Romeo et al (2002) has three internal iterations (27):

$$3 \frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10} \left(\frac{\varepsilon}{3.7065 \cdot D} - \frac{5.0272}{\text{Re}} \cdot \log_{10} \left(\frac{\varepsilon}{3.827 \cdot D} - \frac{4.567}{\text{Re}} \cdot \log_{10} \left(\left(\frac{\varepsilon}{7.7918 \cdot D} \right)^{0.9924} + \left(\frac{5.3326}{208.815 + \text{Re}} \right)^{0.9345} \right) \right) \right) \quad (27)$$

4 The calculation of the parameters of Romeo et al (2002) approximation was done through non-
5 linear multivariable regression. Estimated error of approximation by Romeo, Royo and Monzón
6 is up to 0.13% compared with implicit Colebrook equation (Figure 16).

7
8 Figure 16. Distribution of estimated error of approximation by Romeo, Royo and Monzón
9 compared with implicit Colebrook equation

10 3.16 Sonnad and Goudar approximation

11 Approximation by Sonnad and Goudar (2006) was developed using Lambert W-function (28):

$$12 \frac{1}{\sqrt{\lambda}} \approx 0.8686 \cdot \ln \left(\frac{0.4587 \cdot \text{Re}}{G^{G/(G+1)}} \right) \quad (28)$$

13 Where G is (29):

$$14 G = 0.124 \cdot \text{Re} \cdot \frac{\varepsilon}{D} + \ln(0.4587 \cdot \text{Re}) \quad (29)$$

15 For some details on Sonnad and Goudar (2006) approximation readers can see discussions by
16 Vatankhah and Kouchakzadeh (2008, 2009) and Yıldırım (2008). Paper of Sonnad and Goudar
17 (2007) should be also recommended as reference. Approximation by Sonnad and Goudar (2006)
18 is not suitable for all range of relative roughness and Reynolds numbers (Sonnad and Goudar
19 2004). Estimated error of approximation by Sonnad and Goudar (2006) is up to 0.8% compared
20 with implicit Colebrook equation (Figure 17).

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11 4 Vatankhah and Kouchakzadeh (2008, 2009) rearranged approximation proposed by Sonnad and
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14 5 Goudar (2006) as (29a, 29b), to increase its accuracy with error up to 0.15% (Figure 18). These
15
16 6 two improved equations here will be noted as approximations by Vatankhah and Kouchakzadeh
17
18
19 7 (2008, 2009).
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22 8
$$\frac{1}{\sqrt{\lambda}} \approx 0.8686 \cdot \ln \left(\frac{0.4587 \cdot \text{Re}}{(G - 0.31)^{G/(G+0.9633)}} \right) \quad (29a)$$

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$$\lambda \approx \left(0.8686 \cdot \ln \left(\frac{0.4587 \cdot \text{Re}}{(G - 0.28)^{G/(G+0.98)}} \right) \right)^{-2} \quad (29b)$$

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33 12 compared with implicit Colebrook equation
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38 14 Parameter G in Vatankhah and Kouchakzadeh approximation is actually parameter G from
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40 15 Sonnad and Goudar approximation
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44 17 **3.17 Rao and Kumar approximation**
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46
47 18 Approximation by Rao and Kumar (2007) cannot be recommended to be used because of its
48
49 19 inaccuracy (30). Of course this inaccuracy is valid only apropos standard Colebrook equation.
50

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52 20
$$\frac{1}{\sqrt{\lambda}} \approx 2 \cdot \log_{10} \left(\frac{\left(2 \cdot \frac{\varepsilon}{D} \right)^{-1}}{\left(\frac{0.444 + 0.135 \cdot \text{Re}}{\text{Re}} \right) \cdot \Phi(\text{Re})} \right) \quad (30)$$

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59 21 Where $\Phi(\text{Re})$ is (31):
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$$\Phi(\text{Re}) = 1 - 0.55e^{-0.33 \left(\ln \left(\frac{\text{Re}}{6.5} \right) \right)^2} \tag{31}$$

$\Phi(\text{Re})$ can be neglected in most cases. Some additional details can be seen in paper of Rao and Kumar (2009). Estimated error of Rao and Kumar approximation is up to 82% compared with implicit Colebrook equation (Figure 19).

Figure 19. Distribution of estimated error of Rao and Kumar approximation compared with implicit Colebrook equation

3.18 Buzzelli approximation

Buzzelli (2008) proposed one up to date among most accurate and also relatively simple approximation (32):

$$\frac{1}{\sqrt{\lambda}} = B_1 - \left(\frac{B_1 + 2 \cdot \log_{10} \left(\frac{B_2}{\text{Re}} \right)}{1 + \frac{2.18}{B_2}} \right) \tag{32}$$

Where B is (33):

$$B_1 = \frac{(0.774 \cdot \ln(\text{Re})) - 1.41}{\left(1 + 1.32 \cdot \sqrt{\frac{\varepsilon}{D}} \right)} \tag{33}$$

and B_2 is (34):

$$B_2 = \frac{\varepsilon}{3.7 \cdot D} \cdot \text{Re} + 2.51 \cdot B_1 \tag{34}$$

Estimated error of approximation by Buzzelli (2008) is up to 0.13% compared with implicit Colebrook equation (Figure 20).

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6 2 Colebrook equation
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4 **3.19 Avci and Karagoz approximation**

5 Formula by Avci and Karagoz (2009) have been recently developed from the experimental
6 Princeton super-pipe data (35):
7

$$8 \quad \lambda = \frac{6.4}{\left(\ln(\text{Re}) - \ln \left(1 + 0.01 \cdot \text{Re} \cdot \frac{\varepsilon}{D} \cdot \left(1 + 10 \cdot \sqrt{\frac{\varepsilon}{D}} \right) \right) \right)^{2.4}} \quad (35)$$

9 Estimated error of approximation by Avci and Karagoz (2009) is up to 4.7% compared with
10 implicit Colebrook equation (Figure 21).
11

12 Figure 21. Distribution of estimated error of Avci and Karagoz approximation compared with
13 implicit Colebrook equation
14

15 **3.20 Papaevangelou, Evangelides and Tzimopoulos approximation**

16 Papaevangelou et al (2010) noticed that error values tended to “bend” to negative values in an
17 exponential way for Re lower than 10^6 . According to that they chose parameters in their equation
18 (36):
19

$$20 \quad \lambda = \frac{0.2479 - 0.0000947 \cdot (7 - \log_{10} \text{Re})^4}{\left(\log_{10} \left(\frac{\varepsilon}{3.615 \cdot D} + \frac{7.366}{\text{Re}^{0.9142}} \right) \right)^2} \quad (36)$$

21 Estimated error of approximation by Papaevangelou, Evangelides and Tzimopoulos is up to
22 0.85% compared with implicit Colebrook equation (Figure 22).
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6 2 Tzimopoulos compared with implicit Colebrook equation
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4 3.21 Brkić approximation

5 Brkić (2011a) approximations were developed using Lambert W-function (37, 38):

$$6 \frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10} \left(10^{-0.4343\beta} + \frac{\varepsilon}{3.71 \cdot D} \right) \quad (37)$$

$$7 \frac{1}{\sqrt{\lambda}} \approx -2 \cdot \log_{10} \left(\frac{2.18 \cdot \beta}{Re} + \frac{\varepsilon}{3.71 \cdot D} \right) \quad (38)$$

8 Where β is (39):

$$9 \beta = \ln \frac{Re}{1.816 \cdot \ln \left(\frac{1.1 \cdot Re}{\ln(1 + 1.1 \cdot Re)} \right)} \quad (39)$$

10 Estimated error of Brkić approximation is up to 2.3% compared with implicit Colebrook
11 equation (Figure 23).

12
13 Figure 23. Distribution of estimated error of Brkić approximation compared with implicit
14 Colebrook equation

15
16 Additionally (Barry et al. 2000), error can be reduced using (39a) or (39b):

$$17 \beta_1 \approx 1.4586887 \cdot \beta - 0.4586887 \cdot \ln \left(\frac{Re \cdot 0.917365}{\ln(1 + Re \cdot 0.917365)} \right) \quad (39a)$$

$$18 \beta_1 \approx \ln \left(0.488 \cdot Re \cdot \left[\ln \left(\frac{Re}{\beta} \right) \right]^{-1} \right) \quad (39b)$$

19 For parameter β (39), solution for Lambert W-function by Barry et al. (2000) is used. Procedure
20 by Winitzki (2003) gives parameter β with similar accuracy (40):

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$$\beta \approx \ln(1 + 0.458 \cdot \text{Re}) \cdot \left(1 - \frac{\ln(1 + \ln(1 + 0.458 \cdot \text{Re}))}{2 + \ln(1 + 0.458 \cdot \text{Re})} \right) \quad (40)$$

Parameter β calculated by using procedures by Barry et al. (2000) and by Winitzki (2003) gives similar results in accuracy.

4. Comparative analysis and complexity of available explicit approximations of the Colebrook equation

Although the Colebrook formula itself is not very accurate (Cipra 1996), its accurate resolution is nonetheless an issue for numerical simulations because a too crude resolution may affect the repeatability and comparisons of calculation (Clamond 2009).

There is no special explanation for different coefficients in some equations (e.g. 3.7065 or 3.707 instead of 3.71 etc). Only reasonable explanation can be that this changed coefficients maybe better fit experimental data. Examples for this are e.g. Churchill (1973) approximation (9) and Jain (1976) approximation (11) with slightly different coefficients.

Churchill (1977) relation (12) holds for all values of Reynolds and relative roughness, including laminar regime (Figure 24).

Figure 24. Churchill (1977) approximation includes laminar and highly unstable transient zone from laminar to turbulent

Since Churchill (1977) relation (12) is a continuous function for Reynolds numbers above 0, it also lets one calculate a friction factor in the transient zone, between laminar and smooth

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1 turbulent regime. Of course the accuracy of such a friction factor probably cannot be determined,
2 but it will be reasonable. Regarding this issue, readers also can see paper of Swamee and
3 Swamee (2007).

4
5 In his recent paper, Yıldırım (2009, 2011) compared 12 of here presented more than 20
6 approximations of the implicit Colebrook equation. Ouyang and Aziz (1996) made similar
7 research for the approximations available in that time. Data of Ouyang and Aziz (1996) are also
8 available from paper of Abdolahi et al (2007). Similar comparisons are available from the papers
9 of Gregory and Fogarasi (1985), Zigrang and Sylvester (1985) and papers of Goudar and Sonnad
10 (2007, 2008).

11
12 For the analysis of relative error distribution, 20 points (16 for relative comparisons shown in
13 figures 25-30) are used for relative roughness (ϵ/D) and 37 points for Reynolds number (Re).
14 Points for relative roughness (ϵ/D) used for comparative study are shown in figures 25-30. Points
15 for Reynolds number are between $1 \cdot 10^4 - 1 \cdot 10^8$. This means that grid with 592 check points is
16 formed for comparative study (Figures 25-30) and with 740 check points for error distribution
17 study (Figures 4-23). This comparative study is based on some particular conditions of test grid
18 points which means that presented relative error could be different using different check points.
19 But here presented analysis with 592 and 740 check points give good general picture of accuracy
20 for the presented approximations. Relative error is not distributed systematically over the entire
21 range of Reynolds number and relative roughness which means that real maximal relative error

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1 can be slightly above here reported. MS Excel file is available as electronic appendix² to this
2 paper and in this file any other values for Reynolds number and for relative roughness can be
3 used as input parameters. Modern software, especially in the case of spreadsheets, can be
4 conveniently programmed to solve any implicit equation, with a minimum of programming and
5 details. Some of MS Excel effectiveness for numerical computations comes from a module
6 ‘Solver’. It was originally designed for optimization problems, where one has to find values of a
7 number of different parameters such that some quantity is minimized, usually the sum of errors
8 of a number of equations. With this tool one can find such optimal solutions, or solutions of one
9 or many equations, even if they are nonlinear. In more details, to allow iterative computations in
10 MS Excel 2007, one has to choose ‘Excel options’, and then in ‘Formulas’ to tick box ‘Enable
11 iterative calculation’. In this case, Excel is set to terminate the calculation after maximum $3 \cdot 10^4$
12 iterations or $1 \cdot 10^{-7}$ difference or less between the values of two successive iterations.
13 Consequently, deviations involving explicit equations have been reported to $1 \cdot 10^{-5}\%$. This means
14 that real relative error is presented by sum of calculated relative error and deviation. So deviation
15 in ‘Excel options’ has to be set to be significantly smaller compared to estimated error of
16 observed approximations (‘Maximum Change’ in ‘Formulas’).
17
18 Maximal percentage (relative) error of presented approximations over the entire range of
19 applicability of Colebrook equation is shown in table 1.

20
21 Table 1. Maximal relative error for available approximations for test check points

² MS Excel can be also successfully used in other engineering fields; see electronic appendix in Brkić and Tanasković (2008)

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2 Approximation by Rao and Kumar (2007) is extremely inaccurate compared with standard
3 Colebrook equation (Figure 25). Approximations proposed by Moody (1947), Wood (1966), Eck
4 (1973) and Round (1980) should not be used because they produce significant relative error
5 (Figure 26). Moderate accurate approximations such as proposed by Jain (1976), Swamee and
6 Jain (1976), Churchill (1973, 1977), Manadilli (1997), Brkić (2011a) and Avzi and Karagoz
7 (2009), can be used since they made maximal relative error up to 5% (Figure 27).
8 Approximations by Zigrang and Sylvester (1982) (18), Haaland (1983), Sonnad and Goudar
9 (2006) and Papaevangelou et al (2010) produce maximal error up to 1.5% (Figure 28). Very
10 accurate (Figure 29), with estimated error up to 0.5%, are approximations by Chen (1979), Barr
11 (1981), Zigrang and Sylvester (1982) (19), Serghides (1984), Romeo et al (2002), Buzzelli
12 (2008), and approximations proposed by Vatankhah and Kouchakzadeh approximations (2008,
13 2009). Note that approximations proposed by Vatankhah and Kouchakzadeh (2008, 2009) are
14 actually very successfully improved approximation by Sonnad and Goudar (2006).

15

16 Figure 25. Inaccuracy of approximation by Rao and Kumar (2007)

17

18 Figure 26. Non-advisable approximations

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20 Figure 27. Approximations with estimated error up to 5% (less accurate approximations)

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22 Figure 28. Approximations with estimated error up to 1.5% (moderate accurate approximations)

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1 Figure 29. Approximations with estimated error up to 0.5% (very accurate approximations)

2
3 Extremely accurate approximations from figure 29 can be seen in figure 30 in better resolution.

4
5 Figure 30. Extremely accurate approximations presented in higher resolution

6
7 Numerical solutions for friction factors based on the Colebrook equation can be obtained by to
8 any desired degree of precision. For many applications, the simpler but less accurate explicit
9 equation will be sufficed. Sometime, simplicity is sacrificed for excessive accuracy. To find
10 balance between these two extremes it is appropriate to introduce concept of complexity or
11 complexity index of explicit approximations. Zigrang and Sylvester (1985) defined complexity
12 as the number of algebraic notation calculator key strokes required to solve the equation for
13 $Re=10^5$ and $\epsilon/D=0.001$ (Table 2). Complexity index is defined as quotient of key strokes
14 required for an observed approximation and the least complex one (Table 2).

15
16 Table 2. Complexity and complexity index of available explicit approximations

17
18 Eck (1973) approximation is the least complex with only 27 key strokes required for solution and
19 hence this equation has complexity index 1 (Table 2).

20
21 **5. A note on power-law formulas valid for the same range of turbulent flow as standard**
22 **Colebrook equation**

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1 Derivation of relations for the friction factor is mostly based on the logarithmic or power law
2 formulation of velocity profiles in boundary layers (Zagarola et al 1997). As mentioned in
3 introduction, Colebrook equation is based on the logarithmic formulation. Example of the power-
4 law formula, valuable for the same flow regimes as Colebrook's is old Altshul equation from the
5 Soviet era (Figure 31) (41, 42):

$$\lambda = 0.11 \cdot \left(\frac{\varepsilon}{D} + \frac{68}{\text{Re}} \right)^{\frac{1}{4}} \quad (41)$$

$$\lambda = 0.1 \cdot \left(\frac{1.46 \cdot \varepsilon}{D} + \frac{100}{\text{Re}} \right)^{\frac{1}{4}} \quad (42)$$

8
9 Altshul formula was eliminated from the last wording of the Russian norms. However, it is used
10 as before since other recommendations were not proposed (Sukharev et al 2005).

11
12 Figure 31. Distribution of deviation of Altshul formula (41) compared with implicit Colebrook
13 equation

14
15 Similar are formulas proposed by Chen (1984) (43, 44):

$$\lambda = 0.3164 \cdot \left(\frac{1}{\text{Re}^{0.83}} + 0.11 \cdot \left(\frac{\varepsilon}{D} \right) \right)^{0.3} \quad (43)$$

$$\lambda = 0.184 \cdot \left(\frac{1}{\text{Re}^{0.67}} + 0.7 \cdot \left(\frac{\varepsilon}{D} \right) \right)^{0.3} \quad (44)$$

18 Deviation of presented power-law formulas from Russian practice and by Chen (1984) is shown
19 in figure 32.

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1 Figure 32. Power law formulas as substitution for implicit Colebrook equation

2

3 Readers also can see paper by Chen (1985). In general, approximations by Moody (1947) and by
4 Wood (1966) also belong to power-law formulas.

5

6 **6. Some remarks on further developments**

7 Today, different approach can be used for determination of friction factor. Good example for the
8 era of computerization is approach of Özger and Yildirim (2009). They use adaptive neuro-fuzzy
9 computing technique for determination of turbulent flow friction coefficient. In the paper of Yoo
10 and Singh (2005) are shown two new methods for the computation of commercial pipe friction
11 factor. Today, main problem is not how to calculate friction factor. Problem is how to measure or
12 estimate roughness of pipe (Farshad et al 2001). Most pipes usually have rough inner pipe
13 surface. Resistance to fluid flow offered by rough boundaries is larger than that for smooth one
14 due to the formation of eddies behind protrusions. Colebrook equation is valid for both, smooth
15 and rough turbulent regime including transient zone between them. In principle, a system of
16 partial differential equations known as Navier-Stokes equations describes the exact behavior of
17 the fluid flow in so-called boundary layer, but solving these equations remains beyond current
18 theory and computations. Sletfjerding and Gudmundsson (2003) proposed also methodology for
19 determination of friction factor directly from roughness measurements. In that way they
20 eliminated roughness as a parameter in Colebrook equation (only Reynolds number and pipe
21 diameter are necessary as input parameters). Using a similar approach to that of in Nikuradse's
22 experiment, Sletfjerding and Gudmundsson (2003) related measured roughness values with

1 friction factor, but their equation is implicit in friction factor. In formulation given by
2 Sletfjerd and Gudmundsson (2003) equation is implicit and valid for average steel pipe (45):

$$\frac{1}{\sqrt{\lambda}} = -1.89 \cdot \log_{10} \left(\frac{1.55}{\text{Re} \cdot \sqrt{\lambda}} + \left(\frac{4}{D} \right)^{1.03} \right) \quad (45)$$

4 For other materials of pipes readers can consult paper of Sletfjerd and Gudmundsson (2003).

6 **7. Conclusion**

7 Maybe, it is difficult for many to recall for the time as recently as the 1970's where there were no
8 personal computers or even calculators that could do much more than add or subtract. In that
9 environment an implicit relationship such as Colebrook (1939), which was well-known then, was
10 impractical and some simplification was essential. Today, it is not difficult to solve single
11 Colebrook equation by iteration. But solution of complex looped pipeline problem in such case
12 requires double iterative procedure where first is for the standard implicit Colebrook equation
13 while second one is for Hardy Cross method or similar iterative method used to solve simulation
14 problem in a looped pipe network (Brkić 2009b, 2011b). This double procedure can be serious
15 burden even for today very powerful computers.

17 All shown approximate equations give the friction factor explicitly as a function of Reynolds
18 number and relative roughness. Comparative analysis indicates that almost all approximate
19 equations give a very good prediction of the friction factor and can reproduce the Colebrook
20 equation and its Rouse and Moody plot. Therefore, these approximations for the friction factor
21 provide a rational, accurate, and practically useful method over the entire range of the Moody
22 chart in terms of Reynolds number and relative roughness. Apropos relative complexity at first
23 sight, these approximations can be very easily implemented in a computer code.

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Most available approximations of the Colebrook equation are very accurate. Exceptions are Round (1980), Eck (1973), Moody (1947), Wood (1966), and Rao and Kumar (2007) approximations. The average error of almost all explicit approximations of the Colebrook (1939) relation is up to 3%.

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Appendix. Supporting Information

Supplementary data associated with this article can be found in the online version at doi:

MS Excel spreadsheet file is accompanied as electronic annex with on-line version of this paper. Note that you have to install Office 2007 (Enterprise edition) to inspect this file. File contains all formulas presented in the text. Readers can change values of Reynolds number and relative roughness to calculate Darcy friction factor. To allow necessary implicit calculation in MS Excel 2007 which is used for calculation of implicit Colebrook’s relation, the ‘Office button’ at the upper-left corner of the Excel screen have to be pressed, and in the ‘Excel options’, ‘Formulas’

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1 has to be chosen and finally box ‘Enable iterative calculation’ have to be ticked. This allows
2 implementation of so called ‘Circular references’ into a calculation.

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1 Nomenclature:

- 2 D-inner diameter of pipe (m)
- 3 Re-Reynolds number (-)
- 4 ϵ -absolute roughness (m)
- 5 λ -Darcy (i.e. Moody or Darcy-Weisbach) friction factor (-)
- 6 δ -relative error (%)
- 7 V-auxiliary term in Wood approximation
- 8 C_1, C_2 -auxiliary terms in Churchill approximation
- 9 S_1, S_2, S_3 -auxiliary terms in Serghides approximations
- 10 G-auxiliary term in Sonnad-Goudar and Vatankhah-Kouchakzadeh approximation
- 11 $\Phi(Re)$ -auxiliary term in Rao-Kumar approximation
- 12 B_1, B_2 -auxiliary terms in Buzzelli approximation
- 13 β -auxiliary term in Brkić approximations
- 14 n-explained with Haaland approximation (can be n=1 or n=3, n=1 is valid for flow of liquid
- 15 while n=3 is recommended for use in gas transmission lines)

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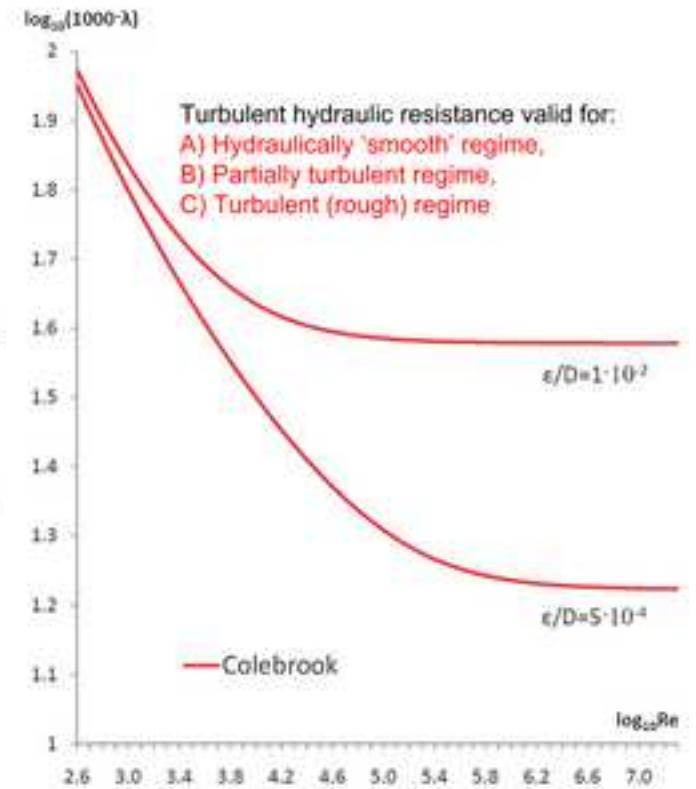
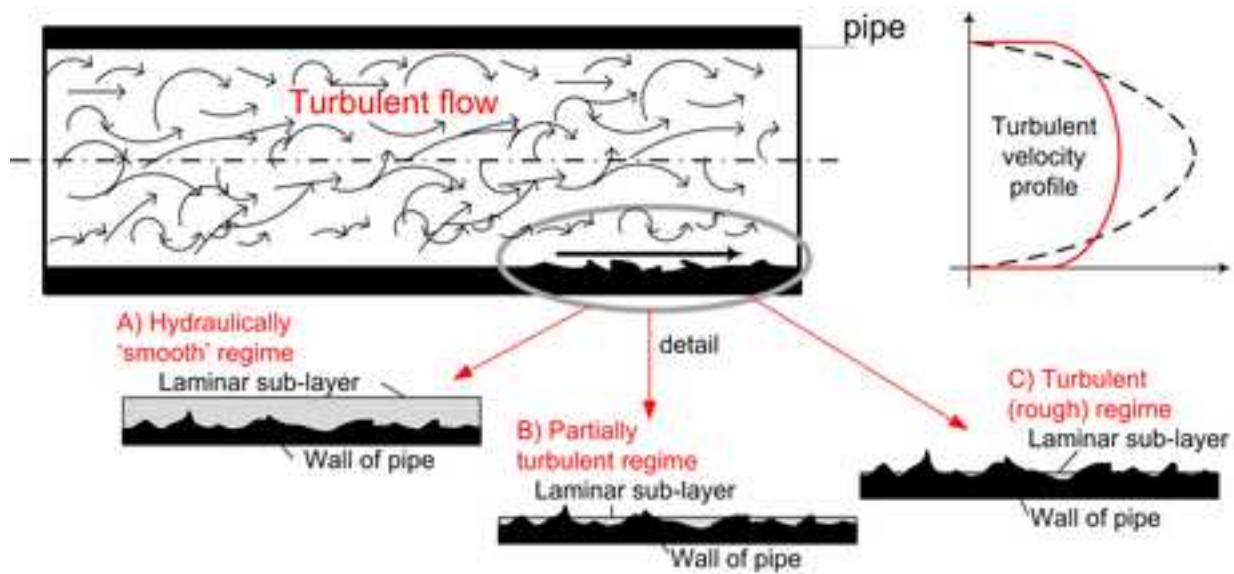
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Research highlights

- Empirical Colebrook equation is an accepted standard for calculation of flow friction factor.
- Colebrook equation is transcendental function implicit in unknown friction coefficient.
- Implicit Colebrook equation cannot be rearranged to derive friction factor directly.
- Colebrook equation has to be solved iteratively or using approximations.
- Iterative calculus can cause problem in simulation of flow in a pipe system.
- Error of almost all explicit approximations of the Colebrook relation is up to 3%.
- An explicit approximation of the Colebrook relation can be very complex but also it can be easily implemented in a computer code.



λ -Darcy friction factor (Also Moody, Darcy-Weisbach)
 Re-Reynolds number

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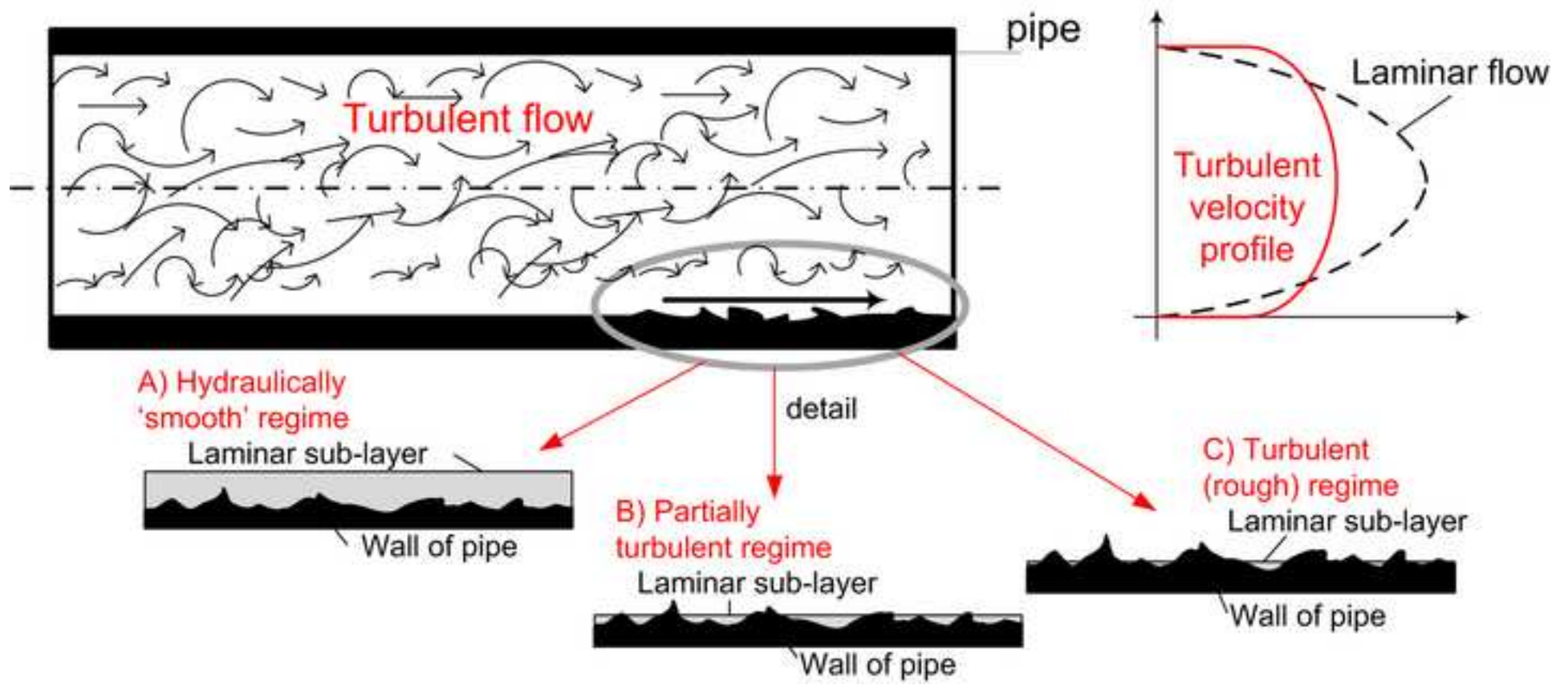


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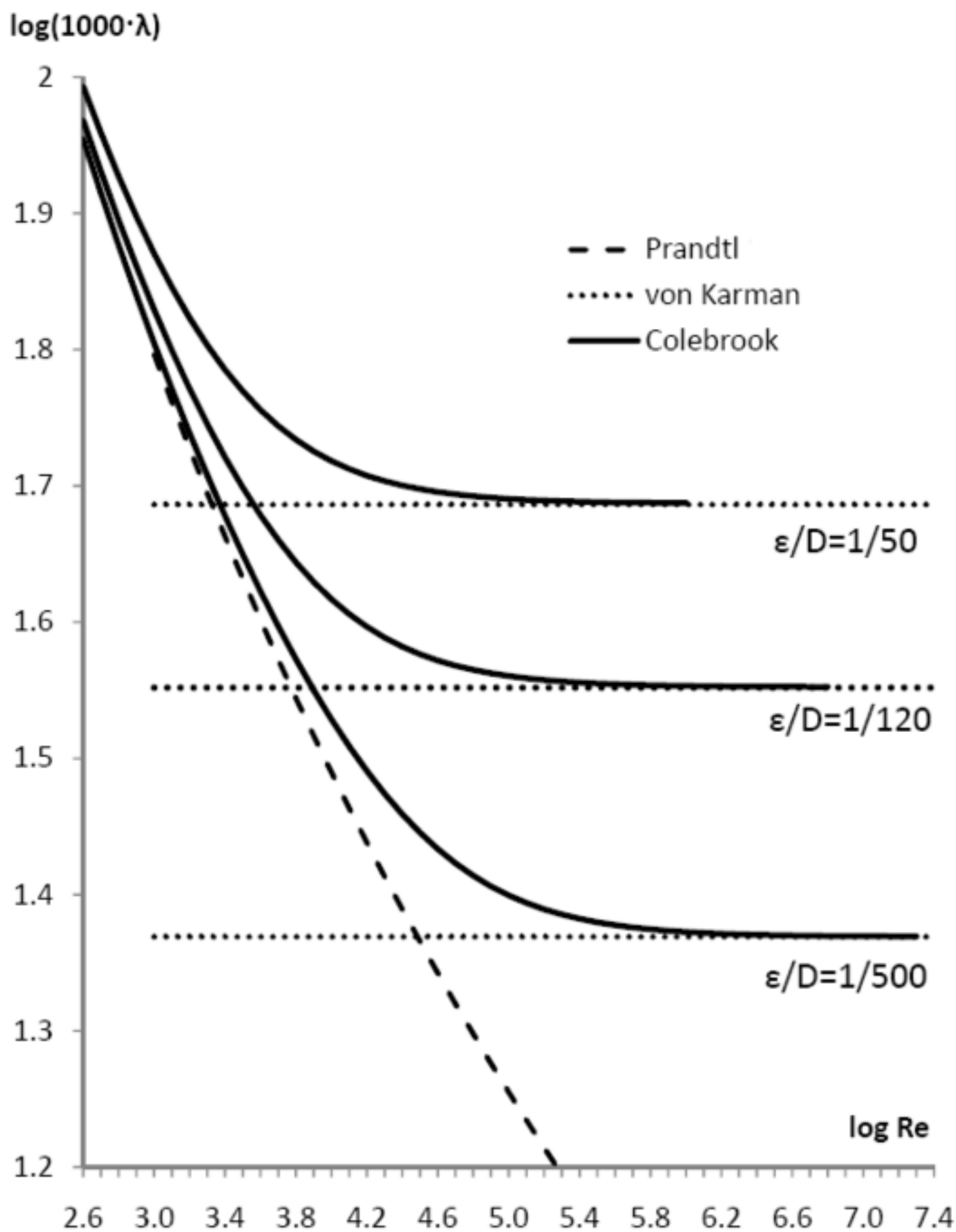


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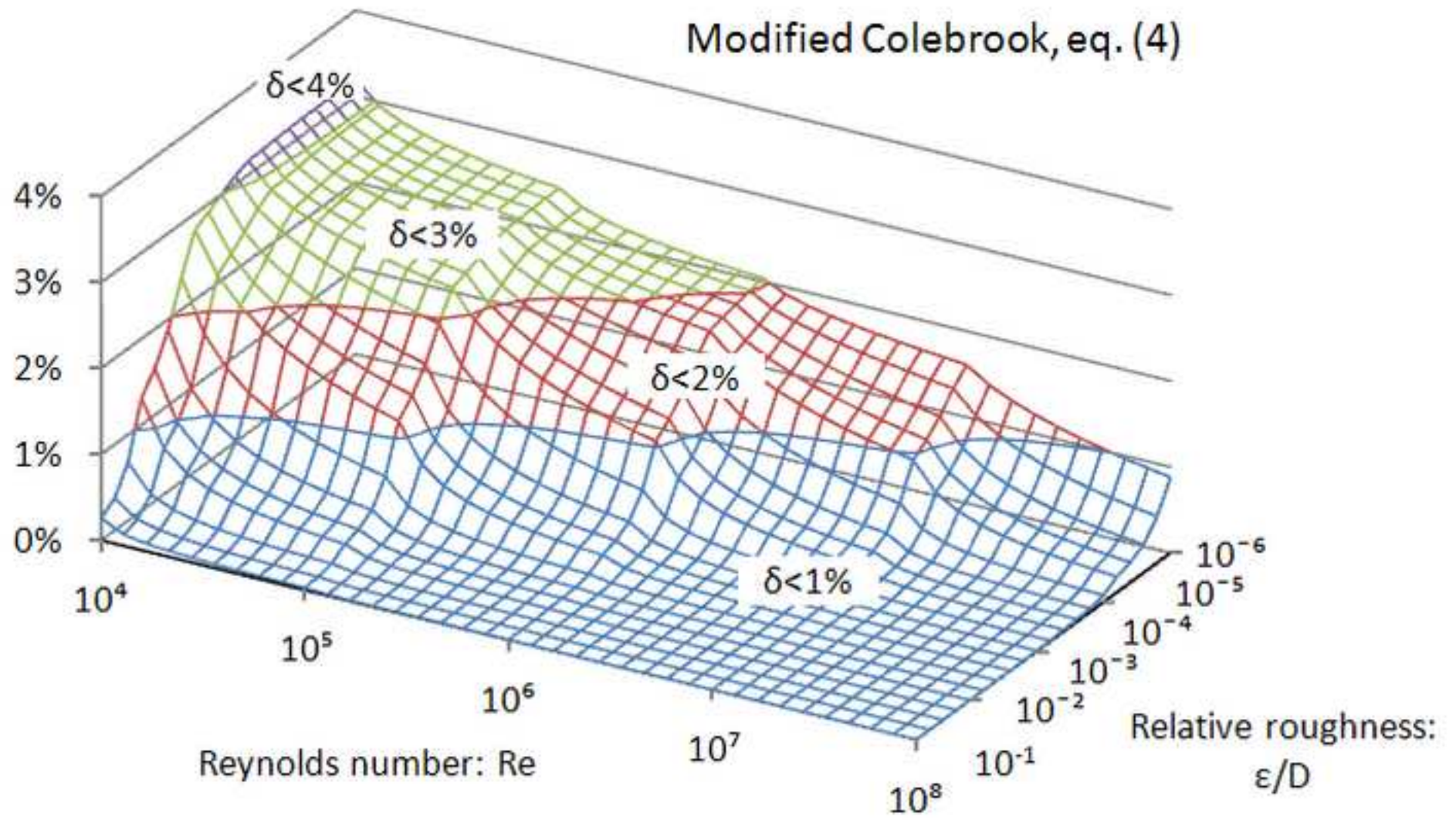


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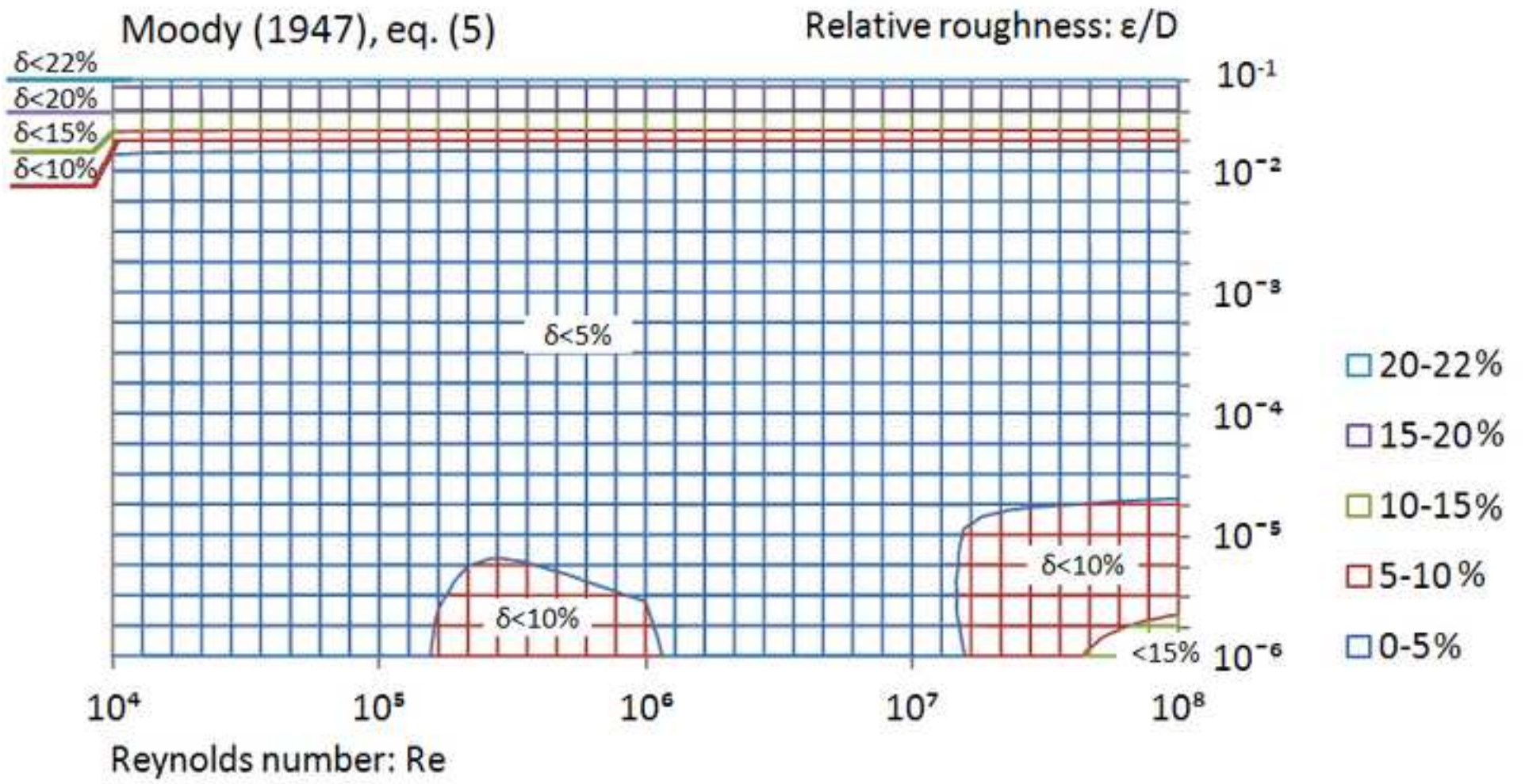


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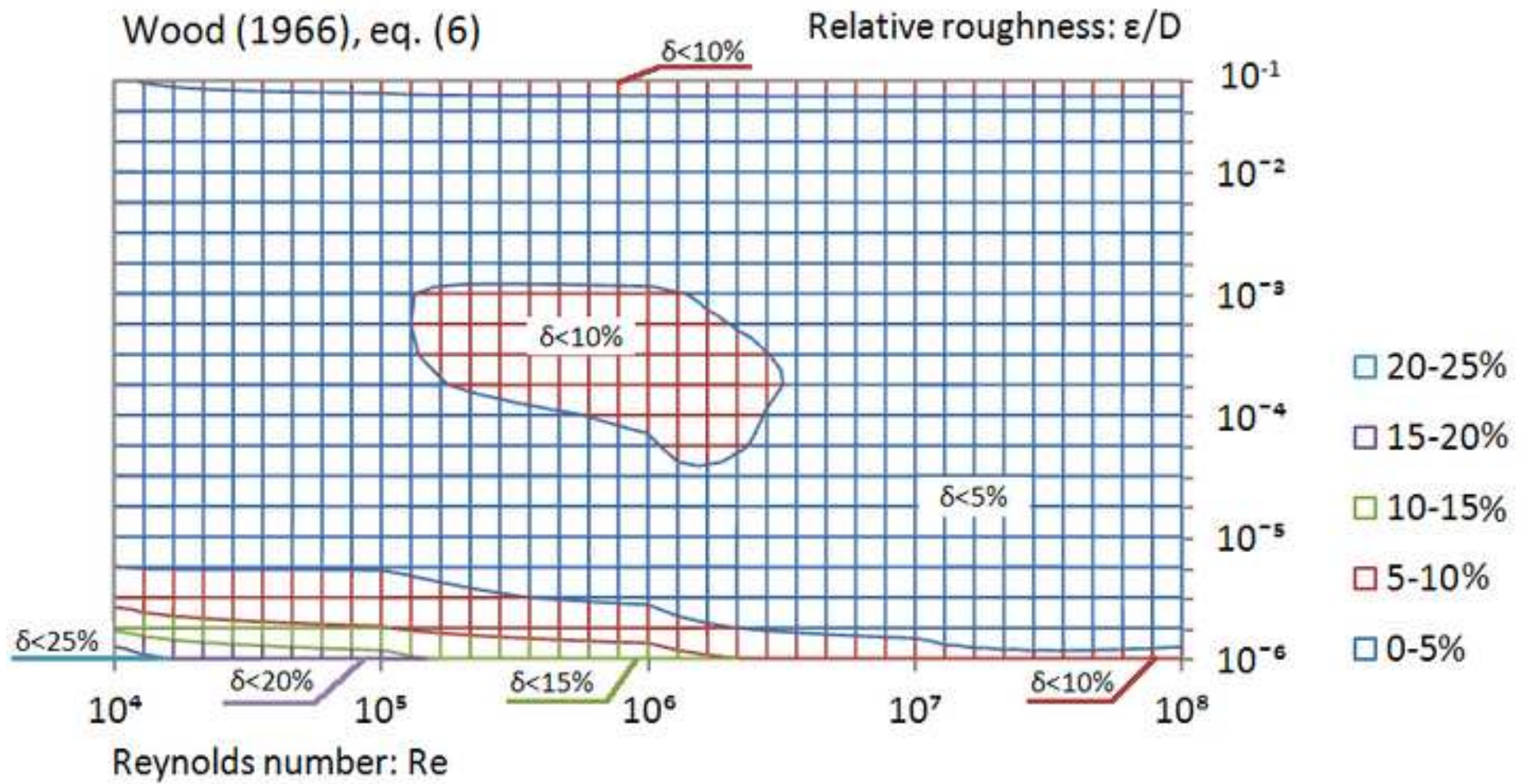


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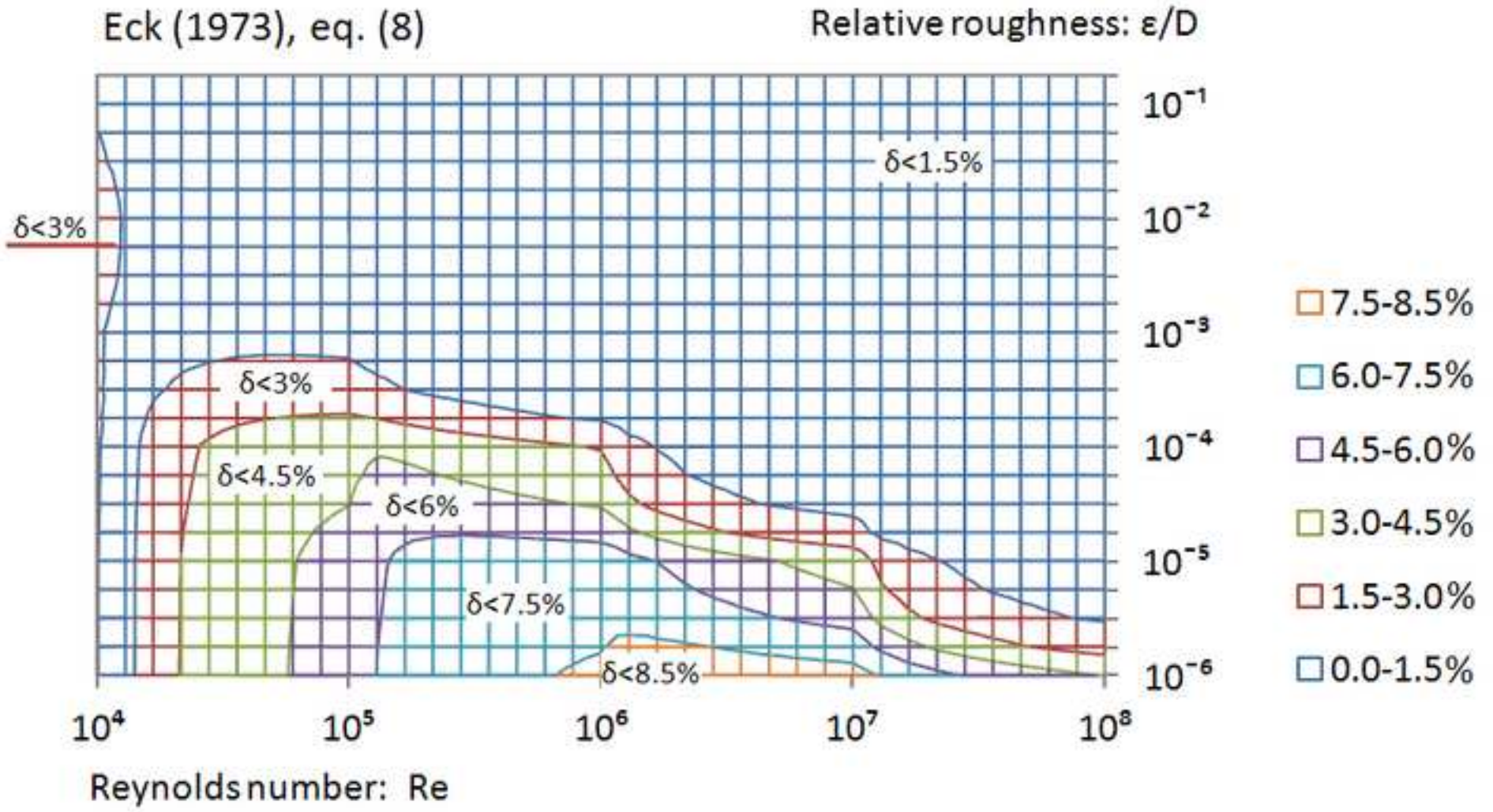


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Churchill (1973), eq. (9);
Swamee and Jain (1976), eq. (10);
Jain (1976), eq. (11);
Churchill (1977), eq. (12)

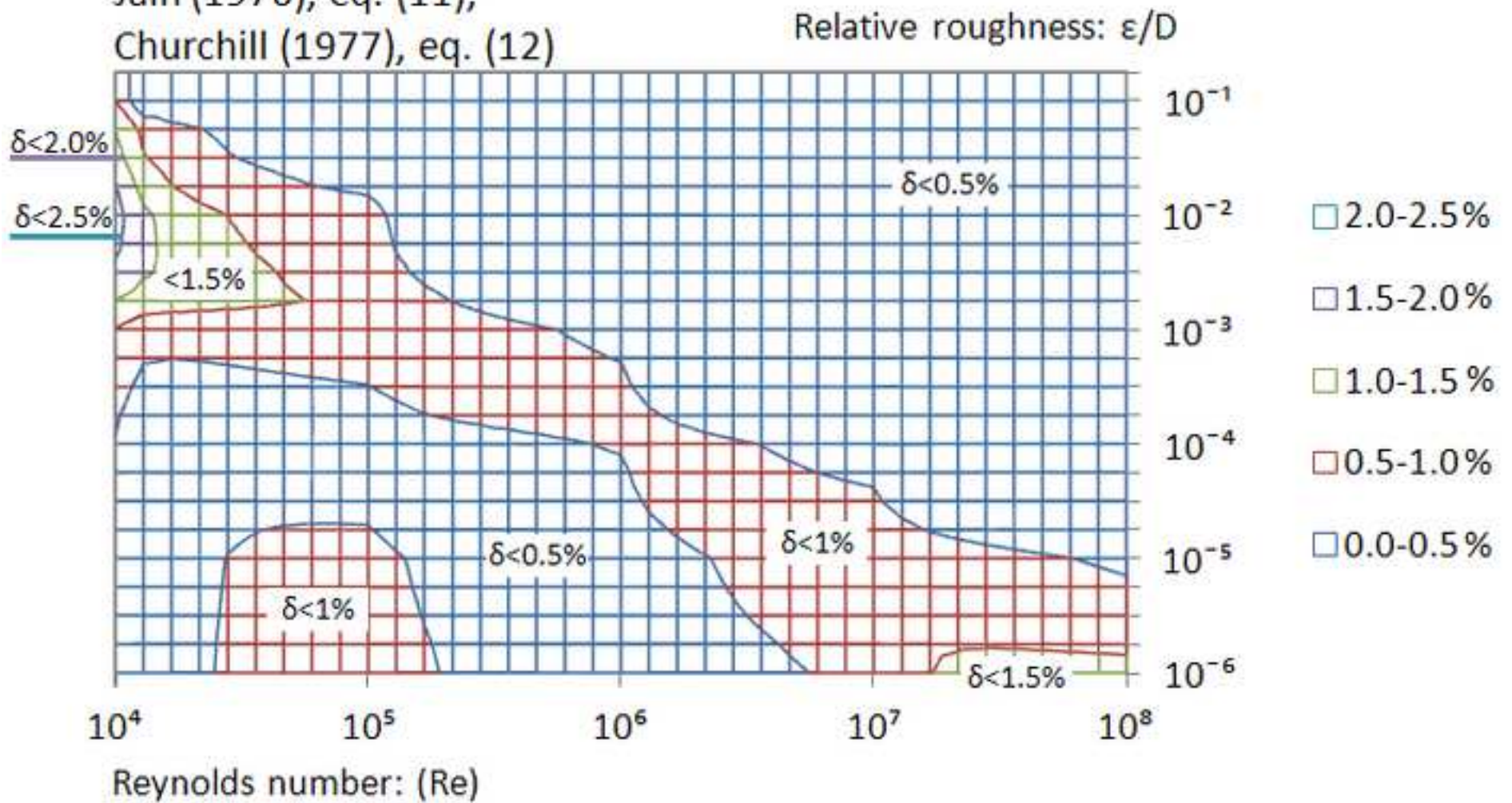


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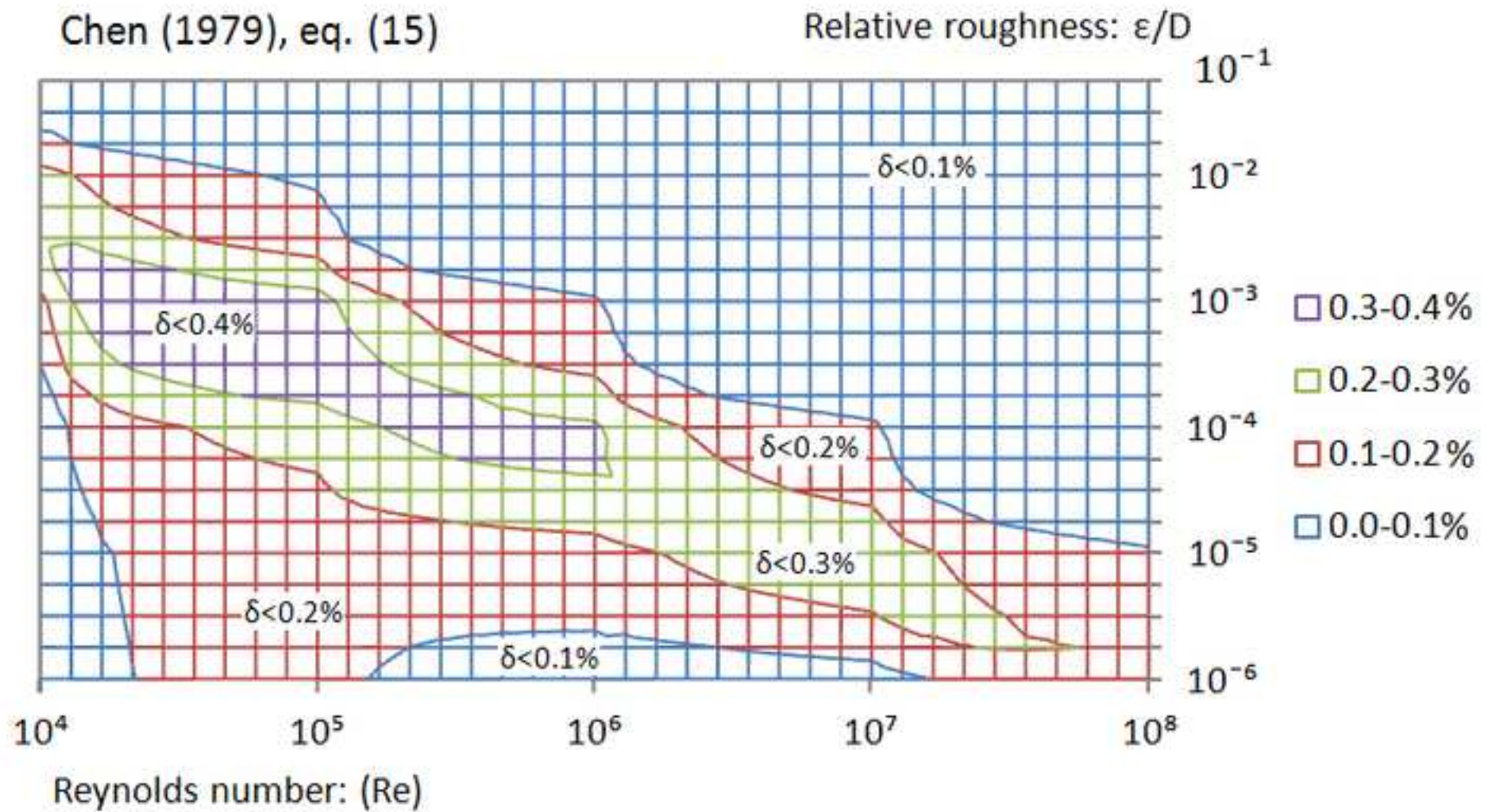


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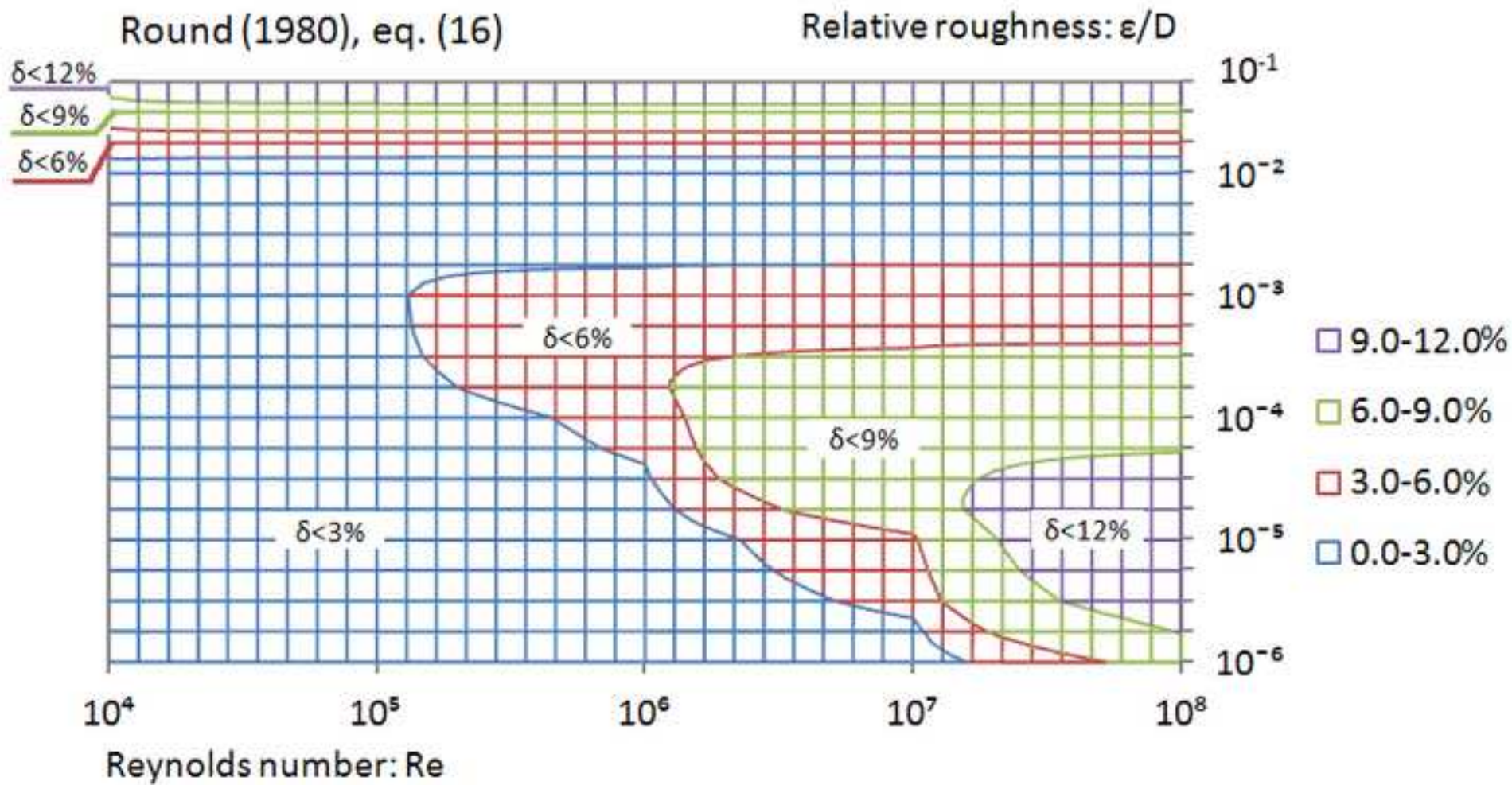


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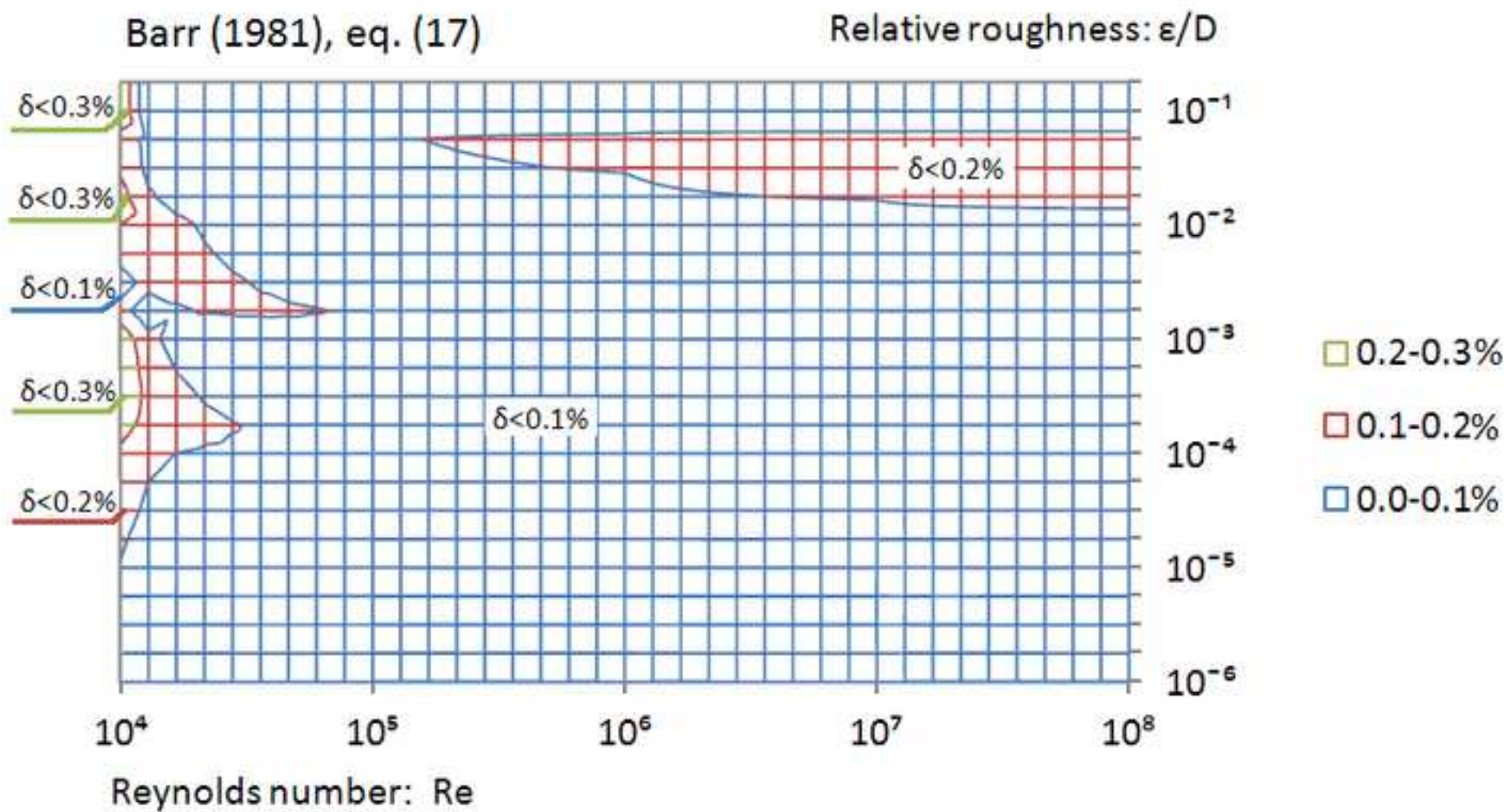


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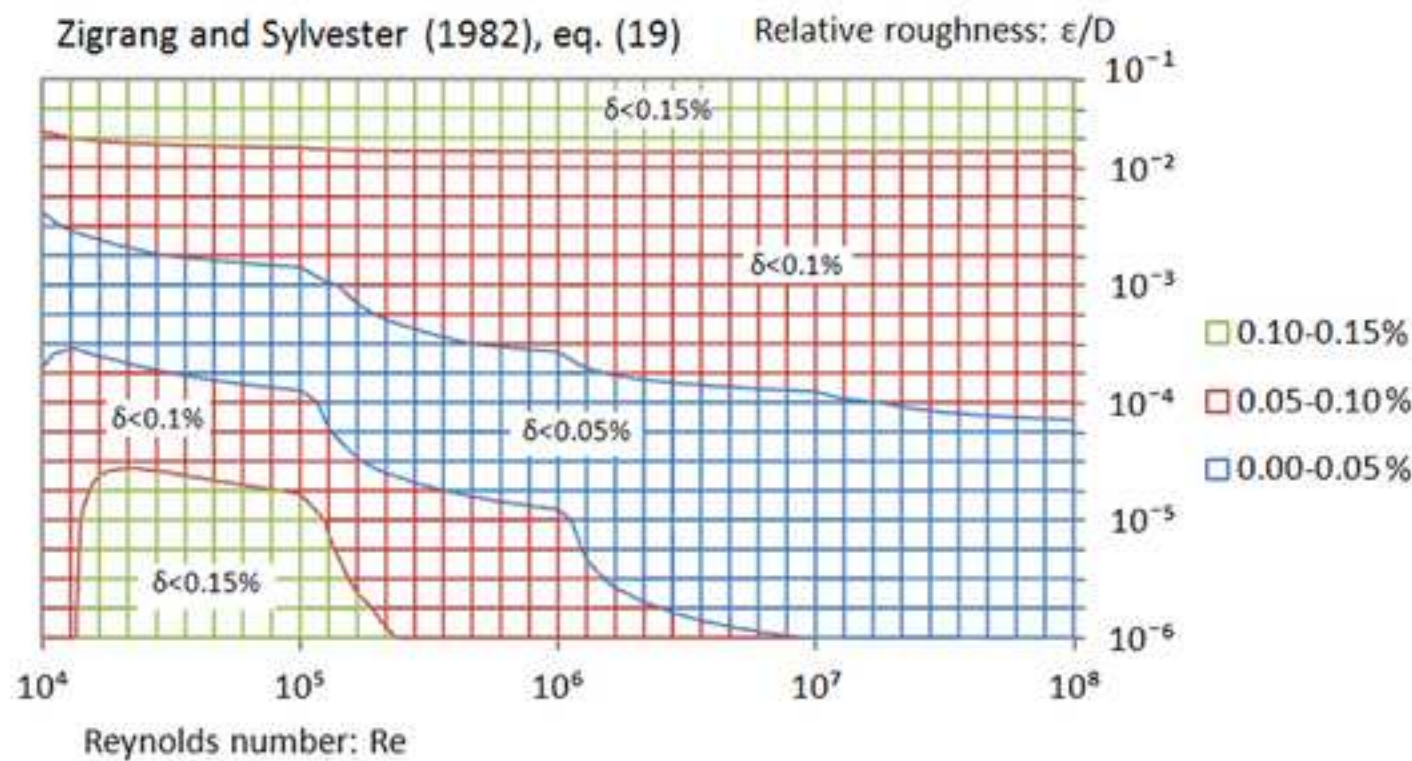
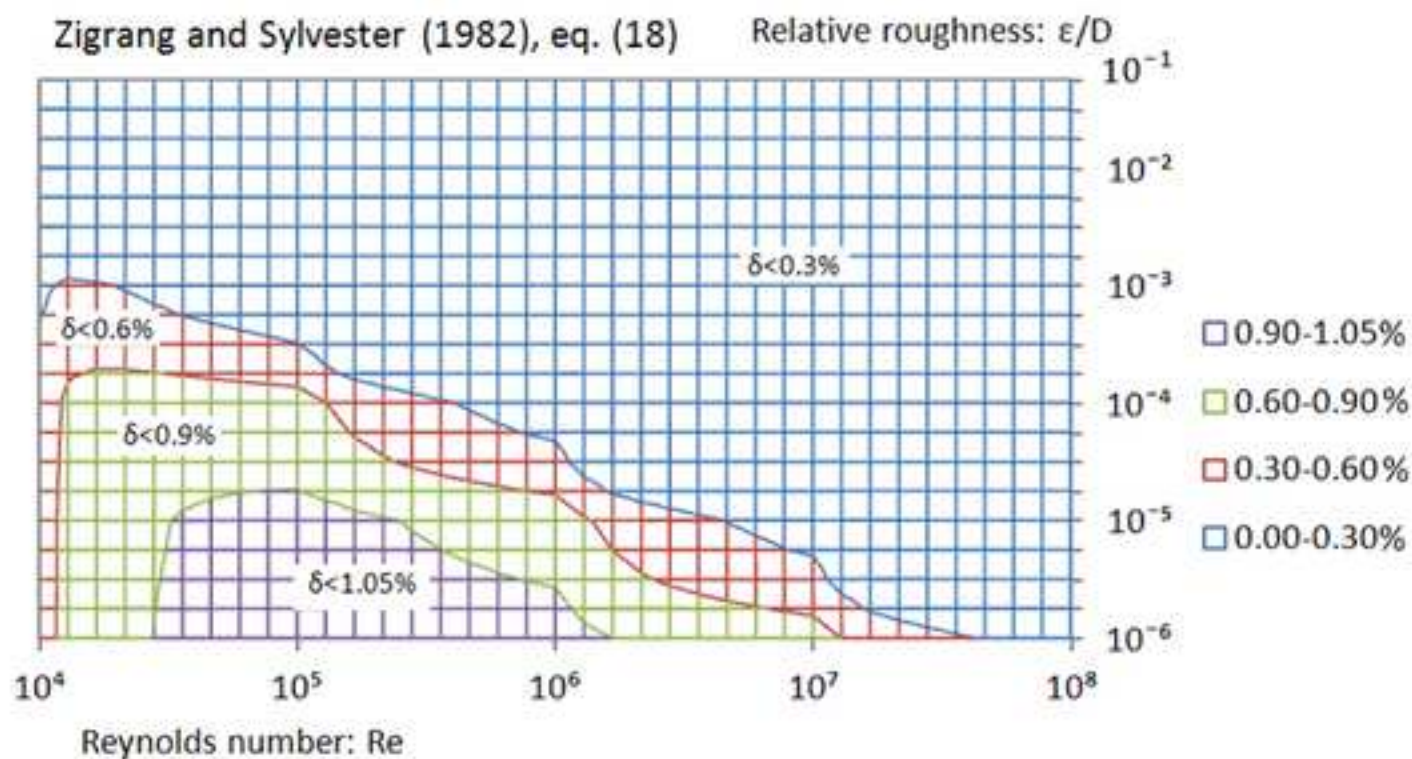


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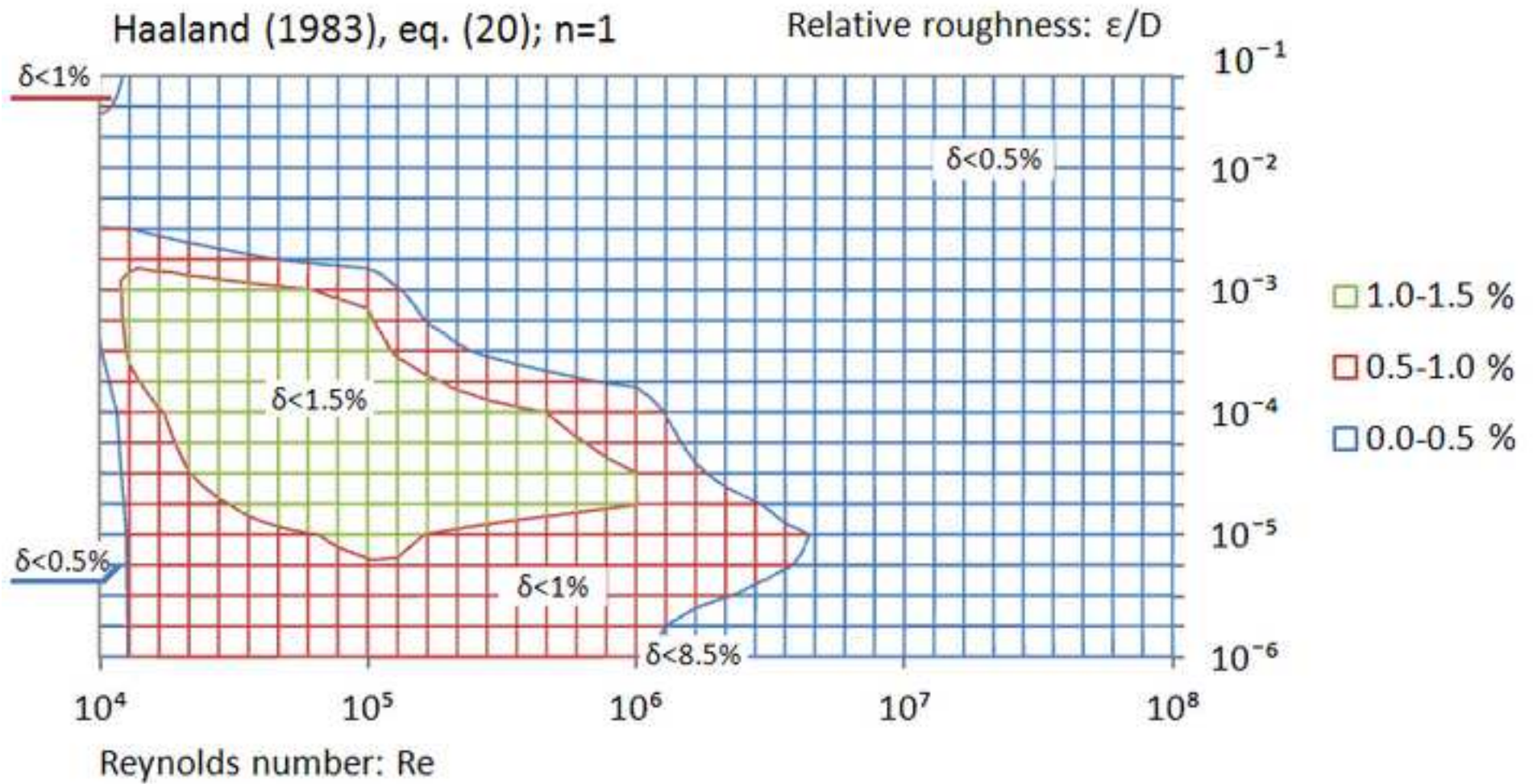
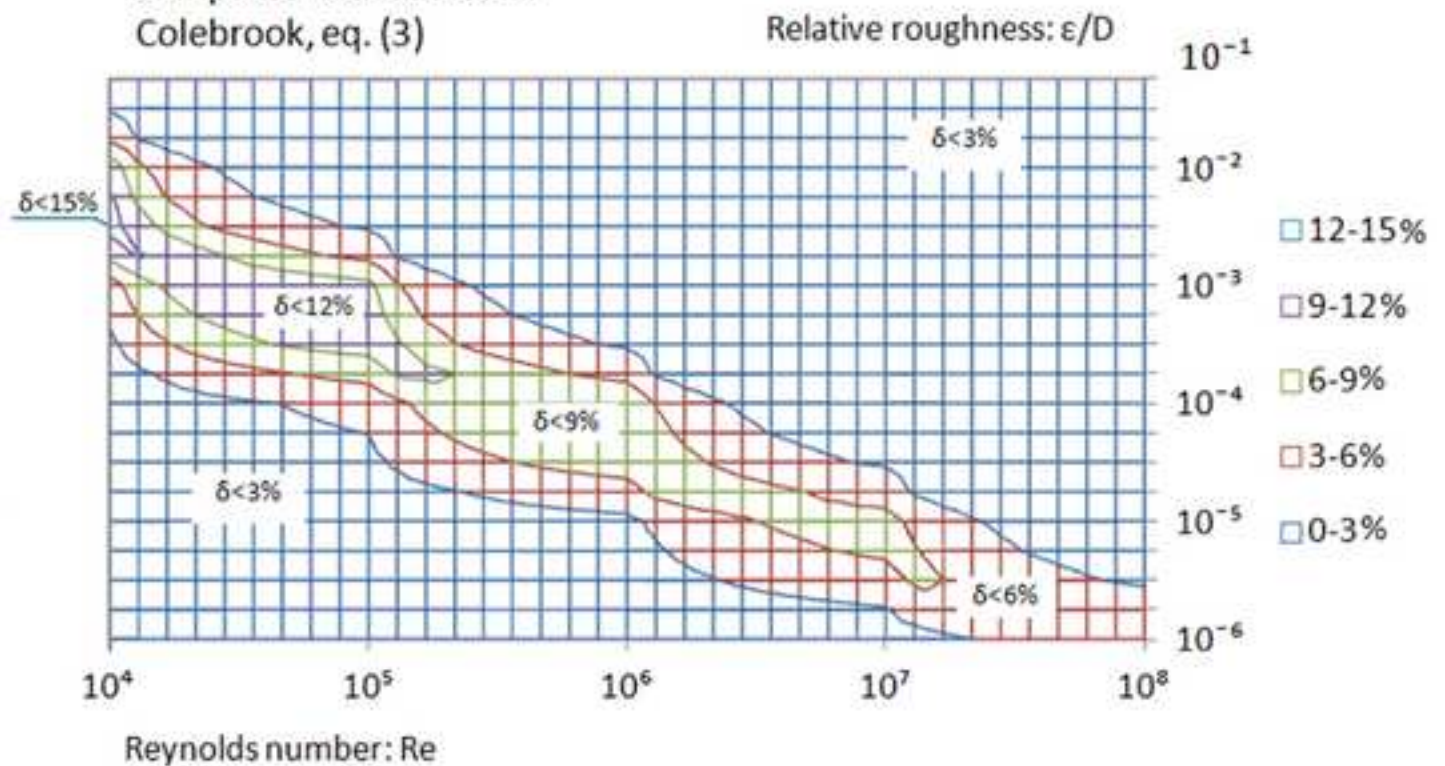


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Haaland (1983), eq. (20); $n=3$,
compared with standard
Colebrook, eq. (3)



Haaland (1983), eq. (20); $n=3$,
compared with eq. (4) proposed by AGA

Relative roughness: ϵ/D

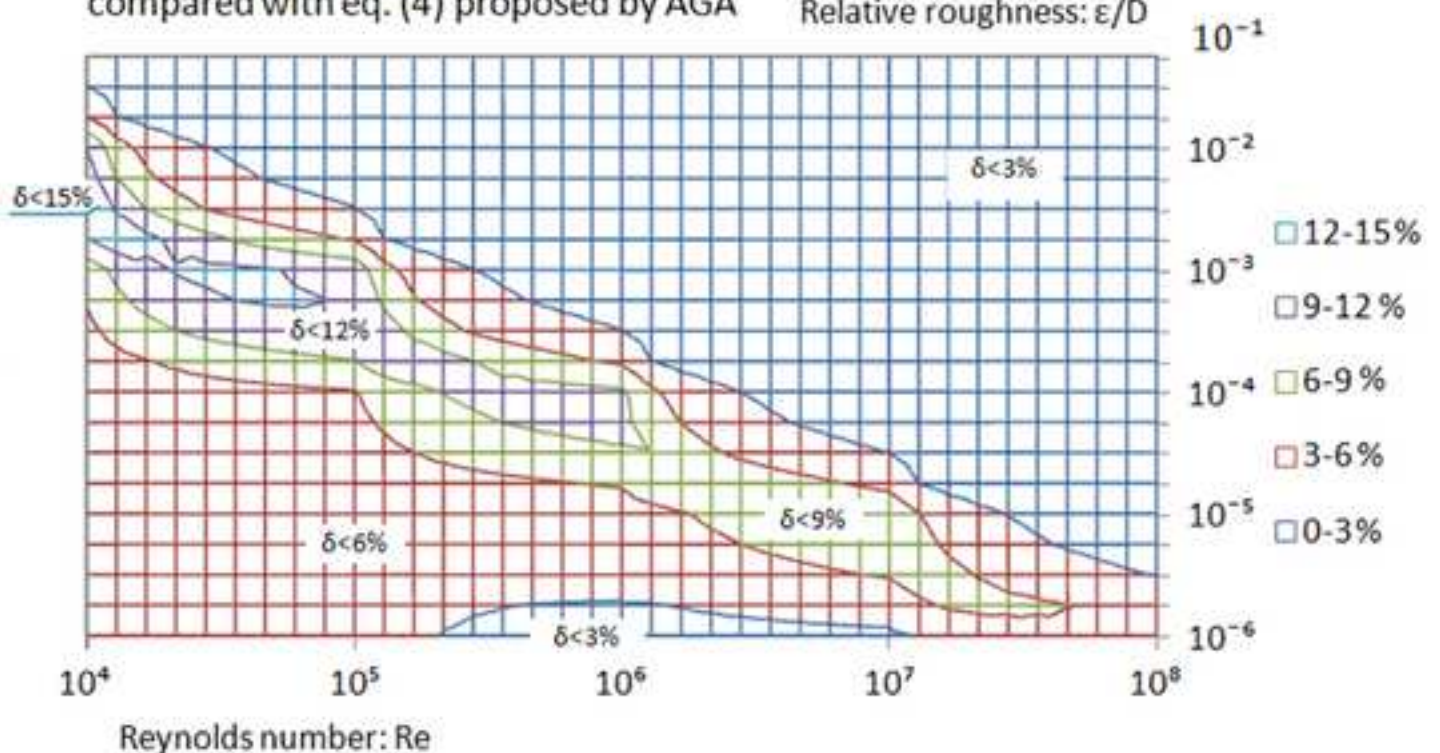


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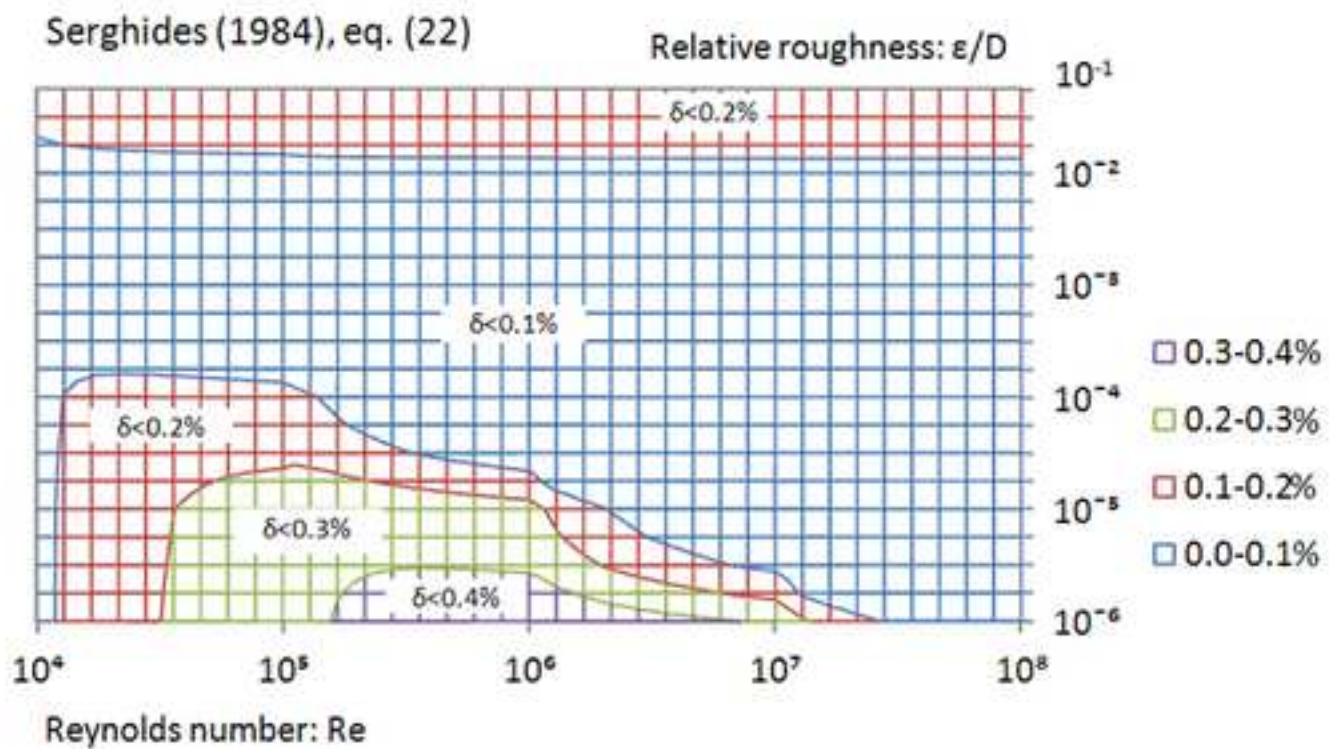
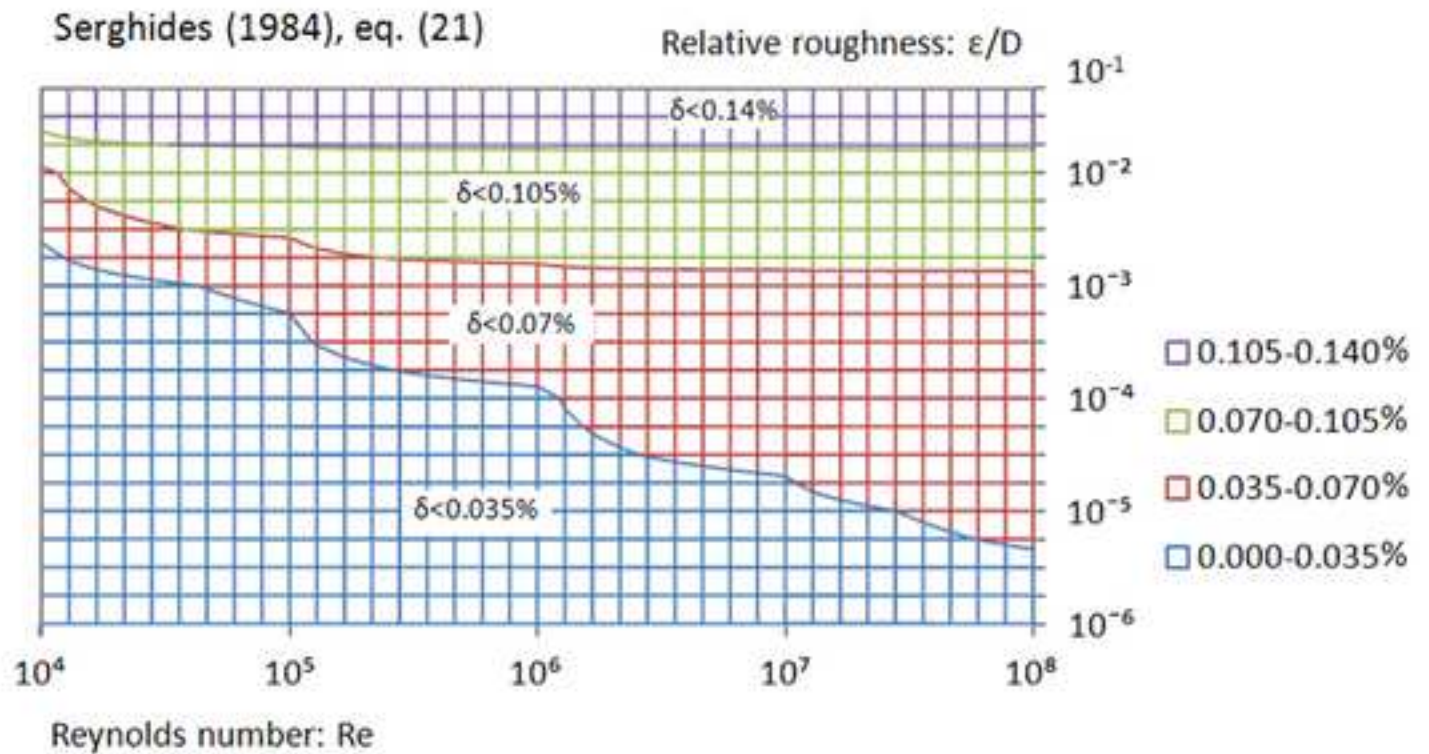


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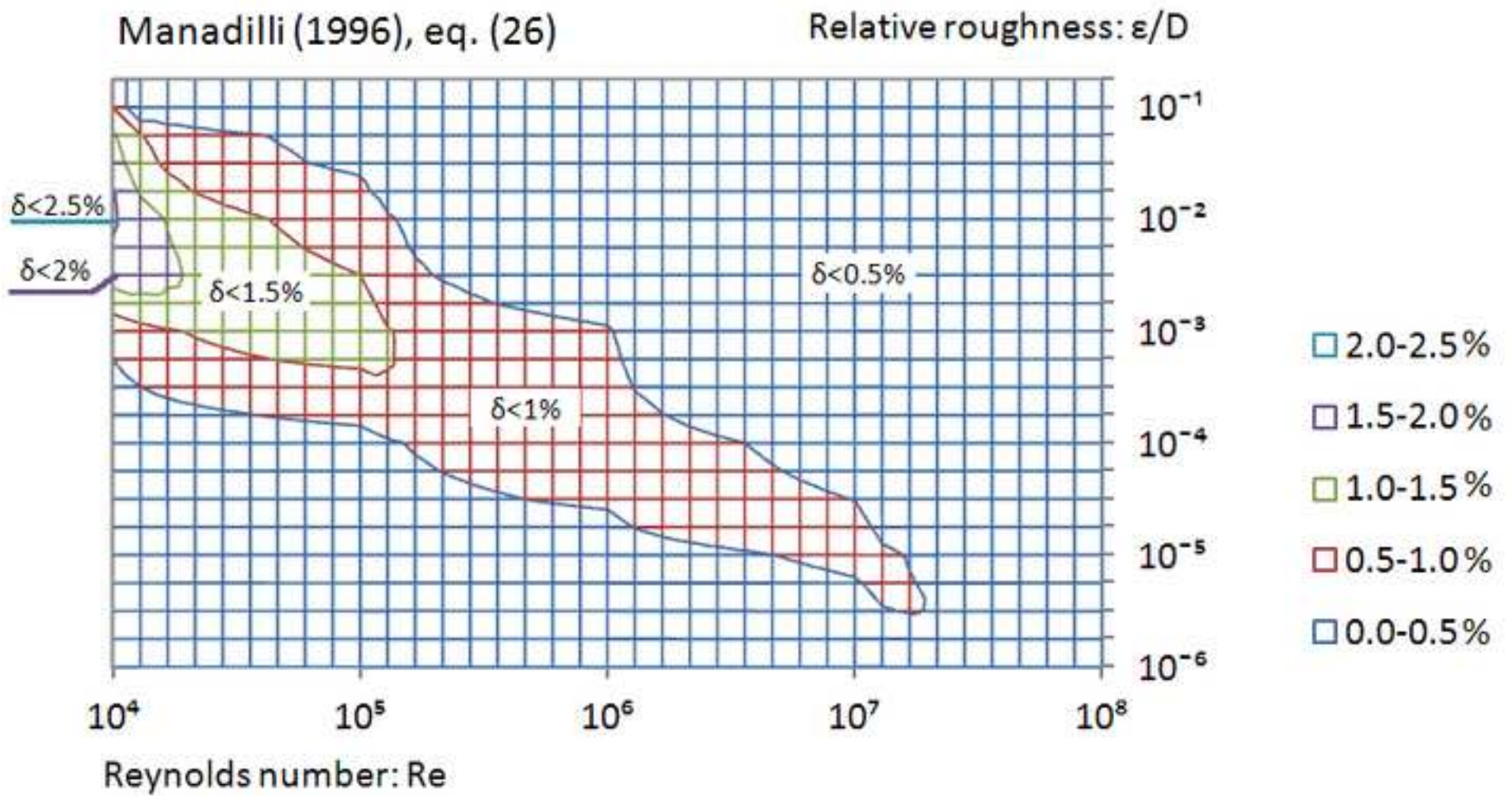


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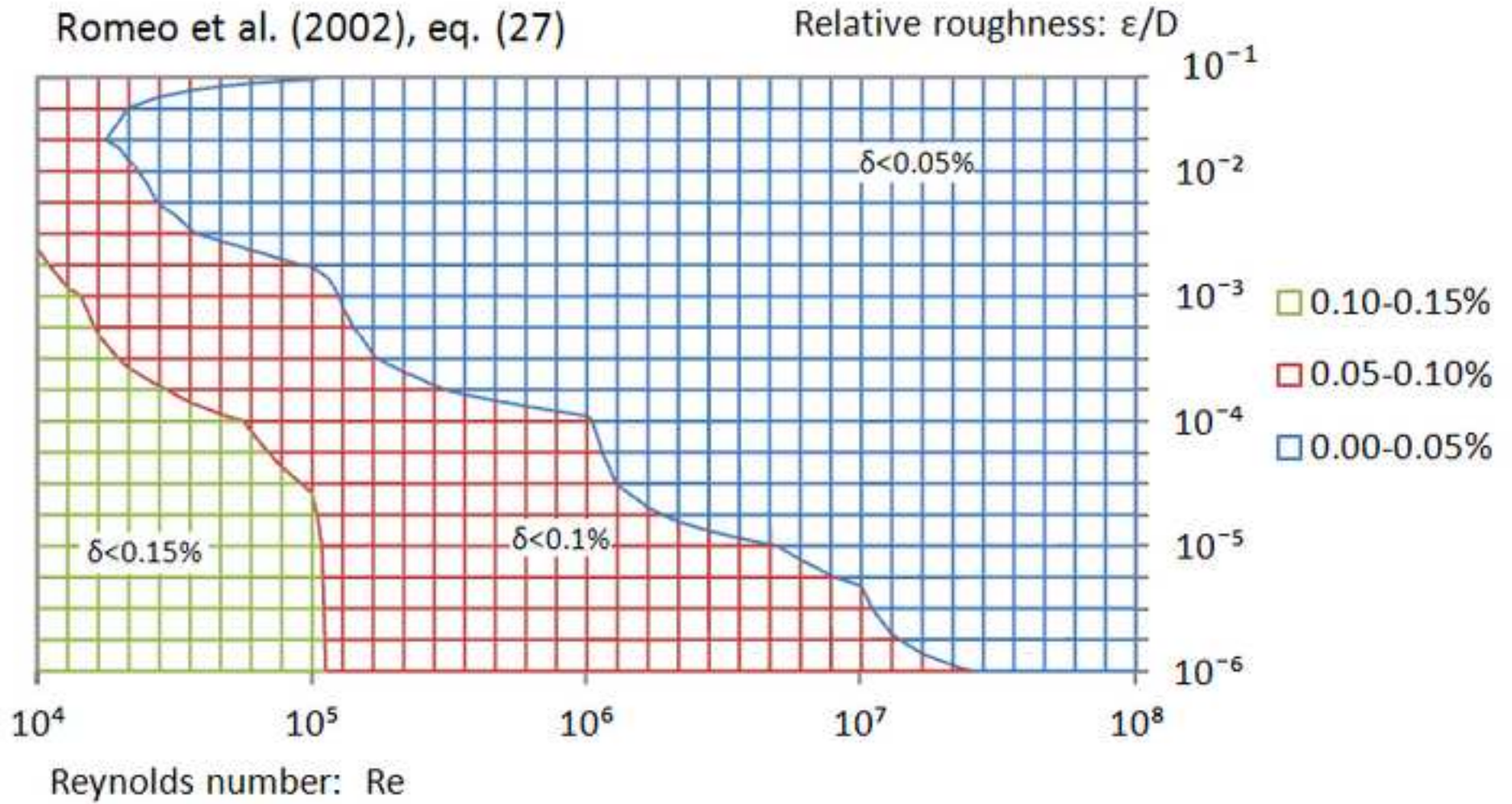


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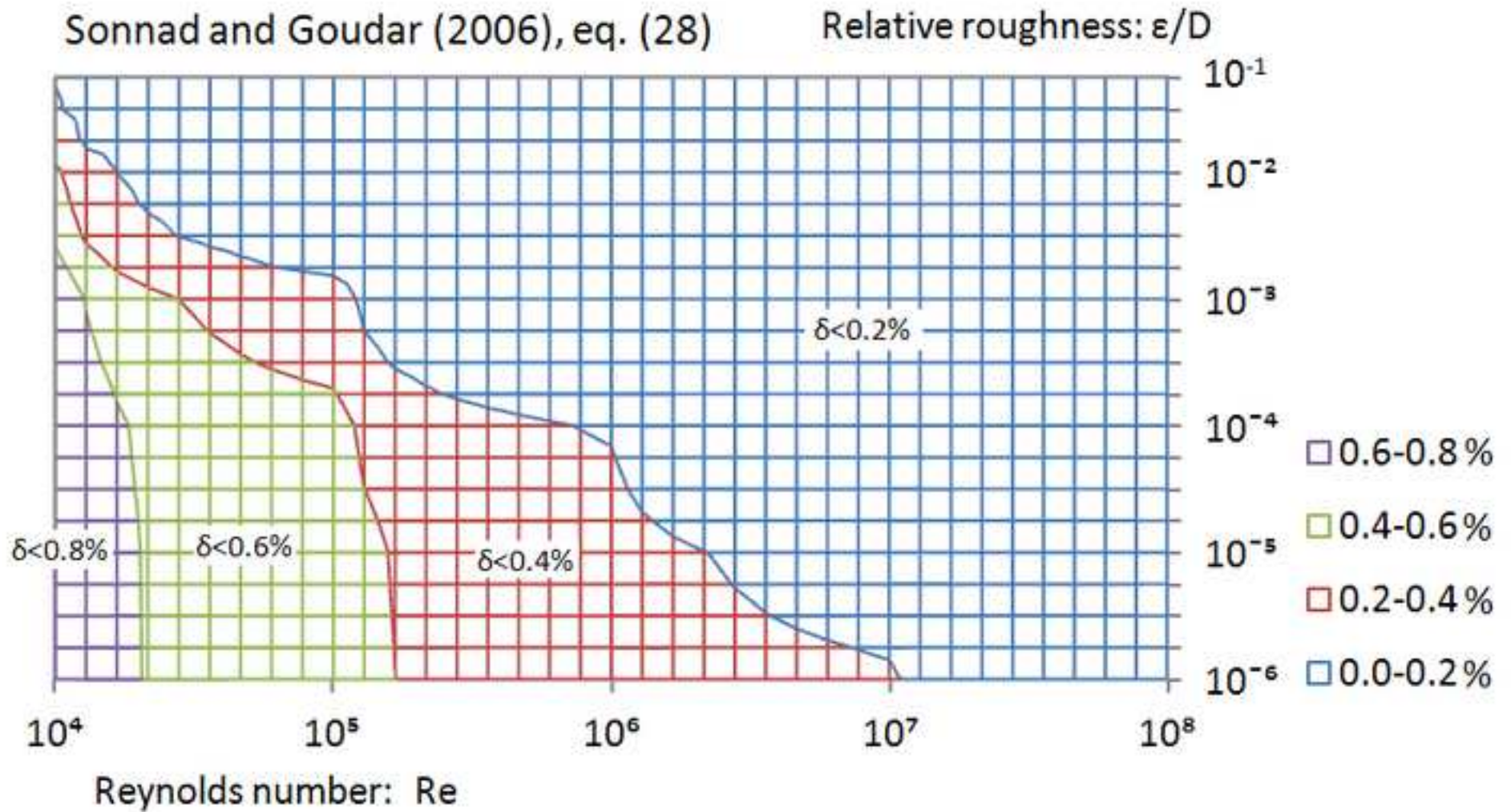


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Vatankhah and Kouchakzadeh
(2008, 2009), eq. (29a)

Relative roughness: ϵ/D

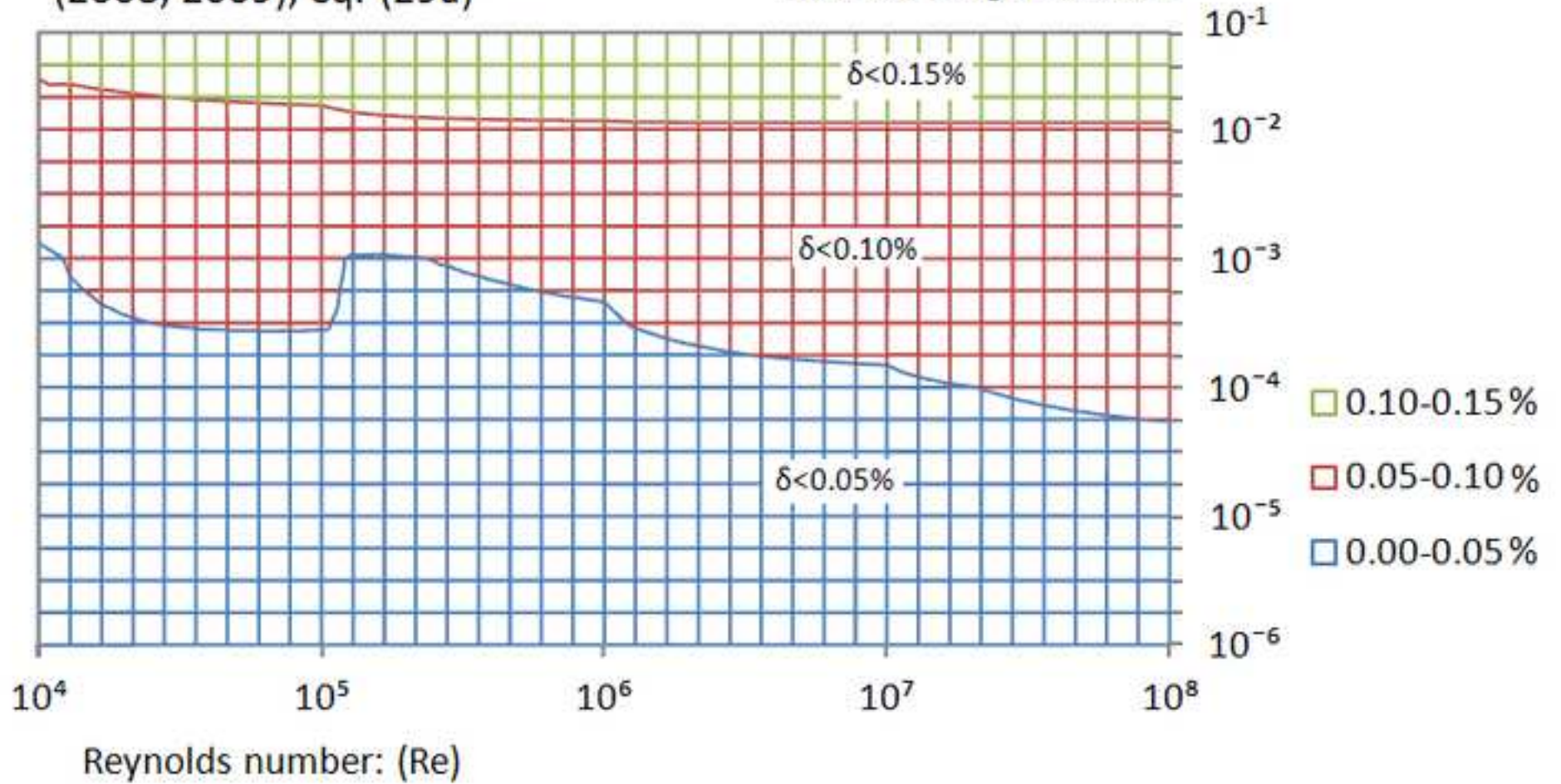


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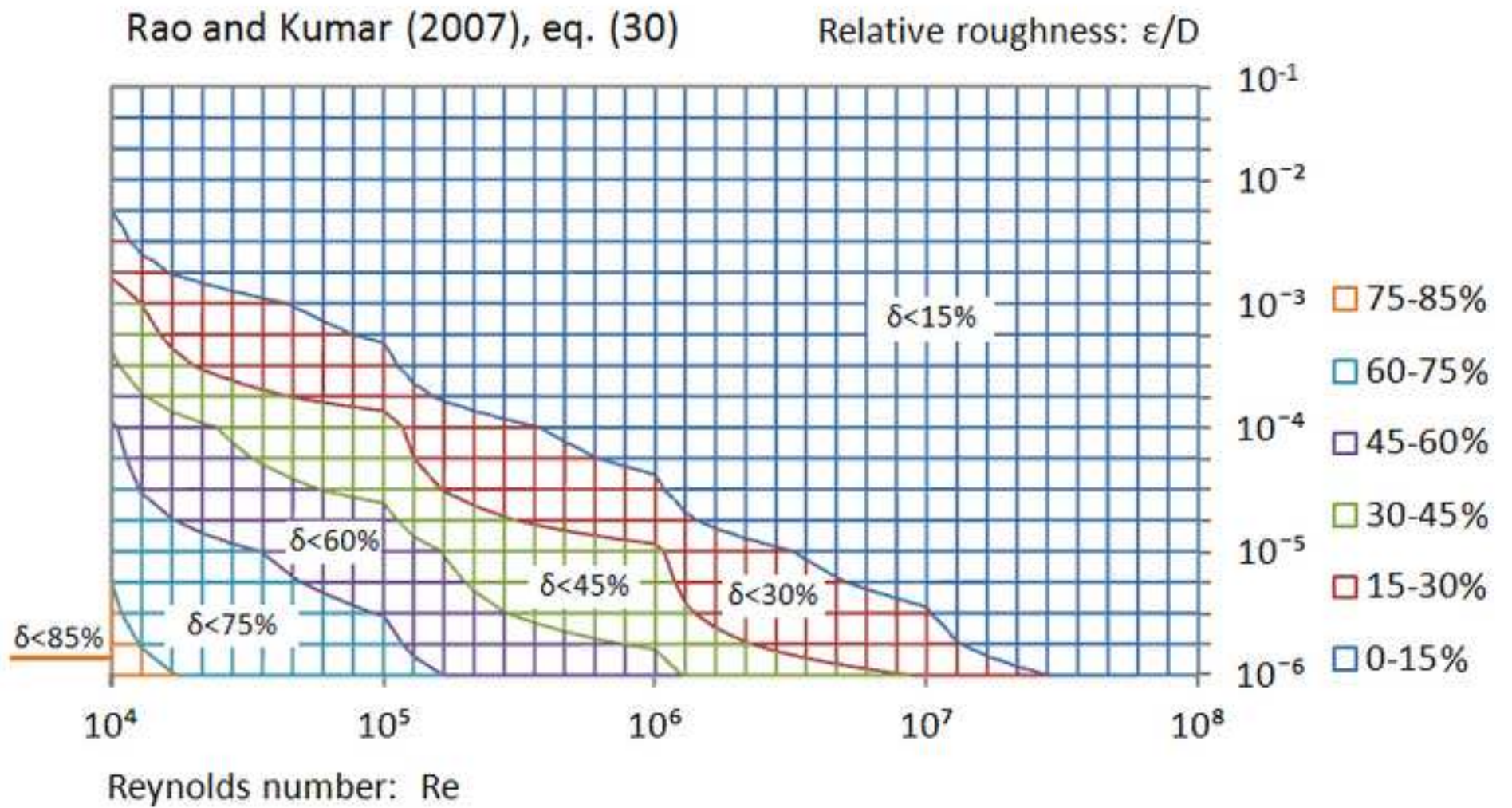


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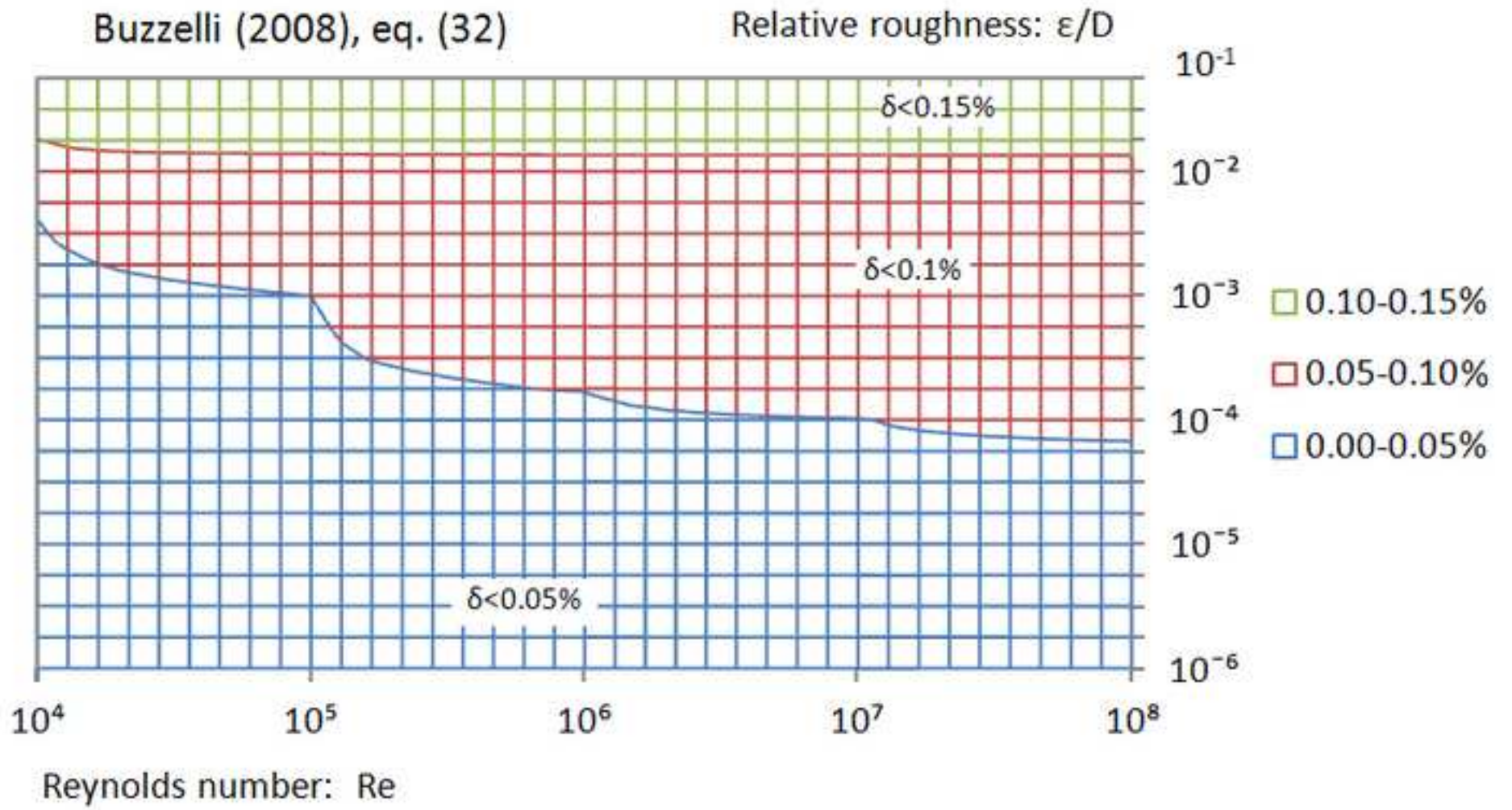


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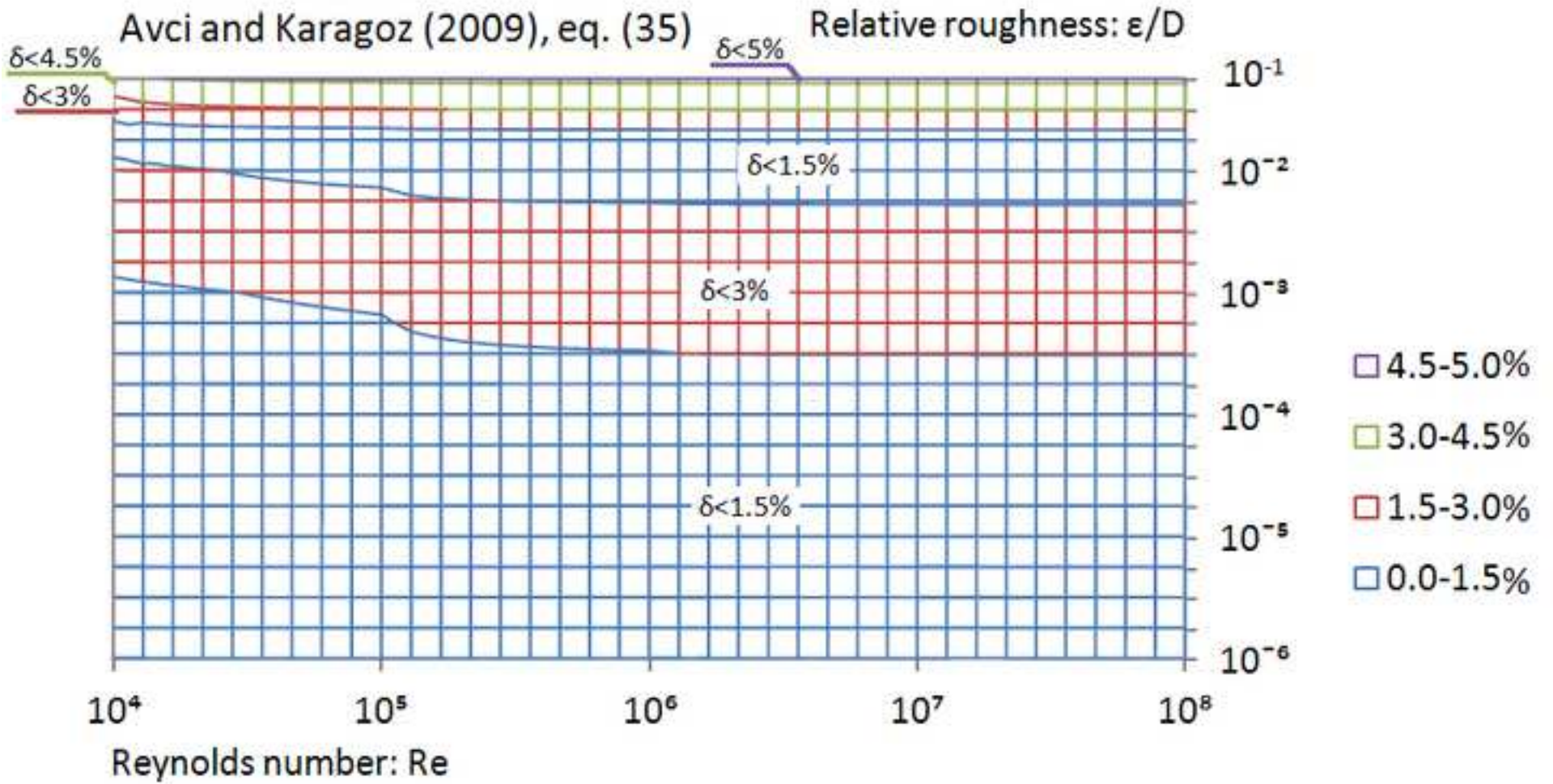


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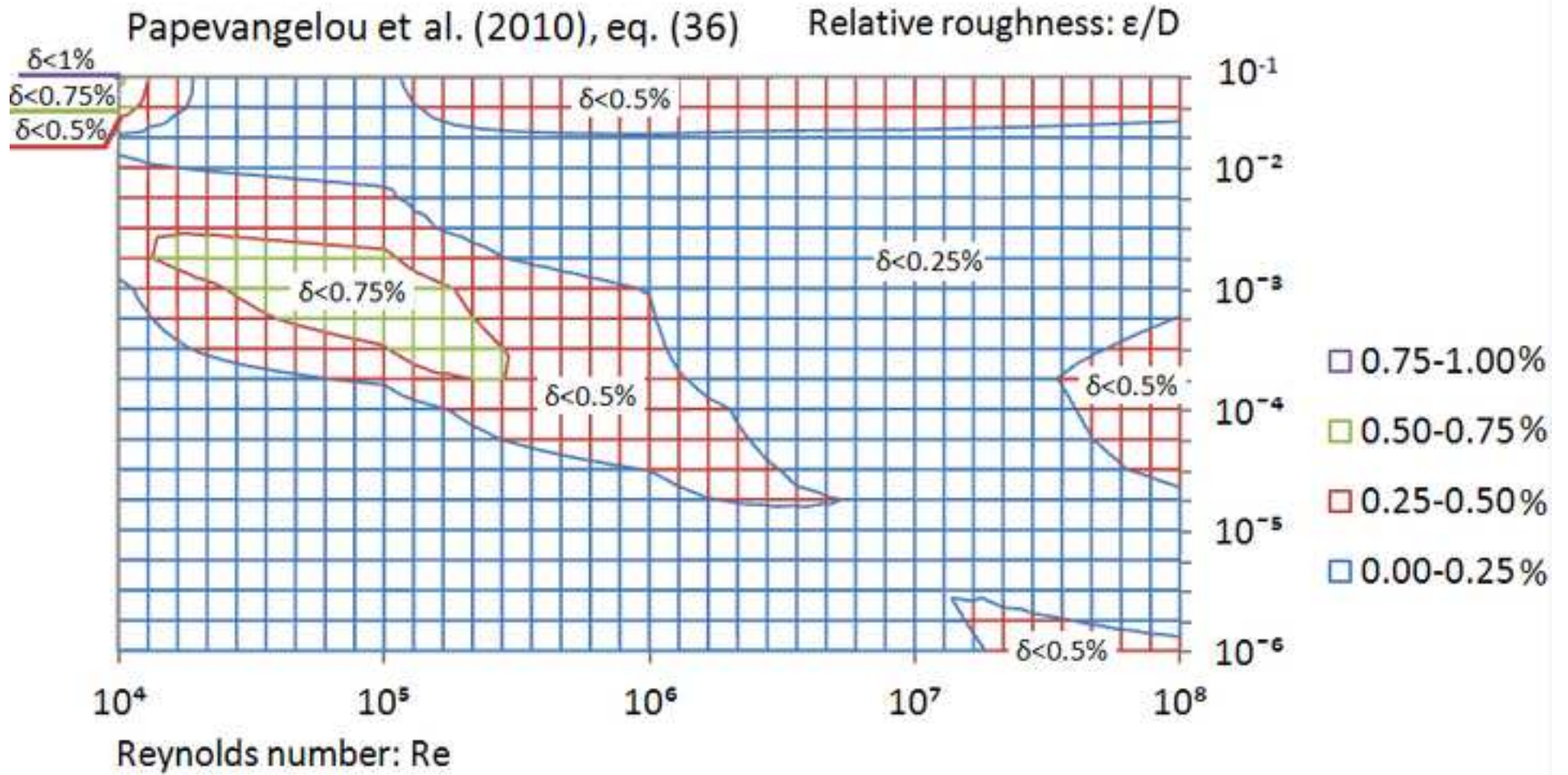


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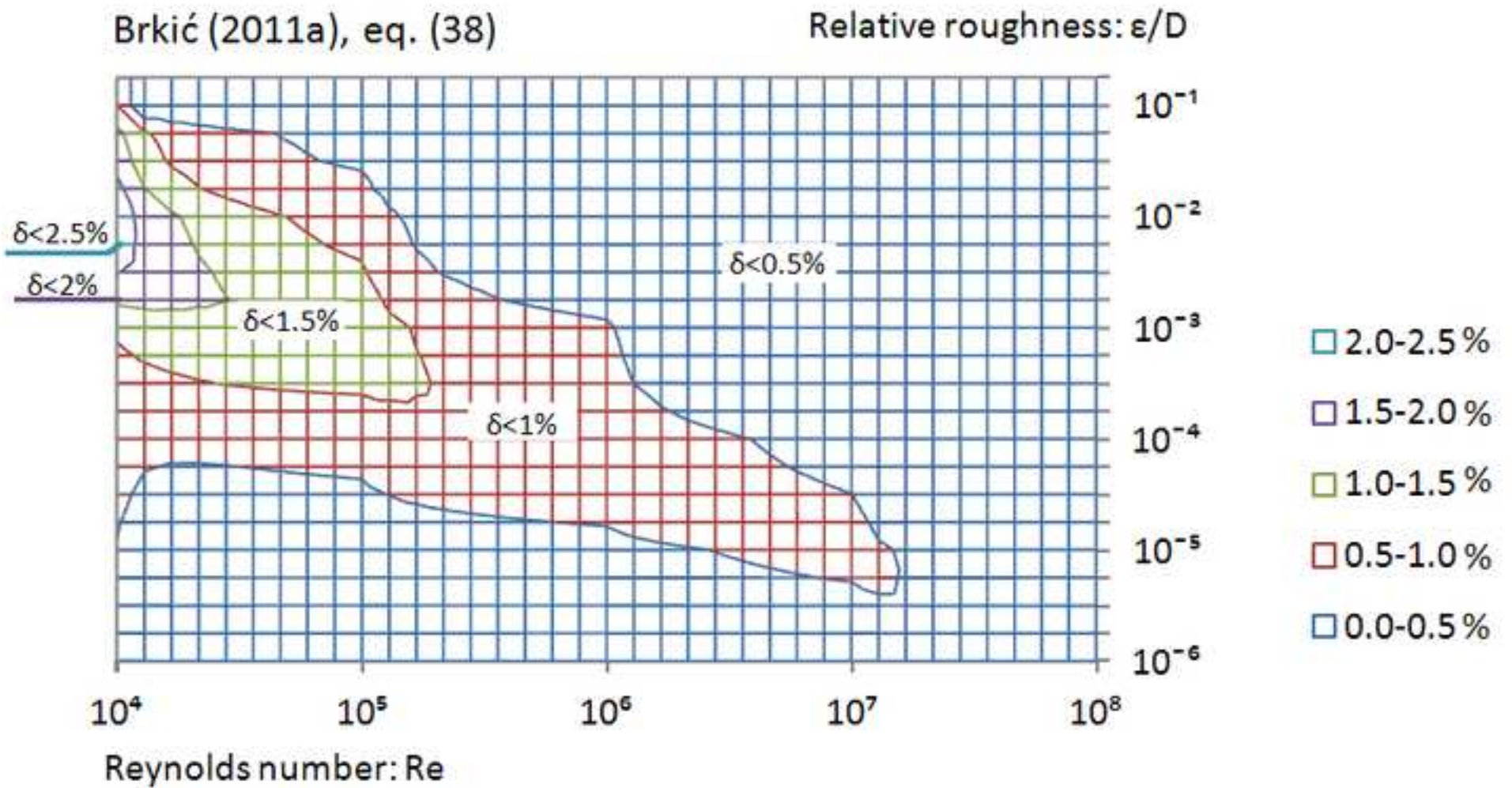


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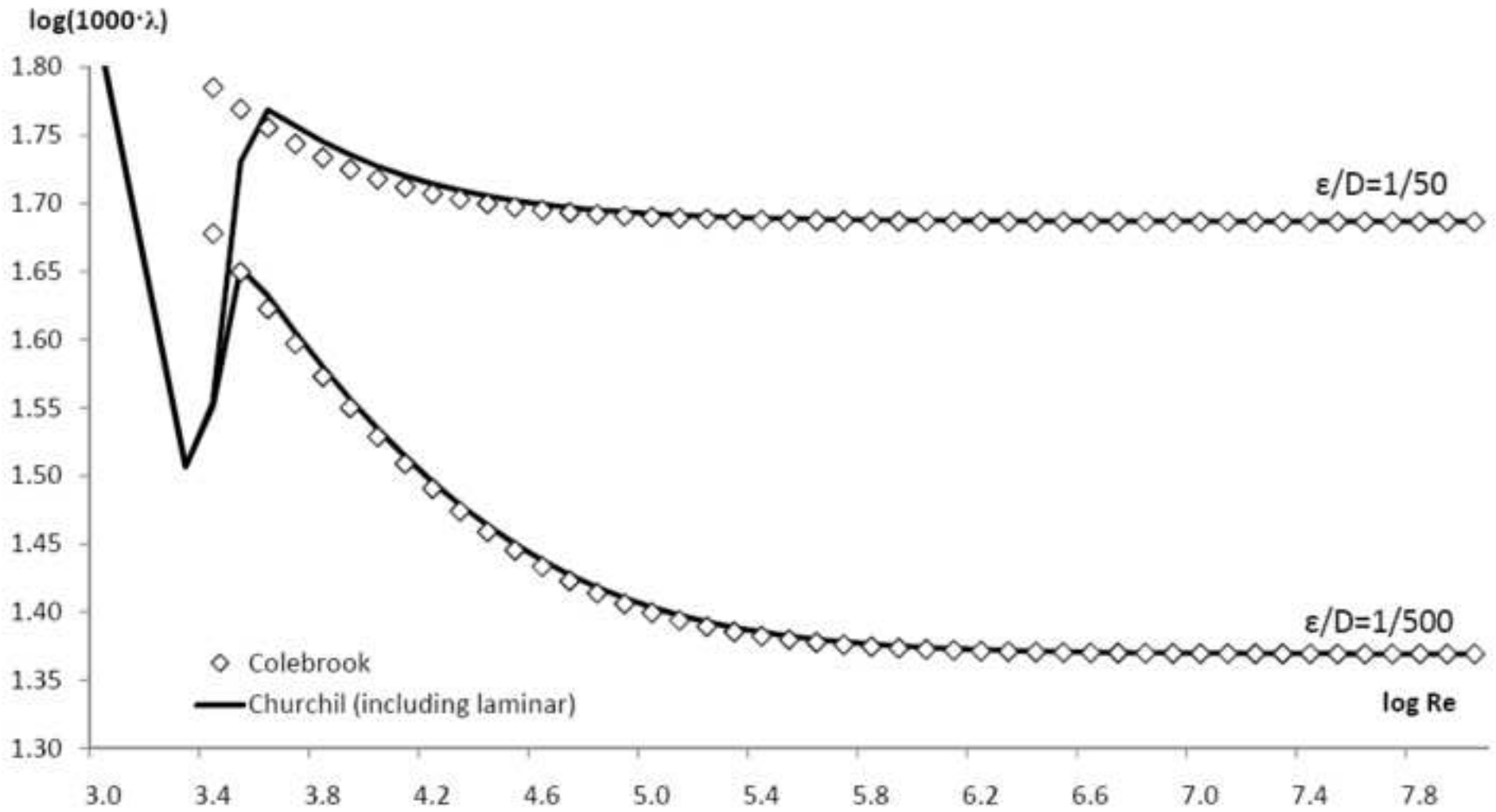


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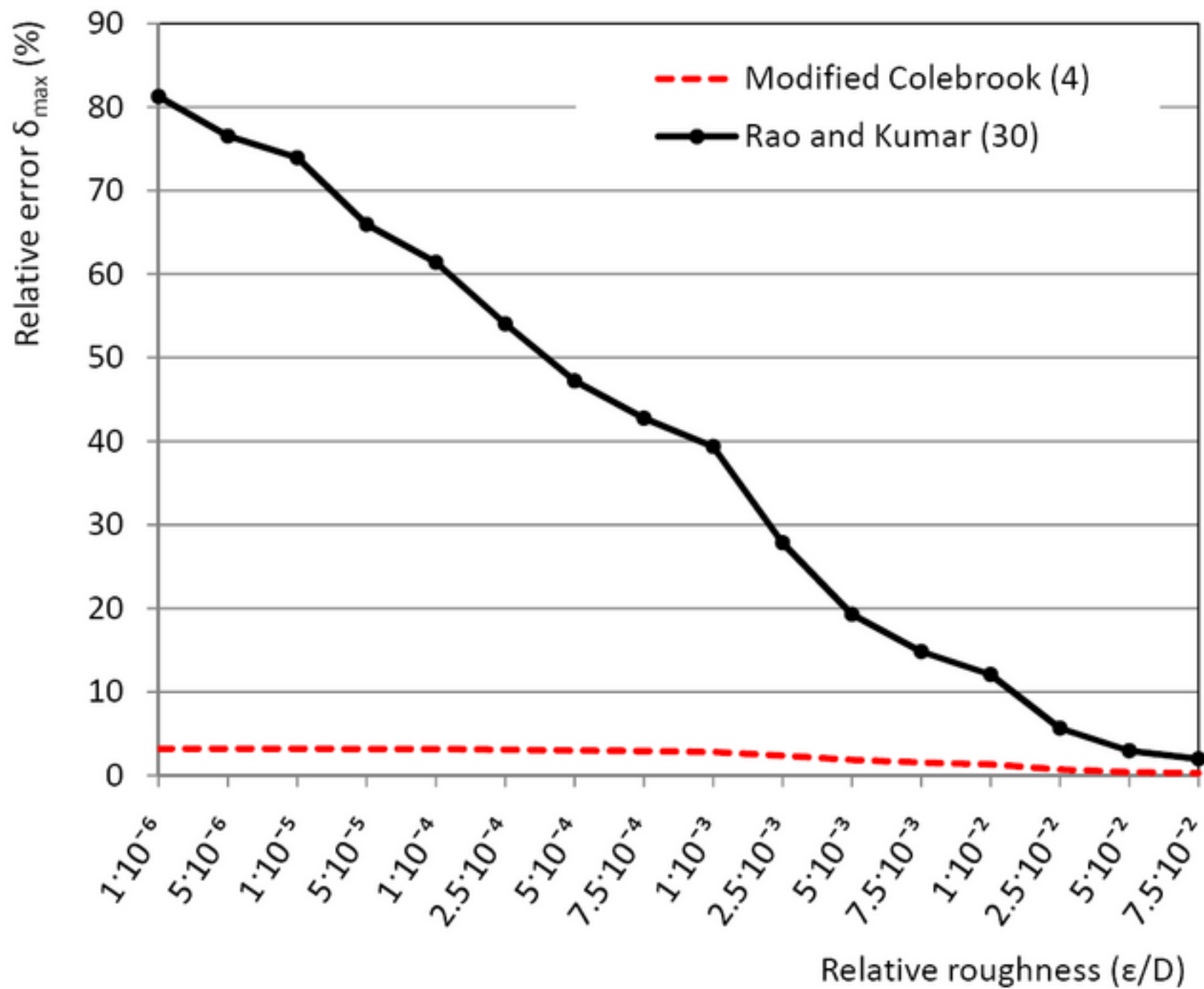
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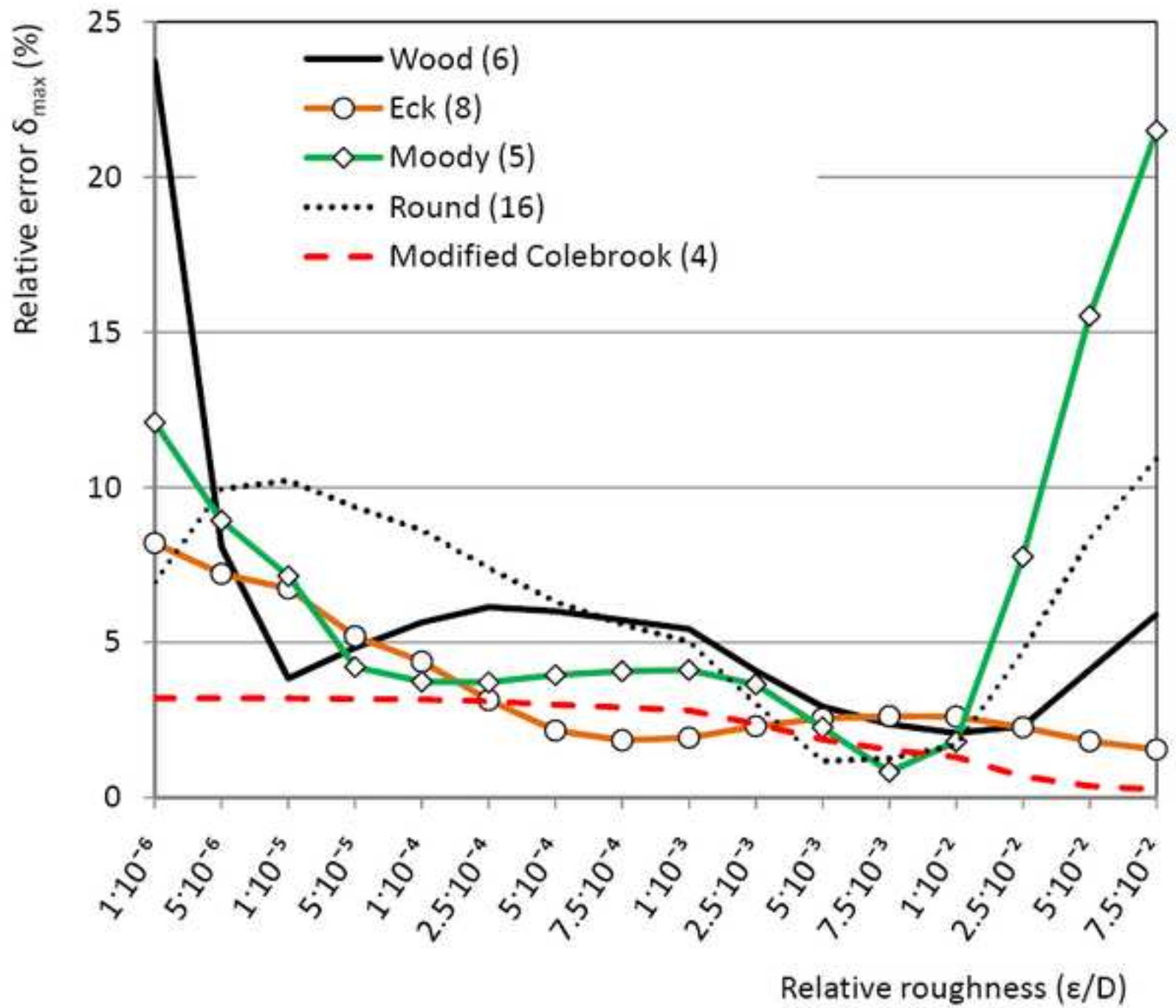


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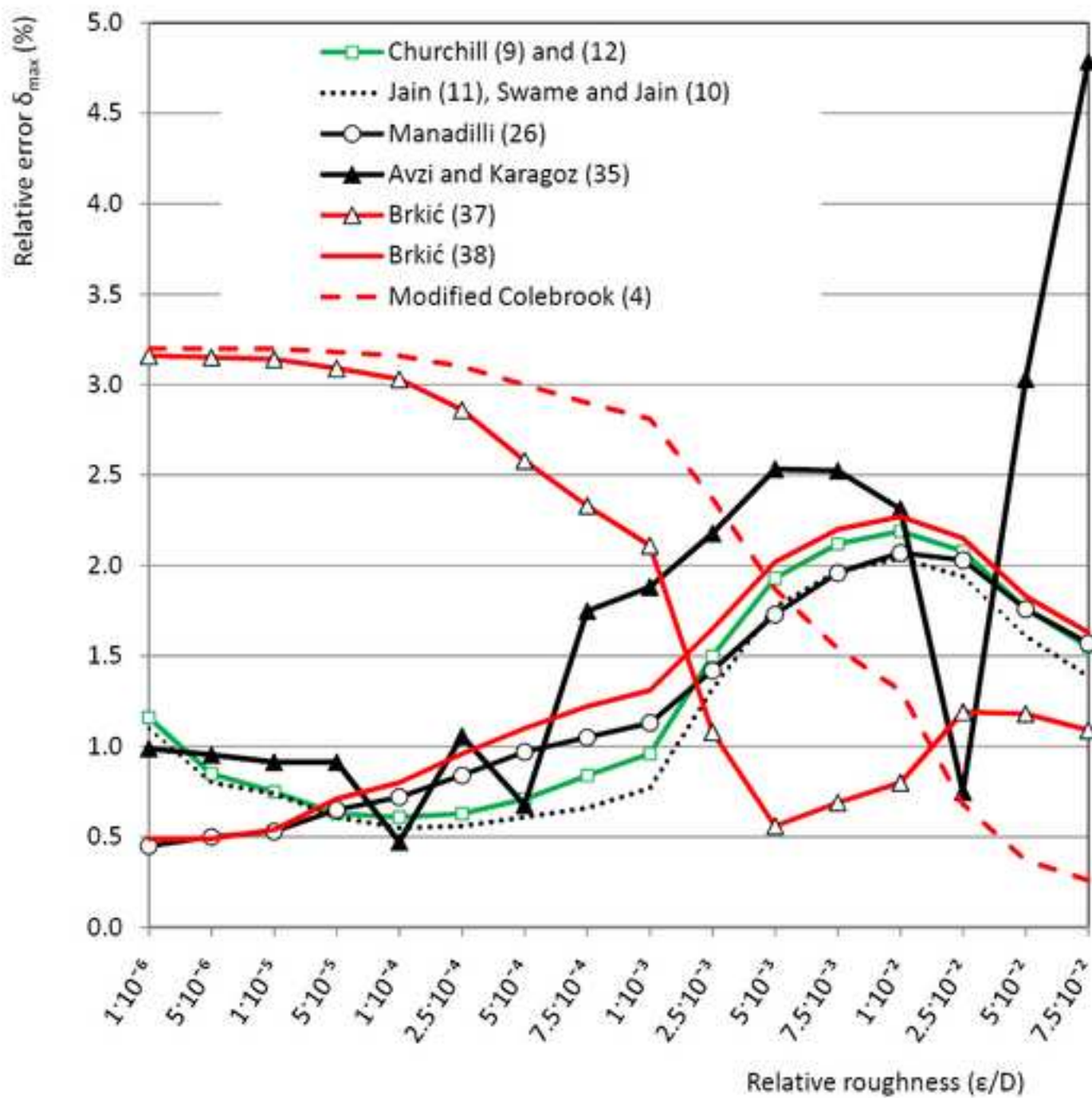


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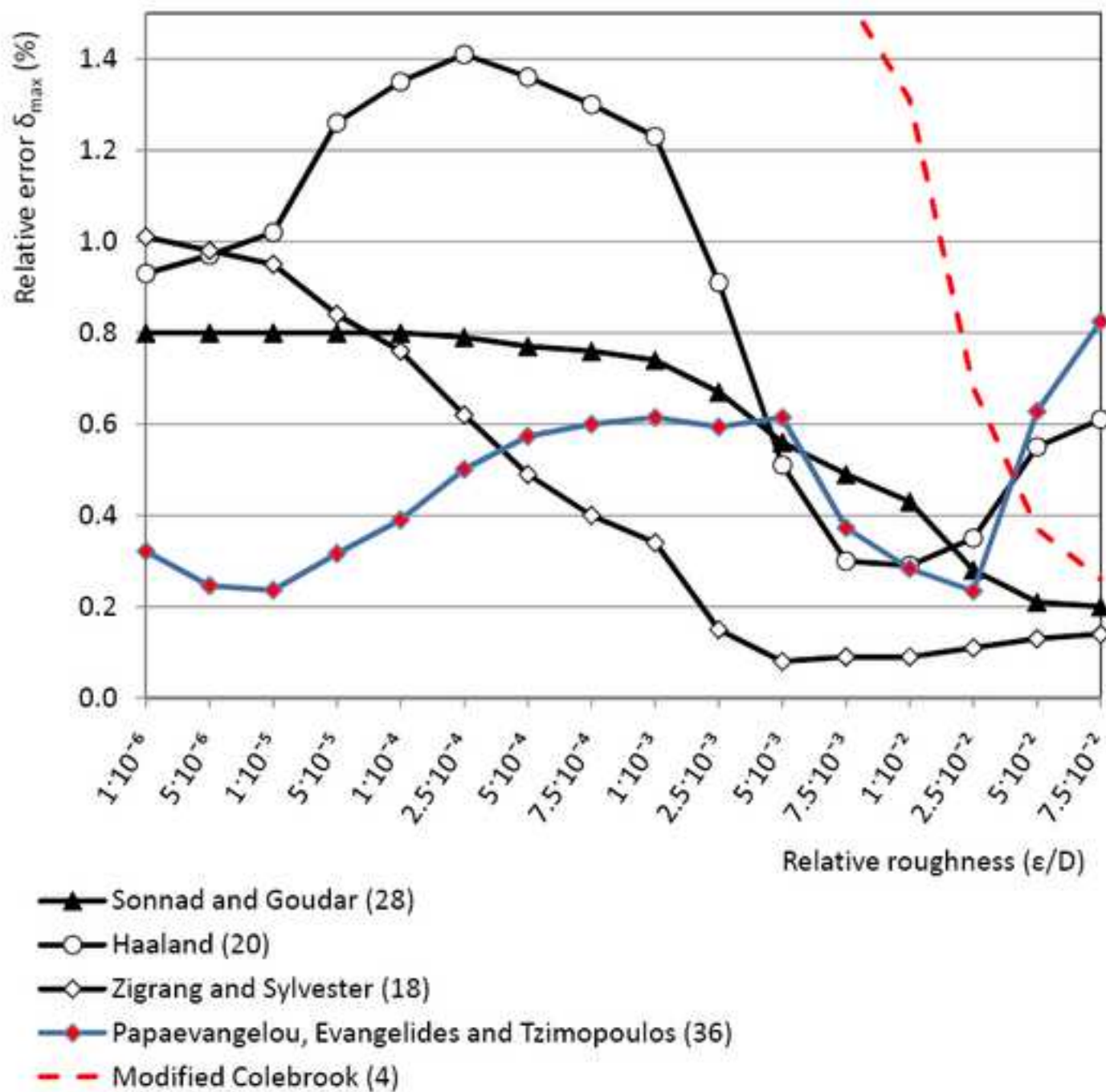


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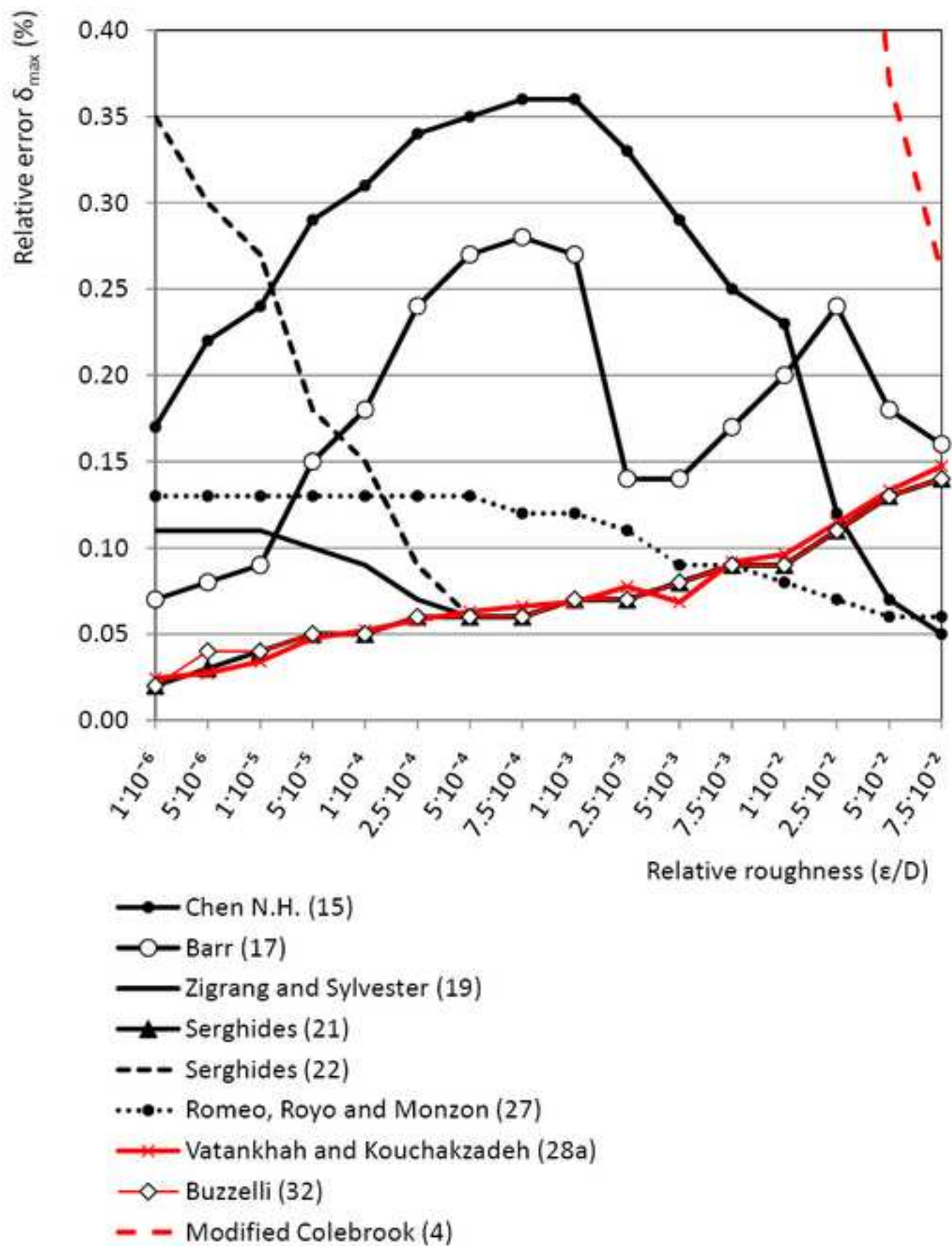


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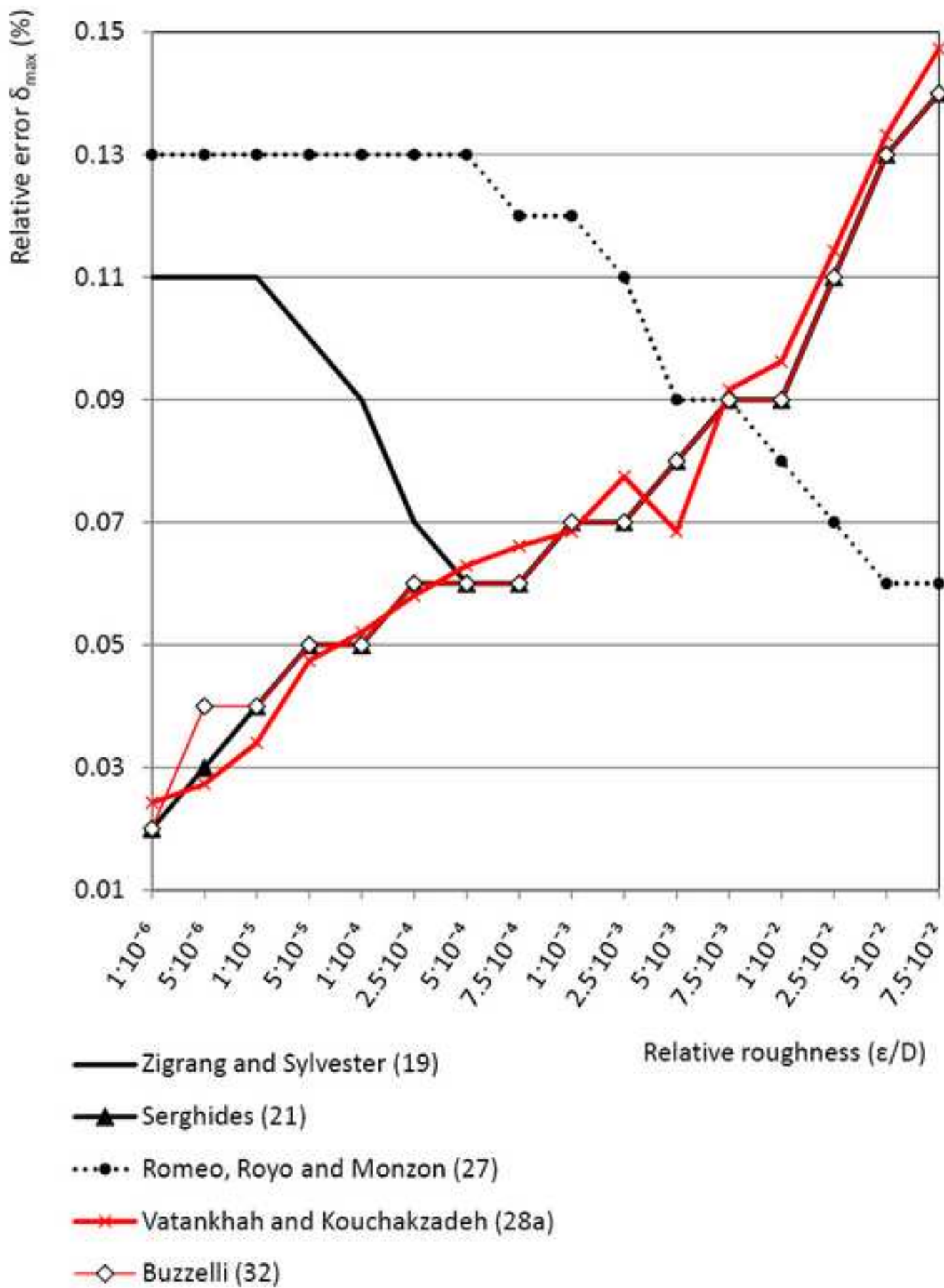


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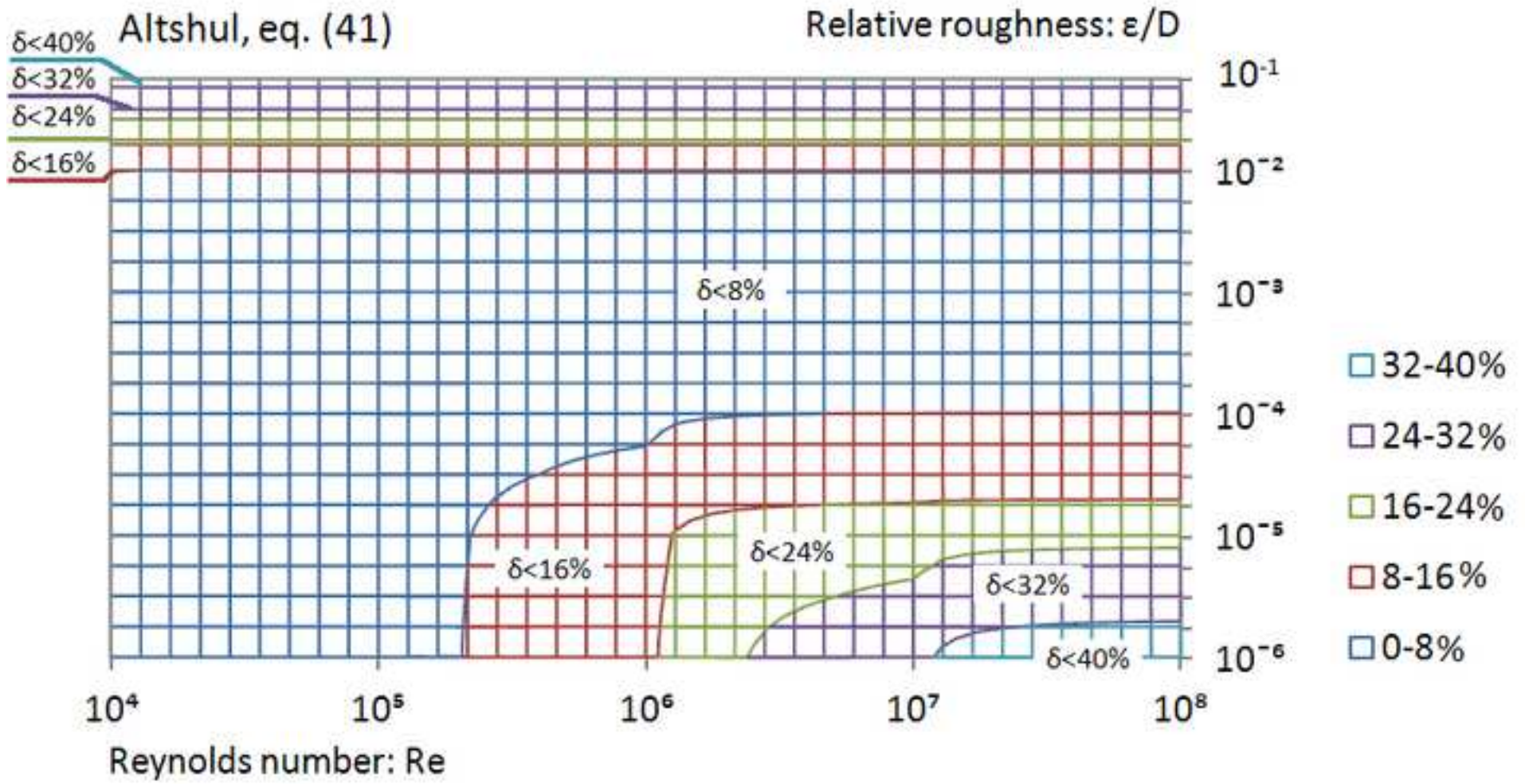


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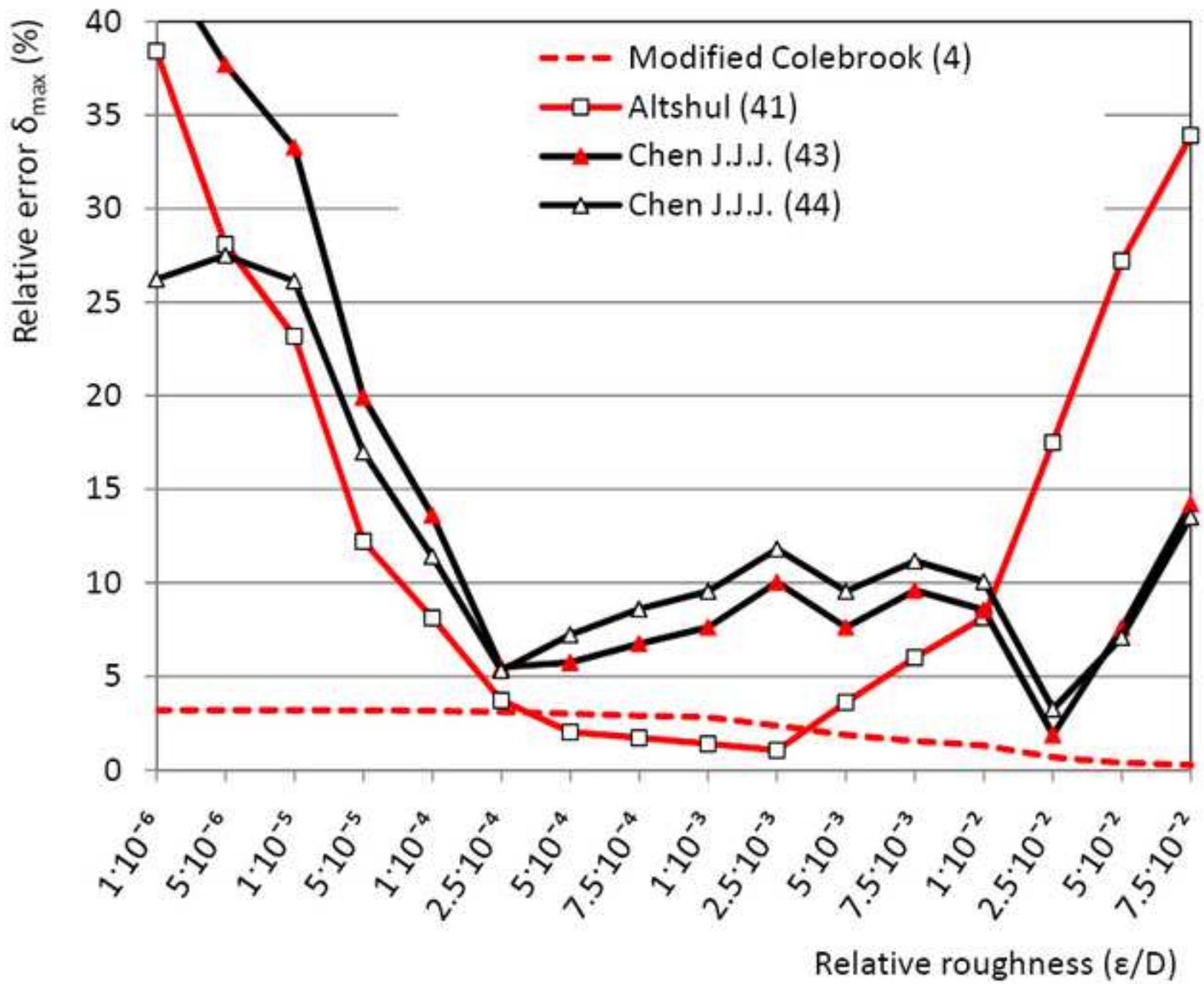


Table 1. Maximal relative error for available approximations for test check points

	δ_{\max} (%)	Relative roughness (ϵ/D)	Reynolds number (Re)
Romeo, Royo and Monzon (27)	-0.1345	$1 \cdot 10^{-6}$	$1 \cdot 10^4$
Buzzelli (32)	-0.1385	$7.5 \cdot 10^{-2}$	$1 \cdot 10^8$
Serghides (21)	-0.1385	$7.5 \cdot 10^{-2}$	$1 \cdot 10^8$
Zigrang and Sylvester (19)	-0.1385	$7.5 \cdot 10^{-2}$	$1 \cdot 10^8$
Vatankhah and Kouchakzadeh (29a)	0.1472	$7.5 \cdot 10^{-2}$	
Barr (17)	+0.2775	$7.5 \cdot 10^{-4}$	$1 \cdot 10^4$
Serghides (22)	+0.3544	$1 \cdot 10^{-6}$	$2 \cdot 10^6$
Chen (15)	-0.3556	$7.5 \cdot 10^{-4}$	$8 \cdot 10^4$
Sonnad and Goudar (28)	-0.8003	$1 \cdot 10^{-5}$	$1 \cdot 10^4$
Papaevangelou, Evangelides and Tzimopoulos (36)	-0.8247	$7.5 \cdot 10^{-2}$	$2 \cdot 10^6$
Zigrang and Sylvester (18)	-1.0074	$1 \cdot 10^{-6}$	$3 \cdot 10^5$
Haaland (20)	+1.4083	$2.5 \cdot 10^{-4}$	$9 \cdot 10^4$
Jain (11)	-2.0437	$1 \cdot 10^{-2}$	$1 \cdot 10^4$
Swame and Jain (10)	-2.0404	$1 \cdot 10^{-2}$	$1 \cdot 10^4$
Manadilli (26)	-2.0651	$1 \cdot 10^{-2}$	$1 \cdot 10^4$
Churchill (9)	-2.1718	$1 \cdot 10^{-2}$	$1 \cdot 10^4$
Churchill (12)	-2.1914	$1 \cdot 10^{-2}$	$1 \cdot 10^4$
Brkić (38)	-2.2719	$1 \cdot 10^{-2}$	$1 \cdot 10^4$
Brkić (37)	+3.1560	$1 \cdot 10^{-6}$	$1 \cdot 10^4$
^a Modified Colebrook (4)	-3.2025	$1 \cdot 10^{-6}$	$1 \cdot 10^4$
Avzi and Karagoz (35)	-4.7857	$7.5 \cdot 10^{-2}$	$1 \cdot 10^8$
Eck (8)	+8.20	$1 \cdot 10^{-6}$	$5 \cdot 10^6$
Round (16)	+10.92	$7.5 \cdot 10^{-2}$	$1 \cdot 10^8$
Moody (5)	+21.49	$7.5 \cdot 10^{-2}$	$1 \cdot 10^4$
Wood (6)	+23.72	$1 \cdot 10^{-6}$	$1 \cdot 10^4$
Rao and Kumar (30)	+81.24	$1 \cdot 10^{-6}$	$1 \cdot 10^4$

^aalso in implicit form (constant 2.51 is replaced with 2.825, recommended by AGA-American Gas Association)

Table 2. Complexity and complexity index of available explicit approximations

	^a Complexity	^b Complexity index
^c Eck (8)	27	1
Moody (5)	29	1.07
Churchil (9)	31	1.14
Haaland (20)	35	1.29
Jain (11)	35	1.29
Swame and Jain (10)	36	1.33
Round (16)	36	1.33
Manadilli (26)	44	1.62
Zigrang and Sylvester (18)	47	1.74
Avzi and Karagoz (35)	47	1.74
Rao and Kumar (30)	61	2.25
Brkić (38)	67	2.48
Sonnad and Goudar (28)	67	2.48
Papaevangelou, Evangelides and Tzimopoulos (36)	67	2.48
Brkić (37)	69	2.55
Zigrang and Sylvester (19)	69	2.55
Vatankhah and Kouchakzadeh (28a)	77	2.85
Barr (17)	80	2.96
Chen (15)	91	3.37
Wood (6)	98	3.62
Buzzelli (32)	104	3.85
Churchil (12)	106	3.92
Serghides (22)	107	3.96
Romeo, Royo and Monzon (27)	125	4.62
Serghides (21)	144	5.33

^anumber of estimated algebraic notation calculator key strokes required to solve the approximation, i.e. to find value of friction factor λ (estimated, i.e. average number of strokes)

^bComplexity index is defined as quotient of key strokes required for an observed approximation and the least complex one; here Eck (8)

^cthe least complex approximation presented here

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