



HAL
open science

Comparison of the Lambert W -function based solutions to the Colebrook equation

Dejan Brkić

► **To cite this version:**

Dejan Brkić. Comparison of the Lambert W -function based solutions to the Colebrook equation. Engineering Computations, 2012, 29 (6), pp.617-630. 10.1108/02644401211246337 . hal-01586546

HAL Id: hal-01586546

<https://hal.science/hal-01586546>

Submitted on 18 Sep 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Comparison of the Lambert W-Function Based Solutions to the Colebrook Equation

Journal:	<i>Engineering Computations</i>
Manuscript ID:	EC-Dec-2010-0166.R2
Manuscript Type:	Research Article
Keywords:	Fluid flow, Friction factor, Turbulent flow, Hydraulic resistances, Colebrook equation, Lambert W-function
Author: Dejan Brkić	

SCHOLARONE™
Manuscripts

review

Comparison of the Lambert W-Function Based Solutions to the Colebrook Equation

Structured abstract:

Purpose: The Colebrook equation for determination of hydraulic resistances is implicit in fluid flow friction factor and hence it has to be approximately solved using iterative procedure or using some of the approximate explicit formulas developed by many authors.

Design/methodology/approach: Alternate mathematically equivalents to the implicit Colebrook equation in explicit form with no approximation involved actually exist.

Findings: These alternate equations were developed using Lambert W-function. The paper compares various implementations of the Lambert W methodology and shows that some of these are less able than others to yield solutions using modern computer hardware. This is because the functions require the evaluation of terms with numerical values outside the ranges that can be expressed on most computers.

Research limitations/implications: Some of existed transformations cannot be applied for high values of relative roughness of inner pipe surface and the Reynolds number. This limitation applied only for computer computations. Other presented transformations do not sufferer of this limitation.

Practical implications: Presented procedures can be easily implemented in a computer code. Recommended solution can be used in all cases that can occur in engineering practice.

Originality/value: Here will be shown some possible practical procedures for solution to the transformed Colebrook equation. Accuracy analysis and comparisons of presented formulas are also performed. Recommendation for use is shown.

Keywords: Fluid flow; Friction factor; Turbulent flow; Hydraulic resistances; Colebrook equation; Lambert W-function

1. Introduction

The empirical Colebrook equation is valid for turbulent regime in smooth and rough pipes with special attention on transitional regime between them (1).

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log\left(\frac{2.51}{\text{Re} \cdot \sqrt{\lambda}} + \frac{\varepsilon}{3.71 \cdot D}\right) \quad (1)$$

The Colebrook equation has two parts, i.e. turbulent smooth and turbulent rough part which can be noted separately. The Colebrook equation unites these two parts in one coherent equation but also Colebrook was not attempting to find some sort of approximation to a sum of the smooth and rough expressions. Actually, Colebrook was seeking a combined equation that is asymptotic to the smooth and rough expressions. This new coherent equation covers transitional zone between smooth and rough regime. First addend in bracket of previous equation is implicitly given and it represents smooth regime (2).

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log\left(\frac{2.51}{\text{Re} \cdot \sqrt{\lambda}}\right) = 2 \cdot \log(\text{Re} \cdot \sqrt{\lambda}) - 0.8 \quad (2)$$

Similarly the rough portion of turbulence is represented by second addend of the Colebrook equation (3).

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log\left(\frac{\varepsilon}{3.71 \cdot D}\right) = 1.74 - 2 \cdot \log\left(\frac{2 \cdot \varepsilon}{D}\right) = 1.14 - 2 \cdot \log\left(\frac{\varepsilon}{D}\right) \quad (3)$$

Mathematically, shown unification is incorrect i.e. $\log(A+B) \neq \log(A) + \log(B)$, but physically this approach gives good results. Problem can be treated as inverse; according to the logarithmic

1
2
3 rules equally is incorrect to split the Colebrook relation into two pieces. How well Colebrook's
4
5 equation smoothing contact between the turbulent smooth and the turbulent rough regime can be
6
7
8 best seen in the graphical interpretation (Figure 1).
9

10
11
12 **Figure 1.** Colebrook relation make transitional curve between hydraulically smooth regime and
13
14 turbulent rough regime
15

16
17
18 Today, the Colebrook equation for calculation of hydraulic flow friction factor is almost
19
20 worldwide accepted as a standard. It was developed in 1930's by Colebrook (1939) who used
21
22 data from the experiment conducted by his colleague, Professor White and himself (Colebrook
23
24 and White 1937). Later in 1940's, Rouse and Moody used Colebrook's equation for their famous
25
26 charts (Moody 1944). Since then, the Colebrook equation has been historically adopted as
27
28 sufficiently accurate for engineering applications although it is often incorrect. Even Moody
29
30 (1944) stated the accuracy of his diagram is about $\pm 5\%$ for smooth pipes and $\pm 10\%$ for rough
31
32 pipes. In spite of that, the Colebrook equation is replaced very slowly by new equations such as
33
34 those developed using the data from the Princeton super-pipe or the Oregon facility (Cordero
35
36 2008). Actually, many possible equations can be found that are asymptotic to the smooth and
37
38 rough expressions. Colebrook chose one and compared it with limited experimental data
39
40 available to him. Since different types of roughness cause different forms of transition from
41
42 rough to smooth, it is clear that the Colebrook equation should not be regarded as special. It was
43
44 the first and that it is now widely used even though hardly anybody seems to acknowledge its
45
46 considerable limitations.
47
48
49
50
51
52
53
54
55
56
57
58
59
60

1
2
3 Alternate mathematically equivalents to the original expression of the Colebrook equation in
4 explicit form with no approximation involved actually exist (More 2006, Sonnad and Goudar
5 2007, Clamond 2009, Brkić 2011a). These alternate equations were developed using the Lambert
6 W-function. Some of the existed transformations cannot be applied for high values of relative
7 roughness of inner pipe surface and the Reynolds number (Sonnad and Goudar 2004). This
8 limitation is applied only for computer computations. Other presented transformations do not
9 sufferer of such limitation as will be shown in this paper.
10
11
12
13
14
15
16
17
18
19
20
21

22 **2. Derivate of explicit Lambert W-function based Colebrook equation**

23
24 The Lambert W-function is define by relation $W(x) \cdot e^{W(x)} = x$, where W is the Lambert function
25 and x is its argument (Corless et al 1996, Hayes 2005). For the real values of the argument x, the
26 W-function has two branches, W_0 (the principal branch) and W_{-1} (the negative branch). Only
27 upper branch W_0 is used for the transformations of Colebrook's equation in explicit form (Figure
28 2).
29
30
31
32
33
34
35
36
37
38
39

40 **Figure 2.** Detailed view of the Lambert W-function
41
42

43 Goudar and Sonnad (2003) show the explicit friction factor correlation for turbulent flow in
44 smooth pipes which was developed using the Lambert W-function and the smooth part of the
45 Colebrook equation. Using this approach (Brkić 2011a), Colebrook's equation can be also
46 expressed in the explicit form where some transformation of its smooth part should be performed
47 as follows (4).
48
49
50
51
52
53
54
55
56
57
58
59
60

$$\left. \begin{aligned}
 \frac{1}{\sqrt{\lambda}} &= -2 \cdot \log\left(\frac{2.51}{\text{Re} \cdot \sqrt{\lambda}}\right) = -a \cdot \ln\left(\frac{2.51}{\text{Re} \cdot \sqrt{\lambda}}\right) = -a \cdot \ln\left(\frac{2.51}{\text{Re}}\right) - a \cdot \ln\left(\frac{1}{\sqrt{\lambda}}\right) \\
 a &= \frac{2}{\ln(10)} \Rightarrow \Phi = \frac{1}{a \cdot \sqrt{\lambda}} \Rightarrow a \cdot \Phi + a \cdot \ln(a \cdot \Phi) = a \cdot \ln\left(\frac{\text{Re}}{2.51}\right) \Rightarrow \Phi + \ln(\Phi) = \ln\left(\frac{\text{Re}}{2.51 \cdot a}\right) \\
 W(x) + \ln[W(x)] &= \ln(x) \Rightarrow \Phi = W\left[\frac{\text{Re}}{2.51 \cdot a}\right] \Rightarrow \frac{1}{\sqrt{\lambda}} = a \cdot W\left[\frac{\text{Re}}{2.51 \cdot a}\right] \Rightarrow \frac{1}{\sqrt{\lambda}} = \frac{2}{\ln(10)} W\left[\frac{\text{Re} \cdot \ln(10)}{2 \cdot 2.51}\right] \\
 \frac{1}{\sqrt{\lambda}} &= 2 \cdot \log\left(\frac{\text{Re} \cdot \sqrt{\lambda}}{2.51}\right) = -2 \cdot \log\left(10^{\frac{-1}{\ln(10)} W\left[\frac{\text{Re} \cdot \ln(10)}{5.02}\right]}\right) \Rightarrow \frac{\text{Re} \cdot \ln(10)}{2 \cdot 2.51} = 10^{-2 \cdot \log\left(\frac{2.51}{\text{Re} \cdot \sqrt{\lambda}}\right)}
 \end{aligned} \right\} \quad (4)$$

After that implicit expression in the Colebrook equation can be replaced with the Lambert W-function while its second so called “rough” addend remains untouched (5).

$$\left. \begin{aligned}
 \frac{1}{\sqrt{\lambda}} &= -2 \cdot \log\left(\frac{5.02 \cdot W\left[\frac{\text{Re} \cdot \ln(10)}{5.02}\right] + \frac{\varepsilon}{3.71 \cdot D}}{\text{Re} \cdot \ln(10)}\right) \\
 x &= \frac{\text{Re} \cdot \ln(10)}{5.02}
 \end{aligned} \right\} \quad (5)$$

Set of equations (5) was developed using the original idea of Colebrook to bridge the gap smoothing the line between hydraulically “smooth” turbulent flow and turbulent flow in rough pipes as shown before in figure 1. No constraints in ranges for values of the Reynolds number and relative roughness in comparisons to the original Colebrook relation do not exist (as will be shown in this paper). But, beside presented reformulations of the Colebrook equation, in the papers of More (2006), Sonnad and Goudar (2007) and Nandakumar (2007), another possible transformation of the Colebrook equation using the Lambert-W function is shown. But relation (6) shown in these papers has limitation in applicability for high values of the Reynolds number and relative roughness as shown in Sonnad and Goudar (2004) because today available computers cannot operate with the extremely large numbers.

$$\left. \begin{aligned} \frac{1}{\sqrt{\lambda}} &= \frac{2}{\ln(10)} \cdot W \left[x \cdot e^{\frac{x \cdot \varepsilon}{3.71 \cdot D}} \right] - \frac{\varepsilon}{3.71 \cdot D} \cdot \frac{\text{Re}}{2.51} \\ x &= \frac{\text{Re} \cdot \ln(10)}{5.02} \\ x_1 &= x \cdot e^{\frac{x \cdot \varepsilon}{3.71 \cdot D}} \end{aligned} \right\} \quad (6)$$

In previous equation (6) argument of the Lambert W-function has changed. It is now noted as x_1 where this new argument contains the old argument x from the set of equations (5). However, the numerical solution of eq. (6) using computers is not possible for all values of relative roughness and the Reynolds number and their combinations which can be occurred in engineering practice. Problem is in the exponential term in the argument x_1 of the W-function which increases rapidly. In fact, infinite combinations of relative roughness and the Reynolds number values exist for which the argument x_1 of the W-function becomes too large even for today very powerful computers. Argument of the Lambert W-function has to be smaller than $1.79 \cdot 10^{308}$ because registers in memory of computers have limited capacity (Sonnad and Goudar 2004).

3. Solutions of alternate equations, accuracy requirements and error distribution

The main aim of this paper is not to investigate accuracy or computational burden (i.e. speed) of presented procedures for solution of the Lambert W based Colebrook equation. Accuracy of alternate representations of the Colebrook equation depends only on accuracy of used approximate solution of the Lambert W-function. Besides the relative simplicity of an explicit form, presented equations allow highly accurate estimation of friction factor as the Lambert W-function can be evaluated accurately. But, the main purpose of this paper is to compare different Lambert W based solutions of the Colebrook equation and to make comparisons among them. Or

better to say, main purpose is to identify some constraints in applicability of certain solutions whereas other solutions do not suffer of such limitations.

Clamond (2009) provides MATLAB and FORTRAN codes for the Colebrook relation expressed in term of the Lambert W-function. Keady (1998) also provides Maple interpretation of here presented problem.

Since the Lambert W is transcendental function, its formal solution can be expressed only in endless form (7). Similar as using iterative solution, accuracy of formal solution depends on number of terms used to form “tower” in eq. (7).

$$W(x) = \ln \frac{x}{\ln \left(\frac{x}{\ln \left(\frac{x}{\ln(\cdot)} \right)} \right)} \quad (7)$$

After Boyd (1998), instead $W=W(x)$ it is convenient to define a new function $\omega=\omega(y)$ as auxiliary tool (8).

$$\left. \begin{aligned} \omega &= W + 1 \\ y &= 1 + x \cdot e^1 \approx 1 + 2.71 \cdot x \\ \xi &= \frac{\sqrt{2 \cdot y}}{\ln(10) - \ln(\ln(10))} \\ \tanh(\xi) &= (e^\xi - e^{-\xi}) / (e^\xi + e^{-\xi}) \\ \omega_0 &= \{ \ln(y + 10) - \ln(\ln(y + 10)) \} \cdot \tanh(\xi) \\ \Omega &= \frac{\left(\ln(y) - \frac{7}{5} \right) \cdot e^{\frac{3}{40} \left(\ln(y) - \frac{7}{5} \right)^2}}{10} \\ \bar{\omega}_0 &= \omega_0 \cdot (1 + \Omega) \\ \omega_{i+1} &= \omega_i - \frac{(\omega_i - 1) - e^{-\omega_i \cdot (y-1)}}{\omega_i} \end{aligned} \right\} \quad (8)$$

Barry et al (2000) proposed approximate solution for the upper branch of the Lambert W-function which is valid for solution of here presented problem (9). Also, improved version eq. (9) can be seen in Barry et al (2000).

$$W_0^+(x) \approx \ln \left(\frac{6}{5} \cdot \frac{x}{\ln \left[\left(\frac{12}{5} \right) \left(\frac{x}{\ln \left(1 + \frac{12 \cdot x}{5} \right)} \right) \right]} \right) \quad (9)$$

Winitzki (2003) also proposed global approximation to the Lambert W-function (10).

$$W(x) \approx \ln(1+x) \cdot \left(1 - \frac{\ln(1+\ln(1+x))}{2+\ln(1+x)} \right) \quad (10)$$

Two numerical examples are solved in tables 1 and 2 using formulas shown in this section of the text. These two examples are marked also in figure 1 and 3 to 9. The implicit Colebrook equation today can be easily solved using some kind of spreadsheet solvers (iterative solution). Here will be used as standard, solution calculated using MS Excel ver. 2007. Since iterative solution after enough number of iterations can be accepted as accurate, it will be used for comparisons in examples 1 and 2. Approach with set of equations (5) will be used in table 1 while approach with set of equations (6) will be used in table 2.

Table 1. Solutions of the explicit, Lambert W-function based Colebrook equation (5)

Limitation of formulas presented by set of equations (6) for high values of the Reynolds number and relative roughness is shown in table 2.

1
2
3
4
5
6 **Table 2.** Solutions of the explicit, Lambert W-function based Colebrook equation (6)
7
8
9

10 Main difference between the calculation in table 1 and table 2 is in the argument of the Lambert
11 W-function x and x_1 , respectively. Argument x is never too large for computer register while in
12 other approach, argument x_1 is. Of course, argument x is not too large only if the Reynolds
13 number and relative roughness are in the range which occurs in engineering practice.
14
15
16
17
18
19

20
21 Error distribution for solution of equations (5) and (6) using procedure by Boyd (1998) is shown
22 in figures 3 and 4, respectively. Similar for the procedure proposed by Barry et al (2000) and
23 Winitzki (2003) error distribution can be seen in figures 5 and 6, and 7 and 8.
24
25
26
27
28
29
30
31

32 **Figure 3.** Distribution of error of solution of explicit Lambert W-function based Colebrook
33 equation (5) by Boyd (1998)
34
35

36 **Figure 4.** Distribution of error of solution of explicit Lambert W-function based Colebrook
37 equation (6) by Boyd (1998)
38
39

40 **Figure 5.** Distribution of error of solution of explicit Lambert W-function based Colebrook
41 equation (5) by Barry et al (2000)
42
43
44

45 **Figure 6.** Distribution of error of solution of explicit Lambert W-function based Colebrook
46 equation (6) by Barry et al (2000)
47
48

49 **Figure 7.** Distribution of error of solution of explicit Lambert W-function based Colebrook
50 equation (5) by Winitzki (2003)
51
52
53
54
55
56
57
58
59
60

Figure 8. Distribution of error of solution of explicit Lambert W-function based Colebrook equation (6) by Winitzki (2003)

The distribution of error shown in figures 3, 5 and 7 are for the set of equations (5). Distribution of error is sensitive and therefore similar set of equations (11) produce different distribution of error (Figure 9) but calculation is still possible for all practical range of the Reynolds number and relative roughness.

$$\left. \begin{aligned} \frac{1}{\sqrt{\lambda}} &= -2 \cdot \log \left(10^{\frac{-1}{\ln(10)} W \left[\frac{\text{Re} \cdot \ln(10)}{5.02} \right]} + \frac{\varepsilon}{3.71 \cdot D} \right) \\ x &= \frac{\text{Re} \cdot \ln(10)}{5.02} \end{aligned} \right\} \quad (11)$$

Figure 9. Distribution of error of solution of explicit Lambert W-function based Colebrook equation (11) by Barry et al (2000)

4. Conclusion

The Colebrook equation can be expressed in explicit form using the Lambert W-function. Related approximate solutions of the Lambert W-function are shown in this paper. Accuracy of some available Lambert W based forms of the Colebrook equation from literature is investigated (More 2006, Sonnad and Goudar 2007, Nandakumar 2007, Brkić 2011a). Also, related approximate solutions of the Lambert W-function are shown. Some already known limitation in use of some of available formulas is confirmed (Sonnad and Goudar 2004). Other available formulas do not suffer of these limitations for practical use (Brkić 2011a). The maximal error of these new formulas compared with iterative solution of Colebrook equation is up to 2% or

1
2
3 slightly above (Figures 3, 5 and 7). Error distribution of new formulas (set of equations 5 and 11)
4
5 is stable or better to say, distribution of error does not change with change of procedure for
6
7 solution of the Lambert W-function. Transformation of the Colebrook equation based on the
8
9 Lambert W-function by More (2006), Sonnad and Goudar (2007), Nandakumar (2007) cannot be
10
11 used for all practical combination of the Reynolds number and relative roughness (Figures 4, 6,
12
13 8). Reason is that typical IEEE-compliant machines cannot operate with numbers larger than
14
15 about 10^{308} (Figure 10).
16
17
18
19
20
21

22 **Figure 10.** Constraints for using of a possible Lambert W function-based explicit Colebrook
23
24 equation
25
26
27
28

29 Review of available explicit approximations to the Colebrook equation can be seen in Brkić
30
31 (2011b). It has to be said that use of an explicit approximation do not necessary mean that the
32
33 computational burden is reduced in comparison to iterative approach (Giustolisi et al 2011).
34
35 Actually, main purpose of this paper is to compare the different formulations of Lambert W
36
37 based Colebrook equation available in literature (More 2006, Sonnad and Goudar 2007,
38
39 Nandakumar 2007, Brkić 2011a). The paper compares various implementations of the Lambert
40
41 W methodology and shows that some of these are less able than others to yield solutions using
42
43 modern computer hardware. This is because the functions require the evaluation of terms with
44
45 numerical values outside the ranges that can be expressed on most computers. Approach by
46
47 Brkić (2011a) does not suffer of that constraints and hence it can be recommended since it can be
48
49 use for the whole range of the Reynolds number and relative roughness that can be occurred in
50
51 engineering practice. Approach by More (2006), Sonnad and Goudar (2007) and Nandakumar
52
53
54
55
56
57
58
59
60

(2007) suffers of certain limitations as it is explained in this paper as well as in Sonnad and Goudar (2004). Decision whether someone will use the Lambert W based solution proposed by Brkić (2011a), iterative solution or some of explicit approximate formulas presented in Brkić (2011b), requires deeper analysis.

Nomenclature

x, x_1 -arguments of the Lambert W-function

W-in this text referred to the positive branch of Lambert function

ω -shifted, auxiliary function proposed by Boyd (1998)

y - argument of the shifted, auxiliary function ω

D-inner pipe diameter (m)

Re-Reynolds number (dimensionless)

ε -roughness of the inner surface of pipe (m)

ε/D -relative roughness of the inner surface of pipe (dimensionless)

λ -Darcy, Darcy-Weisbach or Moody friction factor (dimensionless)

a, ξ, Φ, Ω -auxiliary term defined in the text

i -counter

References

Barry, D.A., Parlange, J.Y., Li, L., Prommer, H., Cunningham, C.J. and Stagnitti, F. (2000), "Analytical approximations for real values of the Lambert W-function", *Mathematics and Computers in Simulation*, Vol. 53 No. 1-2, pp. 95-103.

- 1
2
3 Boyd, J.P. (1998), "Global approximations to the principal real-valued branch of the Lambert W-
4 function", *Applied Mathematics Letters*, Vol. 11 No. 6, pp. 27-31.
5
6
7
8 Brkić, D. (2011a), "W solutions of the CW equation for flow friction", *Applied Mathematics*
9
10 *Letters*, Vol. 24 No. 8, pp. 1379-1383.
11
12 Brkić, D. (2011b), "Review of explicit approximations to the Colebrook relation for flow
13 friction", *Journal of Petroleum Science and Engineering*, Vol. 77 No. 1, pp. 34-48.
14
15 Clamond, D. (2009), "Efficient resolution of the Colebrook equation", *Industrial and*
16
17 *Engineering Chemistry Research*, Vol. 48 No. 7, pp. 3665–3671.
18
19
20 Colebrook, C.F. (1939), "Turbulent flow in pipes with particular reference to the transition
21 region between the smooth and rough pipe laws", *Journal of the Institution of Civil Engineers.*
22
23 *(London)*, Vol. 11 No 4, pp. 133-156.
24
25
26 Colebrook, C.F. and White, C.M. (1937), "Experiments with fluid friction in roughened pipes",
27
28 *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol.
29
30 161, No. 906, pp. 367-381.
31
32
33
34
35 Cordero, G.O. (2008), "An improved experimental correlation for Darcy friction factor",
36
37 *Hydrocarbon Processing*, Vol. 87 No. 7, pp. 97-99.
38
39
40 Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J. and Knuth, D.E. (1996), "On the
41 Lambert W function", *Advanced in Computational Mathematics*, Vol. 5 No. 1, pp. 329–359.
42
43
44 Goudar, C.T. and Sonnad, J.R. (2003), "Explicit friction factor correlation for turbulent flow in
45 smooth pipes", *Industrial and Engineering Chemistry Research*, Vol. 42 No. 12, pp. 2878-2880.
46
47
48 Giustolisi, O., Berardi, L. and Walski, T.M. (2011), "Some explicit formulations of Colebrook–
49 White friction factor considering accuracy vs. computational speed", *Journal of*
50
51 *Hydroinformatics*, Vol. 13 No. 3, pp. 401-418.
52
53
54
55
56
57
58
59
60

1
2
3 Hayes, B. (2005), “Why W?”, *American Scientist*, Vol. 93, No. 2, pp. 104-108.

4
5 Keady, G. (1998), “Colebrook-White formulas for pipe flow”, *Journal of Hydraulic Engineering*
6
7
8
9 *ASCE*, Vol. 124 No. 1, pp. 96-97.

10
11 Moody, L.F. (1944), “Friction factors for pipe flow”, *Transactions of the ASME*, Vol. 66 No. 8,
12
13 pp. 671-684.

14
15 More, A.A. (2006), “Analytical solutions for the Colebrook and White equation and for pressure
16
17 drop in ideal gas flow in pipes”, *Chemical Engineering Science*, Vol. 61 No. 16, pp. 5515-5519.

18
19 Nandakumar, V. (2007), “Analytical calculation of fanning friction factor”, *Hydrocarbon*
20
21
22
23 *Processing*, Vol. 86 No. 1, pp. 97-100.

24
25 Sonnad, J.R. and Goudar, C.T. (2004), “Constraints for using Lambert W Function-based
26
27 explicit Colebrook-White equation”, *Journal of Hydraulic Engineering ASCE*, Vol. 130 No. 9,
28
29 pp. 929-931.

30
31 Sonnad, J.R. and Goudar, C.T. (2007), “Explicit reformulation of the Colebrook–White equation
32
33 for turbulent flow friction factor calculation”, *Industrial and Engineering Chemistry Research*,
34
35
36
37 Vol. 46 No. 8, pp. 2593–2600.

38
39 Winitzki, S. (2003). Uniform approximations for transcendental functions. In: Kumar, V.,
40
41 Gavrilova, M.L., Jeng, C., Tan, K., L'Ecuyer, P. (Eds.), *Lecture Notes in Computer Science*.
42
43 *Computational Science and Its Applications — ICCSA 2003, International Conference*,
44
45
46
47 Montreal, Canada, May 18–21, 2003 Proceedings, Part 1, 2667. Springer, Berlin/Heidelberg, pp.
48
49 780–789.

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60

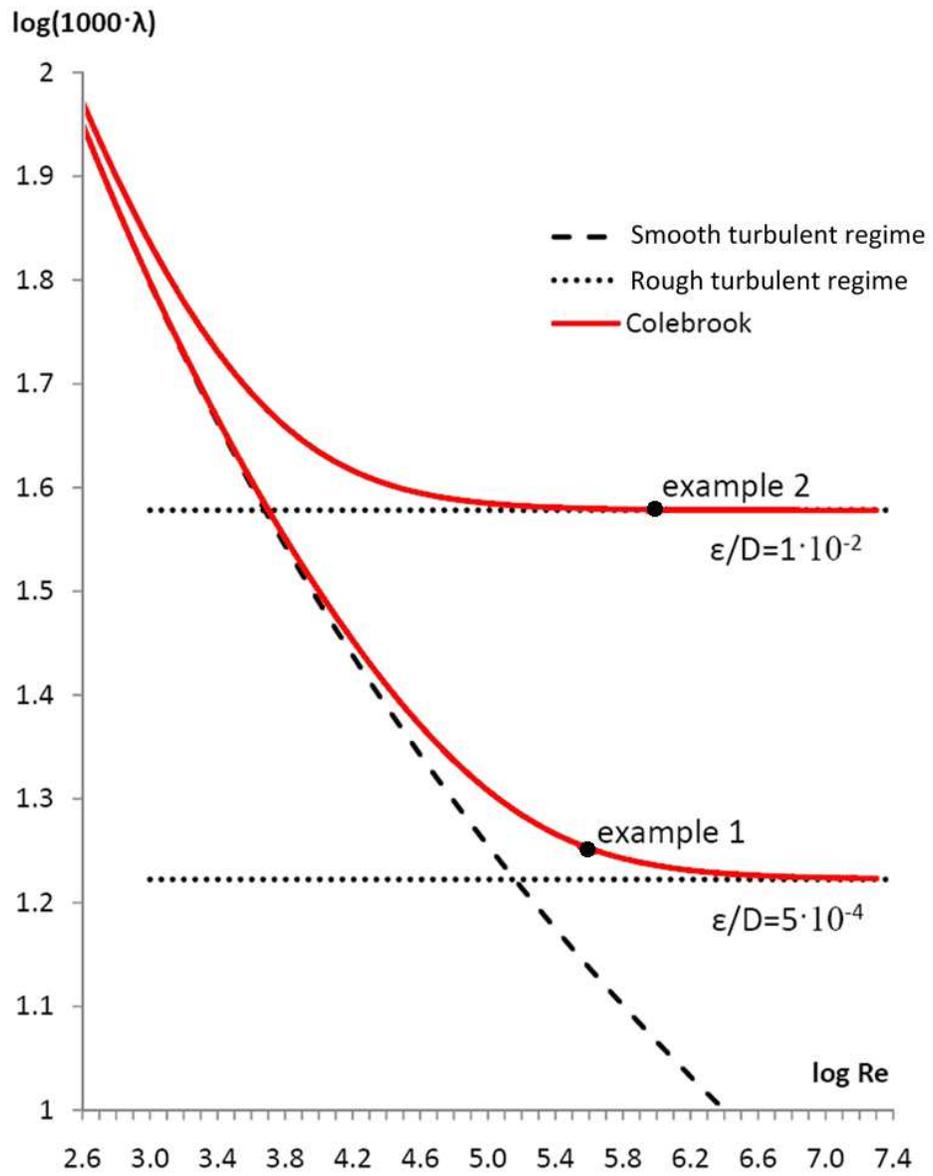


Figure 1. Colebrook relation make transitional curve between hydraulically smooth regime and turbulent rough regime
65x82mm (300 x 300 DPI)

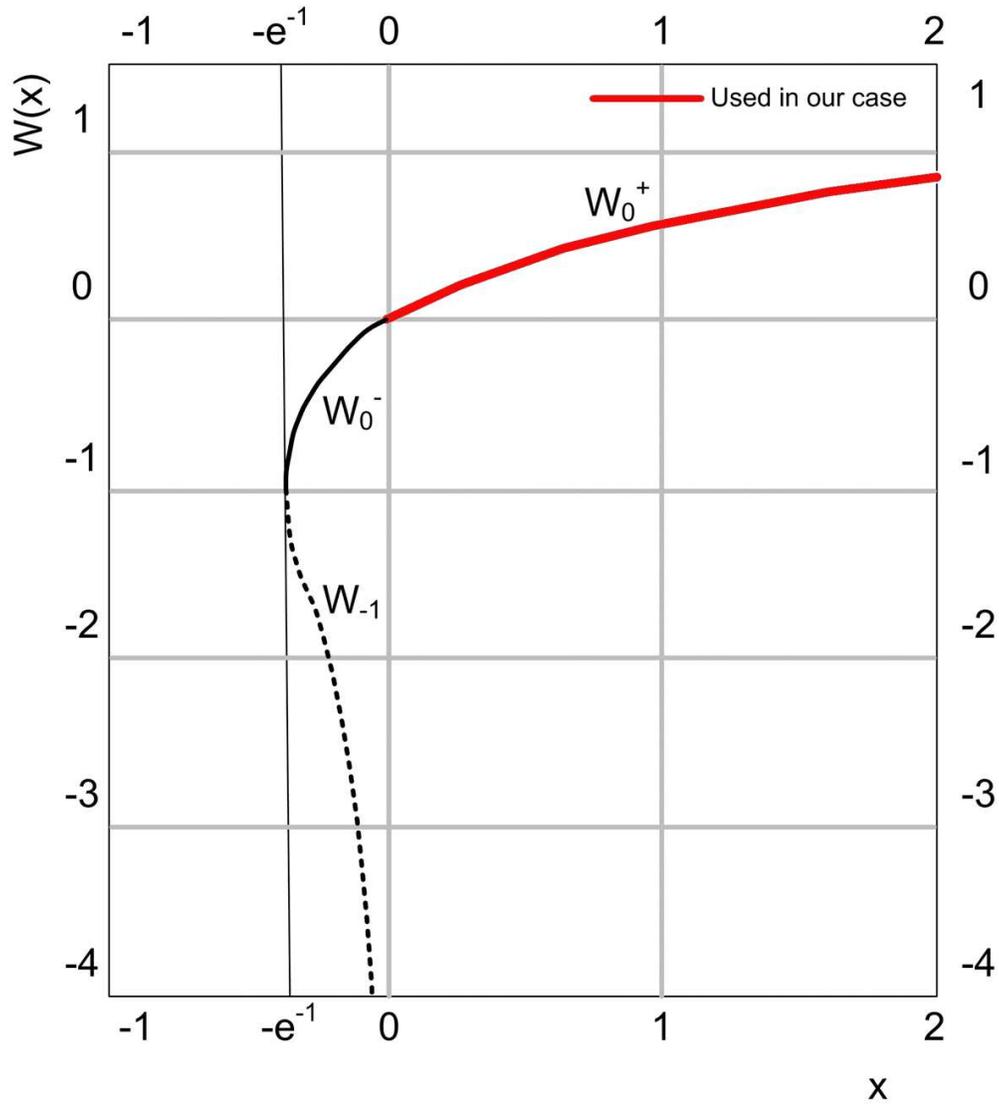


Figure 2. Detailed view of the Lambert W-function
99x111mm (300 x 300 DPI)

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60

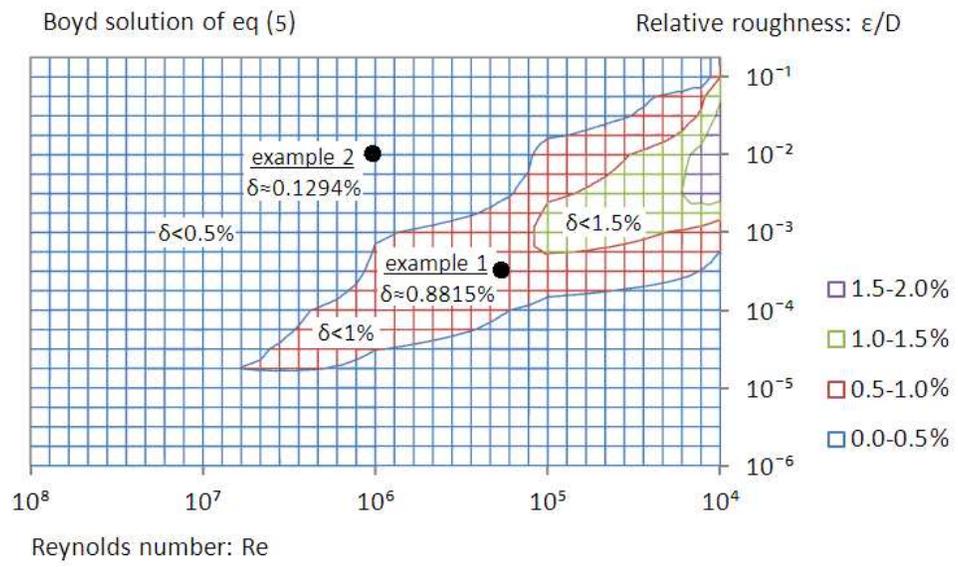


Figure 3. Distribution of error of solution of explicit Lambert W-function based Colebrook equation (5) by Boyd (1998)
65x37mm (300 x 300 DPI)

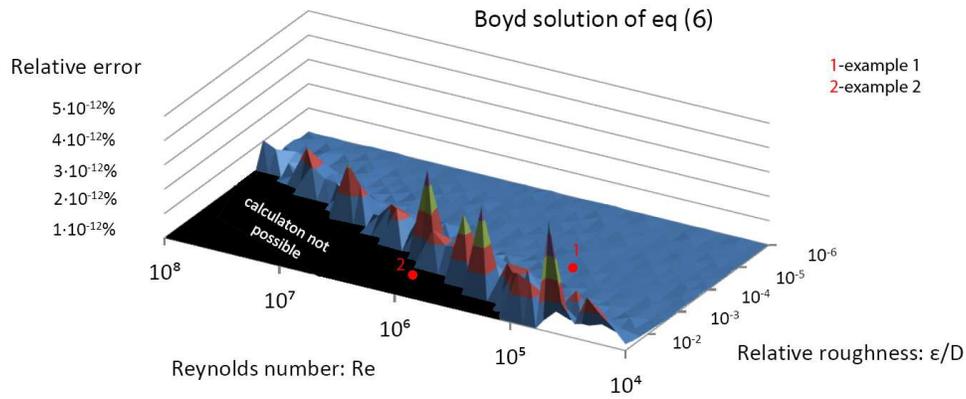


Figure 4. Distribution of error of solution of explicit Lambert W-function based Colebrook equation (6) by Boyd (1998)
118x50mm (300 x 300 DPI)

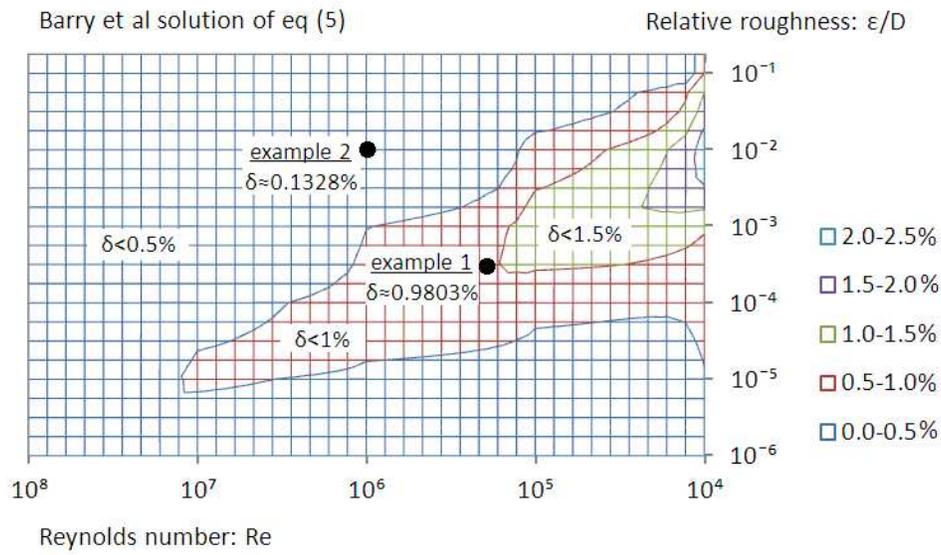


Figure 5. Distribution of error of solution of explicit Lambert W-function based Colebrook equation (5) by Barry et al (2000)
66x38mm (300 x 300 DPI)

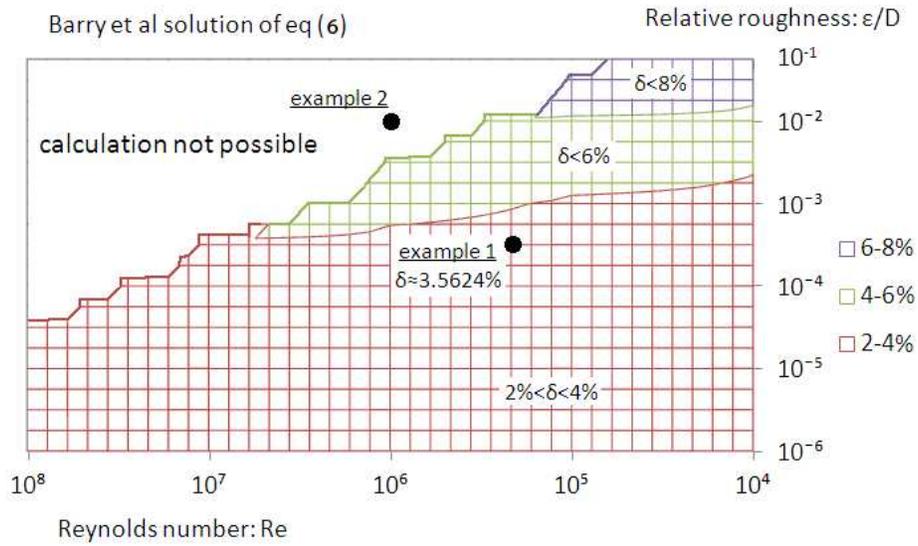


Figure 6. Distribution of error of solution of explicit Lambert W-function based Colebrook equation (6) by Barry et al (2000)
63x36mm (300 x 300 DPI)

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60

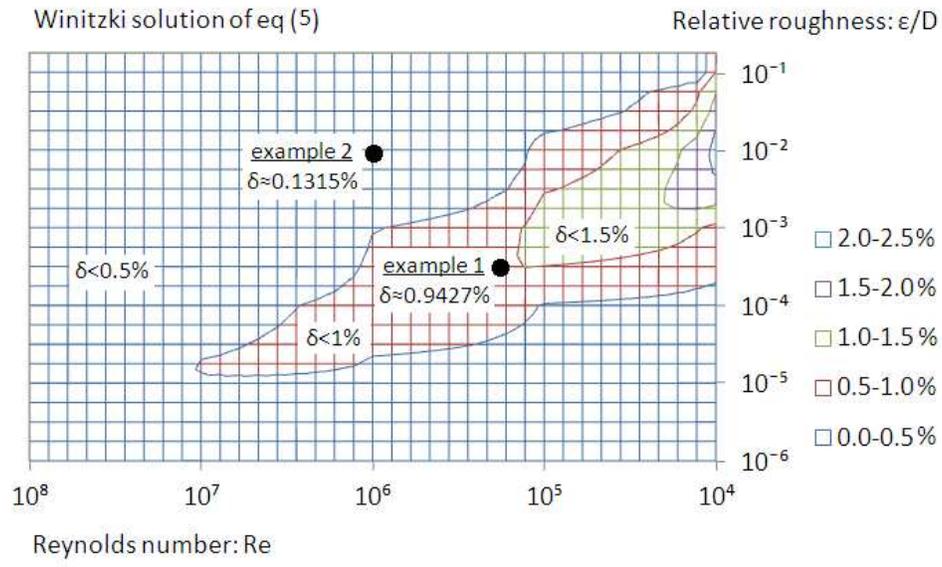


Figure 7. Distribution of error of solution of explicit Lambert W-function based Colebrook equation (5) by Winitzki (2003)
61x36mm (300 x 300 DPI)

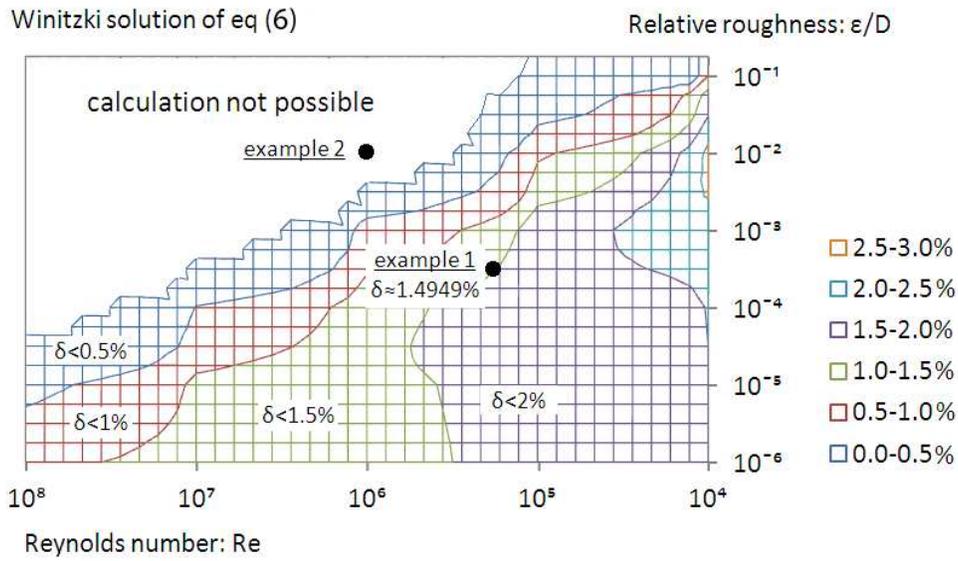


Figure 8. Distribution of error of solution of explicit Lambert W-function based Colebrook equation (6) by Winitzki (2003)
69x40mm (300 x 300 DPI)

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60

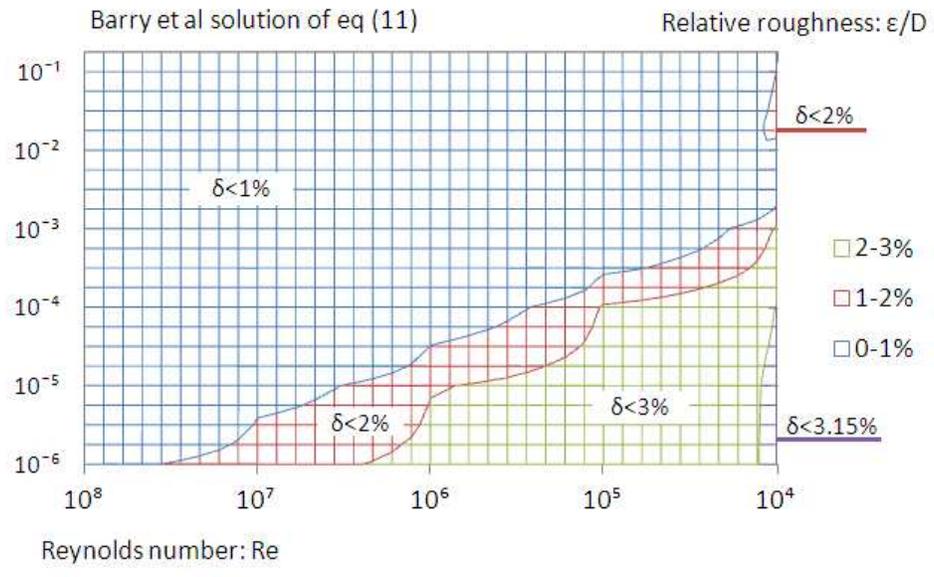


Figure 9. Distribution of error of solution of explicit Lambert W-function based Colebrook equation (11) by Barry et al (2000) 56x33mm (300 x 300 DPI)

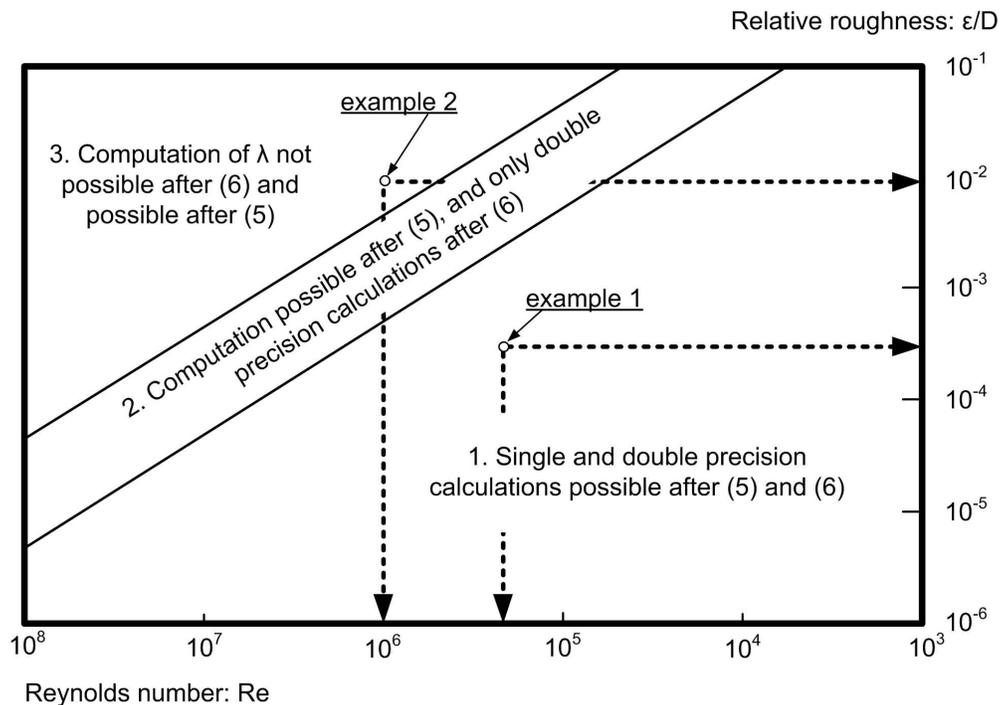


Figure 10. Constraints for using of a possible Lambert W function-based explicit Colebrook equation
166x118mm (300 x 300 DPI)

Review

Table 1. Solutions of the explicit, Lambert W-function based Colebrook equation (5)

	example 1	example 2
	Re=316227, $\varepsilon/D=5 \cdot 10^{-4}$	Re=10 ⁶ , $\varepsilon/D=10^{-2}$
Eq. (7)		
x	145047.72	458682.29
W	9.620883753	10.66879129
$\lambda^{-0.5}$ after eq. (5)	7.393176451	5.13128464
λ	0.018295229	0.037979373
Relative error $\delta\%$	0.8815%	0.1294%
Set of eqs. (8)		
x	145047.72	458682.29
y	394281.59	1246828.73
ω_0	10.32879411	11.39448775
Ω	0.000065130	0.0000088393
$\bar{\omega}_0$	10.32946683	11.39458847
ω_1	10.67279093	11.7140380025
ω_2	10.62232833	11.6698835813
ω_3	10.62088328	11.6687910258
ω_4	10.62088214	11.6687903773
ω_5	10.62088214	11.6687903773
W	9.62088214	10.6687903773
$\lambda^{-0.5}$ after eq. (5)	7.39317599	5.131284633
λ	0.018295232	0.037979373
Relative error $\delta\%$	0.8815%	0.1294%
Eq. (9)		
x	145047.72	458682.29
$W_0^+(x)$	9.743385337	10.79550445
$\lambda^{-0.5}$ after eq. (5)	7.389556742	5.131196922
λ	0.018313157	0.037980671
Relative error $\delta\%$	0.9803%	0.1328%
Eq. (10)		
x	145047.72	458682.29
W(x)	9.696953700	10.74585341
$\lambda^{-0.5}$ after eq. (5)	7.390934613	5.131231501
λ	0.018306330	0.037980159
Relative error $\delta\%$	0.9427%	0.1315%
Iterative solution of eq. (1)		
$\lambda^{-0.5}$	7.425691007	5.134604649
λ_0	0.018135362	0.037930274
Relative error $\delta\%$	-	-
$\delta\% = 100 \cdot \lambda_0 - \lambda / \lambda_0$		

Table 2. Solutions of the explicit, Lambert W-function based Colebrook equation (6)

	example 1	example 2
	Re=316227, $\varepsilon/D=5 \cdot 10^{-4}$	Re=10 ⁶ , $\varepsilon/D=10^{-2}$
Eq. (7)		
x_1	44791026258619.00	
W	28.09735354	
$\lambda^{-0.5}$ after eq. (6)	7.425691011	Computer calculation not possible
λ	0.018135363	
Relative error $\delta\%$	$6.39 \cdot 10^{-8}\%$	
Set of eqs. (8)		
x_1	44791026258619.00	
y	121754632756837.00	
ω_0	28.95385161	
Ω	$1.3882775 \cdot 10^{-31}$	
$\bar{\omega}_0$	28.95385161	
ω_1	29.1085521825	
ω_2	29.0974181464	Computer calculation not possible
ω_3	29.0973535390	
ω_4	29.0973535368	
ω_5	29.0973535368	
W	28.0973535368	
$\lambda^{-0.5}$ after eq. (6)	7.42569100698	
λ	0.01813536298	
Relative error $\delta\%$	$4.63 \cdot 10^{-8}\%$	
Eq. (9)		
x_1	44791026258619.00	
$W_0^+(x)$	28.253824116	
$\lambda^{-0.5}$ after eq. (6)	7.5615996248	Computer calculation not possible
λ	0.0174893086	
Relative error	3.5624%	
Eq. (10)		
x_1	44791026258619.00	
W(x)	28.161979769	
$\lambda^{-0.5}$ after eq. (6)	7.4818246387	Computer calculation not possible
λ	0.0178642566	
Relative error $\delta\%$	0.4949%	

$\delta\% = 100 \cdot |\lambda_0 - \lambda| / \lambda_0$ where λ_0 is iterative solution of eq. (1) from Table 1