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▶ **To cite this version:**

Caterina Calgaro, Meriem Ezzoug, Ezzeddine Zahrouni. Stability and convergence of an hybrid finite volume-finite element method for a multiphasic incompressible fluid model. Communications on Pure and Applied Analysis, 2018, 17 (2), pp.429-448. 10.3934/cpaa.2018024 . hal-01586201

HAL Id: hal-01586201
https://hal.science/hal-01586201

Submitted on 12 Sep 2017

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Stability and convergence of an hybrid finite volume-finite element method for a multiphasic incompressible fluid model

September 12, 2017

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Key Words: Kazhikhov-Smagulov model, Finite Volume method, Finite Element method, stability, convergence.

Abstract

In this paper, we construct a fully discrete numerical scheme for approximating a two-dimensional multiphasic incompressible fluid model, also called the Kazhikhov-Smagulov model. We use a first-order time discretization and a splitting in time to allow us the construction of an hybrid scheme which combines a Finite Volume and a Finite Element method. Consequently, at each time step, one only needs to solve two decoupled problems, the first one for the density and the second one for the velocity and pressure. We will prove the stability of the scheme and the convergence towards the global in time weak solution of the model.

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1 Introduction.

1.1 The model.

The Kazhikhov-Smagulov model, which can be deduced from the compressible Navier-Stokes system, describes the motion of a viscous, incompressible mixture of two fluids having different densities. The mixture is subject to a diffusion effect modeled by the Fick's law, which relates the velocity to the derivatives of the density. We assume that the fluid fills the domain $\Omega \subset \mathbb{R}^2$, a bounded open set with sufficiently regular boundary Γ. We denote by $\boldsymbol{n}$ the unit outward normal on the boundary Γ and by $[0,T]$ the time interval, for $T>0$. With the notations $\mathcal{Q}_T = (0,T)\times\Omega$ and $\Sigma = (0,T)\times\Gamma$, we consider the following model in $\mathcal{Q}_T$:

$$\begin{cases} \partial_t \rho + \operatorname{div}\big(\rho \boldsymbol{u}\big) = \lambda \Delta \rho, \\ \rho\big(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big) - \lambda(\nabla\rho\cdot\nabla)\boldsymbol{u} + \lambda \operatorname{div}\big(\rho \nabla \boldsymbol{u}^T\big) - \mu \boldsymbol{\Delta} \boldsymbol{u} + \nabla P = \rho \boldsymbol{g}, \\ \operatorname{div} \boldsymbol{u} = 0. \end{cases} \tag{1}$$

The unknowns are $\rho : \mathcal{Q}_T \to \mathbb{R}$ the density of the fluid, $\boldsymbol{u} : \mathcal{Q}_T \to \mathbb{R}^2$ the incompressible velocity field and $P : \mathcal{Q}_T \to \mathbb{R}$ the pressure of the fluid (a modified pressure). Moreover, $\boldsymbol{g}$ stands for the gravity acceleration (but it can include further external forces) and the parameters $\lambda > 0$ and $\mu > 0$ represent mass diffusion and dynamic viscosity coefficients, respectively (which are assumed to be constant). Given a vector $\boldsymbol{a} \in \mathbb{R}^d$, we set $\operatorname{div} \boldsymbol{a} = \sum_{i=1}^d \partial_{x_i} \boldsymbol{a}_i$; given a matrix valued function A, we denote $\operatorname{div}(A)$ the vector having components $\sum_{j=1}^d \partial_{x_j} A_{ij}$. This model was derived and analyzed for the first time by Kazhikhov and Smagulov [21].

We complete (1) with the boundary conditions:

$$\frac{\partial \rho}{\partial \boldsymbol{n}}(t,\boldsymbol{x}) = 0, \quad \boldsymbol{u}(t,\boldsymbol{x}) = 0, \qquad (t,\boldsymbol{x}) \in \Sigma, \tag{2}$$

and the initial conditions:

$$\rho(0,\boldsymbol{x}) = \rho_0(\boldsymbol{x}), \quad \boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{3}$$

where $\rho_0 : \Omega \to \mathbb{R}$ and $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^2$ are given functions, with $\operatorname{div} \boldsymbol{u}_0 = 0$. Throughout this paper, we assume the hypothesis:

$$0 < m \le \rho_0(\boldsymbol{x}) \le M < +\infty, \quad \boldsymbol{x} \in \Omega. \tag{4}$$

1.2 Functional setup of the Kazhikhov-Smagulov model.

Let us introduce the following functional spaces (see for instance [22], [26] for their properties):

$$\begin{aligned}
\mathcal{V} &= \left\{ \boldsymbol{u} \in \mathcal{D}(\Omega)^2 : \ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega \right\}, \\
\mathbf{V} &= \left\{ \boldsymbol{u} \in \mathbf{H}_0^1(\Omega) : \ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega \right\}, \\
\mathbf{H} &= \left\{ \boldsymbol{u} \in \mathbf{L}^2(\Omega) : \ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \right\}, \\
L_0^2(\Omega) &= \left\{ p \in L^2(\Omega) : \ \int_\Omega p(\boldsymbol{x}) \, d\boldsymbol{x} = 0 \right\}.
\end{aligned}$$

We recall that $\mathbf{V}$ and $\mathbf{H}$ are the closures of $\mathcal{V}$ in $\mathbf{H}_0^1(\Omega)$ and $\mathbf{L}^2(\Omega)$ respectively and $(.,.)$ denote the scalar product in $\mathbf{L}^2(\Omega)$ or $L^2(\Omega)$. In $\mathbf{V}$, the norms $\| \ \boldsymbol{u} \ \|_{H^1(\Omega)}$ and $\| \ \nabla \boldsymbol{u} \ \|_{L^2(\Omega)}$ are equivalent.

Definition 1.1. A pair of functions $(\rho, \boldsymbol{u})$ is called a weak solution of problem (1)-(2)-(3) on Ω if and only if for any $T > 0$ the following assumptions are satisfied:

1. $\boldsymbol{u} \in L^\infty\big(0,T;\mathbf{H}\big) \cap L^2\big(0,T;\mathbf{V}\big)$,
   $\rho \in L^2\big(0,T;H^1\big) \cap L^\infty(\mathcal{Q}_T)$,
   $0 < m \leq \rho(t,\boldsymbol{x}) \leq M < +\infty$, a.e. $(t,\boldsymbol{x}) \in \mathcal{Q}_T$.

2. For all $\varphi \in \mathcal{C}^1([0,T];H^1(\Omega))$ such that $\varphi(T,.) = 0$, one has:

$$\int_0^T \Big\{ \big(\boldsymbol{u} \cdot \nabla \rho, \varphi\big) + \lambda \big(\nabla \rho, \nabla \varphi\big) - \big(\rho, \partial_t \varphi\big) \Big\} dt = \big(\rho_0, \varphi(0)\big). \tag{5}$$

3. For all $\boldsymbol{\phi} \in \mathcal{C}^1([0,T];\mathbf{V})$ such that $\boldsymbol{\phi}(T,.) = 0$, one has:

$$\begin{aligned}
\int_0^T \Big\{ - \big(\boldsymbol{u}, \rho \partial_t \boldsymbol{\phi} + \big((\rho \boldsymbol{u} - \lambda \nabla \rho) \cdot \nabla\big)\boldsymbol{\phi}\big) + \mu \big(\nabla \boldsymbol{u}, \nabla \boldsymbol{\phi}\big) - \lambda \big(\rho \nabla \boldsymbol{u}^T, \nabla \boldsymbol{\phi}\big) \Big\} dt \\
= \int_0^T \big(\rho \boldsymbol{g}, \boldsymbol{\phi}\big) dt + \big(\rho_0 \boldsymbol{u}_0, \boldsymbol{\phi}(0)\big).
\end{aligned} \tag{6}$$

Moreover, it is convenient to write the variational formulation of the problem. Let us assume that $(\rho, \boldsymbol{u}, P)$ is a sufficiently regular solution of (1)-(2)-(3). Multiplying the equations (1) respectively by arbitrary test functions $(\bar{\rho}, \bar{\boldsymbol{u}}, \bar{p}) \in H^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$, integrating over Ω and using Green's theorem, adding to the momentum equation the density equation where we choose $\bar{\rho} = \frac{1}{2} \boldsymbol{u} \cdot \bar{\boldsymbol{u}}$ and we integrate by parts the convective and diffusive terms, finally we obtain the following formulation for a.e. $t \in (0,T)$:

$$\left\{ \begin{array}{ll}
\dfrac{d}{dt}\big(\rho, \bar{\rho}\big) + b\big(\rho, \bar{\rho}, \boldsymbol{u}\big) + \lambda \ \big(\nabla \rho, \nabla \bar{\rho}\big) = 0, & \forall \bar{\rho} \in H^1(\Omega), \\
\Big(\rho \dfrac{\partial \boldsymbol{u}}{\partial t}, \bar{\boldsymbol{u}}\Big) + \dfrac{1}{2}\Big(\dfrac{\partial \rho}{\partial t} \boldsymbol{u}, \bar{\boldsymbol{u}}\Big) + a\big(\rho, \boldsymbol{u}, \bar{\boldsymbol{u}}\big) + c\big(\rho \boldsymbol{u} - \lambda \nabla \rho, \boldsymbol{u}, \bar{\boldsymbol{u}}\big) + d\big(\bar{\boldsymbol{u}}, P\big) & \\
\qquad = \big(\rho \boldsymbol{g}, \bar{\boldsymbol{u}}\big), & \forall \bar{\boldsymbol{u}} \in \mathbf{H}_0^1(\Omega), \\
d\big(\boldsymbol{u}, \bar{p}\big) = 0, & \forall \bar{p} \in L_0^2(\Omega),
\end{array} \right. \tag{7}$$

where we have used the following notations:

- $b\big(.,.,.\big)$, $a\big(.,.,.\big)$ and $c\big(.,.,.\big)$ are the trilinear forms defined by:

$$
\begin{aligned}
b(\rho,\beta,\boldsymbol{u}) &= \int_\Omega \mathrm{div}(\rho\boldsymbol{u})\,\beta\,d\boldsymbol{x}, \quad \forall\rho\in H^1(\Omega)\cap L^\infty(\Omega),\ \forall\beta\in H^1(\Omega),\forall\boldsymbol{u}\in\mathbf{V},\\
a\big(\rho,\boldsymbol{u},\boldsymbol{v}\big) &= \mu\big(\nabla\boldsymbol{u},\nabla\boldsymbol{v}\big)-\lambda\int_\Omega \rho\nabla\boldsymbol{u}^T:\nabla\boldsymbol{v}\,d\boldsymbol{x},\ \forall\rho\in H^1(\Omega),\forall\boldsymbol{u},\boldsymbol{v}\in\mathbf{H}_0^1(\Omega),\\
c\big(\boldsymbol{\omega},\boldsymbol{u},\boldsymbol{v}\big) &= \frac{1}{2}\Big[\big((\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u},\boldsymbol{v}\big)-\big((\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v},\boldsymbol{u}\big)\Big], \qquad \forall\boldsymbol{\omega}\in\mathbf{V},\forall\boldsymbol{u},\boldsymbol{v}\in\mathbf{H}_0^1(\Omega),
\end{aligned}
$$

- $d\big(.,.\big)$ is the bilinear form defined by:

$$
d\big(\boldsymbol{v},p\big) = -\big(p,\mathrm{div}\boldsymbol{v}\big), \qquad\qquad \forall\boldsymbol{v}\in\mathbf{H}_0^1(\Omega),\forall p\in L_0^2(\Omega).
$$

The trilinear forms verify the following properties of continuity, coercivity and antisymmetric:

$$
\begin{aligned}
&a\big(\rho,\boldsymbol{u},\boldsymbol{u}\big)\geq\mu_1\parallel\nabla\boldsymbol{u}\parallel^2_{L^2(\Omega)}, \quad \text{with } \mu_1=\mu-\lambda M>0, \qquad \forall\boldsymbol{u}\in\mathbf{H}_0^1(\Omega), \qquad (8)\\
&a\big(\rho,\boldsymbol{u},\boldsymbol{v}\big)\leq C\parallel\boldsymbol{u}\parallel_{H^1(\Omega)}\parallel\boldsymbol{v}\parallel_{H^1(\Omega)}, \qquad \forall\boldsymbol{u},\boldsymbol{v}\in\mathbf{H}_0^1(\Omega).\\
&c\big(\boldsymbol{\omega},\boldsymbol{u},\boldsymbol{v}\big)\leq C\parallel\boldsymbol{\omega}\parallel_{L^3(\Omega)}\parallel\boldsymbol{u}\parallel_{H^1(\Omega)}\parallel\boldsymbol{v}\parallel_{H^1(\Omega)},\forall\boldsymbol{\omega}\in\mathbf{L}^3(\Omega),\forall\boldsymbol{u},\boldsymbol{v}\in\mathbf{H}_0^1(\Omega),\\
&c\big(\boldsymbol{\omega},\boldsymbol{u},\boldsymbol{v}\big)=-c\big(\boldsymbol{\omega},\boldsymbol{v},\boldsymbol{u}\big), \qquad \forall\boldsymbol{\omega}\in\mathbf{V},\forall\boldsymbol{u},\boldsymbol{v}\in\mathbf{H}_0^1(\Omega),\\
&c\big(\boldsymbol{\omega},\boldsymbol{u},\boldsymbol{u}\big)=0, \qquad \forall\boldsymbol{\omega}\in\mathbf{V},\forall\boldsymbol{u}\in\mathbf{H}_0^1(\Omega). \qquad (9)
\end{aligned}
$$

1.3 Known results.

Many authors treat the mathematical analysis of the Kazhikhov-Smagulov model in three-dimensional domains. We can refer for instance to [21, 1, 2]. In [21, 1], under assumption (4) and if the constants λ, μ, m, M are such that $\lambda<2\mu(M-m)^{-1}$, the authors prove the existence of a weak solution of the problem (1)-(2)-(3), when $\boldsymbol{u}_0\in\mathbf{H}$, $\rho_0\in H^1(\Omega)$ and $\boldsymbol{g}\in L^2\big(0,T;\mathbf{L}^2(\Omega)\big)$. In [2], the existence and uniqueness of a weak solution of (1)-(2)-(3) is proved under assumption (4) but without any restriction on the constant λ. In [23], under de condition λ/μ small enough, the existence and uniqueness of the global solution is proved in the two-dimensional case. Moreover, it is showed the convergence (as $\lambda\to 0$) towards a weak solution of the Navier-Stokes system for nonhomogeneous fluids in two- and three-dimensional domains. Recently, in [4, 5] the authors prove the existence of a regular (resp. strong) solution of a two-dimensional generalized Kazhikhov-Smagulov model.

Concerning the numerical study, there exists few numerical schemes in order to approximate the problem (1). Some adequate choices can be found in [17, 18, 19, 20, 3], where

the authors propose a fully discrete numerical scheme consists of $\mathcal{C}^0$ finite element spatial approximation for all unknowns (density, velocity and pressure) combined with the backward Euler method in time. In [18] the authors obtain unconditional stability and convergence results for the two-dimensional case, by applying a truncating operator in the terms depending on the density. A conditionally stable and convergent numerical scheme is obtained in [19] for the three-dimensional case. In this work, the authors prove an approximate maximum principle bounding by excess and defect the discrete density with respect to the upper and lower bound of the initial density. Also, they study the asymptotic behavior of the numerical scheme as the diffusion parameter λ goes to zero, obtaining convergence towards a weak solution of the density-dependent Navier-Stokes problem. In [20], under hypotheses of regularity for the data and the exact solution, the authors present optimal error estimates of a linearized fully discrete scheme for the three-dimensional case. For the complete three-dimensional Kazhikhov-Smagulov model with $\mathcal{O}(\lambda^2)$ terms added in $(1)_2$, the existence of regular solutions and some error estimates are given in [17], by assuming smallness conditions on the data. An extension of the results in [19] for the complete model with $\mathcal{O}(\lambda^2)$ terms is obtained in [3], where the treatment of the $\mathcal{O}(\lambda^2)$ terms requires special attention. Finally, another numerical scheme is developed in [12] by using a backward Euler scheme together with the method of characteristics for the volume fraction of the denser fluid, and a mixed finite element method in space for velocity and pressure.

The finite volume schemes are widely used for the numerical resolution of linear or non-linear conservation laws (see [13, 24, 27] for instance). The discrete maximum principle, implying the L^∞-stability of a numerical scheme, is very important in the study of conservation laws. Indeed, the maximum principle is the first fundamental physical property of the problem that a suitable numerical scheme must faithfully reproduce. In [14], a convergence result for the numerical solution of a nonlinear convection-diffusion problem was investigated. The authors use a combined finite volume-finite element scheme, where the nonlinear convective terms are discretized by a monotone finite volume scheme and the diffusion term is approximated using conforming piecewise linear finite elements. In [15], the authors pursue this approach. The discrete maximum principle is necessary and it requires the use of triangulations of a weakly acute type. Under this assumption, the analysis of the error estimates of this combined finite volume-finite element scheme is achieved. Let us mention also the review paper by Droniou [11] which concerns various finite volume methods for solving diffusion equation on general meshes. The development of an hybrid finite volume-finite element scheme was firstly introduced in [7] and used in [6], in order to compute the numerical solution of the variable density incompressible Navier-Stokes system. Using a splitting in time, this hybrid scheme combines a finite volume method to discretize the density equation and a mixed finite element method to compute the velocity field and the pressure. In [6], the L^∞-stability was obtained under an explicit CFL condition by introducing a second order finite volume scheme with multislope gradient reconstruction in order to solve transport equation on unstructured meshes with local refinements. Recently, the hy-

brid finite volume-finite element scheme was extended in [8] to the numerical simulation of powder–snow avalanches by solving the two-dimensional model (1).

1.4 Main results.

The objective of this work is to study the Kazhikhov-Smagulov model (1) in a fully discrete setting by coupling finite volumes to approximate the density and finite elements to approximate the velocity and pressure. Let $h > 0$, we denote by $\mathcal{T}_h$ a partition of Ω composed of conforming and isotropic triangles. We take $\mathcal{W}_h \times \mathbf{V}_h \times \mathcal{Y}_h \subset H^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ the finite elements spaces associated to density, velocity and pressure, respectively. In order to simplify the notations, we restrict our study to the case of a uniform time discretization of $[0, T]$, but all the results given in this work can be extended without any difficulty to the case of a general time discretization. Let N be a positive integer, then we define $\Delta t = T/N$ the time step and $(t^n = n\Delta t)_{n=0}^N$ the partition of $[0, T]$. Moreover, we consider the following stability condition:

$$0 < \Delta t \leq c_0\, h, \tag{10}$$

where $c_0 > 0$ is a constant which is independent of h and Δt, but depends on the velocity field $\boldsymbol{u} \in \mathbf{V}_h$. Obviously, (10) is a typical CFL condition often used for the numerical solution of conservation laws (see [24]). Let $\left(\rho_h^n, \boldsymbol{u}_h^n\right) \in \mathcal{W}_h \times \mathbf{V}_h$ be the approximations of density and velocity at time t^n. We denote by $\rho_{h,\Delta t}$ and $\boldsymbol{u}_{h,\Delta t}$ the piecewise constant functions in time taking values ρ_h^n and $\boldsymbol{u}_h^n$ on $(t^{n-1}, t^n]$, respectively. Thus, the following main result will be proved:

Theorem 1.2. *There exists a convergent subsequence of* $\left(\rho_{h,\Delta t}, \boldsymbol{u}_{h,\Delta t}\right)$ *(denoted in the same way) as* $(h, \Delta t) \to 0$ *towards the (unique) weak solution* $\left(\rho, \boldsymbol{u}\right)$ *of problem* (1)-(2)-(3) *in the sense of Definition 1.1, when* $\mu - \lambda M > 0$ *and* (10) *are verified.*

The outline of the paper is organized as follows. In section 2 we describe the hybrid finite volume-finite element scheme, in particular we introduce the finite volume scheme and its properties in the vertex-based framework. In section 3 we study the stability of the numerical scheme. Then, we deduce the weak and weak$*$ convergences results in section 4. Afterwards, we establish the compactness arguments for the discrete density and velocity that provide the strong convergence results in section 5. Finally, section 6 is devoted to the passage to the limit, concluding the proof of Theorem 1.2.

2 Description of the numerical schemes.

This section is devoted to the development of an hybrid finite volume-finite element scheme. We recall that finite volume schemes are widely used for the numerical solution of conservation laws (see [13]), whereas finite element methods are naturally applied to approximate the solution of diffusive problems (i.e. elliptic or parabolic problems, see [10, 16]). The idea of combined finite volume and finite element methods was used

in various works concerning numerical computation of conservation laws. The idea behind this approach is to match the finite volume discretization of convective terms with the finite element discretization of diffusive terms. For the theoretical analysis of these combined schemes we refer to [15] and references therein. In our case, the hybrid finite volume-finite element scheme combines a finite volume approach to approximate the mass conservation equation and a finite element method to solve the momentum equation and the divergence free condition.

At first, we present the triangular mesh of Ω, the discrete spaces associated to density, velocity and pressure, and we set our assumptions on the discretization of the Kazhikhov-Smagulov model (1). Second, we focus on the development of the finite volume scheme to approximate the convection-diffusion equation, and we introduce the properties of the discrete density. Finally, we define the fully discrete scheme by using finite volume and finite element methods to approximate all unknowns (density, velocity, pressure) of Kazhikhov-Smagulov model.

2.1 Mesh definitions.

The discretization of the Kazhikhov-Smagulov model (1) will be carried out on an unstructured triangular mesh. Let $h > 0$, we denote by $\mathcal{T}_h$ a partition of the polygonal domain Ω composed of conforming and isotropic triangles. The triangulation $\mathcal{T}_h$ is called a *basic (or primal) mesh*. By $h(T)$ we denote the length of the longest side of the triangle $T \in \mathcal{T}_h$, and put

$$h = \max_{T \in \mathcal{T}_h} h(T).$$

We suppose the following assumptions for the triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω [15, 14]:

(A1) Let $\{\mathcal{T}_h\}_{h>0}$ be a *regular* family of triangulations of Ω.

(A2) The triangulations $\mathcal{T}_h$ are of *weakly acute type*. In other words, the magnitude of all angles of all $T \in \mathcal{T}_h$ is less than or equal to $\pi/2$.

(A3) The triangulations $\mathcal{T}_h$ verify the following *inverse assumption*:

$$h \leq c\ h(T), \qquad \forall\ T \in \mathcal{T}_h,$$

where $c > 0$ is constant independent of h.

According to [10, Chap.3, § 3.1, Remark 3.1.3], assumptions (A1) and (A3) imply the existence of a constant $c > 0$ independent of h, such that

$$h^2 \leq c\ |T|, \qquad \forall\ T \in \mathcal{T}_h, \tag{11}$$

where $|T|$ = area of $T \in \mathcal{T}_h$.
Now, let $\mathcal{M}_h = \{M_i, i \in J\}$ be the set of all vertices of the triangulation $\mathcal{T}_h$, (J is a suitable index set, of cardinality $\#J$). The set $\mathcal{E}_h$ of the edges of $\mathcal{T}_h$ is made of

straight segments $[M_iM_j]$ joining two vertices of $\mathcal{M}_h$. Let us construct the *dual mesh* $\mathcal{C}_h = \{\mathcal{C}_i, i \in J\}$ over the basic mesh $\mathcal{T}_h$, which defines a second partition of Ω. The dual finite volume $\mathcal{C}_i$ associated with each vertex $M_i \in \mathcal{M}_h$ is a closed polygon obtained in the following way: we join the barycenter of every triangle $T \in \mathcal{T}_h$ which share the vertex M_i with the middle point of every side of T containing M_i. If $M_i \in \mathcal{M}_h \cap \partial\Omega$, then we complete the boundary of $\mathcal{C}_i$ by the segments joining M_i with the middle point of boundary sides that contain M_i. $\mathcal{C}_i$ is often called the vertex-based control volume around the node M_i. Accordingly, we have

$$\bigcup_{T\in\mathcal{T}_h} T = \bar{\Omega} = \bigcup_{i\in J} \mathcal{C}_i. \tag{12}$$

Moreover, if we denote $|\mathcal{C}_i|$ = area of $\mathcal{C}_i \in \mathcal{C}_h$, then, we have

$$|\mathcal{C}_i| = \sum_{T, M_i\in T} \frac{|T|}{3}. \tag{13}$$

For $i \in J$, let $\mathcal{V}(i) = \{j \in J, \mathcal{C}_j \text{ is a neighbor of } \mathcal{C}_i\}$. Let $i \in J$ and $j \in \mathcal{V}(i)$, we define by $T_{ij,1}$ and $T_{ij,2}$ two neighboring triangles of $\mathcal{T}_h$ sharing the common edge $[M_iM_j]$. We denote B_1 (resp. B_2) the barycenter of $T_{ij,1}$ (resp. $T_{ij,2}$) and M_{ij} the middle of $[M_iM_j]$. Then, we put

$$\Gamma_{ij,1} = [M_{ij}B_1] \qquad \text{and} \qquad \Gamma_{ij,2} = [M_{ij}B_2].$$

If $M_i \in \mathcal{M}_h \cap \partial\Omega$, we have only one neighboring triangle of $\mathcal{T}_h$ sharing the common edge $[M_iM_j]$. In this case, we denote B_1 the barycenter of T_{ij} and B_2 the middle of $[M_iM_j]$. Then, we put $\Gamma_{ij,2} = [M_iB_2] \subset \partial\Omega$.
We denote $\boldsymbol{n}_{ij,1}$ (resp. $\boldsymbol{n}_{ij,2}$) the unit outward normal to $\mathcal{C}_i$ along $\Gamma_{ij,1}$ (resp. $\Gamma_{ij,2}$) and $|\Gamma_{ij,1}|$ (resp. $|\Gamma_{ij,2}|$) the length of the segment $\Gamma_{ij,1}$ (resp. $\Gamma_{ij,2}$). For every $\mathcal{C}_i \in \mathcal{C}_h$, the boundary of $\mathcal{C}_i$ is

$$\partial\mathcal{C}_i = \bigcup_{j\in\mathcal{V}(i)} \Big(\Gamma_{ij,1} \cup \Gamma_{ij,2}\Big). \tag{14}$$

Obviously, we have

$$|\Gamma_{ij,l}| \le \frac{h}{2}, \qquad \text{for } l = 1, 2. \tag{15}$$

Consequently, there exists a constant $c_1 > 0$, such that

$$\big|\partial\mathcal{C}_i\big| \le c_1 h, \qquad \forall i \in J. \tag{16}$$

Also, (13), (11) and (16) imply the existence of a constant $c_2 > 0$, such that

$$\frac{\big|\mathcal{C}_i\big|}{\big|\partial\mathcal{C}_i\big|} \ge c_2 h, \qquad \forall i \in J. \tag{17}$$

2.2 Discrete spaces.

In order to combine the efficiency of the finite volume method for the convection-diffusion equation and the finite element method for the momentum equation, we need to define some discrete spaces associated to the unknowns.

Let us define the following discrete spaces for the approximation of the density over the meshes $\mathcal{T}_h$ and $\mathcal{C}_h$:

$$\mathcal{W}_h = \Big\{ \beta_h \in \mathcal{C}^0(\bar{\Omega});\ \beta_{h|_T} \in \mathbb{P}_1\ \forall\ T \in \mathcal{T}_h \Big\} \subset H^1(\Omega), \tag{18}$$

$$\mathcal{Z}_h = \big\{ \eta \in L^2(\Omega);\ \eta_{|\mathcal{C}_i} = \text{constant}\ \forall\ \mathcal{C}_i \in \mathcal{C}_h \big\} \subset L^2(\Omega). \tag{19}$$

Given a vector $(\beta_{M_i})_{i\in J} \in \mathbb{R}^{\#J}$, there exists a unique $\Pi_h\beta \in \mathcal{W}_h$ and a unique $\mathbb{L}_h\beta \in \mathcal{Z}_h$ such that

$$\Pi_h\beta(M_i) = \mathbb{L}_h\beta(M_i) = \beta_{M_i}, \qquad \forall\ i \in J. \tag{20}$$

As a consequence, there are one-to-one mappings between $\mathbb{R}^{\#J}$, $\mathcal{W}_h$ and $\mathcal{Z}_h$. Here, $\Pi_h : \mathbb{R}^{\#J} \to \mathcal{W}_h$ denotes the Lagrange interpolation operator and $\mathbb{L}_h : \mathcal{C}^0(\bar{\Omega}) \to \mathcal{Z}_h$ is the so-called lumping operator. Obviously, $\mathbb{L}_h$ is a continuous linear operator.

For the approximations of velocity and pressure, there are several choices to define the mixed finite element spaces $\mathbf{V}_h \subset \mathbf{H}_0^1(\Omega)$ and $\mathcal{Y}_h \subset L_0^2(\Omega)$ verifying the Ladyshenskaya-Babuška-Brezzi (LBB) condition (also called "inf-sup" condition) [16]. Following [7, 8], we choose the Taylor-Hood element ($\mathbb{P}_2 \times \mathbb{P}_1$), but others choices are also possibles (the mini-element $\mathbb{P}_1$-bubble $\times\ \mathbb{P}_1$ for instance). Then

$$\mathbf{V}_h = \Big\{ \boldsymbol{v}_h \in \mathcal{C}^0(\bar{\Omega});\ \boldsymbol{v}_{h|_T} \in \mathbb{P}_2\ \forall\ T \in \mathcal{T}_h \Big\} \cap \mathbf{H}_0^1(\Omega), \tag{21}$$

$$\mathcal{Y}_h = \mathcal{W}_h \cap L_0^2(\Omega). \tag{22}$$

Throughout this work, we will suppose the following hypotheses [10, 3, 18, 19]:

(H1) Regularity for the data:
Let $\boldsymbol{u}_0 \in \mathbf{V}$, $\rho_0 \in H^1(\Omega)$ with $0 < m \le \rho_0 \le M$ in Ω and $\boldsymbol{g} \in L^2\big(0,T;\mathbf{L}^2(\Omega)\big)$.

(H2) The triangulation $\mathcal{T}_h$ of Ω and the finite elements space $\mathcal{W}_h$ verify the following inverse inequality:

$$\| \nabla\bar{\rho}_h \|_{L^2(\Omega)} \le C\ h^{-1}\ \| \bar{\rho}_h \|_{L^2(\Omega)}, \qquad \forall \bar{\rho}_h \in \mathcal{W}_h. \tag{23}$$

(H3) Inf-sup condition:
There exists a constant $C > 0$ independent of h, such that

$$\inf_{p_h\in\mathcal{Y}_h} \sup_{\boldsymbol{v}_h\in\mathbf{V}_h\setminus\{\mathbf{0}\}} \frac{d\big(\boldsymbol{v}_h, p_h\big)}{\| p_h \|_{L^2(\Omega)} \| \nabla\boldsymbol{v}_h \|_{L^2(\Omega)}} \ge C.$$

2.3 The finite volume scheme.

Here, we design the finite volume scheme for solving the convection-diffusion equation $(1)_1$ on the dual mesh $\mathcal{C}_h$. With a view to obtain the numerical scheme corresponding to $(7)_1$, we will denote by $\big(.,.\big)_h$ (resp. $\| \, . \, \|_h$) the approximation of the scalar product (resp. norm) on $L^2(\Omega)$, such that

$$\begin{array}{rcll} (\beta_1, \beta_2)_h & = & \displaystyle\int_\Omega \Pi_h(\beta_1 \, \beta_2) \; d\boldsymbol{x}, & \beta_1, \beta_2 \in \mathcal{C}^0(\bar{\Omega}), \\ \| \; \beta \; \|_h & = & (\beta, \beta)_h^{1/2}, & \beta \in \mathcal{C}^0(\bar{\Omega}). \end{array} \tag{24}$$

Also, the approximation $\big(.,.\big)_h$ can be defined with the aid of a numerical integration using the vertices of $T \in \mathcal{T}_h$ as integration points:

$$\begin{array}{rcll} (\beta_1, \beta_2)_h & = & \displaystyle\int_\Omega \mathbb{L}_h(\beta_1)\mathbb{L}_h(\beta_2) \; d\boldsymbol{x}, & \beta_1, \beta_2 \in \mathcal{C}^0(\bar{\Omega}), \\ \| \; \beta \; \|_h & = & \| \; \mathbb{L}_h\beta \; \|_{L^2(\Omega)}, & \beta \in \mathcal{C}^0(\bar{\Omega}). \end{array} \tag{25}$$

In particular, (24) and (25) correspond to the mass lumping technique applied to the mass matrix. Then, there exists constants $\hat{c}_1, \hat{c}_2 > 0$ such that $\forall \, h \in (0, h_0)$,

$$\hat{c}_1 \; \| \; \beta \; \|_{L^2(\Omega)} \leq \| \; \beta \; \|_h \leq \hat{c}_2 \; \| \; \beta \; \|_{L^2(\Omega)} \quad \forall \beta \in \mathcal{W}_h. \tag{26}$$

For the bilinear form associated to the laplacian, we consider the control volume finite element (CVFE) scheme (see [13]). We denote by $(\phi_i)_{i \in J}$ the canonical basis of $\mathcal{W}_h$ characterized by

$$\phi_i(M_j) = \delta_{ij}, \quad \forall \, i \in J, \quad \forall \, M_j \in \mathcal{M}_h.$$

The following geometrical properties hold:

$$\int_\Omega \phi_i(\boldsymbol{x}) \; d\boldsymbol{x} = 3|\mathcal{C}_i|, \quad \forall \, i \in J, \qquad\qquad \sum_{i \in J} \nabla \phi_i(\boldsymbol{x}) = 0, \quad \text{for a.e. } \boldsymbol{x} \in \Omega.$$

For all $[M_i, M_j] \in \mathcal{E}_h$, we define the coefficient

$$a_{ij} = - \int_\Omega \nabla \phi_i(\boldsymbol{x}) \cdot \nabla \phi_j(\boldsymbol{x}) \; d\boldsymbol{x} = a_{ji}.$$

Then, we obtain that

$$-a_{ii} = \sum_{j \neq i} a_{ij} > 0.$$

As consequence, $\forall \rho, \beta \in \mathcal{W}_h$ we have

$$(\nabla \rho, \nabla \beta) = \int_\Omega \nabla \rho(\boldsymbol{x}) \cdot \nabla \beta(\boldsymbol{x}) \; d\boldsymbol{x} = \sum_{[M_i, M_j] \in \mathcal{E}_h} a_{ij}(\rho_i - \rho_j)(\beta_i - \beta_j).$$

Using assumption (A2) on the triangulation $\mathcal{T}_h$, we have

$$a_{ij} \geq 0, \qquad \forall\, [M_i, M_j] \in \mathcal{E}_h,$$

which is a necessary condition for guaranteeing the (discrete) maximum principle of the finite volume scheme.
Moreover, we will construct an approximation b_h of the trilinear form b by using the finite volume approach, according to Feistauer et al. ideas [14, 15]. We assume that the velocity field $\boldsymbol{u} \in \mathbf{V}_h$ is a given function which satisfies the divergence free constraint. Then, using (12), (20), Green's theorem and (14), we write for $\rho, \beta \in \mathcal{W}_h$,

$$\begin{aligned}
\int_\Omega \mathrm{div}\big(\rho \boldsymbol{u}\big)\ \beta\ d\boldsymbol{x} \ &\approx\ \int_\Omega \mathrm{div}\big(\rho \boldsymbol{u}\big)\ \mathbb{L}_h\beta\ d\boldsymbol{x} \\
&= \sum_{i\in J} \int_{\mathcal{C}_i} \mathrm{div}\big(\rho \boldsymbol{u}\big)\ \mathbb{L}_h\beta\ d\boldsymbol{x} \\
&= \sum_{i\in J} \beta_{M_i} \int_{\mathcal{C}_i} \mathrm{div}\big(\rho \boldsymbol{u}\big)\ d\boldsymbol{x} \\
&= \sum_{i\in J} \beta_{M_i} \int_{\partial\mathcal{C}_i} \rho(t,\boldsymbol{x})\boldsymbol{u}(t,\boldsymbol{x})\cdot \boldsymbol{n}(\boldsymbol{x})\ d\sigma \\
&= \sum_{i\in J} \beta_{M_i} \sum_{j\in\mathcal{V}(i)} \sum_{l=1}^{2} \int_{\Gamma_{ij,l}} \rho(t,\boldsymbol{x})\boldsymbol{u}(t,\boldsymbol{x})\cdot \boldsymbol{n}_{ij,l}\ d\sigma
\end{aligned}$$

Finally, we characterize the approximation b_h as follows. Given $\boldsymbol{u} \in \mathbf{V}_h$, we define $\forall \rho, \beta \in \mathcal{W}_h$,

$$b_h(\rho, \beta, \boldsymbol{u}) \ := \sum_{i\in J} \beta_{M_i} \sum_{j\in\mathcal{V}(i)} \sum_{l=1}^{2} |\Gamma_{ij,l}|\, G_{ij_l}\big(\rho_{M_i}, \rho_{M_j}, \boldsymbol{n}_{ij,l}\big), \tag{27}$$

where we have introduced in (27) an upstream numerical flux G_{ij_l} as defined in [7, 6]. More specifically, we introduce the cell-averaged velocity $\boldsymbol{u}^\star$ defined for each $\boldsymbol{u} \in \mathbf{V}_h$:

$$\forall\, T \in \mathcal{T}_h, \quad \boldsymbol{u}^\star_{|_T} = \frac{1}{|T|} \int_T \boldsymbol{u}(t,\boldsymbol{x})\ d\boldsymbol{x}.$$

As shown numerically in [7], the choice of the cell-averaged velocity $\boldsymbol{u}^\star$ is a necessary condition in order to ensure the divergence free constraint at the discrete level. Then, for $l = 1, 2, i \in J, j \in \mathcal{V}(i)$, the constant values $\boldsymbol{u}^\star_{ij,l}$ defined on $\partial\mathcal{C}_i$ are such that

$$\boldsymbol{u}^\star(\boldsymbol{x}) = \left\{ \begin{array}{ll} \boldsymbol{u}^\star_{ij,1} & \text{on } \Gamma_{ij,1}, \\ \boldsymbol{u}^\star_{ij,2} & \text{on } \Gamma_{ij,2}, \end{array} \right. \qquad \forall \boldsymbol{x} \in \partial\mathcal{C}_i = \bigcup_{j\in\mathcal{V}(i)} \Big(\Gamma_{ij,1} \cup \Gamma_{ij,2}\Big).$$

With the value of $\boldsymbol{u}^\star_{ij,l}$ at hand, a simple upwind finite volume scheme lead to

$$G_{ij_l}\big(\rho_1, \rho_2, \boldsymbol{n}_{ij,l}\big) = \left\{ \begin{array}{ll} \rho_1\ \boldsymbol{u}^\star_{ij,l}\cdot \boldsymbol{n}_{ij,l} & \text{if } \boldsymbol{u}^\star_{ij,l}\cdot \boldsymbol{n}_{ij,l} > 0, \\ \rho_2\ \boldsymbol{u}^\star_{ij,l}\cdot \boldsymbol{n}_{ij,l} & \text{if } \boldsymbol{u}^\star_{ij,l}\cdot \boldsymbol{n}_{ij,l} \leq 0. \end{array} \right. \tag{28}$$

Obviously, upwind flux such as (28) restricts the method to first order accuracy. To improve the accuracy, more general fluxes can be designed based on MUSCL strategies. A multislope method on general unstructured dual meshes was introduced in [6], where the L^∞ stability for the convection equation was established. The discrete maximum principle can also be proved for the convection-diffusion equation. As for first order methods, it requires the assumption (A2) on the triangulation $\mathcal{T}_h$ (see [9]).
With the numerical flux (28), some properties are ensured, such as the consistency, conservativity and monotonicity of the numerical flux. Also, $G_{ij_l}(\rho_1, \rho_2, \boldsymbol{n}_{ij,l})$ is locally Lipschitz-continuous with respect to ρ_1, ρ_2:

$$\Big|G_{ij_l}(\rho_1, \rho_2, \boldsymbol{n}_{ij,l}) - G_{ij_l}(\rho_1^*, \rho_2^*, \boldsymbol{n}_{ij,l})\Big| \;\leq\; |\boldsymbol{u}^\star_{ij,l}|\Big(\big|\rho_1 - \rho_1^*\big| + \big|\rho_2 - \rho_2^*\big|\Big).$$

Thereafter, we use the following estimate of the approximation b_h.

Proposition 1. *There exists a constant $C > 0$, such that for $\rho, \beta \in \mathcal{W}_h$ and $\boldsymbol{u} \in \boldsymbol{V}_h$, we have*

$$\big|b_h\big(\rho, \beta, \boldsymbol{u}\big)\big| \leq C \parallel \boldsymbol{u} \parallel_{L^4(\Omega)} \parallel \nabla\rho \parallel_{L^4(\Omega)} \parallel \beta \parallel_{L^2(\Omega)} . \tag{29}$$

Proof. Let $\rho, \beta \in \mathcal{W}_h$ and $\boldsymbol{u} \in \mathbf{V}_h$. We write

$$\begin{aligned} b_h(\rho, \beta, \boldsymbol{u}) \;&=\; \sum_{i\in J} \beta_{M_i} \sum_{j\in\mathcal{V}(i)} \sum_{l=1}^{2} |\Gamma_{ij,l}| G_{ij_l}\big(\rho_{M_i}, \rho_{M_j}, \boldsymbol{n}_{ij,l}\big) \\ &=\; \sum_{j\in J} \beta_{M_j} \sum_{i\in\mathcal{V}(j)} \sum_{l=1}^{2} |\Gamma_{ji,l}| G_{ji_l}\big(\rho_{M_j}, \rho_{M_i}, \boldsymbol{n}_{ji,l}\big). \end{aligned}$$

Using the conservativity of the numerical flux G_{ij_l} and the relations $\Gamma_{ji,l} = \Gamma_{ij,l}$, $\boldsymbol{n}_{ij,l} = -\boldsymbol{n}_{ji,l}$, we obtain

$$b_h(\rho, \beta, \boldsymbol{u}) \;=\; -\sum_{i\in J} \beta_{M_j} \sum_{j\in\mathcal{V}(i)} \sum_{l=1}^{2} |\Gamma_{ij,l}| G_{ij_l}\big(\rho_{M_i}, \rho_{M_j}, \boldsymbol{n}_{ij,l}\big).$$

Then, we find that

$$b_h(\rho, \beta, \boldsymbol{u}) \;=\; \frac{1}{2} \sum_{i\in J} \sum_{j\in\mathcal{V}(i)} \sum_{l=1}^{2} |\Gamma_{ij,l}| G_{ij_l}\big(\rho_{M_i}, \rho_{M_j}, \boldsymbol{n}_{ij,l}\big)\big[\beta_{M_i} - \beta_{M_j}\big].$$

The numerical flux is consistent, then for any constant function ρ, we have

$$\sum_{j\in\mathcal{V}(i)} \sum_{l=1}^{2} |\Gamma_{ij,l}| G_{ij_l}\big(\rho_{M_i}, \rho_{M_i}, \boldsymbol{n}_{ij,l}\big) = 0.$$

Then, we write

$$b_h(\rho,\beta,\boldsymbol{u}) = \frac{1}{2}\sum_{i\in J}\sum_{j\in\mathcal{V}(i)}\sum_{l=1}^{2}|\Gamma_{ij,l}|\Big(G_{ij_l}\big(\rho_{M_i},\rho_{M_j},\boldsymbol{n}_{ij,l}\big)-G_{ij_l}\big(\rho_{M_i},\rho_{M_i},\boldsymbol{n}_{ij,l}\big)\Big)\big(\beta_{M_i}-\beta_{M_j}\big).$$

Let $l=1,2$, if $i\in J$ and $j\in\mathcal{V}(i)$, then the segment $[M_iM_j]\in\mathcal{E}_h$ is the common side of triangles $T_{ij,l}\in\mathcal{T}_h$, such that $\Gamma_{ij,l}\subset T_{ij,l}$ (see Section 2.1). We have

$$\begin{aligned}&\| \overrightarrow{M_iM_j} \|\leq h, \quad |\Gamma_{ij,l}|\leq\frac{h}{2},\\ &\big|\rho_{M_i}-\rho_{M_j}\big|\leq h\ |\nabla\rho_{|T_{ij,l}}|,\\ &\big|\beta_{M_i}-\beta_{M_j}\big|\leq h\ |\nabla\beta_{|T_{ij,l}}|.\end{aligned}$$

Moreover, the numerical flux G_{ij_l} is locally Lipschitz-continuous, then we obtain

$$\begin{aligned}\big|b_h(\rho,\beta,\boldsymbol{u})\big| &\leq \frac{1}{4}\sum_{i\in J}\sum_{j\in\mathcal{V}(i)}\sum_{l=1}^{2}h\ \big|\boldsymbol{u}^{\star}_{ij,l}\big|\ \big|\rho_{M_i}-\rho_{M_j}\big|\ \big|\beta_{M_i}-\beta_{M_j}\big|\\ &\leq \frac{1}{4}\sum_{i\in J}\sum_{j\in\mathcal{V}(i)}\sum_{l=1}^{2}h^3\ \big|\boldsymbol{u}^{\star}_{ij,l}\big|\ \big|\nabla\rho_{|T_{ij,l}}\big|\ \big|\nabla\beta_{|T_{ij,l}}\big|\\ &\leq C\sum_{T\in\mathcal{T}_h}h^3\ |\boldsymbol{u}^{\star}_{|_T}|\ |\nabla\rho_{|_T}|\ |\nabla\beta_{|_T}|,\end{aligned}$$

taking into account that each triangle $T\in\mathcal{T}_h$ appears in the above sum as some $T_{ij,l}$ at most six times. Finally, in virtue of (11), the generalized Hölder inequality and the inverse inequality (23), we arrive at

$$\begin{aligned}\big|b_h(\rho,\beta,\boldsymbol{u})\big| &\leq Ch\sum_{T\in\mathcal{T}_h}|T|\ |\boldsymbol{u}^{\star}_{|_T}|\ |\nabla\rho_{|_T}|\ |\nabla\beta_{|_T}|\\ &= Ch\int_{\Omega}|\boldsymbol{u}|\ |\nabla\rho|\ |\nabla\beta|\ d\boldsymbol{x}\\ &\leq Ch\ \|\ \boldsymbol{u}\ \|_{L^4(\Omega)}\|\ \nabla\rho\ \|_{L^4(\Omega)}\|\ \nabla\beta\ \|_{L^2(\Omega)}\\ &\leq C\ \|\ \boldsymbol{u}\ \|_{L^4(\Omega)}\|\ \nabla\rho\ \|_{L^4(\Omega)}\|\ \beta\ \|_{L^2(\Omega)}\ .\end{aligned}$$

■

In order to complete the time discretization, we consider an Euler type method, which is implicit with respect to the diffusive term and explicit with respect to the convective term. In conclusion, we define the following finite volume scheme for the approximate solution of $(1)_1$.

Initialization: Let $\rho_h^0 = \Pi_h \rho_0 \in \mathcal{W}_h$ be approximation of the initial data ρ_0, with

$$\rho_h^0(M_i) = \frac{1}{|\mathcal{C}_i|} \int_{\mathcal{C}_i} \rho_0(\boldsymbol{x})\ d\boldsymbol{x}, \quad \forall\ i \in J.$$

Time step $n+1$**:** Given $\rho_h^n \in \mathcal{W}_h$ and $\boldsymbol{u}_h^n \in \mathbf{V}_h$, find $\rho_h^{n+1} \in \mathcal{W}_h$ such that, for each $\bar{\rho}_h \in \mathcal{W}_h$:

$$\Big(\frac{\rho_h^{n+1} - \rho_h^n}{\Delta t}, \bar{\rho}_h\Big)_h + b_h\big(\rho_h^n, \bar{\rho}_h, \boldsymbol{u}_h^n\big) + \lambda\big(\nabla \rho_h^{n+1}, \nabla \bar{\rho}_h\big) = 0. \tag{30}$$

2.4 Properties of the density.

In this paragraph, we are interested to introduce some importants properties of the discrete density computed with the finite volume scheme (30). The existence and uniqueness of the solution ρ_h^{n+1} of (30) follows from the well-known Lax-Milgram theorem. Notice that (30) is equivalent to the following linear problem for the unknown vector $(\rho_{M_i}^{n+1})_{i\in J} \in \mathbb{R}^{\#J}$:

$$\rho_{M_i}^{n+1} + \lambda \frac{\Delta t}{|\mathcal{C}_i|} \sum_{j\in\mathcal{V}(i)} a_{ij} \rho_{M_j}^{n+1} = \rho_{M_i}^n - \frac{\Delta t}{|\mathcal{C}_i|} \sum_{j\in\mathcal{V}(i)} \sum_{l=1}^{2} |\Gamma_{ij,l}|\, G_{ijl}\big(\rho_{M_i}^n, \rho_{M_j}^n, \boldsymbol{n}_{ij,l}\big), \quad i \in J.$$

First of all, it is essential to ensure that the previous finite volume scheme preserves the maximum principle. The following proposition clames the L^∞ stability of (30) on unstructured mesh verifying assumption (A2) under an appropriate CFL condition.

Proposition 2. *Let $\boldsymbol{u} \in \boldsymbol{V}_h$ be the velocity field satisfying the divergence free constraint and let the initial density ρ_0 which verifies* (4)*. If Δt and h satisfy the condition:*

$$0 < \Delta t \le c_3\ \frac{|\mathcal{C}_i|}{|\partial\mathcal{C}_i|}, \qquad i \in J, \tag{31}$$

where $c_3 > 0$ is a constant independent of h and Δt, then for each n, $0 \le n \le N-1$, there exists a unique discrete solution ρ_h^{n+1} of finite volume scheme (30) *which verifies the pointwise estimates:*

$$0 < m \le \rho_h^{n+1} \le M < +\infty, \qquad \text{in } \Omega. \tag{32}$$

The proof of Proposition 2 can be found in [14, 13]. Evidently, the stability condition (10) and (17) imply (31).

Now, let $\Delta_h : \mathcal{W}_h \to \mathcal{W}_h$ be the linear operator defined as follows:

$$-\big(\Delta_h \rho_h, \bar{\rho}_h\big)_h = \big(\nabla \rho_h, \nabla \bar{\rho}_h\big), \qquad \forall\ \bar{\rho}_h \in \mathcal{W}_h. \tag{33}$$

Furthermore, we introduce the discrete Gagliardo-Nirenberg inequality for the density. The proof of this inequality can be found in [18, Lemma 10], using also (26).

Proposition 3. *There exists $C = C(\Omega) > 0$ (independent of h) such that, for any $\rho_h \in \mathcal{W}_h$, one has:*

$$\| \nabla \rho_h \|_{L^4(\Omega)} \leq C \| \nabla \rho_h \|^{1/2}_{L^2(\Omega)} \| \Delta_h \rho_h \|^{1/2}_{L^2(\Omega)} . \tag{34}$$

2.5 The hybrid finite volume-finite element scheme.

We aim to design a fully discrete numerical scheme in order to solve the Kazhikhov-Smagulov model (1) by applying an hybrid scheme which combines finite volume and finite element methods. Using a time splitting procedure, we are able to subdivide the global problem into decoupled sub-problems with respect to ρ and $\big(\boldsymbol{u}, P\big)$ at each time step. Concerning the spatial discretization, we use a finite volume scheme for the convection-diffusion equation for ρ (which was described in Section 2.3), and mixed finite elements for the linearized Navier-Stokes problem. The time discretization is obtained considering a backward Euler type scheme which is implicit with respect to the diffusion terms in both equations and explicit (resp. semi-implicit) with respect to the convective term in the density (resp. velocity) equation. Then, we define the numerical scheme as follows.

Initialization: Let $\big(\boldsymbol{u}_h^0, \rho_h^0\big) \in \mathbf{V}_h \times \mathcal{W}_h$ be approximations of $\big(\boldsymbol{u}_0, \rho_0\big)$ as $h \to 0$.

Time step $n+1$: Given $\big(\boldsymbol{u}_h^n, \rho_h^n\big) \in \mathbf{V}_h \times \mathcal{W}_h$,

1. Find $\rho_h^{n+1} \in \mathcal{W}_h$ such that, for each $\bar{\rho}_h \in \mathcal{W}_h$:

$$\Big(\frac{\rho_h^{n+1} - \rho_h^n}{\Delta t}, \bar{\rho}_h\Big)_h + b_h\big(\rho_h^n, \bar{\rho}_h, \boldsymbol{u}_h^n\big) + \lambda\big(\nabla \rho_h^{n+1}, \nabla \bar{\rho}_h\big) = 0. \tag{35}$$

2. Find $\big(\boldsymbol{u}_h^{n+1}, P_h^{n+1}\big) \in \mathbf{V}_h \times \mathcal{Y}_h$ such that, for each $\big(\bar{\boldsymbol{u}}_h, \bar{P}_h\big) \in \mathbf{V}_h \times \mathcal{Y}_h$:

$$\left\{ \begin{array}{l} \Big(\rho_h^n \dfrac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \bar{\boldsymbol{u}}_h\Big) + \dfrac{1}{2}\Big(\dfrac{\rho_h^{n+1} - \rho_h^n}{\Delta t} \boldsymbol{u}_h^{n+1}, \bar{\boldsymbol{u}}_h\Big) + a\big(\rho_h^{n+1}, \boldsymbol{u}_h^{n+1}, \bar{\boldsymbol{u}}_h\big) \\ + c\big(\rho_h^{n+1} \boldsymbol{u}_h^n - \lambda \nabla \rho_h^{n+1}, \boldsymbol{u}_h^{n+1}, \bar{\boldsymbol{u}}_h\big) = \big(P_h^{n+1}, \nabla \cdot \bar{\boldsymbol{u}}_h\big) + \big(\rho_h^{n+1} \boldsymbol{g}^{n+1}, \bar{\boldsymbol{u}}_h\big), \end{array} \right. \tag{36}$$

$$\big(\nabla \cdot \boldsymbol{u}_h^{n+1}, \bar{P}_h\big) = 0, \tag{37}$$

where we have used the following notation:

$$\boldsymbol{g}^{n+1} \;=\; \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \boldsymbol{g}(t)\, dt.$$

In (35)-(36)-(37) we have to solve at each time step two linear systems in order to compute $\big(\rho_h^{n+1}, \boldsymbol{u}_h^{n+1}, P_h^{n+1}\big)$: at first ρ_h^{n+1} as a finite volume approximation of the

convection-diffusion equation, with $\boldsymbol{u}_h^n$ the previous velocity, then $\big(\boldsymbol{u}_h^{n+1}, P_h^{n+1}\big)$ as a mixed finite element approximation of the Navier-Stokes equation with the divergence free condition. On the contrary, in [18, 19, 3], the authors have used a finite element approximation of the convection-diffusion equation, instead of the finite volume scheme (35).

Then, using the discrete Laplacian operator Δ_h defined by (33), the finite volume scheme (35) can be rewritten as:

$$\Big(\frac{\rho_h^{n+1}-\rho_h^n}{\Delta t}, \bar{\rho}_h\Big)_h + b_h\big(\rho_h^n, \bar{\rho}_h, \boldsymbol{u}_h^n\big) - \lambda\big(\Delta_h\rho_h^{n+1}, \bar{\rho}_h\big)_h = 0. \tag{38}$$

In the next section, we shall see that the discrete problem (35)-(36)-(37) is well-posed, that is the existence and uniqueness of a solution holds. Also, we will establish the discrete version of the energy estimate for this hybrid scheme, independent of the discrete parameters.

3 Uniform Estimates.

In this section energy estimates for the velocity and strong estimates for the density will be obtained, using the discrete Laplacian of the density and involving the scheme (38), in order to prove the global stability of the hybrid scheme (35)-(36)-(37).
First of all, we start to establish some useful inequalities for the hybrid scheme (35)-(36)-(37).

Theorem 3.1. *There exists a unique solution* $\big(\rho_h^{n+1}, \boldsymbol{u}_h^{n+1}, P_h^{n+1}\big)$ *of the discrete problem* (35)-(36)-(37) *which verifies:*

$$\begin{aligned} \| \sqrt{\rho_h^{n+1}}\boldsymbol{u}_h^{n+1} \|^2_{L^2(\Omega)} - \| \sqrt{\rho_h^n}\boldsymbol{u}_h^n \|^2_{L^2(\Omega)} + \| \sqrt{\rho_h^n}\big(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\big) \|^2_{L^2(\Omega)} \\ + \mu_1\Delta t \| \nabla\boldsymbol{u}_h^{n+1} \|^2_{L^2(\Omega)} \leq C_1\Delta t \| \boldsymbol{g}^{n+1} \|^2_{L^2(\Omega)}, \end{aligned} \tag{39}$$

$$\begin{aligned} \| \nabla\rho_h^{n+1} \|^2_{L^2(\Omega)} - \| \nabla\rho_h^n \|^2_{L^2(\Omega)} + \| \nabla(\rho_h^{n+1}-\rho_h^n) \|^2_{L^2(\Omega)} + \lambda\,\hat{c}_1^2\,\Delta t \| \Delta_h\rho_h^{n+1} \|^2_{L^2(\Omega)} \\ \leq C_2\Delta t \| \boldsymbol{u}_h^n \|^2_{L^2(\Omega)} \| \nabla\boldsymbol{u}_h^n \|^2_{L^2(\Omega)} \| \nabla\rho_h^n \|^2_{L^2(\Omega)} + \frac{\lambda\,\hat{c}_1^2}{2}\Delta t \| \Delta_h\rho_h^n \|^2_{L^2(\Omega)}, \end{aligned} \tag{40}$$

where C_1, C_2, $\hat{c}_1$ *are positives constants independent of* h, Δt *and* n.

Proof. At first, we prove the inequality (39). We start taking $\bar{\boldsymbol{u}}_h = 2\Delta t\ \boldsymbol{u}_h^{n+1}$ in (36) and $\bar{P}_h = P_h^{n+1}$ in (37):

$$\begin{aligned} 2\Delta t\Big(\rho_h^n\frac{\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{u}_h^{n+1}\Big) + \Delta t\Big(\frac{\rho_h^{n+1}-\rho_h^n}{\Delta t}\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}\Big) + 2\Delta t\ a\big(\rho_h^{n+1}, \boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}\big) \\ + 2\Delta t\ c\big(\rho_h^{n+1}\boldsymbol{u}_h^n - \lambda\nabla\rho_h^{n+1}, \boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}\big) = 2\Delta t\big(\rho_h^{n+1}\boldsymbol{g}^{n+1}, \boldsymbol{u}_h^{n+1}\big). \end{aligned} \tag{41}$$

First of all, we compute the first two terms of the above equality (41), using the identity $(a-b,2a) = \| a \|^2_{L^2(\Omega)} - \| b \|^2_{L^2(\Omega)} + \| a-b \|^2_{L^2(\Omega)}$:

$$
\begin{aligned}
&\Big(\rho_h^n \frac{\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{u}_h^{n+1}\Big) + \frac{1}{2}\Big(\frac{\rho_h^{n+1}-\rho_h^n}{\Delta t}\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}\Big) \\
&= \frac{1}{2}\int_\Omega \frac{\rho_h^n|\boldsymbol{u}_h^{n+1}|^2 - 2\rho_h^n \boldsymbol{u}_h^n\cdot\boldsymbol{u}_h^{n+1} + \rho_h^{n+1}|\boldsymbol{u}_h^{n+1}|^2}{\Delta t}\, d\boldsymbol{x} \\
&= \frac{1}{2}\int_\Omega \frac{\rho_h^{n+1}|\boldsymbol{u}_h^{n+1}|^2 - \rho_h^n|\boldsymbol{u}_h^n|^2}{\Delta t}\, d\boldsymbol{x} + \frac{1}{2}\int_\Omega \rho_h^n \frac{|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n|^2}{\Delta t}\, d\boldsymbol{x} \\
&= \frac{1}{2\Delta t}\Big(\| \sqrt{\rho_h^{n+1}}\boldsymbol{u}_h^{n+1} \|^2_{L^2(\Omega)} - \| \sqrt{\rho_h^n}\boldsymbol{u}_h^n \|^2_{L^2(\Omega)} + \| \sqrt{\rho_h^n}\big(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\big) \|^2_{L^2(\Omega)} \Big).
\end{aligned}
$$

Then, using (8) and (9) in the third and fourth terms of (41), we obtain

$$
\begin{aligned}
\| \sqrt{\rho_h^{n+1}}\boldsymbol{u}_h^{n+1} \|^2_{L^2(\Omega)} - \| \sqrt{\rho_h^n}\boldsymbol{u}_h^n \|^2_{L^2(\Omega)} + \| \sqrt{\rho_h^n}\big(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n\big) \|^2_{L^2(\Omega)} \\
+ 2\mu_1\Delta t \| \nabla \boldsymbol{u}_h^{n+1} \|^2_{L^2(\Omega)} \leq 2\Delta t\big(\rho_h^{n+1}\boldsymbol{g}^{n+1}, \boldsymbol{u}_h^{n+1}\big).
\end{aligned} \tag{42}
$$

By applying the estimates (32), the Poincaré and Young inequalities, the last term of (42) is estimate as follows:

$$
2\big(\rho_h^{n+1}\boldsymbol{g}^{n+1}, \boldsymbol{u}_h^{n+1}\big) \leq \mu_1 \| \nabla \boldsymbol{u}_h^{n+1} \|^2_{L^2(\Omega)} + \frac{M^2}{\mu_1} \| \boldsymbol{g}^{n+1} \|^2_{L^2(\Omega)},
$$

and we get to (39).
Next, to prove (40), we consider (38). Taking $\bar{\rho}_h = -\Delta t\, \Delta_h \rho_h^{n+1}$ in (38), $\bar{\rho}_h = \rho_h^{n+1} - \rho_h^n$ in (33) for $\rho_h = \rho_h^{n+1}$, we find that:

$$
\begin{aligned}
\| \nabla \rho_h^{n+1} \|^2_{L^2(\Omega)} - \| \nabla \rho_h^n \|^2_{L^2(\Omega)} + \| \nabla(\rho_h^{n+1}-\rho_h^n) \|^2_{L^2(\Omega)} + 2\lambda\, \Delta t \| \Delta_h \rho_h^{n+1} \|_h \\
= 2\Delta t\; b_h\big(\rho_h^n, \Delta_h\rho_h^{n+1}, \boldsymbol{u}_h^n\big) \;:=\; I.
\end{aligned} \tag{43}
$$

Taking into account the estimate (29), using the Young inequality and the discrete Gagliardo-Nirenberg inequality (34), we estimate the right-hand side term I of (43) as follows:

$$
\begin{aligned}
I &\leq C\, \Delta t \| \boldsymbol{u}_h^n \|_{L^4(\Omega)} \| \nabla \rho_h^n \|_{L^4(\Omega)} \| \Delta_h \rho_h^{n+1} \|_{L^2(\Omega)} \\
&\leq \lambda\, \hat{c}_1^2 \Delta t \| \Delta_h \rho_h^{n+1} \|^2_{L^2(\Omega)} + \frac{C}{\lambda}\Delta t \| \boldsymbol{u}_h^n \|^2_{L^4(\Omega)} \| \nabla \rho_h^n \|^2_{L^4(\Omega)} \\
&\leq \lambda\, \hat{c}_1^2 \Delta t \| \Delta_h \rho_h^{n+1} \|^2_{L^2(\Omega)} + \frac{C}{\lambda}\Delta t \| \boldsymbol{u}_h^n \|^2_{L^4(\Omega)} \| \nabla \rho_h^n \|_{L^2(\Omega)} \| \Delta_h \rho_h^n \|_{L^2(\Omega)}\,.
\end{aligned} \tag{44}
$$

Finally, using the Young inequality and the 2D Gagliardo-Nirenberg inequality for the velocity in (44), taking into account (26) in the left-hand side of (43), we obtain (40). ∎

Now, by virtue to show the global stability of scheme (35)-(36)-(37), it is easy to obtain, from Theorem 3.1 and from the discrete Gronwall's lemma, the following estimates for the velocity and more regularity for the density.

Lemma 3.2. *Let $\boldsymbol{u}_0 \in \boldsymbol{V}$, $\rho_0 \in H^1(\Omega)$ satisfying (4) and $\boldsymbol{g} \in L^2\big(0,T;\boldsymbol{L}^2(\Omega)\big)$. Then, the solution $\big(\rho_h^{n+1}, \boldsymbol{u}_h^{n+1}\big)$ of the discrete problem (35)-(36)-(37) satisfies the following estimates:*

$$(\boldsymbol{i}) \max_{0\le n\le N} \| \boldsymbol{u}_h^n \|_{L^2(\Omega)} \le C, \qquad (\boldsymbol{ii}) \ \Delta t \sum_{n=0}^{N} \| \nabla \boldsymbol{u}_h^n \|_{L^2(\Omega)}^2 \le C,$$

$$(\boldsymbol{iii}) \sum_{n=0}^{N-1} \| \boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n \|_{L^2(\Omega)}^2 \le C,$$

$$(\boldsymbol{iv}) \max_{0\le n\le N} \big(\| \rho_h^n \|_{L^\infty(\Omega)} + \| \nabla \rho_h^n \|_{L^2(\Omega)} \big) \le C,$$

$$(\boldsymbol{v}) \ \lambda\, \Delta t \sum_{n=0}^{N-1} \| \Delta_h \rho_h^{n+1} \|_{L^2(\Omega)}^2 \le C, \qquad (\boldsymbol{vi}) \sum_{n=0}^{N-1} \| \nabla(\rho_h^{n+1} - \rho_h^n) \|_{L^2(\Omega)}^2 \le C,$$

where $C > 0$ depends on the data $\big(\rho_0, \boldsymbol{u}_0, \boldsymbol{g}, \lambda\big)$ but is independent of h and Δt.

At last, as a consequence of Lemma 3.2, we deduce the following result:

Corollary 1. *Under the assumptions of Lemma 3.2, the following estimate holds:*

$$\Delta t \sum_{n=0}^{N} \| \nabla \rho_h^n \|_{L^4(\Omega)}^4 \le C_\lambda,$$

where $C_\lambda > 0$ is independent of h and Δt.

Proof. This is a direct consequence result of Lemma 3.2 and Proposition 3. ∎

4 Weak convergence.

In view of study the convergence of hybrid finite volume-finite element scheme (35)-(36)-(37) towards the (unique) solution of (1)-(2)-(3), we define the following auxiliary functions:

Definition 4.1. Let $\boldsymbol{u}_{h,\Delta t}$, $\widehat{\boldsymbol{u}}_{h,\Delta t}$, $\rho_{h,\Delta t}$, $\widehat{\rho}_{h,\Delta t}$, and $P_{h,\Delta t}$ be the piecewise constant functions in time taking the values $\boldsymbol{u}_h^{n+1}$, $\boldsymbol{u}_h^n$, ρ_h^{n+1}, ρ_h^n, and P_h^{n+1} on $(t^n, t^{n+1}]$, respectively. Moreover, let $\widetilde{\rho}_{h,\Delta t} \in \mathcal{C}^0\big([0,T];\mathcal{W}_h\big)$ be the continuous piecewise linear function in time defined as

$$\widetilde{\rho}_{h,\Delta t}(t) = \rho_h^{n+1} + \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t}(t - t_{n+1}), \quad t_n < t \le t_{n+1}.$$

Then, using Lemma 3.2 and Corollary 1, we arrive at the following results (see [3, 19, 18]):

Lemma 4.2. *The following estimates (independent of h and Δt) hold:*

$$\{\boldsymbol{u}_{h,\Delta t}\}_{h,\Delta t}, \{\widehat{\boldsymbol{u}}_{h,\Delta t}\}_{h,\Delta t} \quad \textit{are bounded in } L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big)\cap L^2\big(0,T;\boldsymbol{H}^1_0(\Omega)\big),$$
$$\{\rho_{h,\Delta t}\}_{h,\Delta t}, \{\widehat{\rho}_{h,\Delta t}\}_{h,\Delta t}, \{\widetilde{\rho}_{h,\Delta t}\}_{h,\Delta t} \quad \textit{are bounded in } L^\infty\big(0,T;H^1(\Omega)\big)\cap L^\infty(\mathcal{Q}_T),$$
$$\{\nabla\rho_{h,\Delta t}\}_{h,\Delta t} \quad \textit{is bounded in } L^4\big(0,T;L^4(\Omega)\big).$$

Moreover,

$$\| \boldsymbol{u}_{h,\Delta t} - \widehat{\boldsymbol{u}}_{h,\Delta t} \|^2_{L^2(0,T;L^2(\Omega))} \leq C\ \Delta t \quad \textit{and} \quad \| \rho_{h,\Delta t} - \widehat{\rho}_{h,\Delta t} \|^2_{L^2(0,T;H^1(\Omega))} \leq C\ \Delta t. \quad (45)$$

Next, by taking into account the previous estimates given in Lemma 4.2, there exist subsequences (denoted in the same way) with the corresponding weak convergences towards limit functions $\boldsymbol{u}$, $\widehat{\boldsymbol{u}}$, ρ, $\widehat{\rho}$. Thanks to (45), we have the identities of the limits $\boldsymbol{u}=\widehat{\boldsymbol{u}}$ and $\rho=\widehat{\rho}$. Therefore, we have the following result (for the proof see [19, Lemma 5.3]):

Lemma 4.3. *There exists subsequences of $\{\boldsymbol{u}_{h,\Delta t}\}_{h,\Delta t}$, $\{\widehat{\boldsymbol{u}}_{h,\Delta t}\}_{h,\Delta t}$, $\{\rho_{h,\Delta t}\}_{h,\Delta t}$, $\{\widehat{\rho}_{h,\Delta t}\}_{h,\Delta t}$, $\{\widetilde{\rho}_{h,\Delta t}\}_{h,\Delta t}$ (denoted in the same way) and limit functions $\boldsymbol{u}$, ρ verifying the following weak convergences, as $(h,\Delta t)\to 0$:*

$$\boldsymbol{u}_{h,\Delta t}\to\boldsymbol{u}, \quad \widehat{\boldsymbol{u}}_{h,\Delta t}\to\boldsymbol{u} \quad \textit{in} \begin{cases} L^2\big(0,T;\boldsymbol{H}^1_0(\Omega)\big)\textit{-weak}, \\ L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big)\textit{-weak}*, \end{cases}$$
$$\rho_{h,\Delta t}\to\rho, \quad \widehat{\rho}_{h,\Delta t}\to\rho, \quad \widetilde{\rho}_{h,\Delta t}\to\rho \quad \textit{in} \begin{cases} L^\infty\big(0,T;H^1(\Omega)\big)\textit{-weak}*, \\ L^\infty(\mathcal{Q}_T)\textit{-weak}*, \end{cases}$$
$$\nabla\rho_{h,\Delta t}\to\nabla\rho \quad \textit{in } L^4\big(0,T;L^4(\Omega)\big)\textit{-weak}.$$

5 Strong convergence.

As usual in this type of nonlinear system, to obtain the convergence of the hybrid scheme (35)-(36)-(37), we need strong convergence for the approximations in some suitable space in order to make the passage to the limit in the nonlinear terms. To do this, we must get the compactness for the discrete density and velocity.

At first, we establish a time derivative estimate for the discrete density.

Proposition 4. *The following estimate holds:*

$$\Delta t\sum_{n=0}^{N-1} \| \frac{\rho_h^{n+1}-\rho_h^n}{\Delta t} \|^{4/3}_{L^2(\Omega)} \leq C_\lambda,$$

where $C_\lambda>0$ is independent of h and Δt, but depends on the data $(\rho_0,\boldsymbol{u}_0,\boldsymbol{g},\lambda)$.

Proof. We start from (38), and using the Cauchy-Schwarz inequality, the estimate (29), the Sobolev embedding $H_0^1(\Omega) \subset L^4(\Omega)$, and (26), we find that $\forall\, \bar{\rho}_h \in \mathcal{W}_h$,

$$\Big|\Big(\frac{\rho_h^{n+1}-\rho_h^n}{\Delta t}, \bar{\rho}_h\Big)_h\Big| \leq \Big(C \parallel \nabla \boldsymbol{u}_h^n \parallel_{L^2(\Omega)} \parallel \nabla \rho_h^n \parallel_{L^4(\Omega)} + \lambda \parallel \Delta_h \rho_h^{n+1} \parallel_h \Big) \parallel \bar{\rho}_h \parallel_h . \quad (46)$$

Having in mind that $\dfrac{\rho_h^{n+1}-\rho_h^n}{\Delta t} \in \mathcal{W}_h \subset L^2(\Omega)$ and using a continuity argument, we deduce from (46) and (26) that

$$\parallel \frac{\rho_h^{n+1}-\rho_h^n}{\Delta t} \parallel_{L^2(\Omega)} \leq C \parallel \nabla \boldsymbol{u}_h^n \parallel_{L^2(\Omega)} \parallel \nabla \rho_h^n \parallel_{L^4(\Omega)} + \lambda\, \hat{c}_2 \parallel \Delta_h \rho_h^{n+1} \parallel_{L^2(\Omega)} . \quad (47)$$

Using the Minkowsky inequality and summing up (47) for $n = 0, \ldots, N-1$, we get promptly that

$$\begin{aligned} \Delta t \sum_{n=0}^{N-1} \parallel \frac{\rho_h^{n+1}-\rho_h^n}{\Delta t} \parallel_{L^2(\Omega)}^{4/3} \leq C\Big(\Delta t \sum_{n=0}^{N-1} \parallel \nabla \boldsymbol{u}_h^n \parallel_{L^2(\Omega)}^{4/3} \parallel \nabla \rho_h^n \parallel_{L^4(\Omega)}^{4/3} \\ + \lambda^{4/3} \Delta t \sum_{n=0}^{N-1} \parallel \Delta_h \rho_h^{n+1} \parallel_{L^2(\Omega)}^{4/3} \Big). \end{aligned} \quad (48)$$

Finally, by applying the Hölder inequality, the estimates given by Lemma 3.2 and Corollary 1, we get straightforwardly the desired result. ∎

Corollary 2. *As a consequence of Proposition 4, we deduce the following estimate:*

$$\parallel \frac{d}{dt} \widetilde{\rho}_{h,\Delta t} \parallel_{L^{4/3}(0,T;L^2(\Omega))} \leq C_\lambda. \quad (49)$$

From (49), one can deduce the following strong convergences for the density, thanks to a compactness theorem of Aubin-Lions type (see [26, Chap.3, Theorem 2.1]),

$$\rho_{h,\Delta t} \to \rho, \quad \widehat{\rho}_{h,\Delta t} \to \rho \quad \text{in } L^2\big(0,T;L^2(\Omega)\big)\text{-strong as } (h,\Delta t) \to 0. \quad (50)$$

Furthermore, from Lemma 4.2 the discrete density is bounded in $L^\infty(\mathcal{Q}_T)$, then one also gets the strong convergence in $L^p\big(0,T;L^p(\Omega)\big)$ for any $p < \infty$. For $p = \infty$, one can deduce the convergence for at least a subsequence of $\rho_{h,\Delta t}$ or $\widehat{\rho}_{h,\Delta t}$ to ρ for a.e. $(t,x) \in \mathcal{Q}_T$.
Now, we are going to estimate the fractional time derivative for the discrete velocity.

Proposition 5. *The following estimate holds:*

$$\int_0^{T-\delta} \parallel \sqrt{\rho_{h,\Delta t}(t+\delta)}\Big(\boldsymbol{u}_{h,\Delta t}(t+\delta) - \boldsymbol{u}_{h,\Delta t}(t)\Big) \parallel_{L^2(\Omega)}^2 dt \leq C_\lambda \delta^{1/2}, \quad \forall \delta : 0 < \delta < T,$$

with $C_\lambda > 0$ is independent of $(h, \Delta t, \delta)$, but depends on λ.

Proof. We follow exactly the proof done in [18, Proposition 27], where $[\rho_h^n]_T$ is replaced by ρ_h^n for each $n = m, \dots m+r$ and the equation [18, (66)] is replaced by

$$\| \rho_h^m - \rho_h^{m+r} \|_h^2 = \Delta t \sum_{n=m}^{m-1+r} \Big(b_h\big(\rho_h^n, \rho_h^m - \rho_h^{m+r}, \boldsymbol{u}_h^n\big) - \lambda\big(\Delta_h \rho_h^{n+1}, \rho_h^m - \rho_h^{m+r}\big)_h \Big). \quad (51)$$

If in the right side of (51) we use the Cauchy-Schwarz inequality, the estimates (29) and (26), the estimates given by Lemma 3.2 and Corollary 1, then the estimate [18, (67)] is replaced by

$$\max_{0 \le m \le N-r} \| \rho_h^m - \rho_h^{m+r} \|_{L^2(\Omega)} \le C_\lambda (r\Delta t)^{1/2}.$$

■

Finally, based on the previous result and thanks to a compactness theorems of Aubin-Lions type (see [25, Theorem 5]), we obtain the strong convergences for the velocity.

Proposition 6. *One has the following convergences, as* $(h, \Delta t) \to 0$*:*

$$\boldsymbol{u}_{h,\Delta t} \to \boldsymbol{u}, \quad \widehat{\boldsymbol{u}}_{h,\Delta t} \to \boldsymbol{u} \quad \text{in } L^2\big(0,T;\boldsymbol{L}^2(\Omega)\big)\text{-strong.} \quad (52)$$

6 Passing to the limit.

The final step to complete this study is to employ the convergence results obtained throughout this work in order to pass to the limit in the discrete problem. Our goal is to show that the approximate solution $\big(\rho_h^{n+1}, \boldsymbol{u}_h^{n+1}\big)$, obtained with the aid of the hybrid finite volume-finite element scheme (35)-(36)-(37), converges in some sense to the weak solution $\big(\rho, \boldsymbol{u}\big)$ of the Kazhikhov-Smagulov model (1)-(2)-(3), when the space and time parameters $(h, \Delta t)$ tend to zero.

In the case of a convection-diffusion equation, the passage to the limit for the finite volume scheme (35) (as $(h, \Delta t) \to 0$) has been done in details by Feistauer et al. (see [14, section E]). But in our case, the velocity in the convective part of $(7)_1$ depends by $(h, \Delta t)$. To handle this difficulty, we use the following strong-weak convergence result.

Lemma 6.1. *Let* $(\boldsymbol{v}_n)_n \in L^2(0,T;\boldsymbol{L}^2(\Omega))$ *and* $(\eta_n)_n \in L^2(0,T;H^1(\Omega))$ *such that* $\boldsymbol{v}_n \to \boldsymbol{v}$ *in* $L^2(0,T;\boldsymbol{L}^2(\Omega))$*-strong, and* $\eta_n \to \eta$ *in* $L^2(0,T,H^1(\Omega))$*-weak, then for all* $\varphi \in \mathcal{C}^1([0,T];H^1(\Omega))$ *such that* $\varphi(T,.) = 0$*, one has*

$$\int_0^T \int_\Omega \boldsymbol{v}_n \cdot \nabla \eta_n \, \varphi \, d\boldsymbol{x} dt \longrightarrow \int_0^T \int_\Omega \boldsymbol{v} \cdot \nabla \eta \, \varphi \, d\boldsymbol{x} dt \quad \text{when } n \to \infty. \quad (53)$$

Thanks to the previous convergence results, we can prove that the limit function ρ satisfies the weak formulation of the convection-diffusion equation $(7)_1$ in the distribution sense on $(0,T)$.

We focus now, on the passing to the limit in the velocity scheme (36)-(37). In order to eliminate the discrete pressure, we proceed as in [18], considering adequate test functions. Let $\boldsymbol{v} \in \mathcal{C}^1\big([0,T];\mathcal{C}_c^\infty(\Omega)\big)$ be a free divergence function, such that $\nabla\cdot\boldsymbol{v}=0$ and $\boldsymbol{v}(T,.)=0$. We define $\boldsymbol{v}_h^n$ the projection of $\boldsymbol{v}(t^n)$ (by a discrete Stokes problem) onto $\mathbf{V}_h$. Let $\boldsymbol{v}_{h,\Delta t} \in L^\infty\big(0,T;\mathbf{V}_h\big)$ be the piecewise constant function taking the value $\boldsymbol{v}_h^{n+1}$ on $(t^n,t^{n+1}]$ and let $\widetilde{\boldsymbol{v}}_{h,\Delta t} \in \mathcal{C}^0\big([0,T];\mathbf{V}_h\big)$ be the continuous piecewise linear function, such that $\widetilde{\boldsymbol{v}}_{h,\Delta t}(t^n)=\boldsymbol{v}_h^n$. Then, as $(h,\Delta t)\to 0$, one has:

$$\begin{array}{ll}\boldsymbol{v}_{h,\Delta t}\to\boldsymbol{v} & \text{in } L^\infty\big(0,T;\mathbf{H}_0^1(\Omega)\big),\\ \widetilde{\boldsymbol{v}}_{h,\Delta t}\to\boldsymbol{v} & \text{in } W^{1,\infty}\big(0,T;\mathbf{H}_0^1(\Omega)\big).\end{array}$$

Let us write the time derivative of the discrete equation (36) in the conservative form. By adding at the right and left-hand sides of (36) the term $\frac{1}{2}\big(\frac{\rho_h^{n+1}-\rho_h^n}{\Delta t}\boldsymbol{u}_h^{n+1},\bar{\boldsymbol{u}}_h\big)$, we obtain

$$\begin{aligned}\Big(\frac{\rho_h^{n+1}\boldsymbol{u}_h^{n+1}-\rho_h^n\boldsymbol{u}_h^n}{\Delta t},\bar{\boldsymbol{u}}_h\Big)+a\big(\rho_h^{n+1},\boldsymbol{u}_h^{n+1},\bar{\boldsymbol{u}}_h\big)+c\big(\rho_h^{n+1}\boldsymbol{u}_h^n-\lambda\nabla\rho_h^{n+1},\boldsymbol{u}_h^{n+1},\bar{\boldsymbol{u}}_h\big)\\ =\Big(P_h^{n+1},\nabla\cdot\bar{\boldsymbol{u}}_h\Big)+\Big(\rho_h^{n+1}\boldsymbol{g}^{n+1},\bar{\boldsymbol{u}}_h\Big)+\frac{1}{2}\Big(\frac{\rho_h^{n+1}-\rho_h^n}{\Delta t}\boldsymbol{u}_h^{n+1},\bar{\boldsymbol{u}}_h\Big).\end{aligned}\tag{54}$$

Next, taking $\bar{\boldsymbol{u}}_h=\boldsymbol{v}_h^{n+1}$ as test function in (54), multiplying by Δt, summing for $n=0,\dots,N-1$, and using the following discrete integration by parts in time

$$\sum_{n=0}^{N-1}\Big(\rho_h^{n+1}\boldsymbol{u}_h^{n+1}-\rho_h^n\boldsymbol{u}_h^n,\boldsymbol{v}_h^{n+1}\Big)=\Big(\rho_h^N\boldsymbol{u}_h^N,\boldsymbol{v}_h^N\Big)-\sum_{n=0}^{N-1}\Big(\rho_h^n\boldsymbol{u}_h^n,\boldsymbol{v}_h^{n+1}-\boldsymbol{v}_h^n\Big)-\Big(\rho_h^0\boldsymbol{u}_h^0,\boldsymbol{v}_h^0\Big),$$

with $\boldsymbol{v}_h^N=0$ (since $\boldsymbol{v}(T,.)=0$), we arrive at the following conservative form:

$$\begin{aligned}&-\Delta t\sum_{n=0}^{N-1}\Big(\rho_h^n\boldsymbol{u}_h^n,\frac{\boldsymbol{v}_h^{n+1}-\boldsymbol{v}_h^n}{\Delta t}\Big)-\Big(\rho_h^0\boldsymbol{u}_h^0,\boldsymbol{v}_h^0\Big)+\Delta t\sum_{n=0}^{N-1}a\big(\rho_h^{n+1},\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h^{n+1}\big)\\ &+\Delta t\sum_{n=0}^{N-1}c\big(\rho_h^{n+1}\boldsymbol{u}_h^n-\lambda\nabla\rho_h^{n+1},\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h^{n+1}\big)\\ &=\Delta t\sum_{n=0}^{N-1}\Big(\rho_h^{n+1}\boldsymbol{g}^{n+1},\boldsymbol{v}_h^{n+1}\Big)+\frac{\Delta t}{2}\sum_{n=0}^{N-1}\Big(\frac{\rho_h^{n+1}-\rho_h^n}{\Delta t}\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h^{n+1}\Big).\end{aligned}$$

Then, by using Definition 4.1, we obtain the following variational formulation:

$$
\begin{aligned}
&-\int_0^T \Big(\widehat{\rho}_{h,\Delta t}\widehat{\boldsymbol{u}}_{h,\Delta t}, \frac{\partial}{\partial t}\widetilde{\boldsymbol{v}}_{h,\Delta t}\Big)dt - \Big(\rho_h^0\boldsymbol{u}_h^0, \boldsymbol{v}_h^0\Big) + \int_0^T a\big(\rho_{h,\Delta t}, \boldsymbol{u}_{h,\Delta t}, \boldsymbol{v}_{h,\Delta t}\big)dt \\
&+\int_0^T c\big(\rho_{h,\Delta t}\widehat{\boldsymbol{u}}_{h,\Delta t} - \lambda\nabla\rho_{h,\Delta t}, \boldsymbol{u}_{h,\Delta t}, \boldsymbol{v}_{h,\Delta t}\big)dt \qquad (55)\\
&=\int_0^T \Big(\rho_{h,\Delta t}\boldsymbol{g}_{\Delta t}, \boldsymbol{v}_{h,\Delta t}\Big)dt + \frac{1}{2}\int_0^T \Big(\frac{\partial}{\partial t}\widetilde{\rho}_{h,\Delta t}\boldsymbol{u}_{h,\Delta t}, \boldsymbol{v}_{h,\Delta t}\Big)dt.
\end{aligned}
$$

From the variational formulation (55) of the discrete velocity equation (36), we are able to pass to the limit in a standard manner thanks to the convergence results obtained previously. Then, the limit function $(\rho, \boldsymbol{u})$ satisfies the weak formulation (7) of the Kazhikhov-Smagulov model in the distribution sense on $(0, T)$. Consequently, we conclude the proof of Theorem 1.2.

Acknowledgments

The authors would like to thanks the reviewer for his careful reading and his valuable remarks in order to improve the quality of the manuscript. The first and third authors were supported in part by the Labex CEMPI (ANR-11-LABX-0007-01), University of Lille. The second author was supported by the Unité de Recherche Multifractals et Ondelettes UR 11 ES 53 - University of Monastir.

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