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► **To cite this version:**

Axel Kröner, Athena Picarelli, Hasnaa Zidani. Infinite Horizon Stochastic Optimal Control Problems with Running Maximum Cost. 2017. hal-01585766v2

HAL Id: hal-01585766

<https://hal.science/hal-01585766v2>

Preprint submitted on 17 Oct 2017 (v2), last revised 25 Sep 2018 (v3)

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October 16, 2017

INFINITE HORIZON STOCHASTIC OPTIMAL CONTROL PROBLEMS WITH RUNNING MAXIMUM COST

AXEL KRÖNER, ATHENA PICARELLI, AND HASNAA ZIDANI

ABSTRACT. An infinite horizon stochastic optimal control problem with running maximum cost is considered. The value function is characterized as the viscosity solution of a second-order HJB equation with mixed boundary condition. A general numerical scheme is proposed and convergence is established under the assumptions of consistency, monotonicity and stability of the scheme. These properties are verified for a specific semi-Lagrangian scheme.

1. INTRODUCTION

In this paper we consider infinite horizon stochastic optimal control problems with cost in a maximum form of the following type:

$$(1.1) \quad \left\{ \begin{array}{l} \inf_{u \in \mathcal{U}} \mathbb{E} \left[\max_{s \in [0, \infty)} e^{-\lambda s} g(X_x^u(s)) \right], \quad \text{subject to} \\ dX_x^u(s) = b(X_x^u(s), u(s))ds + \sigma(X_x^u(s), u(s))dB(s), \quad s \in [0, \infty), \\ X_x^u(0) = x \in \mathbb{R}^d \end{array} \right.$$

with p -dimensional Brownian motion $B(\cdot)$, set \mathcal{U} of progressively measurable processes with values in a compact set $U \subset \mathbb{R}^m$, functions $g: \mathbb{R}^d \rightarrow \mathbb{R}$, $b: \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \times U \rightarrow \mathbb{R}^{d \times p}$, and discount factor $\lambda > 0$.

This type of problem can be used for the characterization of viable and invariant sets and arise in the study of some path-dependent options in finance (lookback options). In this paper we focus on the derivation of the associated Hamilton-Jacobi-Bellman (HJB) equation and on its numerical approximation.

The corresponding finite horizon problem has been considered in [7], while the infinite horizon case has been previously studied in [17] for the particular case of a Zubov-type equation. Finite horizon problems of this type were also addressed in [4, 5] and, for what concerns their numerical approximation, in [1]. However, in these works the dynamic programming techniques are applied on the L^p -approximation of the L^∞ -cost functional, using that

$$a \vee b \simeq (a^p + b^p)^{\frac{1}{p}} \quad (\text{for } p \rightarrow \infty),$$

The first author acknowledges the hospitality and support of the COMMANDS team at ENSTA ParisTech (Palaiseau, France) where, during a research visit, this project started.

for any $a, b \geq 0$ with $a \vee b := \max(a, b)$. This is different to the approach considered here which follows more closely the one in [7, 17] where the Dynamic Programming equation is directly derived from the original formulation of the problem involving the running maximum cost.

The main contribution of this paper is the following. In the first part we show that the value function of the optimal control problem (1.1) is the unique viscosity solution of a second-order HJB equation with mixed boundary condition involving oblique derivative and Dirichlet conditions. While the oblique derivative boundary condition is understood in the viscosity sense [18], the Dirichlet condition is considered pointwise. The presence of oblique derivative boundary conditions typically arises when dealing with running maximum operators (see [6, 7]). The additional Dirichlet condition is instead introduced in order to ensure the boundedness of the problem and to guarantee uniqueness of the solution (see also [17]).

The second part of the paper is devoted to the numerical approximation. Thereby we follow ideas developed in [7] for the corresponding finite horizon problem. First, a general scheme is proposed to deal with HJB equations with mixed boundary conditions. A convergence result is proven by using the general framework of Barles-Souganidis [3] based on the monotonicity, stability and consistency of the scheme. Then, we introduce a semi-Lagrangian (SL) scheme and prove its convergence. This type of schemes have been introduced in [12] for first order Hamilton-Jacobi equations and then extended to the second order case in [11, 14, 15, 21]. For stationary equations as considered here SL schemes are formulated as fixed-point problems. Therefore, differently from the evolutionary case considered in [7] where the scheme is proven to be stable by deriving at each time step explicit bounds on the numerical solution, in order to guarantee the stability of the scheme the existence of such a fixed-point has to be proven. To deal with the boundary problem, the numerical scheme we propose couples the classical SL scheme with an additional projection step on the boundary taking into account the overstepping of the domain which is typical in such a wide stencil scheme. For this scheme we prove stability, consistency and monotonicity.

The paper is organized as follows: Section 2 introduces the problem and the main assumptions. Sections 3 and 4 are devoted to the characterization of the value function by the appropriate HJB equation: the Dynamic Programming Principle (DPP) is established, the HJB equation is derived and uniqueness proven by a strong comparison principle. In Section 5 the numerical approximation is discussed and a general convergence result is provided. The semi-Lagrangian scheme is presented in Section 5.2 and its convergence properties are investigated.

2. FORMULATION OF THE PROBLEM

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\{\mathcal{F}_t, t \geq 0\}$ a filtration on \mathcal{F} and $B(\cdot)$ a $\{\mathcal{F}_t\}_{t \geq 0}$ -Brownian motion in \mathbb{R}^p , $p \geq 1$. Let \mathcal{U} be a set of progressively

measurable processes with values in $U \subset \mathbb{R}^m$, $m \geq 1$, U compact set. We consider stochastic differential equations of type

$$(2.1) \quad \begin{cases} dX_x^u(s) = b(X_x^u(s), u(s))ds + \sigma(X_x^u(s), u(s))dB(s), & s \in [0, \infty), \\ X_x^u(0) = x \end{cases}$$

with control $u \in \mathcal{U}$.

Throughout the paper we make the following assumptions on the coefficients in (2.1):

(H1) $b: \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \times U \rightarrow \mathbb{R}^{d \times p}$ are continuous functions satisfying, for some constants $C_0 \geq 0$,

$$\begin{aligned} |b(x, u) - b(y, u)| + |\sigma(x, u) - \sigma(y, u)| &\leq C_0|x - y|, \\ |b(x, u)| + |\sigma(x, u)| &\leq C_0 \end{aligned}$$

for any $x, y \in \mathbb{R}^d$ and $u \in U$.

Theorem 2.1. *Let assumption (H1) hold. Then, for any $x \in \mathbb{R}^d$ and $u \in \mathcal{U}$ there exists a unique strong solution $X_x^u(\cdot)$ of (2.1). Moreover, there exists $C \geq 0$ such that*

$$(2.2) \quad \mathbb{E} \left[\sup_{\theta \in [0, T]} |X_x^u(\theta) - X_{x'}^u(\theta)| \right] \leq Ce^{CT}|x - x'|$$

for any $u \in \mathcal{U}$, $T > 0$ and $x, x' \in \mathbb{R}^d$.

For a proof we refer to [23, p. 42, Thm. 6.3] and [22, p.14].

Let the cost function g be given such that

(H2) $g: \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies

$$|g(x) - g(y)| \leq L_g|x - y|, \quad |g(x)| \leq M_g$$

for some $M_g, L_g \geq 0$, for any $x, y \in \mathbb{R}^d$.

For given discount factor $\lambda > 0$ and $x \in \mathbb{R}^d$ the infinite horizon optimal control problem reads as

$$(2.3) \quad \inf_{u \in \mathcal{U}} J(x, u) := \mathbb{E} \left[\max_{s \in [0, \infty)} e^{-\lambda s} g(X_x^u(s)) \right].$$

We will denote by $v: \mathbb{R}^d \rightarrow \mathbb{R}$ the associated value function, i.e.

$$(2.4) \quad v(x) := \inf_{u \in \mathcal{U}} J(x, u).$$

Remark 2.2. *We point out that the study of optimal control problems with cost depending on a running maximum as in (2.3) can be used to characterise the region of viability and invariance of a domain. This results from an application of a level set approach as shown in [7, Section 2.3] for the case of a finite horizon of time.*

3. DYNAMIC PROGRAMMING PRINCIPLE

In the next sections, we aim to characterize the value function v as a viscosity solution of a HJB equation. The presence of the maximum operator inside the expectation makes the cost J in (2.3) non-Markovian and it is well-known that this prevents to establish a Dynamic Programming Principle (DPP), which is the first fundamental result towards the HJB characterisation. A classical strategy to overcome this difficulty consists in adding an auxiliary variable y that, roughly speaking, gets rid of the non-Markovian component of the cost. This has been originally used in [6] in a finite horizon setting. There, an approximation technique of the L^∞ -norm is used, as a consequence the results apply only if the function g in (2.3) is a positive function. We proceed in a different way. Following the approach in [7, 17] we derive directly the HJB equation without making use of any approximation.

Let us define the auxiliary value function $\vartheta: \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$

$$\vartheta(x, y) := \inf_{u \in \mathcal{U}} \mathbb{E} \left[\max_{s \in [0, \infty)} e^{-\lambda s} g(X_x^u(s)) \vee y \right],$$

which satisfies, for any $x \in \mathbb{R}^d$,

$$\vartheta(x, g(x)) = v(x),$$

where v is defined as in (2.4). Consequently, if the value function ϑ of the auxiliary problem is known, one can immediately recover the corresponding value function v of the original problem. Therefore, we consider only ϑ in the sequel of the paper.

We start by proving a continuity result for the value function.

Proposition 3.1. *Let assumptions (H1)-(H2) be satisfied. Then the value function ϑ is uniformly continuous in $\mathbb{R}^d \times \mathbb{R}$. Moreover,*

$$\vartheta(x, M_g) = M_g \quad \text{and} \quad |\vartheta(x, y)| \leq M_g$$

for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}$ with $|y| \leq M_g$.

Proof. The last statement follows directly by the definition of ϑ and the bound on g given by assumption (H2).

By the very definition of ϑ we have that for any $(x, y), (\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}$

$$\begin{aligned} & |\vartheta(x, y) - \vartheta(\xi, \eta)| \\ & \leq \sup_{u \in \mathcal{U}} \mathbb{E} \left[\left| \max_{s \in [0, \infty)} e^{-\lambda s} g(X_x^u(s)) \vee y - \max_{s \in [0, \infty)} e^{-\lambda s} g(X_\xi^u(s)) \vee \eta \right| \right] \\ & \leq \sup_{u \in \mathcal{U}} \mathbb{E} \left[\max_{s \in [0, \infty)} \left| e^{-\lambda s} g(X_x^u(s)) - e^{-\lambda s} g(X_\xi^u(s)) \right| \vee |y - \eta| \right] \\ & \leq \sup_{u \in \mathcal{U}} \mathbb{E} \left[\max_{s \in [0, \infty)} \left| e^{-\lambda s} g(X_x^u(s)) - e^{-\lambda s} g(X_\xi^u(s)) \right| \right] + |y - \eta| \end{aligned}$$

For any $T > 0$ we have that (recalling that g is bounded)

$$\begin{aligned}
 & \mathbb{E} \left[\max_{s \in [0, \infty)} \left| e^{-\lambda s} g(X_x^u(s)) - e^{-\lambda s} g(X_\xi^u(s)) \right| \right] \\
 & \leq \mathbb{E} \left[\max_{s \in [0, T]} \left| e^{-\lambda s} g(X_x^u(s)) - e^{-\lambda s} g(X_\xi^u(s)) \right| \right] + \mathbb{E} \left[\max_{s \in [T, \infty)} \left| e^{-\lambda s} g(X_x^u(s)) - e^{-\lambda s} g(X_\xi^u(s)) \right| \right] \\
 & \leq \mathbb{E} \left[\max_{s \in [0, T]} e^{-\lambda s} \left| g(X_x^u(s)) - g(X_\xi^u(s)) \right| \right] + \mathbb{E} \left[\max_{s \in [T, \infty)} 2M_g e^{-\lambda s} \right] \\
 & = \mathbb{E} \left[\max_{s \in [0, T]} e^{-\lambda s} \left| g(X_x^u(s)) - g(X_\xi^u(s)) \right| \right] + 2M_g e^{-\lambda T}.
 \end{aligned}$$

For the first term we have the classical estimates (see Theorem 2.1)

$$\mathbb{E} \left[\max_{s \in [0, T]} e^{-\lambda s} \left| g(X_x^u(s)) - g(X_\xi^u(s)) \right| \right] \leq L_g \mathbb{E} \left[\sup_{s \in [0, T]} |X_{t,x}^u(s) - X_{t,\xi}^u(s)| \right] \leq L_g C e^{CT} |x - \xi|.$$

So putting everything together we get that for every T

$$|\vartheta(x, y) - \vartheta(\xi, \eta)| \leq L_g C e^{CT} |x - \xi| + 2M_g e^{-\lambda T} + |y - \eta|.$$

Now, for any $\varepsilon > 0$ we can fix $T \equiv T(\varepsilon)$ such that $2M_g e^{-\lambda T} < \varepsilon/3$. It follows that we can choose a $\delta = \delta(\varepsilon, T(\varepsilon))$ such that if $|(x, y) - (\xi, \eta)| \leq \delta$ then

$$|\vartheta(x, y) - \vartheta(\xi, \eta)| < \varepsilon.$$

□

Remark 3.2. *In many cases, under assumptions (H1)-(H2), the value function is expected to be Lipschitz continuous in space and 1/2-Hölder continuous in time. This holds true in the case of integral running costs in both finite and infinite horizon (see [20, Theorem 3.4]) and in the case of maximum running costs in finite horizon (see [7, Proposition 2.2]). For the problem considered in the present paper this property seems not to hold. However, the proof of Proposition 3.1 shows that v is Lipschitz continuous with respect to the auxiliary variable y . This will turn out to be a fundamental property to be preserved when defining a numerical scheme in order to get convergence (see assumption (H3.a) in Section 5).*

Thanks to the presence of the auxiliary variable y we are able to state the Dynamic Programming Principle (DPP) for ϑ . Let us define for $t \geq 0$

$$Y_{x,y}^u(t) := \max_{s \in [0, t]} e^{-\lambda s} g(X_x^u(s)) \vee y.$$

Following the arguments in [10] and thanks to the uniform continuity of ϑ one has:

Theorem 3.3 (DPP). *Let assumptions (H1)-(H2) be satisfied. For $(x, y) \in \mathbb{R}^d \times \mathbb{R}$ and any finite stopping time $\theta \geq 0$ there holds*

$$\vartheta(x, y) = \inf_{u \in \mathcal{U}} \mathbb{E} \left[e^{-\lambda \theta} \vartheta(X_x^u(\theta), e^{\lambda \theta} Y_{x,y}^u(\theta)) \right].$$

Intuitive justification. We only give an intuitive justification for the DPP satisfied by ϑ in the non controlled case.

We can in fact observe that a.s.

$$\begin{aligned} \max_{s \in [0, \infty)} e^{-\lambda s} g(X_x(s)) \vee y &= \max_{s \in [\theta, \infty)} e^{-\lambda s} g(X_x(s)) \vee \underbrace{\max_{s \in [0, \theta]} e^{-\lambda s} g(X_x(s)) \vee y}_{=: Y_{x,y}(\theta)} \\ &= e^{-\lambda \theta} \max_{s \in [0, \infty)} e^{-\lambda s} g(X_{X_x(\theta)}(s)) \vee Y_{x,y}(\theta) \\ &= e^{-\lambda \theta} \left(\max_{s \in [0, \infty)} e^{-\lambda s} g(X_{X_x(\theta)}(s)) \vee e^{\lambda \theta} Y_{x,y}(\theta) \right). \end{aligned}$$

Consequently, by the definition of ϑ (in absence of controls), we have by the tower property of the expectation

$$\begin{aligned} \vartheta(x, y) &= \mathbb{E} \left[\mathbb{E} \left[e^{-\lambda \theta} \left(\max_{s \in [0, \infty)} e^{-\lambda s} g(X_{X_x(\theta)}(s)) \vee e^{\lambda \theta} Y_{x,y}(\theta) \right) \middle| \mathcal{F}_\theta \right] \right] \\ &= \mathbb{E} \left[e^{-\lambda \theta} \mathbb{E} \left[\max_{s \in [0, \infty)} e^{-\lambda s} g(X_{X_x(\theta)}(s)) \vee e^{\lambda \theta} Y_{x,y}(\theta) \middle| \mathcal{F}_\theta \right] \right] \\ &= \mathbb{E} \left[e^{-\lambda \theta} \vartheta(X_x(\theta), e^{\lambda \theta} Y_{x,y}(\theta)) \right]. \end{aligned}$$

Here, we are not considering the additional technical difficulties coming from the presence of controls. We refer to [10] and [22] for more detailed arguments. \square

4. THE HAMILTON-JACOBI-BELLMAN EQUATION

The value function ϑ can be characterized in terms of a solution of a second-order HJB equation.

Theorem 4.1 (Second-order HJB equation). *Under assumptions (H1)-(H2), the value function ϑ is a continuous viscosity solution of*

$$(4.1) \quad \begin{aligned} \lambda \vartheta + H(x, y, D_x \vartheta, \partial_y \vartheta, D_x^2 \vartheta) &= 0 \quad \text{for } x \in \mathbb{R}^d, y > g(x), \\ -\partial_y \vartheta &= 0 \quad \text{for } x \in \mathbb{R}^d, y = g(x) \end{aligned}$$

with $\lambda > 0$ and Hamiltonian $H: \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$ defined by

$$(4.2) \quad H(x, y, p, q, P) = \sup_{u \in U} \left(-b(x, u) \cdot p - \frac{1}{2} \text{Tr}[(\sigma \sigma^T)(x, u)P] \right) - \lambda y q.$$

We recall the notion of viscosity solution for second-order HJB equation.

Definition 4.2. [13, Definition 7.4] *Let \mathcal{O} be a locally compact set in \mathbb{R}^{d+1} . A USC function $\underline{\vartheta}$ (resp. LSC function $\bar{\vartheta}$) on $\bar{\mathcal{O}}$ is a viscosity sub-solution (resp. super-solution) of (4.1), if for every function $\varphi \in C^2(\bar{\mathcal{O}})$ at every*

maximum (resp. minimum) point x of $\underline{\vartheta} - \varphi$ (resp. $\bar{\vartheta} - \varphi$) the following inequalities hold

$$\begin{cases} \lambda\varphi + H(x, y, D_x\varphi, \partial_y\varphi, D_x^2\varphi) \leq 0 & \text{in } \mathcal{O}, \\ \min(\lambda\varphi + H(x, y, D_x\varphi, \partial_y\varphi, D_x^2\varphi), -\partial_y\varphi) \leq 0 & \text{on } \partial\mathcal{O}, \end{cases}$$

(resp.

$$\begin{cases} \lambda\varphi + H(x, y, D_x\varphi, \partial_y\varphi, D_x^2\varphi) \geq 0 & \text{in } \mathcal{O}, \\ \max(\lambda\varphi + H(x, y, D_x\varphi, \partial_y\varphi, D_x^2\varphi), -\partial_y\varphi) \geq 0 & \text{on } \partial\mathcal{O}. \end{cases}$$

A continuous function ϑ is called a viscosity solution of (4.1) if it is both a sub- and super-solution.

Proof of Theorem 4.1. Let

$$\bar{\mathcal{O}} := \{ (x, y) \in \mathbb{R}^{d+1} : y \geq g(x) \}.$$

We first check that ϑ is a viscosity sub-solution. Let $\varphi \in C^2(\bar{\mathcal{O}})$ such that $\vartheta - \varphi$ attains a maximum at point $(\bar{x}, \bar{y}) \in \bar{\mathcal{O}}$. Without loss of generality we can assume that (\bar{x}, \bar{y}) is a strict local maximum point (let us say in a ball of radius $r > 0$ centered in (\bar{x}, \bar{y})) and $\vartheta(\bar{x}, \bar{y}) = \varphi(\bar{x}, \bar{y})$. Thanks to Theorem 3.3, for any $u \in \mathcal{U}$ and for any sufficiently small stopping time $\theta = \theta^u$, we have:

$$\begin{aligned} \varphi(\bar{x}, \bar{y}) = \vartheta(\bar{x}, \bar{y}) &\leq \mathbb{E} \left[e^{-\lambda\theta} \vartheta(X_{\bar{x}}^u(\theta), e^{\lambda\theta} Y_{\bar{x}, \bar{y}}^u(\theta)) \right] \\ (4.3) \quad &\leq \mathbb{E} \left[e^{-\lambda\theta} \varphi(X_{\bar{x}}^u(\theta), e^{\lambda\theta} Y_{\bar{x}, \bar{y}}^u(\theta)) \right]. \end{aligned}$$

Two cases will be considered depending on if the point (\bar{x}, \bar{y}) belongs to the boundary of $\bar{\mathcal{O}}$ or not.

— **Case 1:** $g(\bar{x}) < \bar{y}$. Consider a constant control $u(s) \equiv u \in U$. From the continuity of g and the a.s. continuity of the sample paths it follows that for a.e. $\omega \in \Omega$ there exists $\bar{s}(\omega) > 0$ such that $g(X_{\bar{x}}^u(s))e^{-\lambda s} < \bar{y}$ if $s \in [0, \bar{s}(\omega))$. Given $h > 0$, let $\bar{\theta}$ be the following stopping time:

$$\begin{aligned} (4.4) \quad \bar{\theta} &:= \inf \{ s > 0 : (X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s)) \notin B((\bar{x}, \bar{y}), r) \} \wedge h \\ &\quad \wedge \inf \{ s > 0 : e^{-\lambda s} g(X_{\bar{x}}^u(s)) \geq \bar{y} \} \end{aligned}$$

(where $B((\bar{x}, \bar{y}), r)$ denotes the ball of radius $r > 0$ centered at (\bar{x}, \bar{y})). One can easily observe that a.s. $\bar{\theta} > 0$ and $Y_{\bar{x}, \bar{y}}^u(\bar{\theta}) = \bar{y}$, then by (4.3)

$$\varphi(\bar{x}, \bar{y}) \leq \mathbb{E} \left[e^{-\lambda\bar{\theta}} \varphi(X_{\bar{x}}^u(\bar{\theta}), e^{\lambda\bar{\theta}} \bar{y}) \right] \quad \forall u \in U.$$

By applying Ito's formula [23, Theorem 5.5], and thanks to the smoothness of φ , we get:

$$\begin{aligned} & \mathbb{E} \left[\int_0^{\bar{\theta}} d \left(e^{-\lambda s} \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right) \right] \\ &= \mathbb{E} \left[\int_0^{\bar{\theta}} \left\{ \lambda \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) - b(X_{\bar{x}}^u(s), u) D_x \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right. \right. \\ & \quad \left. \left. - \frac{1}{2} \text{Tr}[\sigma \sigma^T(X_{\bar{x}}^u(s), u) D_x^2 \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y})] - \lambda \bar{y} e^{\lambda s} \partial_y \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right\} e^{-\lambda s} ds \right] \leq 0. \end{aligned}$$

Observing that the stopping times

$$\inf \left\{ s > 0 : (X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s)) \notin B_r(\bar{x}, \bar{y}) \right\} \text{ and } \inf \left\{ s > 0 : e^{-\lambda s} g(X_{\bar{x}}^u(s)) \geq \bar{y} \right\}$$

are a.s. strictly greater than 0, for a.e. fixed ω one obtains $\bar{\theta} = h$ for a sufficiently small h in (4.4). Dividing by $h > 0$, one gets

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{h} \int_0^{\bar{\theta}} \left\{ \lambda \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) - b(X_{\bar{x}}^u(s), u) D_x \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right. \right. \\ & \quad \left. \left. - \frac{1}{2} \text{Tr}[\sigma \sigma^T(X_{\bar{x}}^u(s), u) D_x^2 \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y})] - \lambda \bar{y} e^{\lambda s} \partial_y \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right\} e^{-\lambda s} ds \right] \leq 0. \end{aligned}$$

By the dominate convergence theorem, taking the limit for h going to 0 inside the expectation and applying the mean value theorem, it follows

$$\lambda \varphi(\bar{x}, \bar{y}) - b(\bar{x}, u) D_x \varphi(\bar{x}, \bar{y}) - \frac{1}{2} \text{Tr}[\sigma \sigma^T(\bar{x}, u) D_x^2 \varphi(\bar{x}, \bar{y})] - \lambda \bar{y} \partial_y \varphi(\bar{x}, \bar{y}) \leq 0.$$

Finally, thanks to the arbitrariness of $u \in U$:

$$\lambda \varphi(\bar{x}, \bar{y}) + \sup_{u \in U} \left(-b(\bar{x}, u) D_x \varphi(\bar{x}, \bar{y}) - \frac{1}{2} \text{Tr}[\sigma \sigma^T(\bar{x}, u) D_x^2 \varphi(\bar{x}, \bar{y})] \right) - \lambda \bar{y} \partial_y \varphi(\bar{x}, \bar{y}) \leq 0.$$

— **Case 2:** $g(\bar{x}) = \bar{y}$. Let us assume that $-\partial_y \varphi(\bar{x}, \bar{y}) > 0$, otherwise the conclusion is straightforward.

As in the previous case, we consider a constant control $u(s) \equiv u \in U$. Thanks to the continuity of the sample paths and the smoothness of φ , for a.e. ω there is a time $\bar{s}(\omega) > 0$ and $\eta > 0$ such that:

$$\varphi(X_{\bar{x}}^u(s), e^{\lambda s} y) \leq \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \quad \forall s \in [0, \bar{s}], y \in [\bar{y}, \bar{y} + \eta].$$

Let $\bar{\theta}$ be the stopping time given by:

$$\begin{aligned} \bar{\theta} := & \inf \left\{ s > 0 : (X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s)) \notin B((\bar{x}, \bar{y}), r) \right\} \wedge \inf \left\{ s > 0 : \partial_y \varphi(X_{\bar{x}}^u(s), \bar{y}) \geq 0 \right\} \\ & \wedge \inf \left\{ s > 0 : e^{-\lambda s} g(X_{\bar{x}}^u(s)) \notin [\bar{y}, \bar{y} + \eta] \right\} \wedge h. \end{aligned}$$

By (4.3) we have

$$\varphi(\bar{x}, \bar{y}) \leq \mathbb{E} \left[e^{-\lambda \bar{\theta}} \varphi(X_{\bar{x}}^u(\bar{\theta}), e^{\lambda \bar{\theta}} \bar{y}) \right],$$

which implies (as we have seen for Case 1):

$$\lambda \varphi(\bar{x}, \bar{y}) + \sup_{u \in U} \left\{ -b(\bar{x}, u) D_x \varphi(\bar{x}, \bar{y}) - \frac{1}{2} \text{Tr}[\sigma \sigma^T(\bar{x}, u) D_x^2 \varphi(\bar{x}, \bar{y})] \right\} - \lambda \bar{y} \partial_y \varphi(\bar{x}, \bar{y}) \leq 0.$$

In conclusion at $(\bar{x}, \bar{y}) \in \partial\mathcal{O}$ we have

$$\min(\lambda\varphi + H(\bar{x}, \bar{y}, D_x\varphi, \partial_y\varphi, D_x^2\varphi), -\partial_y\varphi) \leq 0,$$

and ϑ is a viscosity sub-solution of equation (4.1).

It remains to prove that ϑ is a viscosity super-solution of (4.1). Let $\varphi \in C^2(\overline{\mathcal{O}})$ be such that $\vartheta - \varphi$ attains a minimum at point $(\bar{x}, \bar{y}) \in \overline{\mathcal{O}}$. Without loss of generality we can always assume that (\bar{x}, \bar{y}) is a strict local minimum point in a ball $B((\bar{x}, \bar{y}), r)$ and $\vartheta(\bar{x}, \bar{y}) = \varphi(\bar{x}, \bar{y})$. We consider again the two cases:

— **Case 1:** $g(\bar{x}) < \bar{y}$. We assume by contradiction that

$$\lambda\varphi + H(\bar{x}, \bar{y}, D_x\varphi, \partial_y\varphi, D_x^2\varphi) < 0.$$

By using continuity arguments we can also state that

$$(4.5) \quad \lambda\varphi + H(\cdot, \cdot, D_x\varphi, \partial_y\varphi, D_x^2\varphi) \leq 0$$

in a neighborhood $B((\bar{x}, \bar{y}), r_1)$ of (\bar{x}, \bar{y}) for some $r_1 > 0$. Moreover, thanks to the continuity of g , if $\bar{y} - g(\bar{x}) =: \rho > 0$ we can find $r_2(\rho) > 0$ and $T(\rho) > 0$ such that

$$\max_{\substack{t \in [0, T], \\ x \in B(\bar{x}, r_2)}} e^{-\lambda t} g(x) - g(\bar{x}) \leq \frac{\rho}{2}$$

and we have

$$\max_{\substack{t \in [0, T], \\ x \in B(\bar{x}, r_2)}} e^{-\lambda t} g(x) \vee \bar{y} = \bar{y}.$$

For any $u \in \mathcal{U}$ we define the stopping time θ^u as the first exit time of the process $(X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s))$ from the ball $B((\bar{x}, \bar{y}), R)$ for $R := \min(r, r_1, r_2) > 0$, i.e.

$$\theta^u := \inf \left\{ s > 0 : (X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s)) \notin B((\bar{x}, \bar{y}), R) \right\} \wedge T.$$

Applying Ito's formula and taking the expectation we get

$$\begin{aligned} & \varphi(\bar{x}, \bar{y}) - \mathbb{E} \left[e^{-\lambda\theta^u} \varphi(X_{\bar{x}}^u(\theta^u), e^{\lambda\theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \right] \\ &= \varphi(\bar{x}, \bar{y}) - \mathbb{E} \left[e^{-\lambda\theta^u} \varphi(X_{\bar{x}}^u(\theta^u), e^{\lambda\theta^u} \bar{y}) \right] \\ &= \mathbb{E} \left[\int_0^{\theta^u} \left\{ \lambda\varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) - b(X_{\bar{x}}^u(s), u) D_x\varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right. \right. \\ & \quad \left. \left. - \frac{1}{2} Tr[\sigma\sigma^T(X_{\bar{x}}^u(s), u) D_x^2\varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y})] - \lambda\bar{y} e^{\lambda s} \partial_y\varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right\} e^{-\lambda s} ds \right] \leq 0, \end{aligned}$$

that leads to

$$\vartheta(\bar{x}, \bar{y}) = \varphi(\bar{x}, \bar{y}) \leq \mathbb{E} \left[e^{-\lambda\theta^u} \varphi(X_{\bar{x}}^u(\theta^u), e^{\lambda\theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \right].$$

The continuity of the sample paths implies that

$$\begin{cases} \text{either } \theta^u = T \text{ or} \\ (X_{\bar{x}}^u(\theta^u), e^{\lambda\theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \in \partial B((\bar{x}, \bar{y}), R) \text{ a.s.} \end{cases} .$$

Since (\bar{x}, \bar{y}) is a strict minimum point we have

$$\min \left\{ (\vartheta - \varphi) : (x, y) \in \partial B((\bar{x}, \bar{y}), R) \cup \left(B((\bar{x}, \bar{y}), R) \cap y = e^{\lambda T} \bar{y} \right) \right\} =: \eta > 0,$$

and hence

$$\vartheta(\bar{x}, \bar{y}) \leq \varphi(X_{\bar{x}}^u(\theta^u), e^{\lambda\theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \leq \vartheta(\theta^u, X_{\bar{x}}^u(\theta^u), e^{\lambda\theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) + \eta.$$

Since η does not depend on u and u is arbitrary, this contradicts the DPP.

— **Case 2:** $g(\bar{x}) = \bar{y}$. Assume by contradiction that

$$-\partial_y \varphi(\bar{x}, \bar{y}) < 0 \quad \text{and} \quad \lambda \varphi + H(\bar{x}, \bar{y}, D_x \varphi, \partial_y \varphi, D_x^2 \varphi) < 0.$$

We can again define $r_1 > 0$ such that (4.5) is satisfied in $B((\bar{x}, \bar{y}), r_1)$. Moreover there exists $\tilde{r}_2 > 0$ such that

$$\varphi(\xi, \zeta) \leq \varphi(\xi, \zeta')$$

for any $(\xi, \zeta), (\xi, \zeta') \in B((\bar{x}, \bar{y}), \tilde{r}_2)$ such that $\zeta \leq \zeta'$. For any $u \in \mathcal{U}$ we define the stopping time θ^u as the first exit time of the process $(X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s))$ and $(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y})$ from the ball $B((\bar{x}, \bar{y}), \tilde{R})$ for $\tilde{R} := \min(r, r_1, \tilde{r}_2) > 0$. As for Case 1, we can still say that a.s.

$$\vartheta(X_{\bar{x}}^u(\theta^u), Y_{\bar{x}, \bar{y}}^u(\theta^u)) \geq \varphi(\theta^u, X_{\bar{x}}^u(\theta^u), Y_{\bar{x}, \bar{y}}^u(\theta^u)) + \eta'.$$

for some $\eta' > 0$ not depending on u . Therefore, observing that

$$e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s) \geq e^{\lambda s} \bar{y}$$

for any $s \geq 0$ and using Ito's formula, we get

$$\begin{aligned} \vartheta(\bar{x}, \bar{y}) &= \varphi(\bar{x}, \bar{y}) \\ &\leq \mathbb{E} \left[\varphi(\theta^u, X_{\bar{x}}^u(\theta^u), e^{\lambda\theta^u} \bar{y}) \right] \\ &\leq \mathbb{E} \left[\varphi(\theta^u, X_{\bar{x}}^u(\theta^u), e^{\lambda\theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \right] \\ &\leq \mathbb{E} \left[\vartheta(\theta^u, X_{\bar{x}}^u(\theta^u), e^{\lambda\theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \right] - \eta', \end{aligned}$$

which yields again to a contradiction of the DPP. \square

In the sequel we will restrict our domain to

$$D := \{(x, y) \in \mathbb{R}^{d+1} : y > g(x), y \in (-M_g, M_g)\}.$$

Indeed, this is sufficient to characterize ϑ since we have that:

$$(4.6) \quad \begin{aligned} \vartheta(x, y) &= y \quad \text{for any } y \geq M_g \\ \vartheta(x, y) &= \vartheta(x, g(x)) \quad \text{for any } y \leq g(x). \end{aligned}$$

Based on this observation we are going to prove uniqueness in the domain \bar{D} by adding the Dirichlet boundary condition, given by Proposition 3.1,

$$\vartheta(x, M_g) = M_g \quad \text{for any } x \in \mathbb{R}^d.$$

This will give a HJB equation with mixed Dirichlet and derivative boundary conditions. Let

$$(4.7) \quad \Gamma_1 := \{ (x, y) \in \bar{D} : y = M_g \}; \quad \Gamma_2 := \{ (x, y) \in \bar{D} : y = g(x) \}.$$

Then the value function ϑ is a viscosity solution of the following problem

$$(4.8a) \quad \lambda\vartheta + H(x, y, D_x\vartheta, \partial_y\vartheta, D_x^2\vartheta) = 0 \quad \text{in } D,$$

$$(4.8b) \quad -\partial_y\vartheta = 0 \quad \text{on } \Gamma_2$$

and satisfies additional

$$(4.8c) \quad \vartheta = M_g \quad \text{on } \Gamma_1$$

in a strong sense. We point out that in our case the strong constant Dirichlet boundary condition on Γ_1 is compatible with the homogeneous derivative condition on Γ_2 . This prevents possible problems related with mixed boundary conditions at the junctions where different components of the boundary cross.

We can prove the following comparison result:

Theorem 4.3. *Let assumptions (H1)-(H2) be satisfied and let $u \in USC(\bar{D})$, $v \in LSC(\bar{D})$ respectively a bounded sub- and super-solution to equation (4.8) in \bar{D} such that*

$$u \leq M_g \quad \text{and} \quad v \geq M_g \quad \text{on } \Gamma_1.$$

Then $u \leq v$ in \bar{D} .

The proof can be obtained by a slight modification of the arguments in [16, Theorem 2.1], see also [17]. We report here the main steps.

Sketch of the proof. We set

$$\tilde{b}(x, y, u) := \begin{pmatrix} b(x, u) \\ \lambda y \end{pmatrix} \in \mathbb{R}^{d+1} \quad \text{and} \quad \tilde{\sigma}(x, y, u) := \begin{pmatrix} \sigma(x, u) \\ 0 \dots 0 \end{pmatrix} \in \mathbb{R}^{(d+1) \times p}.$$

Further, let x denote the variable in the augmented state space \mathbb{R}^M with $M := d + 1$, i.e. $x \equiv (x, y) \in \mathbb{R}^M$ and $x_M = y$. Then the Hamiltonian H in (4.2) is given by

$$H(x, p, P) := \sup_{u \in U} \left(-\tilde{b}(x, u) \cdot p - \frac{1}{2} \text{Tr}[\tilde{\sigma}\tilde{\sigma}^T(x, u)P] \right).$$

Thanks to the Lipschitz continuity of the function g that defines the boundary Γ_2 , we can observe that there exists $\mu > 0$ such that for any $z \in \Gamma_2$ we have

$$(4.9) \quad \bigcup_{0 \leq \xi \leq \mu} B(z - \xi, \xi\mu) \subset D^C,$$

where D^C denotes the complementary of the set D ; this corresponds to [16, condition (2.9)]. By the same but strongly simplified (thanks to the constancy of the derivative direction) arguments as in [16, Corollary 2.3], we obtain the existence of a function $\zeta \in C^2(\overline{D})$ which satisfies

$$\zeta \geq 0 \text{ in } \overline{D}, \quad -\partial_{x_M}\zeta \geq 1 \text{ on } \Gamma_2 \quad \text{and} \quad \|D\zeta\|_\infty, \|D^2\zeta\|_\infty \leq K$$

for some constant $K \geq 0$. Let us define for positive α and β the functions

$$u_{\alpha,\beta}(x) := u(x) - \alpha\zeta(x) - \beta, \quad v_{\alpha,\beta}(x) := v(x) + \alpha\zeta(x) + \beta.$$

With an abuse of notation, we denote by Du, Dv, D^2u, D^2v the corresponding elements of the semijets of u and v , see [13, Section 2]. We have

$$\lambda u_{\alpha,\beta} + H(x, Du_{\alpha,\beta}, D^2u_{\alpha,\beta}) - \lambda u - H(x, Du, D^2u) \leq -\lambda\beta + \alpha M$$

and

$$\lambda v_{\alpha,\beta} + H(x, Dv_{\alpha,\beta}, D^2v_{\alpha,\beta}) - \lambda v - H(x, Dv, D^2v) \geq \lambda\beta - \alpha M$$

for some constant $M \geq 0$ depending on the bounds on b, σ and $\|D\zeta\|_\infty, \|D^2\zeta\|_\infty$. Moreover, we have

$$\begin{aligned} -\partial_{x_M}u_{\alpha,\beta} &= -\partial_{x_M}u + \alpha\partial_{x_M}\zeta \leq -\partial_{x_M}u - \alpha, \\ -\partial_{x_M}v_{\alpha,\beta} &= -\partial_{x_M}v - \alpha\partial_{x_M}\zeta \geq -\partial_{x_M}v + \alpha \end{aligned}$$

and thanks to the non-negativity of ζ we also have

$$u_{\alpha,\beta} \leq M_g \quad \text{and} \quad v_{\alpha,\beta} \geq M_g$$

on Γ_1 . Therefore for any $\beta > 0$ and $\alpha = \alpha(\beta) > 0$ small enough, $u_{\alpha,\beta}$ (resp. $v_{\alpha,\beta}$) is a sub-solution (resp. super-solution) to (4.8) with the following modified boundary condition on Γ_2 :

$$(4.10) \quad -\partial_{x_M}\vartheta + \alpha \leq 0 \quad (\text{resp. } -\partial_{x_M}\vartheta - \alpha \geq 0).$$

In the sequel we prove a comparison principle between the sub-solution $u_{\alpha,\beta}$ and super-solution $v_{\alpha,\beta}$ of the modified problem. The comparison result between u and v follows from taking the limit of $\alpha, \beta \rightarrow 0$. In order to simplify the notation we denote the sub- and super-solution of the modified problem also by u and v .

Let

$$\Phi_\gamma(x) := u(x) - v(x) - 2\gamma(1 + |x|^2).$$

Thanks to the boundedness and the upper semicontinuity of $u - v$, the function Φ_γ admits a maximum point $\hat{x}_\gamma = \hat{x}$ in \overline{D} .

Let us assume that there exists a sequence $\{\gamma_k\}_{k \geq 0}$ such that $\gamma_k \rightarrow 0$ and the points \hat{x}_{γ_k} approach the boundary Γ_1 . In this case for any $x \in \overline{D}$ we have

$$(4.11) \quad \begin{aligned} u(x) - v(x) &= \limsup_{k \rightarrow \infty} u(x) - v(x) - 2\gamma_k(1 + |x|^2) \\ &\leq \limsup_{k \rightarrow \infty} u(\hat{x}) - v(\hat{x}) \leq 0. \end{aligned}$$

the last inequality follows by the fact that any convergent subsequence of points $\{\hat{x}_{\gamma_k}\}$ converges to a point of Γ_1 where one has $u - v \leq 0$ by assumption. We obtain the same result if $\Phi_{\gamma_k}(\hat{x}) \leq 0$ for all k .

Therefore, in the sequel, we assume that there exists a $\bar{\gamma}$ small enough such that $\text{dist}(\hat{x}, \Gamma_1) = \rho > 0$ and $u(\hat{x}) - v(\hat{x}) \geq \Phi_{\bar{\gamma}}(\hat{x}) > \rho$ for any $\gamma \leq \bar{\gamma}$ and some $\rho > 0$.

Then by the boundedness of u and v , we deduce that

$$(4.12) \quad \gamma|\hat{x}| \rightarrow 0 \quad \text{as } \gamma \rightarrow 0.$$

Thanks to the property (4.9) of the domain, the existence of a family of C^2 test functions $\{w_\varepsilon\}_{\varepsilon>0}$ can be proven as in [16, Theorem 4.1]. Among the other properties, $\{w_\varepsilon\} : \mathbb{R}^M \times \mathbb{R}^M \rightarrow \mathbb{R}$ satisfies:

$$(4.13) \quad w_\varepsilon(x, x) \leq \varepsilon$$

$$(4.14) \quad w_\varepsilon(x, y) \geq C \frac{|x - y|^2}{\varepsilon}$$

$$(4.15) \quad -\partial_{x_M} w_\varepsilon(x, y) \geq -C \frac{|x - y|^2}{\varepsilon} \quad \text{if } x \in \Gamma_2 \cap B(\hat{x}, \eta), y \in B(\hat{x}, \eta)$$

$$(4.16) \quad -\partial_{y_M} w_\varepsilon(x, y) \geq 0 \quad \text{if } y \in \Gamma_2 \cap B(\hat{x}, \eta), x \in B(\hat{x}, \eta)$$

for $\varepsilon > 0$ and some $\eta > 0$ small enough.

Applying the doubling variables procedure we define

$$\Phi_\varepsilon(x, y) := u(x) - v(y) - \gamma(1 + |x|^2) - \gamma(1 + |y|^2) - w_\varepsilon(x, y) - |x - \hat{x}|^4.$$

and we denote by $(x_\varepsilon, y_\varepsilon)$ its maximum point. Thanks to properties (4.13) and (4.14), we have for $\varepsilon \rightarrow 0$ that

$$(4.17) \quad x_\varepsilon, y_\varepsilon \rightarrow \hat{x} \quad \text{and} \quad \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \rightarrow 0.$$

Thus for ε small enough we can assume that $x_\varepsilon, y_\varepsilon \notin \Gamma_1$. Taking ε small enough we can also say that $x_\varepsilon, y_\varepsilon \in B(\hat{x}, \eta)$ and then we can make use of properties (4.15) and (4.16).

If $x_\varepsilon \in \Gamma_2$, using (4.12) and (4.17), we have that for ε small enough

$$\begin{aligned} & -\partial_{x_M} \left(w_\varepsilon(x_\varepsilon, y_\varepsilon) + \gamma(1 + |x_\varepsilon|^2) + |x_\varepsilon - \hat{x}|^4 \right) \\ & \geq -C \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} + 2\gamma|x_\varepsilon| - 4|x_\varepsilon - \hat{x}|^3 > -\alpha. \end{aligned}$$

Similarly if $y_\varepsilon \in \Gamma_2$ one has

$$-\partial_{y_M} \left(-w_\varepsilon(x_\varepsilon, y_\varepsilon) - \gamma(1 + |y_\varepsilon|^2) \right) \leq 0 < \alpha$$

for ε small enough.

This means that, considering respectively $w_\varepsilon(\cdot, y_\varepsilon) + \gamma(1 + |\cdot|^2) + |\cdot - \hat{x}|^4$ and $-w_\varepsilon(x_\varepsilon, \cdot) - \gamma(1 + |\cdot|^2)$ as test functions for u and v , for sufficiently small

values of ε the derivative boundary conditions in x_ε and y_ε can be neglected and one can only consider in $T_2 \cup D$

$$\lambda u + H(x_\varepsilon, Du, D^2u) \leq 0 \quad \text{and} \quad \lambda v + H(y_\varepsilon, Dv, D^2v) \geq 0$$

in the viscosity sense. Thanks to the properties of H , this leads to a contradiction using the arguments in [16]. \square

Corollary 4.4. *The value function ϑ is the unique bounded continuous viscosity solution to equation (4.8) in \bar{D} .*

We extend ϑ in a unique way to the full space $\mathbb{R}^d \times \mathbb{R}$ by (4.6). However, we can observe that, in order to characterize the original value function v , the way we extend ϑ for $y > M_g$ does not matter and only the values of ϑ for $y = g(x) \leq M_g$ are important.

5. NUMERICAL APPROXIMATION

In this section, convergence is proven for a general class of numerical schemes. Then, we show that a fully-discrete semi-Lagrangian method applied to the problem under consideration belongs to this class. Let BC denote the space of bounded and continuous functions in \mathbb{R}^{d+1} equipped with the L^∞ -norm. Further, we introduce the discretization parameters

$$\Delta x = (\Delta x_1, \dots, \Delta x_d) \in (\mathbb{R}^{>0})^d, \quad \Delta y > 0, \quad \rho = (\Delta x, \Delta y)$$

and denote the corresponding mesh by

$$\mathcal{G}^\rho := \{ (x_i, y_j) = (i\Delta x, j\Delta y), (i, j) \in \mathbb{Z}^d \times \mathbb{Z} \},$$

where $i\Delta x = (i_1\Delta x_1, \dots, i_N\Delta x_N)$, $i \in \mathbb{Z}^d$. We set

$$(5.1) \quad j_x := \min \{ j \in \mathbb{Z} \mid j\Delta y \geq g(x) \},$$

$$(5.2) \quad j_M := \min \{ j \in \mathbb{Z} \mid j\Delta y \geq M_g \}$$

and introduce a projection operator

$$(5.3) \quad \Pi^{\mathcal{G}^\rho}(\phi)(x_i, y_j) := \begin{cases} y_j & \text{if } j \geq j_M \\ \phi(x_i, y_j) & \text{if } j_M > j \geq j_{x_i} \\ \phi(x_i, y_{j_{x_i}}) & \text{if } j < j_{x_i} \end{cases}$$

for functions $\phi \in BC$. We aim to define a general approximation of the value function ϑ . We start considering a general scheme

$$S^\rho(x_i, y_j, \phi_{i,j}, \phi) = 0$$

that approximates the equation

$$\lambda\phi + H(x, y, D_x\phi, \partial_y\phi, D_x^2\phi) = 0$$

at node (x_i, y_j) , with H defined by (4.2). Here $\phi_{i,j} = \phi(x_i, y_j)$ and ϕ denote the values of ϕ at nodes different from (x_i, y_j) . S^ρ may represent a finite difference operator (see [9, 8, 19]), or a semi-Lagrangian (SL) scheme ([21, 11, 15]). The main idea of the numerical method described here is to mix

the use of a standard scheme for (4.8a), together with a projection step on ∂D in order to get the desired boundary conditions. Let us point out that a similar method was introduced for treating oblique derivative boundary conditions, i.e. the condition we have in Γ_2 , in [2] for the case $g(x) \equiv |x|$ and in [7] for a general Lipschitz continuous function g . In addition, here the numerical solution has also to satisfy the Dirichlet condition on Γ_1 .

We define V on \mathcal{G}^ρ such that

$$(5.4) \quad \begin{cases} V_{i,j} = y_j & \text{if } y_j \geq M_g, \\ S^\rho(x_i, y_j, V_{ij}, \Pi^{\mathcal{G}^\rho}(V)) = 0 & \text{if } g(x_i) \leq y_j < M_g, \\ V_{i,j} = V_{i,j_{x_i}} & \text{if } y_j < g(x_i) \end{cases}$$

and we denote by V^ρ its continuous extension by bilinear interpolation. We assume the grid \mathcal{G}^ρ aligned with the domain $[-M_g, M_g]$, so that we have $V^\rho = M_g$ on Γ_1 .

5.1. General convergence result. In order to prove the convergence of V^ρ to the unique viscosity solution ϑ of (4.8) we will make use of the arguments introduced by Barles and Souganidis in [3]. These make use of the properties of stability, consistency and monotonicity of the scheme. We assume that the scheme defined by (5.4) satisfies the following assumptions:

(H3.a) Stability: for any ρ the scheme (5.4) admits a solution $V^\rho \in BC(\bar{D})$. Moreover, there exist $M, L \geq 0$ such that

$$V^\rho(x, y) \leq M \quad \text{and} \quad |V^\rho(x, y) - V^\rho(x, y')| \leq L|y - y'|$$

for any $(x, y), (x, y') \in \bar{D}$ and $\rho > 0$.

(H3.b) Consistency: the scheme S^ρ is consistent with respect to (4.1) in \bar{D} , i.e. for all $(x, y) \in \bar{D}$ and every $\phi \in C^2(\bar{D})$ there holds

$$\lim_{\substack{\rho \rightarrow 0, \\ \bar{D} \ni (\xi, \gamma) \rightarrow (x, y), \\ \zeta \rightarrow 0}} S^\rho(\xi, \gamma, \phi(\xi, \gamma) + \zeta, \phi + \zeta) = \lambda\phi + H(x, D_x\phi, D_x^2\phi).$$

(H3.c) Monotonicity: for every $\rho, r \in \mathbb{R}$, $(x, y) \in \bar{D}$, $S^\rho(x, y, r, \phi)$ depends only on the values of ϕ in a neighborhood $B_{\eta(\rho)}(x, y)$ of (x, y) with $\eta(\rho) \geq 0$ such that $\eta(\rho) \rightarrow 0$ for $\rho \rightarrow 0$. For all function $\phi_1, \phi_2: \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ with $\phi_1 \geq \phi_2$ on $B_\rho(x, y)$, there holds

$$(5.5) \quad S^\rho(x, y, r, \phi_1) \leq S^\rho(x, y, r, \phi_2).$$

Remark 5.1. *Stability, consistency and monotonicity are the classical requirements to prove convergence of numerical schemes in the framework of viscosity solutions. However, in our case some slight modifications with respect to the original assumptions considered in [3] are necessary. In particular, the additional regularity assumption with respect to the variable y turns out to be fundamental to deal with Dirichlet boundary conditions, while the reason for our formulation of the monotonicity condition is the fact that*

the numerical scheme S^ρ defined on \bar{D} may use some values of the function V^ρ outside the domain \bar{D} .

Theorem 5.2. *Let assumptions (H1)-(H2) be satisfied and let the scheme (5.4) satisfy assumption (H3). Then for $\rho \rightarrow 0$ the solution V^ρ converges to the unique viscosity solution to (4.1).*

Proof. Let us define for $(x, y) \in \bar{D}$

$$\begin{aligned}\bar{V}(x, y) &:= \limsup_{\substack{\bar{D} \ni (\xi, \gamma) \rightarrow (x, y) \\ \rho \rightarrow 0}} V^\rho(\xi, \gamma), \\ \underline{V}(x, y) &:= \liminf_{\substack{\bar{D} \ni (\xi, \gamma) \rightarrow (x, y) \\ \rho \rightarrow 0}} V^\rho(\xi, \gamma)\end{aligned}$$

Observe that the semi-limits are well-defined thanks to the boundedness of V^ρ uniformly with respect to ρ . We start by proving that \bar{V} is a viscosity sub-solution to equation (4.1).

Thanks to the Lipschitz continuity of V^ρ with respect to y (uniform with respect to ρ) and the fact that, by definition, $V^\rho(x, y) = M_g$ on Γ_1 one has for any $(\xi, \gamma) \in \bar{D}$

$$|V^\rho(\xi, \gamma) - M_g| = |V^\rho(\xi, \gamma) - V^\rho(\xi, M_g)| \leq L|\gamma - M_g|,$$

so that

$$\lim_{\substack{\bar{D} \ni (\xi, \gamma) \rightarrow (x, y) \\ \rho \rightarrow 0}} V^\rho(\xi, \gamma) = M_g.$$

Hence, $\bar{V} = M_g$ on Γ_1 and the Dirichlet condition is satisfied.

Let $\varphi \in C^2(\bar{D})$ and let (\bar{x}, \bar{y}) be a local maximum point for $\bar{V} - \varphi$ on \bar{D} . Without loss of generality we can assume that (\bar{x}, \bar{y}) is a strict local maximum in $B_r(\bar{x}, \bar{y}) \cap \bar{D}$ for a certain $r > 0$ and $\varphi \geq 2 \sup_\rho \|W^\rho\|_\infty$ outside the ball $B_r(\bar{x}, \bar{y})$. We claim that

$$\begin{cases} \lambda u + H(\bar{x}, \bar{y}, D_x \varphi, \partial_y \varphi, D_x^2 \varphi) \leq 0 & \text{if } (\bar{x}, \bar{y}) \in D, \\ \min(\lambda u + H(\bar{x}, \bar{y}, D_x \varphi, \partial_y \varphi, D_x^2 \varphi), -\partial_y \varphi) \leq 0 & \text{if } (\bar{x}, \bar{y}) \in \Gamma_2. \end{cases}$$

We follow the argument in Barles and Souganidis [3]. There exists a sequence $\rho_k, (x_k, y_k)$ such that $\rho_k \rightarrow 0, (x_k, y_k) \rightarrow (\bar{x}, \bar{y})$ for $k \rightarrow \infty$, and

$$(5.6) \quad (V^{\rho_k} - \varphi)(x_k, y_k) = \max_D (V^{\rho_k} - \varphi) = \delta_k \rightarrow 0, \quad \text{as } k \rightarrow \infty$$

and

$$V^{\rho_k}(x_k, y_k) \rightarrow \bar{V}(\bar{x}, \bar{y}), \quad \text{as } k \rightarrow \infty.$$

–**Case 1:** $(\bar{x}, \bar{y}) \in D$. For k large enough $(x_k, y_k) \in D$. Since g is continuous, for ρ_k small enough we can assume that $y > g(x)$ for any $(x, y) \in B_{\eta(\rho_k)}(x_k, y_k)$ (where $B_{\eta(\rho_k)}(x_k, y_k)$ is the neighborhood that appears in assumption (H3.c)). Consequently, for k big enough

$$\Pi^{\mathcal{G}_{\rho_k}}(V^{\rho_k}) = V^{\rho_k} \quad \text{and} \quad V^{\rho_k} < \varphi + \delta_k$$

in $B_{\eta(\rho_k)}(x_k, y_k)$. By the monotonicity of the scheme (assumption (H3.c)) we further deduce

$$\begin{aligned} 0 &= S^{\rho_k}(x_k, y_k, V^{\rho_k}(x_k, y_k), \Pi^{\mathcal{G}_{\rho_k}}(V^{\rho_k})) \\ &\geq S^{\rho_k}(x_k, y_k, \varphi(x_k, y_k) + \delta_k, \varphi + \delta_k) \end{aligned}$$

and by the consistency assumption (H3.b) we obtain that as $k \rightarrow \infty$

$$\lambda\varphi + H(\bar{x}, \bar{y}, D_x\varphi, \partial_y\varphi, D_x^2\varphi) \leq 0.$$

– **Case 2:** $(\bar{x}, \bar{y}) \in \Gamma_2$. If $-\partial_y\varphi(\bar{x}, \bar{y}) \leq 0$ the sub-solution property on Γ_2 is automatically satisfied. Let us assume that $-\partial_y\varphi(\bar{x}, \bar{y}) > 0$.

We point out that if $\bar{y} = g(\bar{x})$, (x_k, y_k) can also be on Γ_2 and the scheme may involve values $V_{m,n}$ on some point (x_m, y_n) which is not in \bar{D} .

If $-\partial_y\varphi(\bar{x}, \bar{y}) > 0$, there exists a neighbourhood \mathcal{V} of (\bar{x}, \bar{y}) where $\partial_y\varphi$ is well defined and $-\partial_y\varphi > 0$. Therefore,

$$(5.7) \quad y \leq y' \Rightarrow \varphi(x, y) \geq \varphi(x, y') \quad \forall (x, y), (x, y') \in \mathcal{V}.$$

and, taking k large enough, $B_{\eta(\rho_k)}(x_k, y_k) \subset \mathcal{V}$. Let $(x, y) \in B_{\eta(\rho_k)}(x_k, y_k)$. If $y \geq g(x)$, we have

$$\Pi^{\mathcal{G}_{\rho}}(V^{\rho_k}(x, y)) = V^{\rho_k}(x, y) \leq \varphi(x, y) + \delta_k.$$

If $y < g(x)$, $\Pi^{\mathcal{G}_{\rho}}(V^{\rho_k}(x, y)) = V^{\rho_k}(x, y_{j_x})$ and we have

$$(5.8) \quad \begin{aligned} \Pi^{\mathcal{G}_{\rho}}(V^{\rho_k}(x, y)) &= V^{\rho_k}(x, y_{j_x}) \\ &\leq \varphi(x, y_{j_x}) + \delta_k \quad \text{using (5.6)} \\ &\leq \varphi(x, y) + \delta_k \quad \text{using (5.7)} \end{aligned}$$

For the last inequality one also need to observe that if $(x, y) \in \mathcal{V}$ and $y < g(x)$, thanks to the continuity of g one can choose k big enough so that also $(x, y_{j_x}) \in \mathcal{V}$.

Consequently, we have $\Pi^{\mathcal{G}_{\rho}}(V^{\rho_k}) \leq \varphi + \delta_k$ on $B_{\eta(\rho_k)}(x_k, y_k)$. Thus by monotonicity we have

$$\begin{aligned} 0 &= S^{\rho_k}(x_k, y_k, V^{\rho_k}(x_k, y_k), \Pi^{\mathcal{G}_{\rho}}(V^{\rho_k})) \\ &\geq S^{\rho_k}(x_k, y_k, \varphi(x_k, y_k) + \delta_k, \varphi + \delta_k) \end{aligned}$$

and using consistency when $k \rightarrow \infty$ we have

$$(5.9) \quad \lambda\varphi + H(\bar{x}, \bar{y}, D_x\varphi, \partial_y\varphi, D_x^2\varphi) \leq 0.$$

This proves that \bar{V} is a viscosity subsolution to (4.1). Analogously one can show that \underline{V} is a viscosity sub-solution. Applying the comparison principle (Theorem 4.3), it follows that $\underline{V} \geq \bar{V}$ on \bar{D} . Since $\underline{V} \leq \bar{V}$ is always true by definition, it is possible to conclude that $\bar{V} = \underline{V}$ on \bar{D} , which proves the assertion. \square

5.2. Semi-Lagrangian scheme. In this section we introduce a semi-Lagrangian scheme and verify that it satisfies assumptions (H3).

5.2.1. *Semi-discretization.* To simplify the presentation, we consider first a semi-discrete version of the scheme. Let $h > 0$ be the time discretization parameter of the dynamics. We define the mapping $\mathcal{T}_0: BC \rightarrow BC$ with

$$\mathcal{T}_0(\phi)(x, y) := \min_{u \in U} \left\{ \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} \phi \left(x + hb(x, u) + \sqrt{hp}(-1)^k \sigma_{\lfloor \frac{k+1}{2} \rfloor}(x, u), \frac{y}{1 - \lambda h} \right) \right\}$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Furthermore, let $X(M_g)$ be the subset of BC given by

$$X(M_g) := \left\{ \phi \in BC \cap Lip_y \mid \begin{array}{l} \phi(x, y) = y \text{ for all } (x, y) \in \mathbb{R}^d \times \mathbb{R} \\ \text{with } M_g \leq y \leq 2M_g(1 - \lambda h) \end{array} \right\},$$

where Lip_y denotes the set of functions that are Lipschitz continuous with respect to y . We introduce the operator $T_0: X(M_g) \rightarrow X(M_g)$ defined by

$$(5.10) \quad T_0(\phi)(x, y) := \mathcal{T}_0(\Pi(\phi))(x, y)$$

where Π is the the continuous version of the projection $\Pi^{\mathcal{G}^\rho}$, i.e. :

$$\Pi(\phi)(x, y) := \begin{cases} y \wedge 2M_g & \text{if } y \geq M_g, \\ \phi(x, y) & \text{if } g(x) \leq y < M_g, \\ \phi(x, g(x)) & \text{if } y \leq g(x). \end{cases}$$

Remark 5.3. *Compared with (5.3) here we have modified the value of the operator Π for $y \geq M_g$ in order to guarantee the boundedness of the operator T . It is possible to verify that this does not affect the proof of the main convergence result which only concerns neighborhoods of points in \bar{D} .*

Observe that for functions ϕ in $X(M_g)$ the operator Π simplifies to

$$(5.11) \quad \Pi(\phi)(x, y) = \phi(x, g(x) \vee (y \wedge 2M_g)).$$

Lemma 5.4. *The operator T_0 is well-defined.*

Proof. For every $\phi \in X(M_g)$ the image $T_0(\phi)$ is bounded and continuous. For $L \geq 0$ let $Lip_y(L) \subset Lip_y$ be the subset of functions with Lipschitz constant L . Then we have for $\phi \in Lip_y(L)$

$$\begin{aligned} & |T_0(\phi)(x, y) - T_0(\phi)(x, y')| \\ & \leq \max_{u \in U} \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} \left| \Pi(\phi) \left(X_x^{u, k, p}(h), Y_y^u(h) \right) - \Pi(\phi) \left(X_x^{u, k, p}(h), Y_{y'}^u(h) \right) \right| \end{aligned}$$

where we use the notation

$$\begin{aligned} X_x^{u, k, p}(h) &:= x + hb(x, u) + \sqrt{hp}(-1)^k \sigma_{\lfloor \frac{k+1}{2} \rfloor}(x, u), \\ Y_y^u(h) &:= \frac{y}{1 - \lambda h} \end{aligned}$$

for any $(x, y) \in \mathbb{R}^{d+1}$ and $u \in U$. From (5.11) we obtain

$$\begin{aligned} |\Pi(\phi)(x, y) - \Pi(\phi)(x, y')| &= |\phi(x, y \vee (g(x) \wedge 2M_g)) - \phi(x, g(x) \vee (y' \wedge 2M_g))| \\ &\leq L|(y \vee (g(x) \wedge 2M_g)) - (y' \vee (g(x) \wedge 2M_g))| \\ &\leq L|y - y'|; \end{aligned}$$

here we used the classical inequalities

$$(5.12) \quad \begin{aligned} |a \vee b - c \vee d| &\leq |a - c| \vee |b - d|, \\ |a \wedge b - c \wedge d| &\leq |a - c| \vee |b - d|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |T_0(\phi)(x, y) - T_0(\phi)(x, y')| &\leq \max_{u \in U} \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} L |Y_y^u(h) - Y_{y'}^u(h)| \\ &\leq \max_{u \in U} \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} L \frac{|y - y'|}{(1 - \lambda h)} \end{aligned}$$

which implies $T_0(\phi) \in Lip_y(L)$.

Moreover, for $y \geq M_g$ one has $y/(1 - \lambda h) > M_g$ and then

$$T_0(\phi)(x, y) = \min_{u \in U} \left\{ \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} \left(\frac{y}{1 - \lambda h} \right) \wedge 2M_g \right\} = y \wedge 2M_g(1 - \lambda h)$$

which proves that $T_0(\phi) \in X(M_g)$. \square

Next we verify the properties of stability, consistency and monotonicity (in the sense of assumption (H3)) for the scheme

$$(5.13) \quad S^h(x, y, \phi(x, y), \phi) := \frac{1}{h} \left(\phi(x, y) - \mathcal{T}_0(\phi)(x, y) \right) \quad \text{for } \phi \in X(M_g).$$

The fact of dealing with an infinite horizon problem and therefore with a stationary PDE requires the use of a fixed point argument in order to prove the existence of a solution for (5.13). We point out that, due to the presence of the projection operator, $X(M_g)$ is the suitable space to guarantee the continuity of $T(\phi)$.

Lemma 5.5 (Stability). *For $0 \leq \lambda h < 1$ there exists a fixed point of the equation*

$$(5.14) \quad S^h(x, y, \phi(x, y), \Pi(\phi)) = \frac{1}{h} \left(\phi(x, y) - T_0(\phi)(x, y) \right)$$

in $X(M_g)$ and the scheme S^h is stable in the sense of assumption (H3.a).

Proof. T_0 is a contraction on $X(M_g)$, since

$$\|T_0(v) - T_0(w)\|_\infty \leq \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} \|(\Pi v) - (\Pi w)\|_\infty \leq (1 - \lambda h) \|v - w\|_\infty.$$

Furthermore, since $(X(M_g), \|\cdot\|_\infty)$ is a closed subset of $(BC, \|\cdot\|_\infty)$, it is a complete metric space. Thus we obtain existence by the Banach-Caccioppoli fixed point theorem. Stability in the sense of assumption (H3.a) follows from the fact that the bound and Lipschitz constant of the fixed point do not depend on h . \square

The lemma implies in particular, that for any $(x, y) \in \overline{D}$ there exists a solution in $X(M_g)$ of the scheme

$$S^h(x, y, \phi(x, y), \Pi(\phi)) = 0,$$

which we denote by V^h in the sequel. Observe that V^h automatically satisfies the boundary condition $V^h(x, M_g) = M_g$.

Lemma 5.6 (Consistency). *For any smooth and bounded function ϕ , there exists some $C > 0$ such that*

$$\left| \frac{1}{h} (\phi(x, y) - \mathcal{T}_0(\phi)(x, y)) - \left(\lambda\phi + H(x, y, D_x\phi, \partial_y\phi, D_x^2\phi) \right) \right| \leq Ch.$$

Proof. The assertion follows straightforward by a Taylor expansion. For simplicity we show the results in the one dimensional case. Observing that

$$\frac{1}{1 - \lambda h} = 1 + \lambda h + O(h^2),$$

we have

$$\begin{aligned} & \phi(x, y) - \mathcal{T}_0(\phi)(x, y) \\ &= \phi(x, y) - \frac{(1 - \lambda h)}{2} \min_{u \in U} \left\{ \phi \left(x + hb(x, u) + \sqrt{h}\sigma(x, u), \frac{y}{1 - \lambda h} \right) \right. \\ & \quad \left. + \phi \left(x + hb(x, u) - \sqrt{h}\sigma(x, u), \frac{y}{1 - \lambda h} \right) \right\} \\ &= \lambda h \phi(x, y) - \frac{(1 - \lambda h)}{2} \min_{u \in U} \left\{ \phi_x(x, y)(hb(x, u) + \sqrt{h}\sigma(x, u)) + \phi_y(x, y)(\lambda h y) \right. \\ & \quad + \frac{1}{2} \phi_{xx}(x, y)(hb(x, u) + \sqrt{h}\sigma(x, u))^2 + \phi_{xy}(x, y)(hb(x, u) + \sqrt{h}\sigma(x, u))(\lambda h y) \\ & \quad + \frac{1}{6} \phi_{xxx}(x, y)(hb(x, u) + \sqrt{h}\sigma(x, u))^3 \\ & \quad + \phi_x(x, y)(hb(x, u) - \sqrt{h}\sigma(x, u)) + \phi_y(x, y)(\lambda h y) \\ & \quad + \frac{1}{2} \phi_{xx}(x, y)(hb(x, u) - \sqrt{h}\sigma(x, u))^2 + \phi_{xy}(x, y)(hb(x, u) - \sqrt{h}\sigma(x, u))(\lambda h y) \\ & \quad \left. + \frac{1}{6} \phi_{xxx}(x, y)(hb(x, u) - \sqrt{h}\sigma(x, u))^3 \right\} + O(h^2) \\ &= \lambda h \phi(x, y) - \min_{u \in U} \left\{ hb(x, u) \phi_x(x, y) + h \lambda y \phi_y(x, y) + \frac{1}{2} h \sigma^2(x, u) \phi_{xx}(x, y) \right\} + O(h^2). \end{aligned}$$

\square

Lemma 5.7 (Monotonicity). *For any $h > 0$ with $0 \leq \lambda h < 1$ the scheme is monotone in the sense of assumption (H3.c).*

Proof. For $(x, y) \in \bar{D}$ the definition of the operator T involves only values at points $(x + hb(x, u) + \sqrt{hp}(-1)^k \sigma_{\lfloor \frac{k+1}{2} \rfloor}(x, u), y/(1 - \lambda h))$ which remain in a neighborhood $B_{\eta(h)}$ of (x, y) , with $\eta(h) = O(\sqrt{h})$. Moreover, by definition we have

$$S^h(x, y, \phi(x, y), \phi) = \frac{1}{h} \left(\phi(x, y) - \mathcal{T}_0(\phi)(x, y) \right)$$

and the assertion follows immediately, since for $\phi_1 \leq \phi_2$ we have

$$\mathcal{T}_0(\phi_1)(x, y) \leq \mathcal{T}_0(\phi_2)(x, y).$$

□

Having verified the assumptions (H3), we can apply Theorem 5.2 and state a convergence result for the semi-discrete scheme.

Theorem 5.8. *Let assumptions (H1)-(H2) be satisfied. Then for $h \rightarrow 0$ the solution V^h of the scheme (5.14) converges uniformly to ϑ in \bar{D} .*

5.2.2. *Full-discretization.* Now, we consider the scheme on the discretized space. This requires to introduce an interpolation step in order to define the value of the solution at points

$$\left(x_i + hb(x_i, u) + \sqrt{hp}(-1)^k \sigma_{\lfloor \frac{k+1}{2} \rfloor}(x_i, u), \frac{y_j}{1 - \lambda h} \right)$$

that may not belong to the grid \mathcal{G}^ρ ($\rho \equiv (\Delta x, \Delta y)$). We denote by $[\cdot]$ the monotone, P_1 interpolation operator, satisfying the following properties:

$$(5.15) \quad \begin{aligned} (i) & \quad [\phi](x_i, y_j) = \phi(x_i, y_j), \quad \forall (x_i, y_j) \in \mathcal{G}^\rho; \\ (ii) & \quad |[\phi](x, y) - \phi(x, y)| \leq L|(\Delta x, \Delta y)|, \quad \forall \phi \in Lip(L); \\ (iii) & \quad |[\phi](x, y) - \phi(x, y)| \leq C(|\Delta x|^2 + \Delta y^2) \|D^2 \phi\|_\infty \quad \forall \phi \in C^2; \\ (iv) & \quad \phi_1 \leq \phi_2 \Rightarrow [\phi_1] \leq [\phi_2]. \end{aligned}$$

The fully discrete operator is then defined by

$$\mathcal{F}(\phi)(x_i, y_j) := \min_{u \in U} \left\{ \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} [\phi] \left(x_i + hb(x_i, u) + \sqrt{hp}(-1)^k \sigma_{\lfloor \frac{k+1}{2} \rfloor}(x_i, u), \frac{y_j}{1 - \lambda h} \right) \right\}$$

and the fully discrete scheme reads:

$$(5.16) \quad S^\rho(x_i, y_j, \phi(x_i, y_j), \Pi^{\mathcal{G}^\rho} \phi) := \frac{1}{h} \left(\phi(x_i, y_j) - \mathcal{F}(\Pi^{\mathcal{G}^\rho} \phi)(x_i, y_j) \right)$$

where

$$\Pi^{\mathcal{G}^\rho}(\phi)(x_i, y_j) := \begin{cases} y_j \wedge 2M_g & \text{if } y_j \geq M_g, \\ \phi(x_i, y_j) & \text{if } g(x_i) \leq y_j < M_g, \\ \phi(x_i, g(x_i)) & \text{if } y_j \leq g(x_i). \end{cases}$$

We associate to each grid function $S^\rho(\cdot, \cdot, \phi(\cdot, \cdot), \Pi^{\mathcal{G}^\rho} \phi)$ the continuous function obtained by bilinear interpolation in the grid points and denote it by the same expression.

Theorem 5.9. *Let $0 \leq \lambda h < 1$ and let the following condition be satisfied*

$$\frac{\Delta x^2 + \Delta y^2}{h} \rightarrow 0 \quad \text{as } \rho, h \rightarrow 0.$$

Then scheme (5.16) is stable in the subspace of piecewise linear functions in $X(M_g)$, consistent and monotone in the sense of definition (H3).

Proof. Stability follows by similar arguments as in the semi-discrete case. Indeed, observing that the bilinear interpolation preserves the Lipschitz constant, the fix point theorem still holds if we consider piecewise bilinear functions in $X(M_g)$. Consistency can follow by (5.15)(iii). Indeed, when interpolation is taken into account, in the estimates obtained proving Lemma 5.6 one gets the extra term

$$\frac{\Delta x^2 + \Delta y^2}{h} \|\phi_{xx}\|$$

which goes to zero guaranteeing the consistency property. Monotonicity is ensured by the use of a linear interpolation. \square

Corollary 5.10. *Let $0 \leq \lambda h < 1$ and let the following condition be satisfied*

$$\frac{\Delta x^2 + \Delta y^2}{h} \rightarrow 0 \quad \text{as } \rho, h \rightarrow 0.$$

Then, as $\rho, h \rightarrow 0$ the solution V^ρ of the fully discrete scheme (5.16) converges in \bar{D} to the unique viscosity of equation (4.8).

Proof. The proof follows from Theorem 5.9 and Theorem 5.2. \square

6. CONCLUSIONS

In this paper we have studied infinite horizon stochastic optimal control problems with cost in maximum form. By the introduction of an auxiliary Markovian problem and dynamic programming arguments we have characterized the associated value function by means of a HJB equation with mixed Dirichlet-derivative boundary conditions. We have proposed a general numerical scheme which incorporates the treatment of the boundary condition and proved its convergence to the unique viscosity solution of the HJB equation under the assumptions of monotonicity, consistency and stability. Furthermore, we have shown that a particular semi-Lagrangian scheme satisfies such assumptions and therefore can be used to approximate the value function of the original problem.

Further directions of work might involve the application of our scheme to the computation of viable and invariant sets as well as the theoretical proof of the rate of convergence associated to our scheme.

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