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# INFINITE HORIZON STOCHASTIC OPTIMAL CONTROL PROBLEMS WITH RUNNING MAXIMUM COST

AXEL KRÖNER, ATHENA PICARELLI, AND HASNAA ZIDANI

ABSTRACT. An infinite horizon stochastic optimal control problem with running maximum cost is considered. The value function is characterized as the viscosity solution of a second-order HJB equation with mixed boundary condition. A general numerical scheme is proposed and convergence is established under the assumptions of consistency, monotonicity and stability of the scheme. A convergent semi-Lagrangian scheme is presented in detail.

## 1. INTRODUCTION

In this paper we consider infinite horizon stochastic control problems with cost in a maximum form of type

$$(1.1) \quad \left\{ \begin{array}{l} \inf_{u \in \mathcal{U}} \mathbb{E} \left[ \max_{s \in [0, \infty)} e^{-\lambda s} g(X_x^u(s)) \right], \quad \text{subject to} \\ dX_x^u(s) = b(X_x^u(s), u(s))ds + \sigma(X_x^u(s), u(s))dB(s), \quad s \in [0, \infty), \\ X_x^u(0) = x \in \mathbb{R}^d. \end{array} \right.$$

As we will discuss in the paper, this type of problem can be used for the characterization of viable and invariant sets. Another application of control problem with maximum running cost comes from the study of some path-dependent options in finance (lookback options).

The contribution of this paper is to consider stochastic control problems with maximum cost in infinite horizon. We focus on the derivation of the associated Hamilton-Jacobi-Bellman (HJB) equation and on its numerical approximation. Problems with running maximum in infinite horizon have been previously studied in [17] in the particular case of Zubov-type equations. For the numerical aspects we follow ideas developed in [7] for the corresponding finite time horizon problem. Other important contributions to the study of this kind of problems in finite horizon can be also found in [5, 4] and, concerning the numerical aspects, in [1]. However, in these works the dynamic programming techniques are applied on the  $L^p$ -approximation

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of the  $L^\infty$ -cost functional, using that

$$a \vee b \simeq (a^p + b^p)^{\frac{1}{p}} \quad (\text{for } p \rightarrow \infty),$$

for any  $a, b \geq 0$ , where  $a \vee b := \max(a, b)$  whereas, following more closely the approach in [17, 7], we directly derive the Dynamic Programming equation working on the original formulation of the cost involving the running maximum.

By this way, the optimal control problem (1.1) is connected to the solution of a second-order Hamilton-Jacobi-Bellman (HJB) equation with mixed boundary condition involving oblique derivative and Dirichlet conditions. The presence of oblique derivative boundary conditions, to be considered in the viscosity sense [18], typically arises when dealing with running maximum operators (see [6, 7]). The additional Dirichlet condition is instead introduced in order to ensure the boundedness of the problem and to guarantee uniqueness of the solution (see also [17]). The last part of the paper is devoted to numerical aspects. First, a general scheme is proposed to deal with HJB equations with mixed derivative boundary conditions. Then, a convergence result is proved by using the general framework of Barles-Souganidis [3] based on the monotonicity, stability, consistency of the scheme. Then, we focus on a semi-Lagrangian scheme and prove its convergence. Let us recall that the semi-Lagrangian schemes have been introduced for first order Hamilton-Jacobi equations in [12]. The extensions to the second order case have been studied in several papers [20, 11, 14, 15]. For stationary equations, which is the case studied in this paper, semi-Lagrangian schemes are formulated as fix point problems. In our context, the numerical scheme that will be studied couples the classical semi-Lagrangian scheme with an additional projection step on the boundary taking into account the overstepping of the domain which is typical in such a wide stencil scheme.

The paper is organized as follows: Section 2 introduces the problem and some motivations. Sections 3 and 4 are devoted to the characterization of the value function by the appropriate HJB equation: the Dynamic Programming Principle (DPP) is established, the HJB equation derived and uniqueness proved through a strong comparison principle. In Section 5 the numerical approximation is discussed and a general convergence result is provided. The semi-Lagrangian scheme is presented in Section 5.2 and the properties of this scheme are investigated.

## 2. FORMULATION OF THE PROBLEM

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $\{\mathcal{F}_t, t \geq 0\}$  a filtration on  $\mathcal{F}$  and  $B(\cdot)$  a  $\{\mathcal{F}_t\}_{t \geq 0}$ -Brownian motion in  $\mathbb{R}^p$ ,  $p \geq 1$ . Let  $\mathcal{U}$  be a set of progressively measurable processes with values in  $U \subset \mathbb{R}^m$ ,  $m \geq 1$ ,  $U$  compact set. We

consider stochastic differential equations of type

$$(2.1) \quad \begin{cases} dX_x^u(s) = b(X_x^u(s), u(s))ds + \sigma(X_x^u(s), u(s))dB(s), & s \in [0, \infty), \\ X_x^u(0) = x \end{cases}$$

with control  $u \in \mathcal{U}$ .

Throughout the paper we make the following assumptions on the coefficients in (2.1):

**(H1)**  $b: \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$  and  $\sigma: \mathbb{R}^d \times U \rightarrow \mathbb{R}^{d \times p}$  are continuous functions satisfying, for some constants  $C_0 \geq 0$ ,

$$\begin{aligned} |b(x, u) - b(y, u)| + |\sigma(x, u) - \sigma(y, u)| &\leq C_0|x - y|, \\ |b(x, u)| + |\sigma(x, u)| &\leq C_0 \end{aligned}$$

for any  $x, y \in \mathbb{R}^d$  and  $u \in U$ .

**Theorem 2.1.** *Under assumption (H1), for any  $x \in \mathbb{R}^d$  there exists a unique strong solution of (2.1). Moreover, denoted by  $X_x^u(\cdot)$  such a solution, there exists a constant  $C \geq 0$  such that*

$$(2.2) \quad \mathbb{E} \left[ \sup_{\theta \in [0, T]} |X_x^u(\theta) - X_{x'}^u(\theta)| \right] \leq Ce^{CT}|x - x'|$$

for any  $u \in \mathcal{U}$ ,  $T > 0$  and  $x, x' \in \mathbb{R}^d$ .

For a proof see [22, p. 42, Thm. 6.3] and [21, p.14].

To formulate the optimal control problem we introduce the cost function  $g$  such that

**(H2)**  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies

$$|g(x) - g(y)| \leq L_g|x - y|, \quad |g(x)| \leq M_g$$

for some  $M_g, L_g \geq 0$ , for any  $x, y \in \mathbb{R}^d$ .

Given a discount factor  $\lambda > 0$ , we aim to solve the infinite horizon optimal control problem of minimizing over the set of controls  $\mathcal{U}$  the following cost

$$(2.3) \quad J(x, u) = \mathbb{E} \left[ \max_{s \in [0, \infty)} e^{-\lambda s} g(X_x^u(s)) \right].$$

We will denote by  $v: \mathbb{R}^d \rightarrow \mathbb{R}$  the associated value function, that is

$$v(x) = \inf_{u \in \mathcal{U}} J(x, u).$$

For given  $x \in \mathbb{R}^d$  a control  $\bar{u} \in \mathcal{U}$  that minimizes (2.3) is called optimal and the corresponding trajectory  $(s, X_x^{\bar{u}}(s))$  an optimal trajectory.

**Remark 2.2.** *We point out that the study of optimal control problems with cost depending on a running maximum as in (2.3) can be used to characterise the region of viability/invariance of a domain. This results by the application of a level set approach as shown, for the case of a finite horizon of time, in [7, Section 2.3].*

## 3. DYNAMIC PROGRAMMING PRINCIPLE

In the next sections, strongly motivated by numerical approximation purposes, we aim to characterize  $v$  as a solution, in the viscosity sense, of a HJB equation. The presence of the maximum operator inside the expectation, makes the cost  $J$  in (2.3) non-Markovian and it is well known that this prevents to establish a Dynamic Programming Principle (DPP), which is the first fundamental result towards the HJB characterisation. A classical strategy to overcome this difficulty consists in adding an auxiliary variable  $y$  that, roughly speaking, gets rid of the non-Markovian component of the cost. This has been originally used in [6], in finite horizon. Here, an approximation technique of the  $L^\infty$ -norm is used, as a consequence the results apply only if the function  $g$  in (2.3) is a positive function. We proceed in a different way, following the approach in [7, 17] and directly deriving the HJB equation without making use of any approximation.

Let us define the auxiliary value function  $\vartheta: \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$

$$\vartheta(x, y) := \inf_{u \in \mathcal{U}} \mathbb{E} \left[ \max_{s \in [0, \infty)} e^{-\lambda s} g(X_x^u(s)) \vee y \right],$$

which satisfies, for any  $x \in \mathbb{R}^d$ ,

$$\vartheta(x, g(x)) = v(x).$$

Consequently, if  $\vartheta$  is known, one can immediately recover the corresponding value of  $v$ , so that from now on we will only consider  $\vartheta$  as value function of our problem.

We start by proving the continuity of the value function  $\vartheta$ .

**Proposition 3.1.** *Let assumptions (H1)-(H2) be satisfied. Then the value function  $\vartheta$  is uniformly continuous in  $\mathbb{R}^d \times \mathbb{R}$ . Moreover,*

$$\vartheta(x, M_g) = M_g \quad \text{and} \quad |\vartheta(x, y)| \leq M_g$$

for any  $(x, y) \in \mathbb{R}^d \times \mathbb{R}$  with  $|y| \leq M_g$ .

*Proof.* The last statement follows directly by the definition of  $\vartheta$  and the bound on  $g$  given by assumption (H2).

By the very definition of  $\vartheta$  we have that for any  $(x, y), (\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}$

$$\begin{aligned} & |\vartheta(x, y) - \vartheta(\xi, \eta)| \\ & \leq \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \left| \max_{s \in [0, \infty)} e^{-\lambda s} g(X_x^u(s)) \vee y - \max_{s \in [0, \infty)} e^{-\lambda s} g(X_\xi^u(s)) \vee \eta \right| \right] \\ & \leq \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \max_{s \in [0, \infty)} \left| e^{-\lambda s} g(X_x^u(s)) - e^{-\lambda s} g(X_\xi^u(s)) \right| \vee |y - \eta| \right] \\ & \leq \sup_{u \in \mathcal{U}} \mathbb{E} \left[ \max_{s \in [0, \infty)} \left| e^{-\lambda s} g(X_x^u(s)) - e^{-\lambda s} g(X_\xi^u(s)) \right| \right] + |y - \eta| \end{aligned}$$

For any  $T > 0$  we have that (recalling that  $g$  is bounded)

$$\begin{aligned}
 & \mathbb{E} \left[ \max_{s \in [0, \infty)} \left| e^{-\lambda s} g(X_x^u(s)) - e^{-\lambda s} g(X_\xi^u(s)) \right| \right] \\
 & \leq \mathbb{E} \left[ \max_{s \in [0, T]} \left| e^{-\lambda s} g(X_x^u(s)) - e^{-\lambda s} g(X_\xi^u(s)) \right| \right] + \mathbb{E} \left[ \max_{s \in [T, \infty)} \left| e^{-\lambda s} g(X_x^u(s)) - e^{-\lambda s} g(X_\xi^u(s)) \right| \right] \\
 & \leq \mathbb{E} \left[ \max_{s \in [0, T]} e^{-\lambda s} \left| g(X_x^u(s)) - g(X_\xi^u(s)) \right| \right] + \mathbb{E} \left[ \max_{s \in [T, \infty)} 2M_g e^{-\lambda s} \right] \\
 & = \mathbb{E} \left[ \max_{s \in [0, T]} e^{-\lambda s} \left| g(X_x^u(s)) - g(X_\xi^u(s)) \right| \right] + 2M_g e^{-\lambda T}.
 \end{aligned}$$

For the first term we have the classical estimates (see Theorem 2.1)

$$\mathbb{E} \left[ \max_{s \in [0, T]} e^{-\lambda s} \left| g(X_x^u(s)) - g(X_\xi^u(s)) \right| \right] \leq L_g \mathbb{E} \left[ \sup_{s \in [0, T]} |X_{t,x}^u(s) - X_{t,\xi}^u(s)| \right] \leq L_g C e^{CT} |x - \xi|.$$

So putting everything together we get that for every  $T$

$$|\vartheta(x, y) - \vartheta(\xi, \eta)| \leq L_g C e^{CT} |x - \xi| + 2M_g e^{-\lambda T} + |y - \eta|.$$

Now, for any  $\varepsilon > 0$  we can fix  $T \equiv T(\varepsilon)$  such that  $2M_g e^{-\lambda T} < \varepsilon/3$ . It follows that we can choose a  $\delta = \delta(\varepsilon, T(\varepsilon))$  such that if  $|(x, y) - (\xi, \eta)| \leq \delta$  then

$$|\vartheta(x, y) - \vartheta(\xi, \eta)| < \varepsilon.$$

□

Thanks to the presence of the auxiliary variable  $y$  we are able to state the Dynamic Programming Principle (DPP) for  $\vartheta$ . Let us define for  $t \geq 0$

$$Y_{x,y}^u(t) := \max_{s \in [0, t]} e^{-\lambda s} g(X_x^u(s)) \vee y.$$

Following the arguments in [10] and thanks to the uniform continuity of  $\vartheta$  one has:

**Theorem 3.2 (DPP).** *Let assumptions (H1)-(H2) be satisfied. For  $(x, y) \in \mathbb{R}^d \times \mathbb{R}$  and any finite stopping time  $\theta \geq 0$  there holds*

$$\vartheta(x, y) = \inf_{u \in \mathcal{U}} \mathbb{E} \left[ e^{-\lambda \theta} \vartheta(X_x^u(\theta), e^{\lambda \theta} Y_{x,y}^u(\theta)) \right].$$

*Intuitive justification.* We only give an intuitive justification for the DPP satisfied by  $\vartheta$  in the non controlled case.

We can in fact observe that a.s.

$$\begin{aligned}
 \max_{s \in [0, \infty)} e^{-\lambda s} g(X_x(s)) \vee y &= \max_{s \in [0, \infty)} e^{-\lambda s} g(X_x(s)) \bigvee \underbrace{\max_{s \in [0, \theta]} e^{-\lambda s} g(X_x(s)) \vee y}_{=: Y_{x,y}(\theta)} \\
 &= e^{-\lambda \theta} \max_{s \in [0, \infty)} e^{-\lambda s} g(X_{X_x(\theta)}(s)) \bigvee Y_{x,y}(\theta) \\
 &= e^{-\lambda \theta} \left( \max_{s \in [0, \infty)} e^{-\lambda s} g(X_{X_x(\theta)}(s)) \bigvee e^{\lambda \theta} Y_{x,y}(\theta) \right).
 \end{aligned}$$

Consequently, by the definition of  $\vartheta$  (in absence of controls), we have by the tower property of the expectation

$$\begin{aligned} \vartheta(x, y) &= \mathbb{E} \left[ \mathbb{E} \left[ e^{-\lambda\theta} \left( \max_{s \in [0, \infty)} e^{-\lambda s} g(X_{X_x(\theta)}(s)) \vee e^{\lambda\theta} Y_{x,y}(\theta) \right) \middle| \mathcal{F}_\theta \right] \right] \\ &= \mathbb{E} \left[ e^{-\lambda\theta} \mathbb{E} \left[ \max_{s \in [0, \infty)} e^{-\lambda s} g(X_{X_x(\theta)}(s)) \vee e^{\lambda\theta} Y_{x,y}(\theta) \middle| \mathcal{F}_\theta \right] \right] \\ &= \mathbb{E} \left[ e^{-\lambda\theta} \vartheta(X_x(\theta), e^{\lambda\theta} Y_{x,y}(\theta)) \right]. \end{aligned}$$

Here, we are not considering the additional technical difficulties coming from the presence of controls. We refer to [10] and [21] for more detailed arguments.  $\square$

#### 4. THE HAMILTON-JACOBI-BELLMAN EQUATION

The value function  $\vartheta$  can be characterized in terms of a solution of a second-order HJB equation.

**Theorem 4.1** (Second-order HJB equation). *Under assumptions (H1)-(H2), the value function  $\vartheta$  is a continuous viscosity solution of*

$$(4.1) \quad \begin{aligned} \lambda\vartheta + H(x, y, D_x\vartheta, \partial_y\vartheta, D_x^2\vartheta) &= 0 & x \in \mathbb{R}^d, y > g(x), \\ -\partial_y\vartheta &= 0 & x \in \mathbb{R}^d, y = g(x) \end{aligned}$$

with Hamiltonian  $H: \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$  defined by

$$(4.2) \quad H(x, y, p, q, P) = \sup_{u \in U} \left( -b(x, u) \cdot p - \frac{1}{2} \text{Tr}[(\sigma\sigma^T)(x, u)P] \right) - \lambda y q.$$

We recall the notion of viscosity solution for second-order HJB equation.

**Definition 4.2.** [13, Definition 7.4] *Let  $\mathcal{O}$  be a locally compact set in  $\mathbb{R}^{d+1}$ . A USC function  $\underline{\vartheta}$  (resp. LSC function  $\bar{\vartheta}$ ) on  $\bar{\mathcal{O}}$  is a viscosity sub-solution (resp. super-solution) of (4.1), if for every function  $\varphi \in C^2(\bar{\mathcal{O}})$  at every maximum (resp. minimum) point  $x$  of  $\underline{\vartheta} - \varphi$  (resp.  $\bar{\vartheta} - \varphi$ ) the following inequalities hold*

$$\begin{cases} \lambda\varphi + H(x, y, D_x\varphi, \partial_y\varphi, D_x^2\varphi) \leq 0 & \text{in } \mathcal{O}, \\ \min(\lambda\varphi + H(x, y, D_x\varphi, \partial_y\varphi, D_x^2\varphi), -\partial_y\varphi) \leq 0 & \text{on } \partial\mathcal{O}, \end{cases}$$

(resp.

$$\begin{cases} \lambda\varphi + H(x, y, D_x\varphi, \partial_y\varphi, D_x^2\varphi) \geq 0 & \text{in } \mathcal{O}, \\ \max(\lambda\varphi + H(x, y, D_x\varphi, \partial_y\varphi, D_x^2\varphi), -\partial_y\varphi) \geq 0 & \text{on } \partial\mathcal{O}.) \end{cases}$$

A continuous function  $\vartheta$  is called a viscosity solution of (4.1) if it is both a sub- and super-solution.

*Proof of Theorem 4.1.* Let

$$\bar{\mathcal{O}} := \{ (x, y) \in \mathbb{R}^{d+1} : y \geq g(x) \}.$$

We first check that  $\vartheta$  is a viscosity sub-solution. Let  $\varphi \in C^2(\bar{\mathcal{O}})$  such that  $\vartheta - \varphi$  attains a maximum at point  $(\bar{x}, \bar{y}) \in \bar{\mathcal{O}}$ . Without loss of generality we can always assume that  $(\bar{x}, \bar{y})$  is a strict local maximum point (let us say in a ball of radius  $r > 0$  centered in  $(\bar{x}, \bar{y})$ ) and  $\vartheta(\bar{x}, \bar{y}) = \varphi(\bar{x}, \bar{y})$ . Thanks to Theorem 3.2, for any  $u \in \mathcal{U}$  and for any sufficiently small stopping time  $\theta = \theta^u$ , we have:

$$\begin{aligned} \varphi(\bar{x}, \bar{y}) &= \vartheta(\bar{x}, \bar{y}) \leq \mathbb{E} \left[ e^{-\lambda\theta} \vartheta(X_{\bar{x}}^u(\theta), e^{\lambda\theta} Y_{\bar{x}, \bar{y}}^u(\theta)) \right] \\ (4.3) \quad &\leq \mathbb{E} \left[ e^{-\lambda\theta} \varphi(X_{\bar{x}}^u(\theta), e^{\lambda\theta} Y_{\bar{x}, \bar{y}}^u(\theta)) \right]. \end{aligned}$$

Two cases will be considered depending on if the point  $(\bar{x}, \bar{y})$  belongs to the boundary of  $\bar{\mathcal{O}}$  or not.

— **Case 1:**  $g(\bar{x}) < \bar{y}$ . Consider a constant control  $u(s) \equiv u \in U$ . From the continuity of  $g$  and the a.s. continuity of the sample paths it follows that for a.e.  $\omega \in \Omega$  there exists  $\bar{s}(\omega) > 0$  such that  $g(X_{\bar{x}}^u(s))e^{-\lambda s} < \bar{y}$  if  $s \in [0, \bar{s}(\omega))$ . Given  $h > 0$ , let  $\bar{\theta}$  be the following stopping time:

$$(4.4) \quad \begin{aligned} \bar{\theta} &:= \inf \{ s > 0 : (X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s)) \notin B((\bar{x}, \bar{y}), r) \} \wedge h \\ &\quad \wedge \inf \{ s > 0 : e^{-\lambda s} g(X_{\bar{x}}^u(s)) \geq \bar{y} \} \end{aligned}$$

(where  $B((\bar{x}, \bar{y}), r)$  denotes the ball of radius  $r > 0$  centered at  $(\bar{x}, \bar{y})$ ). One can easily observe that a.s.  $\bar{\theta} > 0$  and  $Y_{\bar{x}, \bar{y}}^u(\bar{\theta}) = \bar{y}$ , then by (4.3)

$$\varphi(\bar{x}, \bar{y}) \leq \mathbb{E} \left[ e^{-\lambda\bar{\theta}} \varphi(X_{\bar{x}}^u(\bar{\theta}), e^{\lambda\bar{\theta}} \bar{y}) \right] \quad \forall u \in U.$$

By applying the Ito's formula [22, Theorem 5.5], and thanks to the smoothness of  $\varphi$ , we get:

$$\begin{aligned} &\mathbb{E} \left[ \int_0^{\bar{\theta}} d \left( e^{-\lambda s} \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right) \right] \\ &= \mathbb{E} \left[ \int_0^{\bar{\theta}} \left\{ \lambda \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) - b(X_{\bar{x}}^u(s), u) D_x \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right. \right. \\ &\quad \left. \left. - \frac{1}{2} Tr[\sigma \sigma^T(X_{\bar{x}}^u(s), u) D_x^2 \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y})] - \lambda \bar{y} e^{\lambda s} \partial_y \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right\} e^{-\lambda s} ds \right] \leq 0. \end{aligned}$$

Observing that the stopping times

$$\inf \{ s > 0 : (X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s)) \notin B_r(\bar{x}, \bar{y}) \} \quad \text{and} \quad \inf \{ s > 0 : e^{-\lambda s} g(X_{\bar{x}}^u(s)) \geq \bar{y} \}$$



are a.s. strictly greater than 0, for a.e. fixed  $\omega$  one obtains  $\bar{\theta} = h$  for a sufficiently small  $h$  in (4.4). Dividing by  $h > 0$ , one gets

$$\mathbb{E} \left[ \frac{1}{h} \int_0^{\bar{\theta}} \left\{ \lambda \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) - b(X_{\bar{x}}^u(s), u) D_x \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right. \right. \\ \left. \left. - \frac{1}{2} \text{Tr}[\sigma \sigma^T(X_{\bar{x}}^u(s), u) D_x^2 \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y})] - \lambda \bar{y} e^{\lambda s} \partial_y \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right\} e^{-\lambda s} ds \right] \leq 0.$$

By the dominate convergence theorem, taking the limit for  $h$  going to 0 inside the expectation and applying the mean value theorem, it follows

$$\lambda \varphi(\bar{x}, \bar{y}) - b(\bar{x}, u) D_x \varphi(\bar{x}, \bar{y}) - \frac{1}{2} \text{Tr}[\sigma \sigma^T(\bar{x}, u) D_x^2 \varphi(\bar{x}, \bar{y})] - \lambda \bar{y} \partial_y \varphi(\bar{x}, \bar{y}) \leq 0.$$

Finally, thanks to the arbitrariness of  $u \in U$ :

$$\lambda \varphi(\bar{x}, \bar{y}) + \sup_{u \in U} \left( -b(\bar{x}, u) D_x \varphi(\bar{x}, \bar{y}) - \frac{1}{2} \text{Tr}[\sigma \sigma^T(\bar{x}, u) D_x^2 \varphi(\bar{x}, \bar{y})] \right) - \lambda \bar{y} \partial_y \varphi(\bar{x}, \bar{y}) \leq 0.$$

— **Case 2:**  $g(\bar{x}) = \bar{y}$ . Let us assume that  $-\partial_y \varphi(\bar{x}, \bar{y}) > 0$ , otherwise the conclusion is straightforward.

As in the previous case, we consider a constant control  $u(s) \equiv u \in U$ . Thanks to the continuity of the sample paths and the smoothness of  $\varphi$ , for a.e.  $\omega$  there is a time  $\bar{s}(\omega) > 0$  and  $\eta > 0$  such that:

$$\varphi(X_{\bar{x}}^u(s), e^{\lambda s} y) \leq \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \quad \forall s \in [0, \bar{s}], y \in [\bar{y}, \bar{y} + \eta].$$

Let  $\bar{\theta}$  be the stopping time given by:

$$\bar{\theta} := \inf \{s > 0: (X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s)) \notin B((\bar{x}, \bar{y}), r)\} \wedge \inf \{s > 0: \partial_y \varphi(X_{\bar{x}}^u(s), \bar{y}) \geq 0\} \\ \wedge \inf \{s > 0: e^{-\lambda s} g(X_{\bar{x}}^u(s)) \notin [\bar{y}, \bar{y} + \eta]\} \wedge h.$$

By (4.3) one has

$$\varphi(\bar{x}, \bar{y}) \leq \mathbb{E} \left[ e^{-\lambda \bar{\theta}} \varphi(X_{\bar{x}}^u(\bar{\theta}), e^{\lambda \bar{\theta}} \bar{y}) \right],$$

which implies (as we have seen for Case 1):

$$\lambda \varphi(\bar{x}, \bar{y}) + \sup_{u \in U} \left\{ -b(\bar{x}, u) D_x \varphi(\bar{x}, \bar{y}) - \frac{1}{2} \text{Tr}[\sigma \sigma^T(\bar{x}, u) D_x^2 \varphi(\bar{x}, \bar{y})] \right\} - \lambda \bar{y} \partial_y \varphi(\bar{x}, \bar{y}) \leq 0.$$

In conclusion at point  $(\bar{x}, \bar{y}) \in \partial \mathcal{O}$  one has

$$\min(\lambda \varphi + H(\bar{x}, \bar{y}, D_x \varphi, \partial_y \varphi, D_x^2 \varphi), -\partial_y \varphi) \leq 0,$$

and  $\vartheta$  is a viscosity sub-solution of equation (4.1).

It remains to prove that  $\vartheta$  is a viscosity super-solution of (4.1). Let  $\varphi \in C^2(\bar{\mathcal{O}})$  be such that  $\vartheta - \varphi$  attains a minimum at point  $(\bar{x}, \bar{y}) \in \bar{\mathcal{O}}$ . Without loss of generality we can always assume that  $(\bar{x}, \bar{y})$  is a strict local minimum point in a ball  $B((\bar{x}, \bar{y}), r)$  and  $\vartheta(\bar{x}, \bar{y}) = \varphi(\bar{x}, \bar{y})$ . We consider again the two cases:

— **Case 1:**  $g(\bar{x}) < \bar{y}$ . We assume by contradiction that

$$\lambda \varphi + H(\bar{x}, \bar{y}, D_x \varphi, \partial_y \varphi, D_x^2 \varphi) < 0.$$

By using continuity arguments we can also state that

$$(4.5) \quad \lambda\varphi + H(\cdot, \cdot, D_x\varphi, \partial_y\varphi, D_x^2\varphi) \leq 0$$

in a neighborhood  $B((\bar{x}, \bar{y}), r_1)$  of  $(\bar{x}, \bar{y})$  for some  $r_1 > 0$ . Moreover, thanks to the continuity of  $g$ , if  $\bar{y} - g(\bar{x}) =: \rho > 0$  we can find  $r_2(\rho) > 0$  and  $T(\rho) > 0$  such that

$$\max_{\substack{t \in [0, T], \\ x \in B(\bar{x}, r_2)}} e^{-\lambda t} g(x) - g(\bar{x}) \leq \frac{\rho}{2}$$

and we have

$$\max_{\substack{t \in [0, T], \\ x \in B(\bar{x}, r_2)}} e^{-\lambda t} g(x) \vee \bar{y} = \bar{y}.$$

For any  $u \in \mathcal{U}$  we define the stopping time  $\theta^u$  as the first exit time of the process  $(X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s))$  from the ball  $B((\bar{x}, \bar{y}), R)$  for  $R := \min(r, r_1, r_2) > 0$ , i.e.

$$\theta^u := \inf \left\{ s > 0 : (X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s)) \notin B((\bar{x}, \bar{y}), R) \right\} \wedge T.$$

Applying the Ito's formula and taking the expectation we get

$$\begin{aligned} & \varphi(\bar{x}, \bar{y}) - \mathbb{E} \left[ e^{-\lambda \theta^u} \varphi(X_{\bar{x}}^u(\theta^u), e^{\lambda \theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \right] \\ &= \varphi(\bar{x}, \bar{y}) - \mathbb{E} \left[ e^{-\lambda \theta^u} \varphi(X_{\bar{x}}^u(\theta^u), e^{\lambda \theta^u} \bar{y}) \right] \\ &= \mathbb{E} \left[ \int_0^{\theta^u} \left\{ \lambda \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) - b(X_{\bar{x}}^u(s), u) D_x \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right. \right. \\ & \quad \left. \left. - \frac{1}{2} Tr[\sigma \sigma^T(X_{\bar{x}}^u(s), u) D_x^2 \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y})] - \lambda \bar{y} e^{\lambda s} \partial_y \varphi(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y}) \right\} e^{-\lambda s} ds \right] \leq 0, \end{aligned}$$

that leads to

$$\vartheta(\bar{x}, \bar{y}) = \varphi(\bar{x}, \bar{y}) \leq \mathbb{E} \left[ e^{-\lambda \theta^u} \varphi(X_{\bar{x}}^u(\theta^u), e^{\lambda \theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \right].$$

The continuity of the sample paths implies that

$$\begin{cases} \text{either } \theta^u = T \text{ or} \\ (X_{\bar{x}}^u(\theta^u), e^{\lambda \theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \in \partial B((\bar{x}, \bar{y}), R) \text{ a.s.} \end{cases}$$

Being  $(\bar{x}, \bar{y})$  a strict minimum point on has

$$\min \left\{ (\vartheta - \varphi) : (x, y) \in \partial B((\bar{x}, \bar{y}), R) \cup \left( B((\bar{x}, \bar{y}), R) \cap y = e^{\lambda T} \bar{y} \right) \right\} =: \eta > 0,$$

hence

$$\vartheta(\bar{x}, \bar{y}) \leq \varphi(X_{\bar{x}}^u(\theta^u), e^{\lambda \theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \leq \vartheta(\theta^u, X_{\bar{x}}^u(\theta^u), e^{\lambda \theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) + \eta.$$

Since  $\eta$  does not depends on  $u$  and  $u$  is arbitrary, this contradicts the DPP.

— **Case 2:**  $g(\bar{x}) = \bar{y}$ . Assume by contradiction that

$$-\partial_y \varphi(\bar{x}, \bar{y}) < 0 \quad \text{and} \quad \lambda\varphi + H(\bar{x}, \bar{y}, D_x\varphi, \partial_y\varphi, D_x^2\varphi) < 0.$$

We can again define  $r_1 > 0$  such that (4.5) is satisfied in  $B((\bar{x}, \bar{y}), r_1)$ . Moreover there exists  $\tilde{r}_2 > 0$  such that

$$\varphi(\xi, \zeta) \leq \varphi(\xi, \zeta')$$

for any  $(\xi, \zeta), (\xi, \zeta') \in B((\bar{x}, \bar{y}), \tilde{r}_2)$  such that  $\zeta \leq \zeta'$ . For any  $u \in \mathcal{U}$  we define the stopping time  $\theta^u$  as the first exit time of the process  $(X_{\bar{x}}^u(s), e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s))$  and  $(X_{\bar{x}}^u(s), e^{\lambda s} \bar{y})$  from the ball  $B((\bar{x}, \bar{y}), \tilde{R})$  for  $\tilde{R} := \min(r, r_1, \tilde{r}_2) > 0$ . As for Case 1, we can still say that a.s.

$$\vartheta(X_{\bar{x}}^u(\theta^u), Y_{\bar{x}, \bar{y}}^u(\theta^u)) \geq \varphi(\theta^u, X_{\bar{x}}^u(\theta^u), Y_{\bar{x}, \bar{y}}^u(\theta^u)) + \eta'.$$

for some  $\eta' > 0$  not depending on  $u$ . Therefore, observing that

$$e^{\lambda s} Y_{\bar{x}, \bar{y}}^u(s) \geq e^{\lambda s} \bar{y}$$

for any  $s \geq 0$  and using Ito's formula, we get

$$\begin{aligned} \vartheta(\bar{x}, \bar{y}) = \varphi(\bar{x}, \bar{y}) &\leq \mathbb{E} \left[ \varphi(\theta^u, X_{\bar{x}}^u(\theta^u), e^{\lambda \theta^u} \bar{y}) \right] \leq \mathbb{E} \left[ \varphi(\theta^u, X_{\bar{x}}^u(\theta^u), e^{\lambda \theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \right] \\ &\leq \mathbb{E} \left[ \vartheta(\theta^u, X_{\bar{x}}^u(\theta^u), e^{\lambda \theta^u} Y_{\bar{x}, \bar{y}}^u(\theta^u)) \right] - \eta', \end{aligned}$$

the yields again to a contradiction of the DPP.  $\square$

In the sequel we will restrict our domain to

$$D := \{(x, y) \in \mathbb{R}^{d+1} : y > g(x), y \in (-M_g, M_g)\}.$$

Indeed, this is sufficient to characterize  $\vartheta$  since we have that:

$$(4.6) \quad \begin{aligned} \vartheta(x, y) &= y \quad \text{for any } y \geq M_g \\ \vartheta(x, y) &= \vartheta(x, g(x)) \quad \text{for any } y \leq g(x). \end{aligned}$$

Based on this observation we are going to prove uniqueness in the domain  $\bar{D}$  by adding the Dirichlet boundary condition, given by Proposition 3.1,

$$\vartheta(x, M_g) = M_g \quad \text{for any } x \in \mathbb{R}^d.$$

This will give a HJB equation with mixed Dirichlet and derivative boundary conditions. Let

$$(4.7) \quad \Gamma_1 := \{(x, y) \in \bar{D} : y = M_g\}; \quad \Gamma_2 := \{(x, y) \in \bar{D} : y = g(x)\}.$$

Then the value function  $\vartheta$  is a viscosity solution of the following problem

$$(4.8a) \quad \lambda \vartheta + H(x, y, D_x \vartheta, \partial_y \vartheta, D_x^2 \vartheta) = 0 \quad \text{in } D,$$

$$(4.8b) \quad -\partial_y \vartheta = 0 \quad \text{on } \Gamma_2$$

and satisfies additional

$$(4.8c) \quad \vartheta = M_g \quad \text{on } \Gamma_1$$

in a strong sense. We point out that in our case the strong constant Dirichlet boundary condition on  $\Gamma_1$  is compatible with the homogeneous derivative condition on  $\Gamma_2$ . This prevents possible problems related with mixed boundary conditions at the junctions where different components of the boundary

cross.

We can prove the following comparison result:

**Theorem 4.3.** *Let assumptions (H1)-(H2) be satisfied and let  $u \in USC(\bar{D})$ ,  $v \in LSC(\bar{D})$  respectively a bounded sub- and super-solution to equation (4.8) in  $\bar{D}$  such that*

$$u \leq M_g \quad \text{and} \quad v \geq M_g \quad \text{on } \Gamma_1.$$

*Then  $u \leq v$  in  $\bar{D}$ .*

The proof can be obtained by a slight modification of the arguments in [16, Theorem 2.1]. See also [17]. We report here the main steps.

*Sketch of the proof.* Before starting the proof of the result, we introduce a more compact notation. Let us start defining

$$\tilde{b}(x, y, u) := \begin{pmatrix} b(x, u) \\ \lambda y \end{pmatrix} \in \mathbb{R}^{d+1} \quad \text{and} \quad \tilde{\sigma}(x, y, u) := \begin{pmatrix} \sigma(x, u) \\ 0 \dots 0 \end{pmatrix} \in \mathbb{R}^{(d+1) \times p}.$$

In what follow we will directly denote with  $x$  the variable in the augmented state space  $\mathbb{R}^M$  for  $M := d + 1$ , that is  $x \equiv (x, y) \in \mathbb{R}^M$  and  $x_M = y$ . Using this notation we can write the Hamiltonian  $H$  in (4.2) in the following compact form

$$H(x, p, P) := \sup_{u \in U} \left( -\tilde{b}(x, u) \cdot p - \frac{1}{2} \text{Tr}[\tilde{\sigma} \tilde{\sigma}^T(x, u) P] \right).$$

Thanks to the Lipschitz continuity of the function  $g$  that defines the boundary  $\Gamma_2$ , we can observe that there exists  $\mu > 0$  such that for any  $z \in \Gamma_2$  one has

$$(4.9) \quad \bigcup_{0 \leq \xi \leq \mu} B(z - \xi, \xi \mu) \subset D^C.$$

(where  $D^C$  denotes the complementary of the set  $D$ ). This corresponds to condition (2.9) in [16] and, by the same arguments as in [16, Corollary 2.3] (strongly simplified in our case thanks to the constancy of the derivative direction), the existence of a function  $\zeta \in C^2(\bar{D})$  follows such that

$$\zeta \geq 0 \text{ in } \bar{D}, \quad -\partial_{x_M} \zeta \geq 1 \text{ on } \Gamma_2 \quad \text{and} \quad \|D\zeta\|_\infty, \|D^2\zeta\|_\infty \leq K$$

for some constant  $K \geq 0$ .

Let us define for  $\alpha, \beta > 0$

$$u_{\alpha, \beta}(x) := u(x) - \alpha \zeta(x) - \beta$$

and

$$v_{\alpha, \beta}(x) := v(x) + \alpha \zeta(x) + \beta.$$

One has (with an abuse of notation  $Du, Dv, D^2u, D^2v$  will denote a corresponding elements of the semijets of  $u$  and  $v$ , see [13, Section 2])

$$\begin{aligned} & \lambda u_{\alpha, \beta} + H(x, Du_{\alpha, \beta}, D^2u_{\alpha, \beta}) - \lambda u - H(x, Du, D^2u) \\ & \leq -\lambda \beta + \alpha M \end{aligned}$$

and

$$\begin{aligned} & \lambda v_{\alpha,\beta} + H(x, Dv_{\alpha,\beta}, D^2v_{\alpha,\beta}) - \lambda v - H(x, Dv, D^2v) \\ & \geq \lambda\beta - \alpha M \end{aligned}$$

for some constant  $M \geq 0$  depending on the bounds on  $b$ ,  $\sigma$  and  $\|D\zeta\|_\infty, \|D^2\zeta\|_\infty$ . Moreover

$$\begin{aligned} -\partial_{x_M} u_{\alpha,\beta} &= -\partial_{x_M} u + \alpha \partial_{x_M} \zeta \leq -\partial_{x_M} u - \alpha \\ -\partial_{x_M} v_{\alpha,\beta} &= -\partial_{x_M} v - \alpha \partial_{x_M} \zeta \geq -\partial_{x_M} v + \alpha \end{aligned}$$

and thanks to the non negativity of  $\zeta$  we also have that

$$u_{\alpha,\beta} \leq M_g \quad \text{and} \quad v_{\alpha,\beta} \geq M_g$$

on  $\Gamma_1$ . It follows that for any  $\beta > 0$  and for  $\alpha = \alpha(\beta) > 0$  small enough,  $u_{\alpha,\beta}$  (resp.  $v_{\alpha,\beta}$ ) is a sub-solution (resp. super-solution) to (4.8) with the following modified boundary condition on  $\Gamma_2$ :

$$(4.10) \quad -\partial_{x_M} \vartheta + \alpha \leq 0 \quad (\text{resp. } -\partial_{x_M} \vartheta - \alpha \geq 0).$$

In the sequel we are going to prove a comparison between  $u_{\alpha,\beta}$  and  $v_{\alpha,\beta}$  respectively sub and super solution of the modified problem. The comparison result between  $u$  and  $v$  follows taking the limit for  $\alpha, \beta$  going to 0. In order to simplify the notation we will continue to denote by  $u$  and  $v$  the sub and super solution of the modified problem.

Let us consider

$$\Phi_\gamma(x) := u(x) - v(x) - 2\gamma(1 + |x|^2).$$

Thanks to the boundedness and the upper semicontinuity of  $u - v$ ,  $\Phi_\gamma$  admits a maximum point  $\hat{x}_\gamma = \hat{x}$  in  $\overline{D}$ .

Let us assume that there exists a sequence  $\{\gamma_k\}_{k \geq 0}$  such that  $\gamma_k \rightarrow 0$  and the points  $\hat{x}$  approach the boundary  $\Gamma_1$ , that is  $\lim_{k \rightarrow \infty} \text{dist}(\hat{x}, \Gamma_1) = 0$ . In this case for any  $x \in \overline{D}$  one has

$$\begin{aligned} u(x) - v(x) &= \limsup_{k \rightarrow \infty} u(x) - v(x) - 2\gamma_k(1 + |x|^2) \\ &\leq \limsup_{k \rightarrow \infty} u(\hat{x}) - v(\hat{x}) \leq 0 \end{aligned}$$

(the last inequality follows by the fact that any convergent subsequence of points  $\{\hat{x}_{\gamma_k}\}$  converges to a point of  $\Gamma_1$  where one has  $u - v \leq 0$  by assumption. The limsup is then the sup of all the limits of the convergent subsequence that will result also  $\leq 0$ ) and the proof is concluded. The same result also follows if for the sequence  $\{\gamma_k\}_{k \geq 0}$  one assumes  $\Phi_{\gamma_k}(\hat{x}) \leq 0$ .

Therefore, in the sequel, we assume that there exists a  $\bar{\gamma}$  small enough so that  $\text{dist}(\hat{x}, \Gamma_1) = \rho > 0$  and  $u(\hat{x}) - v(\hat{x}) \geq \Phi_\gamma(\hat{x}) > \rho$  for any  $\gamma \leq \bar{\gamma}$  for some  $\rho > 0$ .

In this case, by the boundedness of  $u, v$ , we can deduce that

$$(4.11) \quad \gamma|\hat{x}| \rightarrow 0 \quad \text{as } \gamma \rightarrow 0.$$

Thanks to the property (4.9) of our domain, the existence of a family of  $C^2$  test functions  $\{w_\varepsilon\}_{\varepsilon>0}$  as in [16, Theorem 4.1] can be proved. Among the other properties,  $\{w_\varepsilon\} : \mathbb{R}^M \times \mathbb{R}^M \rightarrow \mathbb{R}$  satisfies:

$$(4.12) \quad w_\varepsilon(x, x) \leq \varepsilon$$

$$(4.13) \quad w_\varepsilon(x, y) \geq C \frac{|x - y|^2}{\varepsilon}$$

$$(4.14) \quad -\partial_{x_M} w_\varepsilon(x, y) \geq -C \frac{|x - y|^2}{\varepsilon} \quad \text{if } x \in \Gamma_2 \cap B(\hat{x}, \eta), y \in B(\hat{x}, \eta)$$

$$(4.15) \quad -\partial_{y_M} w_\varepsilon(x, y) \geq 0 \quad \text{if } y \in \Gamma_2 \cap B(\hat{x}, \eta), x \in B(\hat{x}, \eta)$$

for  $\varepsilon > 0$  and some  $\eta > 0$  small enough.

Applying the doubling variables procedure we define

$$\Phi_\varepsilon(x, y) := u(x) - v(y) - \gamma(1 + |x|^2) - \gamma(1 + |y|^2) - w_\varepsilon(x, y) - |x - \hat{x}|^4.$$

and we denote by  $(x_\varepsilon, y_\varepsilon)$  its maximum point. Thanks to properties (4.12) and (4.13), it is possible to prove that for  $\varepsilon$  going to 0

$$(4.16) \quad x_\varepsilon, y_\varepsilon \rightarrow \hat{x} \quad \text{and} \quad \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \rightarrow 0.$$

It follows that for  $\varepsilon$  small enough we can assume that  $x_\varepsilon, y_\varepsilon \notin \Gamma_1$ . Taking  $\varepsilon$  small enough we can also say that  $x_\varepsilon, y_\varepsilon \in B(\hat{x}, \eta)$  and then we can make use of properties (4.14) and (4.15).

If  $x_\varepsilon \in \Gamma_2$ , using (4.11) and (4.16), we have that for  $\varepsilon$  small enough

$$\begin{aligned} & -\partial_{x_M} \left( w_\varepsilon(x_\varepsilon, y_\varepsilon) + \gamma(1 + |x_\varepsilon|^2) + |x_\varepsilon - \hat{x}|^4 \right) \\ & \geq -C \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} + 2\gamma|x_\varepsilon| - 4|x_\varepsilon - \hat{x}|^3 > -\alpha. \end{aligned}$$

Similarly if  $y_\varepsilon \in \Gamma_2$  one has

$$-\partial_{y_M} \left( -w_\varepsilon(x_\varepsilon, y_\varepsilon) - \gamma(1 + |y_\varepsilon|^2) \right) \leq 0 < \alpha$$

for  $\varepsilon$  small enough.

This means that, considering respectively  $w_\varepsilon(\cdot, y_\varepsilon) + \gamma(1 + |\cdot|^2) + |\cdot - \hat{x}|^4$  and  $-w_\varepsilon(x_\varepsilon, \cdot) - \gamma(1 + |\cdot|^2)$  as test functions for  $u$  and  $v$ , for sufficiently small values of  $\varepsilon$  the derivative boundary conditions in  $x_\varepsilon, y_\varepsilon$  can be neglected and one can only consider in  $\Gamma_2 \cup D$

$$\lambda u + H(x_\varepsilon, Du, D^2u) \leq 0 \quad \text{and} \quad \lambda v + H(y_\varepsilon, Dv, D^2v) \geq 0$$

in the viscosity sense. Thanks to the properties of  $H$ , this leads to a contradiction using the arguments in [16].  $\square$

**Corollary 4.4.** *The value function  $\vartheta$  is the unique bounded continuous viscosity solution to equation (4.8) in  $\bar{D}$ .*

We extend  $\vartheta$  in a unique way to the full space  $\mathbb{R}^d \times \mathbb{R}$  by (4.6). However, we can observe that, in order to characterize the original value function  $v$ ,

the way we extend  $\vartheta$  for  $y > M_g$  does not matter and only the values of  $\vartheta$  for  $y = g(x) \leq M_g$  are important.

## 5. NUMERICAL APPROXIMATION

In this section, convergence is proved for a general class of numerical schemes. Then, we show that a fully-discrete semi-Lagrangian method applied to the problem under consideration belongs to this class.

Let  $BC$  the space of bounded and continuous functions in  $\mathbb{R}^{d+1}$  equipped with the  $L^\infty$ -norm.

Let the discretization parameters be given by

$$\Delta x = (\Delta x_1, \dots, \Delta x_d) \in (\mathbb{R}^{>0})^d, \quad \Delta y > 0, \quad \rho = (\Delta x, \Delta y).$$

The corresponding mesh is denoted by

$$\mathcal{G}^\rho := \{ (x_i, y_j) = (i\Delta x, j\Delta y), (i, j) \in \mathbb{Z}^d \times \mathbb{Z} \},$$

where  $i\Delta x = (i_1\Delta x_1, \dots, i_N\Delta x_N)$ ,  $i \in \mathbb{Z}^d$ . Further we define

$$(5.1) \quad j_x := \min \{ j \in \mathbb{Z} \mid j\Delta y \geq g(x) \},$$

$$(5.2) \quad j_M := \min \{ j \in \mathbb{Z} \mid j\Delta y \geq M_g \}$$

and introduce a projection operator

$$(5.3) \quad \Pi^{\mathcal{G}^\rho}(\phi)(x_i, y_j) := \begin{cases} y_j & \text{if } j \geq j_M \\ \phi(x_i, y_j) & \text{if } j_M > j \geq j_{x_i} \\ \phi(x_i, y_{j_{x_i}}) & \text{if } j < j_{x_i} \end{cases}$$

for  $\phi \in BC$ . We aim to define a general approximation of the value function  $\vartheta$ . We start considering a general scheme

$$S^\rho(x_i, y_j, \phi_{i,j}, \phi) = 0$$

that approximates the equation

$$\lambda\phi + H(x, y, D_x\phi, \partial_y\phi, D_x^2\phi) = 0$$

at node  $(x_i, y_j)$ , with  $H$  defined by (4.2). Here  $\phi_{i,j} = \phi(x_i, y_j)$  and  $\phi$  denote all the values of  $\phi$  at nodes different from  $(x_i, y_j)$ .  $S^\rho$  may represent a finite difference operator (see [9, 8, 19]), or a semi-Lagrangian (SL) scheme ([20, 11, 15]). The main idea of the numerical method described here is to mix the use of a standard scheme for (4.8a), together with a “projection step” on  $\partial D$  in order to get the desired boundary conditions. Let us point out that a similar method was introduced for treating oblique derivative boundary conditions, i.e. the condition we have in  $\Gamma_2$ , in [2] for the case  $g(x) \equiv |x|$  and in [7] for a general Lipschitz continuous function  $g$ . In addition, here the numerical solution has also to satisfy the Dirichlet condition on  $\Gamma_1$ .

We define  $V$  on  $\mathcal{G}^\rho$  such that

$$(5.4) \quad \begin{cases} V_{i,j} = y_j & \text{if } y_j \geq M_g, \\ S^\rho(x_i, y_j, V_{ij}, \Pi^{\mathcal{G}^\rho}(V)) = 0 & \text{if } g(x_i) \leq y_j < M_g, \\ V_{i,j} = V_{i,j_{x_i}} & \text{if } y_j < g(x_i) \end{cases}$$

and we denote by  $V^\rho$  its continuous extension by bilinear interpolation. We assume the grid  $\mathcal{G}^\rho$  aligned with the domain  $[-M_g, M_g]$ , so that we have  $V^\rho = M_g$  on  $\Gamma_1$ .

**5.1. General convergence result.** In order to prove the convergence of  $V^\rho$  to  $\vartheta$ , unique viscosity solution to (4.8), we will make use of the arguments introduced by Barles and Souganidis in [3]. These make use of the properties of stability, consistency and monotonicity of the scheme. The scheme defined by (5.4) satisfies the following assumptions:

**(H3.a)** Stability: for any  $\rho$  the scheme (5.4) admits a solution  $V^\rho \in BC(\bar{D})$ . Moreover, there exist  $M, L \geq 0$  such that

$$V^\rho(x, y) \leq M \quad \text{and} \quad |V^\rho(x, y) - V^\rho(x, y')| \leq L|y - y'|$$

for any  $(x, y), (x, y') \in \bar{D}$  and  $\rho > 0$ .

**(H3.b)** Consistency: the scheme  $S^\rho$  is consistent with respect to (4.1) in  $\bar{D}$ , i.e. for all  $(x, y) \in \bar{D}$  and every  $\phi \in C^2(\bar{D})$  there holds

$$\lim_{\substack{\rho \rightarrow 0, \\ \bar{D} \ni (\xi, \gamma) \rightarrow (x, y), \\ \zeta \rightarrow 0}} S^\rho(\xi, \gamma, \phi(\xi, \gamma) + \zeta, \phi + \zeta) = \lambda\phi + H(x, D_x\phi, D_x^2\phi).$$

**(H3.c)** Monotonicity: for every  $\rho, r \in \mathbb{R}$ ,  $(x, y) \in \bar{D}$ ,  $S^\rho(x, y, r, \phi)$  depends only on the values of  $\phi$  in a neighborhood  $B_{\eta(\rho)}(x, y)$  of  $(x, y)$  with  $\eta(\rho) \geq 0$  such that  $\eta(\rho) \rightarrow 0$  for  $\rho \rightarrow 0$ . For all function  $\phi_1, \phi_2: \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$  with  $\phi_1 \geq \phi_2$  on  $B_\rho(x, y)$ , there holds

$$(5.5) \quad S^\rho(x, y, r, \phi_1) \leq S^\rho(x, y, r, \phi_2).$$

The reason for this formulation of the monotonicity condition is the fact, that the numerical scheme  $S^\rho$  defined on  $\bar{D}$  may use some values of the function  $V^\rho$  outside the domain  $\bar{D}$ .

**Theorem 5.1.** *Let assumptions (H1)-(H2) be satisfied and let the scheme (5.4) satisfy assumption (H3). Then for  $\rho \rightarrow 0$  the solution  $V^\rho$  converges to the unique viscosity solution to (4.1).*

*Proof.* Let us define for  $(x, y) \in \bar{D}$

$$\bar{V}(x, y) := \limsup_{\substack{\bar{D} \ni (\xi, \gamma) \rightarrow (x, y) \\ \rho \rightarrow 0}} V^\rho(\xi, \gamma),$$

$$\underline{V}(x, y) := \liminf_{\substack{\bar{D} \ni (\xi, \gamma) \rightarrow (x, y) \\ \rho \rightarrow 0}} V^\rho(\xi, \gamma)$$

Observe that the semi-limits are well defined thanks to the boundedness of  $V^\rho$  uniformly with respect to  $\rho$ . We start by proving that  $\bar{V}$  is a viscosity sub-solution to equation (4.1).

Thanks to the Lipschitz continuity of  $V^\rho$  with respect to  $y$  (uniform with



respect to  $\rho$ ) and the fact that, by definition,  $V^\rho(x, y) = M_g$  on  $\Gamma_1$  one has for any  $(\xi, \gamma) \in \overline{D}$

$$|V^\rho(\xi, \gamma) - M_g| = |V^\rho(\xi, \gamma) - V^\rho(\xi, M_g)| \leq L|\gamma - M_g|,$$

so that

$$\lim_{\substack{\overline{D} \ni (\xi, \gamma) \rightarrow (x, y) \\ \rho \rightarrow 0}} V^\rho(\xi, \gamma) = M_g.$$

Hence,  $\overline{V} = M_g$  on  $\Gamma_1$  and the Dirichlet condition is satisfied.

Let  $\varphi \in C^2(\overline{D})$  and let  $(\bar{x}, \bar{y})$  be a local maximum point for  $\overline{V} - \varphi$  on  $\overline{D}$ . Without loss of generality we can assume that  $(\bar{x}, \bar{y})$  is a strict local maximum in  $B_r(\bar{x}, \bar{y}) \cap \overline{D}$  for a certain  $r > 0$  and  $\varphi \geq 2 \sup_\rho \|W^\rho\|_\infty$  outside the ball  $B_r(\bar{x}, \bar{y})$ . We claim that

$$\begin{cases} \lambda u + H(\bar{x}, \bar{y}, D_x \varphi, \partial_y \varphi, D_x^2 \varphi) \leq 0 & \text{if } (\bar{x}, \bar{y}) \in D, \\ \min(\lambda u + H(\bar{x}, \bar{y}, D_x \varphi, \partial_y \varphi, D_x^2 \varphi), -\partial_y \varphi) \leq 0 & \text{if } (\bar{x}, \bar{y}) \in \Gamma_2. \end{cases}$$

We follow the argument in Barles and Souganidis [3]. There exists a sequence  $\rho_k, (x_k, y_k)$  such that  $\rho_k \rightarrow 0, (x_k, y_k) \rightarrow (\bar{x}, \bar{y})$  for  $k \rightarrow \infty$ , and

$$(5.6) \quad (V^{\rho_k} - \varphi)(x_k, y_k) = \max_{\overline{D}}(V^{\rho_k} - \varphi) = \delta_k \rightarrow 0, \quad \text{as } k \rightarrow \infty$$

and

$$V^{\rho_k}(x_k, y_k) \rightarrow \overline{V}(\bar{x}, \bar{y}), \quad \text{as } k \rightarrow \infty.$$

–**Case 1:**  $(\bar{x}, \bar{y}) \in D$ . For  $k$  large enough  $(x_k, y_k) \in D$ . Since  $g$  is continuous, for  $\rho_k$  small enough we can assume that  $y > g(x)$  for any  $(x, y) \in B_{\eta(\rho_k)}(x_k, y_k)$  (where  $B_{\eta(\rho_k)}(x_k, y_k)$  is the neighborhood that appears in assumption (H3.c)). Consequently, for  $k$  big enough

$$\Pi^{\mathcal{G}_{\rho_k}}(V^{\rho_k}) = V^{\rho_k} \quad \text{and} \quad V^{\rho_k} < \varphi + \delta_k$$

in  $B_{\eta(\rho_k)}(x_k, y_k)$ . By the monotonicity of the scheme (assumption (H3.c)) we further deduce

$$\begin{aligned} 0 &= S^{\rho_k}(x_k, y_k, V^{\rho_k}(x_k, y_k), \Pi^{\mathcal{G}_{\rho_k}}(V^{\rho_k})) \\ &\geq S^{\rho_k}(x_k, y_k, \varphi(x_k, y_k) + \delta_k, \varphi + \delta_k) \end{aligned}$$

and by the consistency assumption (H3.b) we obtain that as  $k \rightarrow \infty$

$$\lambda \varphi + H(\bar{x}, \bar{y}, D_x \varphi, \partial_y \varphi, D_x^2 \varphi) \leq 0.$$

–**Case 2:**  $(\bar{x}, \bar{y}) \in \Gamma_2$ . If  $-\partial_y \varphi(\bar{x}, \bar{y}) \leq 0$  the sub-solution property on  $\Gamma_2$  is automatically satisfied. Let us assume that  $-\partial_y \varphi(\bar{x}, \bar{y}) > 0$ .

We point out that if  $\bar{y} = g(\bar{x})$ ,  $(x_k, y_k)$  can also be on  $\Gamma_2$  and the scheme may involve values  $V_{m,n}$  on some point  $(x_m, y_n)$  which is not in  $\overline{D}$ .

If  $-\partial_y \varphi(\bar{x}, \bar{y}) > 0$ , there exists a neighbourhood  $\mathcal{V}$  of  $(\bar{x}, \bar{y})$  where  $\partial_y \varphi$  is well defined and  $-\partial_y \varphi > 0$ . Therefore,

$$(5.7) \quad y \leq y' \Rightarrow \varphi(x, y) \geq \varphi(x, y') \quad \forall (x, y), (x, y') \in \mathcal{V}.$$

and, taking  $k$  large enough,  $B_{\eta(\rho_k)}(x_k, y_k) \subset \mathcal{V}$ . Let  $(x, y) \in B_{\eta(\rho_k)}(x_k, y_k)$ . If  $y \geq g(x)$ , we have

$$\Pi^{\mathcal{G}_\rho}(V^{\rho_k}(x, y)) = V^{\rho_k}(x, y) \leq \varphi(x, y) + \delta_k.$$

If  $y < g(x)$ ,  $\Pi^{\mathcal{G}_\rho}(V^{\rho_k}(x, y)) = V^{\rho_k}(x, y_{j_x})$  and we have

$$\begin{aligned} \Pi^{\mathcal{G}_\rho}(V^{\rho_k}(x, y)) &= V^{\rho_k}(x, y_{j_x}) \\ (5.8) \quad &\leq \varphi(x, y_{j_x}) + \delta_k \quad \text{using (5.6)} \\ &\leq \varphi(x, y) + \delta_k \quad \text{using (5.7)} \end{aligned}$$

For the last inequality one also need to observe that if  $(x, y) \in \mathcal{V}$  and  $y < g(x)$ , thanks to the continuity of  $g$  one can choose  $k$  big enough so that also  $(x, y_{j_x}) \in \mathcal{V}$ .

Consequently, we have  $\Pi^{\mathcal{G}_\rho}(V^{\rho_k}) \leq \varphi + \delta_k$  on  $B_{\eta(\rho_k)}(x_k, y_k)$ . Thus by monotonicity we have

$$\begin{aligned} 0 &= S^{\rho_k}(x_k, y_k, V^{\rho_k}(x_k, y_k), \Pi^{\mathcal{G}_\rho}(V^{\rho_k})) \\ &\geq S^{\rho_k}(x_k, y_k, \varphi(x_k, y_k) + \delta_k, \varphi + \delta_k) \end{aligned}$$

and using consistency when  $k \rightarrow \infty$  we have

$$(5.9) \quad \lambda\varphi + H(\bar{x}, \bar{y}, D_x\varphi, \partial_y\varphi, D_x^2\varphi) \leq 0.$$

This proves that  $\bar{V}$  is a viscosity subsolution to (4.1). Analogously one can show that  $\underline{V}$  is a viscosity sub-solution. Applying the comparison principle (Theorem 4.3), it follows that  $\underline{V} \geq \bar{V}$  on  $\bar{D}$ . Since  $\underline{V} \leq \bar{V}$  is always true by definition, it is possible to conclude that  $\bar{V} = \underline{V}$  on  $\bar{D}$ , which proves the assertion.  $\square$

**5.2. Semi-Lagrangian scheme.** In this section, a semi-Lagrangian scheme is introduced and it is shown that it satisfies the conditions in assumption (H3).

**5.2.1. Semi-discretization.** In order to simplify the presentation, we start considering a semi-discrete version of the scheme, introducing a parameter  $h > 0$  which describes a discretization in time of the dynamics. The space discretization, i.e. the space grid  $\mathcal{G}^\rho$ , is not taken into account for the moment.

We define the mapping  $\mathcal{T}_0: BC \rightarrow BC$  by

$$\mathcal{T}_0(\phi)(x, y) := \min_{u \in U} \left\{ \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} \phi \left( x + hb(x, u) + \sqrt{hp}(-1)^k \sigma_{\lfloor \frac{k+1}{2} \rfloor}(x, u), \frac{y}{1 - \lambda h} \right) \right\}$$

where  $\lfloor \cdot \rfloor$  denotes the integer part. We consider the following subset of  $BC$ :

$$X(M_g) := \left\{ \phi \in BC \cap Lip_y : \phi(x, y) = y, \forall x \in \mathbb{R}^d, M_g \leq y \leq 2M_g(1 - \lambda h) \right\}$$

where  $Lip_y$  denotes the set of function that are Lipschitz continuous with respect to  $y$  (we also denote by  $Lip_y(L)$  the subset of  $Lip_y$  of functions with

Lipschitz constant  $L \geq 0$ ). We introduce the operator  $T_0: X(M_g) \rightarrow X(M_g)$  defined by

$$(5.10) \quad T_0(\phi)(x, y) := \mathcal{T}_0(\Pi(\phi))(x, y)$$

where  $\Pi$  is the the continuous version of the projection  $\Pi^{\mathcal{G}^\rho}$ , i.e. :

$$\Pi(\phi)(x, y) := \begin{cases} y \wedge 2M_g & \text{if } y \geq M_g \\ \phi(x, y) & \text{if } g(x) \leq y < M_g \\ \phi(x, g(x)) & \text{if } y \leq g(x) \end{cases}$$

(compared with (5.3) here we have modified the value of the operator  $\Pi$  for  $y \geq M_g$  in order to guarantee the boundedness of the operator  $T$ . It is possible to verify that this does not affect the proof of the main convergence result which only concerns neighborhood of points in  $\overline{D}$ ). Observe that if functions  $\phi$  in  $X(M_g)$  are considered, the operator  $\Pi$  simply reads:

$$(5.11) \quad \Pi(\phi)(x, y) = \phi(x, g(x) \vee (y \wedge 2M_g)).$$

**Lemma 5.2.** *The operator  $T_0$  is well defined on the class of functions  $\phi \in X(M_g)$ .*

*Proof.* For every  $\phi \in X(M_g)$  one has  $T_0(\phi)$  is bounded and continuous. Let  $\phi \in Lip_y(L)$ . One has

$$\begin{aligned} & |T_0(\phi)(x, y) - T_0(\phi)(x, y')| \\ & \leq \max_{u \in U} \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} \left| \Pi(\phi) \left( X_x^{u, k, p}(h), Y_y^u(h) \right) - \Pi(\phi) \left( X_x^{u, k, p}(h), Y_{y'}^u(h) \right) \right| \end{aligned}$$

where for any  $(x, y) \in \mathbb{R}^{d+1}$ ,  $u \in U$  we denoted

$$X_x^{u, k, p}(h) := x + hb(x, u) + \sqrt{hp}(-1)^k \sigma_{\lfloor \frac{k+1}{2} \rfloor}(x, u), \quad Y_y^u(h) := \frac{y}{1 - \lambda h}.$$

From (5.11), one gets

$$\begin{aligned} & |\Pi(\phi)(x, y) - \Pi(\phi)(x, y')| \\ & = |\phi(x, y \vee (g(x) \wedge 2M_g)) - \phi(x, g(x) \vee (y' \wedge 2M_g))| \\ & \leq L|(y \vee (g(x) \wedge 2M_g)) - (y' \vee (g(x) \wedge 2M_g))| \\ & \leq L|y - y'| \end{aligned}$$

where we used the classical inequalities

$$(5.12) \quad \begin{aligned} |a \vee b - c \vee d| & \leq |a - c| \vee |b - d|, \\ |a \wedge b - c \wedge d| & \leq |a - c| \vee |b - d|. \end{aligned}$$

Therefore

$$\begin{aligned} |T_0(\phi)(x, y) - T_0(\phi)(x, y')| &\leq \max_{u \in U} \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} L |Y_y^u(h) - Y_{y'}^u(h)| \\ &\leq \max_{u \in U} \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} L \frac{|y - y'|}{(1 - \lambda h)} \end{aligned}$$

and  $T_0(\phi) \in Lip_y(L)$  follows.

Moreover, for  $y \geq M_g$  one has  $y/(1 - \lambda h) > M_g$  and then

$$T_0(\phi)(x, y) = \min_{u \in U} \left\{ \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} \left( \frac{y}{1 - \lambda h} \right) \wedge 2M_g \right\} = y \wedge 2M_g(1 - \lambda h)$$

which proves that  $T_0(\phi) \in X(M_g)$ .  $\square$

We are going to verify the properties of stability, consistency and monotonicity (in the sense of assumption (H3)) of the scheme defined by

$$(5.13) \quad S^h(x, y, \phi(x, y), \phi) := \frac{1}{h} \left( \phi(x, y) - \mathcal{T}_0(\phi)(x, y) \right).$$

The fact of dealing with an infinite horizon problem and therefore with a stationary PDE, requires the use of a fixed point argument in order to prove the existence of a solution for (5.13). In order to guarantee the continuity of  $T(\phi)$  we had to restrict to functions  $\phi \in X(M_g)$ , so this is the space where we have to look for proving the stability result.

**Lemma 5.3** (Stability). *For  $0 \leq \lambda h < 1$ , there exists a fixed point of the equation*

$$(5.14) \quad S^h(x, y, \phi(x, y), \Pi(\phi)) = \frac{1}{h} \left( \phi(x, y) - T_0(\phi)(x, y) \right)$$

in  $X(M_g)$  and the scheme is stable in the sense of assumption (H3.a).

*Proof.* We show that the map  $T_0$  is a contraction on  $X(M_g)$ . There holds

$$\|T_0(v) - T_0(w)\|_\infty \leq \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} \|(\Pi v) - (\Pi w)\|_\infty \leq (1 - \lambda h) \|v - w\|_\infty.$$

Being  $(X(M_g), \|\cdot\|_\infty)$  a complete metric space (completeness follows by the fact that it is a closed subset of  $(BC, \|\cdot\|_\infty)$ ) by the Banach-Caccioppoli fix point theorem the result follows.

The stability in the sense of assumption (H3.a) then follows observing that the bound and Lipschitz constant of the solution of the fix point problem do not depend on  $h$ .  $\square$

In particular, this means that for any  $(x, y) \in \bar{D}$  there exists a solution in  $X(M_g)$  of the scheme

$$S^h(x, y, \phi(x, y), \Pi(\phi)) = 0.$$

We will denote such a solution by  $V^h$ . Observe that  $V^h$  automatically satisfies the boundary condition  $V^h(x, M_g) = M_g$ .

**Lemma 5.4** (Consistency). *For any smooth and bounded function  $\phi$ , there exists some  $C > 0$  such that*

$$\left| \frac{1}{h} (\phi(x, y) - \mathcal{T}_0(\phi)(x, y)) - \left( \lambda\phi + H(x, y, D_x\phi, \partial_y\phi, D_x^2\phi) \right) \right| \leq Ch.$$

*Proof.* The assertion follows straightforward by a Taylor's expansion. For simplicity we show the results in the one dimensional case. In fact, in this case, observing that

$$\frac{1}{1 - \lambda h} = 1 + \lambda h + O(h^2),$$

one has

$$\begin{aligned} & \phi(x, y) - \mathcal{T}_0(\phi)(x, y) \\ &= \phi(x, y) - \frac{(1 - \lambda h)}{2} \min_{u \in U} \left\{ \phi \left( x + hb(x, u) + \sqrt{h}\sigma(x, u), \frac{y}{1 - \lambda h} \right) \right. \\ & \quad \left. + \phi \left( x + hb(x, u) - \sqrt{h}\sigma(x, u), \frac{y}{1 - \lambda h} \right) \right\} \\ &= \lambda h \phi(x, y) - \frac{(1 - \lambda h)}{2} \min_{u \in U} \left\{ \phi_x(x, y)(hb(x, u) + \sqrt{h}\sigma(x, u)) + \phi_y(x, y)(\lambda h y) \right. \\ & \quad + \frac{1}{2} \phi_{xx}(x, y)(hb(x, u) + \sqrt{h}\sigma(x, u))^2 + \phi_{xy}(x, y)(hb(x, u) + \sqrt{h}\sigma(x, u))(\lambda h y) \\ & \quad + \frac{1}{6} \phi_{xxx}(x, y)(hb(x, u) + \sqrt{h}\sigma(x, u))^3 \\ & \quad + \phi_x(x, y)(hb(x, u) - \sqrt{h}\sigma(x, u)) + \phi_y(x, y)(\lambda h y) \\ & \quad + \frac{1}{2} \phi_{xx}(x, y)(hb(x, u) - \sqrt{h}\sigma(x, u))^2 + \phi_{xy}(x, y)(hb(x, u) - \sqrt{h}\sigma(x, u))(\lambda h y) \\ & \quad \left. + \frac{1}{6} \phi_{xxx}(x, y)(hb(x, u) - \sqrt{h}\sigma(x, u))^3 \right\} + O(h^2) \\ &= \lambda h \phi(x, y) - \min_{u \in U} \left\{ hb(x, u) \phi_x(x, y) + h \lambda y \phi_y(x, y) + \frac{1}{2} h \sigma^2(x, u) \phi_{xx}(x, y) \right\} + O(h^2). \end{aligned}$$

□

**Lemma 5.5** (Monotonicity). *For any  $h > 0$  such that  $0 \leq \lambda h < 1$ , the scheme is monotone in the sense of assumption (H3.c).*

*Proof.* Taken  $(x, y) \in \bar{D}$ , it is clear by the very definition of the operator  $T$  that it involves only values at points  $(x + hb(x, u) + \sqrt{hp}(-1)^k \sigma_{\lfloor \frac{k+1}{2} \rfloor}(x, u), \frac{y}{1 - \lambda h})$  which remain in a neighborhood  $B_{\eta(h)}$  of  $(x, y)$ . Moreover, by definition one has

$$S^h(x, y, \phi(x, y), \phi) := \frac{1}{h} (\phi(x, y) - \mathcal{T}_0(\phi)(x, y))$$

and the assertion follows immediately, since for  $\phi_1 \leq \phi_2$  one has

$$\mathcal{T}_0(\phi_1)(x, y) \leq \mathcal{T}_0(\phi_2)(x, y).$$

□

Having proved all the properties in assumption (H3), we are now ready to apply Theorem 5.1 and state a convergence result for our semi-discrete scheme:

**Theorem 5.6.** *Let assumptions (H1)-(H2) be satisfied. Then for  $h \rightarrow 0$  the solution  $V^h$  of the scheme (5.14) converges uniformly to  $\vartheta$  in  $\bar{D}$ .*

5.2.2. *Full-discretization.* We pass now to consider the scheme on the discretized space. This requires to introduce an interpolation step in order to define the value of the solution at points

$$\left( x_i + hb(x_i, u) + \sqrt{hp}(-1)^k \sigma_{\lfloor \frac{k+1}{2} \rfloor}(x_i, u), \frac{y_j}{1 - \lambda h} \right)$$

that may not belong to the grid  $\mathcal{G}^\rho$  ( $\rho \equiv (\Delta x, \Delta y)$ ). We denote by  $[\cdot]$  the monotone,  $P_1$  interpolation operator, satisfying the following properties:

$$(5.15) \quad \begin{aligned} (i) & \quad [\phi](x_i, y_j) = \phi(x_i, y_j), \quad \forall (x_i, y_j) \in \mathcal{G}^\rho; \\ (ii) & \quad |[\phi](x, y) - \phi(x, y)| \leq L|(\Delta x, \Delta y)|, \quad \forall \phi \in Lip(L); \\ (iii) & \quad |[\phi](x, y) - \phi(x, y)| \leq C(|\Delta x|^2 + \Delta y^2) \|D^2 \phi\|_\infty \quad \forall \phi \in C^2; \\ (iv) & \quad \phi_1 \leq \phi_2 \Rightarrow [\phi_1] \leq [\phi_2]. \end{aligned}$$

The fully discrete operator is then defined by

$$\mathcal{F}(\phi)(x_i, y_j) := \min_{u \in U} \left\{ \frac{(1 - \lambda h)}{2p} \sum_{k=1}^{2p} [\phi] \left( x_i + hb(x_i, u) + \sqrt{hp}(-1)^k \sigma_{\lfloor \frac{k+1}{2} \rfloor}(x_i, u), \frac{y_j}{1 - \lambda h} \right) \right\}$$

The fully discrete scheme reads:

$$(5.16) \quad S^\rho(x_i, y_j, \phi(x_i, y_j), \Pi^{\mathcal{G}^\rho} \phi) := \frac{1}{h} \left( \phi(x_i, y_j) - \mathcal{F}(\Pi^{\mathcal{G}^\rho} \phi)(x_i, y_j) \right).$$

where

$$\Pi^{\mathcal{G}^\rho}(\phi)(x_i, y_j) := \begin{cases} y_j \wedge 2M_g & \text{if } y_j \geq M_g, \\ \phi(x_i, y_j) & \text{if } g(x_i) \leq y_j < M_g, \\ \phi(x_i, g(x_i)) & \text{if } y_j \leq g(x_i). \end{cases}$$

In particular, the value of the solution at grid points is prescribed by (5.16) and we extend it to a continuous function by linear interpolation.

**Theorem 5.7.** *Let  $0 \leq \lambda h < 1$  and let the following condition be satisfied*

$$\frac{\Delta x^2 + \Delta y^2}{h} \rightarrow 0 \quad \text{as } \rho, h \rightarrow 0.$$

*Then scheme (5.16) is stable in the space of piecewise linear functions in  $X(M_g)$ , consistent and monotone in the sense of definition (H3).*

*Proof.* Stability follows by similar arguments as in the semi-discrete case. Indeed, observing that the linear interpolation preserves the Lipschitz constant, the fix point theorem still holds if we consider piecewise linear functions in  $X(M_g)$ . Consistency can follow by (5.15)(iii). Indeed, when interpolation is taken into account, in the estimates obtained proving Lemma 5.4 one gets the extra term

$$\frac{\Delta x^2 + \Delta y^2}{h} \|\phi_{xx}\|$$

which goes to zero guaranteeing the consistency property. Monotonicity is ensured by the use of a linear interpolation.  $\square$

**Corollary 5.8.** *Let  $0 \leq \lambda h < 1$  and let the following condition be satisfied*

$$\frac{\Delta x^2 + \Delta y^2}{h} \rightarrow 0 \quad \text{as } \rho, h \rightarrow 0.$$

*Then, as  $\rho, h \rightarrow 0$  the sol  $V^\rho$  of the fully discrete scheme (5.16) converges in  $\bar{D}$  to the unique viscosity of equation (4.8).*

## 6. CONCLUSIONS

We have studied infinite horizon stochastic optimal control problems with cost in a maximum form. By the introduction of an auxiliary Markovian problem and dynamic programming arguments we have characterized the associated value function by means of a HJB equation with mixed Dirichlet-derivative boundary conditions. We have proposed a general numerical scheme which incorporates the treatment of the boundary condition and proved its convergence to the unique viscosity solution of the HJB equation under the assumptions of monotonicity consistency and stability. We have shown that a particular semi-Lagrangian scheme satisfies such assumptions and therefore can be used to approximate the value function of the original problem.

Further directions of work might involve the application of our scheme to the computation of viable and invariant sets as well as the theoretical proof of the rate of convergence associated to our scheme.

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