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# Uncertainty propagation in a nonlinear reduced-order model in internal elasto-acoustics.

E. CAPIEZ-LERNOUT<sup>a</sup>, C. SOIZE<sup>a</sup>, Q. AKKAOUT<sup>a</sup>, R. OHAYON<sup>b</sup>

a. Laboratoire de Modelisation et Simulation Multi Echelle (MSME) UMR 8208 CNRS  
Universite Paris-Est, 5 Bd Descartes, 77454 Marne-la-Vallée, Cedex 2, France,  
evangeline.capiez-lernout@u-pem.fr

b. Laboratoire de Mécanique des Structures et des Systèmes Couplés (LMSCC) EA 3196  
Conservatoire National des Arts et Métiers, Paris.

## Résumé

*La présente recherche concerne la propagation des incertitudes en élasto-acoustique interne en tenant compte des non-linéarités géométriques induites par les grands déplacements et les grandes déformations de la structure. La structure est couplée à une cavité interne remplie d'un fluide acoustique linéaire. Dans un premier temps, un modèle réduit non-linéaire adapté au problème d'élasto-acoustique interne est construit pour réduire le nombre d'inconnues du problème. La formulation du problème est effectuée avec les inconnues de déplacement structure et de pression fluide. Les incertitudes sont implémentées avec l'approche probabiliste non-paramétrique à partir de ce modèle réduit non-linéaire. En particulier, un opérateur de rigidité dédié contenant l'ensemble de l'information relative aux termes linéaires et non-linéaires de rigidité est construit. Une attention particulière est portée sur la modélisation de cet opérateur au travers d'une seconde réduction effectuée localement de manière à ce que le germe aléatoire soit de taille raisonnable et soit le même dans le cas où les incertitudes portent sur les termes linéaires ou/et non-linéaires de la rigidité. La méthodologie est présentée au travers d'un modèle numérique simple de coque cylindre remplie de fluide.*

## Abstract

*The present research concerns the uncertainty propagation in elasto-acoustics, taking into account the geometrical nonlinearities induced by the large displacements/deformations of the structure and assuming the internal acoustic fluid occupying an internal cavity coupled to the structure to remain in a linear range of vibration. The problem is formulated with structural displacements and fluid pressures unknowns. Uncertainties are implemented from a mean nonlinear reduced-order model using the non-parametric probabilistic approach. More particularly, a dedicated stiffness operator self-containing all the information concerning both linear and nonlinear stiffness terms is constructed. A particular attention concerns the modeling of such stiffness operator through a second local reduction so that the size of the random germ be of reasonable size and be identical whether uncertainties are investigated on only linear or nonlinear stiffness terms. A numerical application is presented.*

**Keywords : Fluid-structure interaction, Geometric nonlinearities, Uncertainty Quantification**

## 1 Introduction

Nowadays, a major challenge in many industrial areas consists in developing advanced methodologies in order to construct predictive numerical simulation tools, which are representative of the observed dynamical behaviour of the mechanical systems. In particular, it is important to quantify how inherent uncertainties propagate on the considered system. Furthermore, an essential aspect is to pay attention to the various nonlinear effects that can subsequently modify the dynamical response of the response. The present work proposes an extension of the uncertainty quantification for the nonlinear dynamical response of fluid-structure systems, for which the structural part present consequent geometrical nonlinear effects and for which the fluid is assumed to be a linear inviscid compressible fluid. First, we are interested by constructing a nonlinear-reduced-order model [10, 8, 7, 6]. A nonlinear mean reduced-order model of the coupled system is constructed by projecting the finite element operators on a chosen projection basis [3]. Then, uncertainties are implemented through the nonparametric probabilistic approach [13, 9], which has been recently improved for the structural part of the nonlinear operators [1]. A numerical example is then presented.

## 2 Mean nonlinear reduced-order model of the structural-acoustic system

The structural-acoustic system under consideration is made up of a tank structure filled with a linear inviscid compressible fluid. A linear elastic constitutive equation is considered for the structure. It is also assumed that the structure undergoes sufficiently large deformations and large displacements in order to consider the geometrical nonlinear effects, but also sufficiently moderate so that the fluid behavior remains linear. A total lagrangian formulation around a static equilibrium state taken as a reference configuration is used.

The three-dimensional bounded domain of the physical space  $\mathbb{R}^3$  occupied by the structure is denoted by  $\Omega_S$  with boundary  $\partial\Omega_S = \Gamma_{S,0} \cup \Gamma_S \cup \Sigma$ . The structure is subjected to a body force field  $\mathbf{f}_{\text{vol}}(\mathbf{x}, t)$ . It is fixed on  $\Gamma_{S,0}$  and subjected to a surface force field  $\mathbf{f}_{\text{surf}}(\mathbf{x})$  on  $\Gamma_S$ . The internal fluid occupies a three-dimensional bounded domain  $\Omega_F$  of  $\mathbb{R}^3$  with boundary  $\partial\Omega_F = \Gamma_F \cup \Sigma$ . It is coupled to the structure through boundary  $\Sigma$  and has a free surface condition on  $\Gamma_F$ . Let  $\mathbf{n}_S$  and  $\mathbf{n}_F$  be the outward unit normals to  $\partial\Omega_S$  and  $\partial\Omega_F$ . Note that  $\mathbf{n}_S = -\mathbf{n}_F$  on  $\Sigma$ . Let  $\mathbf{x}$  be the generic point of  $\mathbb{R}^3$ . The forced response is formulated in the time domain. A formulation in terms of displacements field  $\mathbf{u}(\mathbf{x}, t)$  for the structure and in terms of pressure field  $p(\mathbf{x}, t)$  for the internal fluid is chosen. The equations related to the mean structural-acoustic system [11] are then written as

$$\rho_S \frac{\partial^2 \mathbf{u}}{\partial t^2} - \text{div}(\mathbb{F} \mathbb{S}) = \mathbf{f}_{\text{vol}} \quad \text{in } \Omega_S \quad , \quad (1)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{S,0} \quad , \quad (2)$$

$$(\mathbb{F} \mathbb{S}) \cdot \mathbf{n}_S = \mathbf{f}_{\text{surf}} \quad \text{on } \Gamma_S \quad , \quad (3)$$

$$(\mathbb{F} \mathbb{S}) \cdot \mathbf{n}_S = p \mathbf{n}_S \quad \text{on } \Sigma \quad , \quad (4)$$

$$\frac{1}{\rho_F c_F^2} \frac{\partial^2 p}{\partial t^2} - \frac{1}{\rho_F} \Delta p = 0 \quad \text{in } \Omega_F \quad , \quad (5)$$

$$p = 0 \quad \text{on } \Gamma_F \quad , \quad (6)$$

$$\frac{1}{\rho_F} \frac{\partial p}{\partial \mathbf{n}_F} = -\frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \mathbf{n}_F \quad \text{on } \Sigma \quad , \quad (7)$$

in which the deformation gradient tensor  $\mathbb{F}$  is defined by  $F_{ij} = u_{i,j} + \delta_{ij}$ , in which  $\delta_{ij}$  is the Kronecker symbol such that  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  otherwise, and where the second Piola-Kirchhoff symmetric stress tensor  $\mathbb{S}$  is written, for a linear elastic material, as  $S_{ij} = a_{ijkl} E_{kl}$ . The fourth-order elasticity tensor  $\mathfrak{a} = \{a_{ijkl}\}_{ijkl}$  satisfies the usual symmetry and positive-definiteness properties. The Green strain tensor  $\mathbb{E} = \{E_{ij}\}_{ij}$  is then written as the sum of linear and nonlinear terms such that  $E_{ij} = \varepsilon_{ij} + \eta_{ij}$ , in which  $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$  and  $\eta_{ij} = \frac{1}{2}u_{s,i}u_{s,j}$ . Quantities  $\rho_S$ ,  $\rho_F$  and  $c_F$  denote the mass density of the structure, the mass density of the fluid, and the sound velocity respectively.

The structural-acoustic system is then discretized with the finite element method assuming that the finite element meshes of the structure and of the internal fluid are compatible on the coupling interface  $\Sigma$ . A mean nonlinear reduced matrix model of the structural-acoustic system is then constructed. Let  $\mathbf{u}$  be the  $\mathbb{C}^{n_S}$ -vector of the  $n_S$  DOF of the structure and let  $\mathbf{p}$  be the  $\mathbb{C}^{n_F}$ -vector corresponding to the finite element discretization of the pressure field of the internal acoustic fluid. The projection basis of the structural-acoustic problem is calculated as follows : (1) The  $N_S$  structural modes related to the first  $N_S$  positive structural eigenfrequencies of the structure are stored in the  $n_S \times N_S$  modal matrix  $[\Phi_S]$ . (2) The  $N_F$  acoustic modes related to the first  $N_F$  positive acoustic eigenfrequencies of the internal fluid are stored in the  $n_F \times N_F$  modal matrix  $[\Phi_F]$ . The projection basis allowing the mean reduced matrix model to be constructed is given by

$$\begin{bmatrix} \mathbf{u}(t) \\ \mathbf{p}(t) \end{bmatrix} = \begin{bmatrix} [\Phi_S] & [0] \\ [0] & [\Phi_F] \end{bmatrix} \begin{bmatrix} \mathbf{q}_S(t) \\ \mathbf{q}_F(t) \end{bmatrix}, \quad (8)$$

in which  $\mathbf{q}_S$  and  $\mathbf{q}_F$  are the  $\mathbb{C}^{N_S}$ -vector and the  $\mathbb{C}^{N_F}$ -vector of the generalized coordinates related to the structure and to the internal fluid. From such projection basis, the linear reduced operators  $[\mathcal{M}_S]$ ,  $[\mathcal{M}_F]$ ,  $[\mathcal{K}_S]$  and  $[\mathcal{K}_F]$  and the coupling reduced operator  $[\mathcal{C}]$  are constructed and the nonlinear quadratic and cubic reduced operators  $\mathcal{K}_{S,\alpha\beta\gamma}^{(2)}$  and  $\mathcal{K}_{S,\alpha\beta\gamma\delta}^{(3)}$  are explicitly constructed according [3]. The generalized coordinates are then solution of the matrix equation

$$\begin{bmatrix} [\mathcal{M}_S] & 0 \\ -[\mathcal{C}]^T & [\mathcal{M}_F] \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_S(t) \\ \ddot{\mathbf{q}}_F(t) \end{bmatrix} + \begin{bmatrix} [\mathcal{D}_S] & 0 \\ 0 & [\mathcal{D}_F] \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_S(t) \\ \dot{\mathbf{q}}_F(t) \end{bmatrix} + \begin{bmatrix} [\mathcal{K}_S] & [\mathcal{C}] \\ 0 & [\mathcal{K}_F] \end{bmatrix} \begin{bmatrix} \mathbf{q}_S(t) \\ \mathbf{q}_F(t) \end{bmatrix} + \begin{bmatrix} \mathcal{F}^{NL}(\mathbf{q}_S(t)) \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{F}_S(t) \\ 0 \end{bmatrix}, \quad (9)$$

in which the nonlinear reduced force  $\mathcal{F}^{NL}(\mathbf{q}_S)$  is defined from the nonlinear quadratic and cubic stiffness operators such that

$$\mathcal{F}_\alpha^{NL}(\mathbf{q}_S) = \mathcal{K}_{S,\alpha\beta\gamma}^{(2)} q_{S,\beta} q_{S,\gamma} + \mathcal{K}_{S,\alpha\beta\gamma\delta}^{(3)} q_{S,\beta} q_{S,\gamma} q_{S,\delta}. \quad (10)$$

For the application presented in this paper, the damping model is written as  $[\mathcal{D}_S] = b_S [\mathcal{K}_S]$  and  $[\mathcal{D}_F] = b_F [\mathcal{K}_F]$ . Furthermore, the vector  $\mathcal{F}_S(t)$  is the external reduced load vector of the structure, assuming that there is no external source for the fluid. The presence of the geometric nonlinearities yields the equations of the mean nonlinear reduced-order model to be considered in the time domain. The forced response of the structural-acoustic problem is then investigated in the time-domain by considering an equivalent time-evolution problem with zero initial conditions over a finite time interval, which includes almost all of the signal energy of the excitation. The strategy is to simultaneously and uniformly excite all the frequencies of a chosen frequency band of excitation so that only one computation of the nonlinear dynamical problem is required.

### 3 Stochastic nonlinear reduced-order model of the structural-acoustic system

The uncertainty propagation of the nonlinear dynamical response of the structural-acoustic system is then investigated by implementing the uncertainties through the nonparametric probabilistic approach whose review can be found in [13]. More precisely, we are interested in the modeling of the stiffness linear and nonlinear terms issued from the linear elasticity constitutive equation. The main idea introduced in [9] consist in globally introducing the uncertainties on all linear and nonlinear stiffness terms through a stiffness operator self-containing the linear, quadratic and cubic stiffness terms. Such reduced operator is represented by the  $(P \times P)$  real matrix  $[\mathcal{K}]$  with  $P = N_S(N_S + 1)$ . It has been shown in [9] that matrix  $[\mathcal{K}]$  is a symmetric and positive-definite matrix. Consequently, the nonparametric probabilistic approach can be easily extended to the geometrically nonlinear context. The random matrix  $[\mathcal{K}]$  can be written as  $[\mathcal{K}] = [L_K]^T [\mathbf{G}_K(\delta_K)] [L_K]$ , in which the  $(P \times P)$  matrix  $[L_K]$  is issued from the Choleski factorization of matrix  $[\mathcal{K}]$  and where  $[\mathbf{G}_K(\delta_K)]$  is a full random matrix with values in the set of all the positive-definite symmetric  $(P \times P)$  matrices. The main drawbacks of this method is that the dimension  $P$  of the random germ  $[\mathbf{G}_K(\delta_K)]$  drastically increases with  $N$  and that a scale effect prevents to compare the influence of the uncertainties on the nonlinear stiffness terms with respect to the linear stiffness terms in the nonlinear dynamical analysis [4]. The main idea proposed in [1] is to use another factorization of matrix  $[\mathcal{K}]$ , which involves a rectangular matrix of dimension  $(N \times P)$  with  $N \lll P$ . This is achieved by performing a reduction of matrix  $[\mathcal{K}]$ . Let us then consider the eigenvalue problem

$$[\mathcal{K}] \psi_\alpha = \lambda_\alpha \psi_\alpha \quad . \quad (11)$$

Matrix  $[\mathcal{K}]$  can then be approximated by the  $(P \times P)$  matrix  $[\tilde{\mathcal{K}}]$  such that  $[\tilde{\mathcal{K}}] = [\tilde{L}_K]^T [\tilde{\mathbf{G}}_K] [\tilde{L}_K]$  , in which  $[\tilde{L}_K]$  is the full  $(N \times P)$  matrix defined by  $[\tilde{L}_K] = [\Lambda^N]^{\frac{1}{2}} [\Psi^N]^T$ , where  $[\Lambda^N]$  is the  $(N \times N)$  diagonal matrix such that  $[\Lambda^N]_{\alpha\alpha} = \lambda_\alpha$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$  and where the columns of the  $(P \times N)$  matrix  $[\Psi^N]$  are the eigenvector  $\psi_\alpha$ ,  $\alpha \in \{1, \dots, N\}$  related to eigenvalues  $\lambda_\alpha$ . Random matrix  $[\mathcal{K}]$  is then replaced by the random matrix  $[\tilde{\mathcal{K}}]$  such that

$$[\tilde{\mathcal{K}}] = [\tilde{L}_K]^T [\tilde{\mathbf{G}}_K(\delta_K)] [\tilde{L}_K] + ([\tilde{\mathcal{K}}] - [\mathcal{K}]) \quad , \quad (12)$$

in which  $[\tilde{\mathbf{G}}_K(\delta_K)]$  is a  $(N \times N)$  random matrix with values in the set of the symmetric positive-definite matrices. Note that Eq. (12) ensures that stiffness operator  $[\tilde{\mathcal{K}}]$  is almost surely positive definite. The construction of the random generator of each random  $(N \times N)$  matrix related to each operator is detailed in [12, 13]. Finally, the uncertainty level of the coupled fluid-structure system is entirely controlled by the  $\mathbb{R}^7$ -valued hyperparameter  $\delta = (\delta_{M_S}, \delta_{D_S}, \delta_C, \delta_{M_F}, \delta_{D_F}, \delta_{K_F}, \delta_K)$ , belonging to the admissible set  $\Delta^7$ . Concerning the numerical procedure for computing the response of the stochastic nonlinear reduced-order model in the time domain, the Monte Carlo numerical simulation is used with an implicit and unconditionally stable integration scheme (Newmark method with the averaging acceleration scheme) combined with either the fixed point method or with an adapted efficient algorithm based on the arc-length method [5, 4, 2], depending on the nonlinearity rate. A posterior nonlinear dynamical analysis is then performed in the frequency domain by using Fast Fourier Transform.

## 4 Numerical application

The fluid-structure coupled dynamical system under consideration is a cylindrical tank partially filled with a linear inviscid acoustic fluid that is described in a global cartesian coordinate system  $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ , where  $O$  is the center of the cylinder basis and where the cylinder axis is defined along  $\mathbf{e}_3$ . Its geometry is characterized by the mean radius  $r_m = 0.93 m$ , thickness  $e = 0.14 m$ ,  $h = 2 m$ , and bottom thickness  $h_b = 0.5 m$ . The bottom of the cylindrical shell is clamped. The material is a linear isotropic homogeneous elastic material for which the Young modulus  $E = 2.1 \times 10^{11} N \times m^{-2}$ , the Poisson ratio  $\nu = 0.3$ , and the mass density  $\rho_S = 7860 Kg \times m^{-3}$ . The internal fluid occupies a cylindrical volume with radius  $r_f = 0.86 m$  and height  $h_F = 0.75 m$ . It has a free surface. The fluid has mass density  $\rho_F = 1000 Kg.m^{-3}$  and sound velocity  $c_F = 1480 m \times s^{-1}$ . The finite element model of the system has 1024 three-dimensional solid finite elements with 8 nodes with  $n_S = 3750$  Dofs. The finite element model of the fluid has 768 three-dimensional fluid finite elements with 8 nodes with  $n_F = 819$  Dofs and is represented by Fig. 1. The structure is subjected to a transverse load such that all the nodes located at the top of the cylindrical shell are uniformly excited along  $\mathbf{e}_2$ . The damping model is characterized by  $b_S = 1.5 \times 10^{-5}$  and  $b_F = 10^{-5}$ .

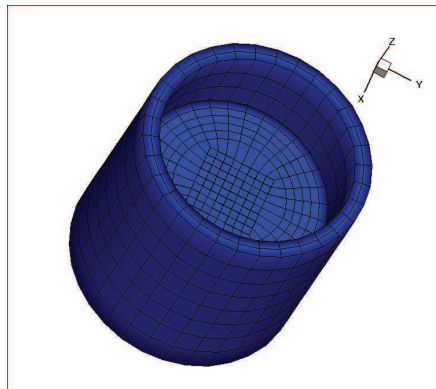


FIGURE 1 – Finite element model of the mean fluid-structure system.

The frequency band of excitation is  $\mathbb{B}_{exc} = [200, 1400] Hz$  such that  $s = 2/3$  and  $\Delta\nu = 1200 Hz$  according to Eq.(14). The stochastic reduced-order model is constructed with  $N_S = 50$  and  $N_F = 40$ , for which a convergence analysis concerning the displacements of the structure and the pressure in the fluid has been made. The chosen observations are the displacements  $U_x, U_y, U_z$  along directions  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  of the point of coordinates  $(-1, 0, 2)$  located at the top of the structure and the pressure  $P$  of the point of coordinated  $(-0.64, 0, 1)$ , located in the fluid. Figures 2 and 3 compare the confidence region of the nonlinear stochastic response of the fluid-structure system at these observation points when uncertainty is located on the stiffness structure only  $\delta_K = 0.2$  or on the stiffness fluid only  $\delta_{K_S} = 0.2$ .

## 5 Conclusion

A numerical method for computing the nonlinear dynamical behaviour of an uncertain coupled fluid-structure system has been presented. A numerical application demonstrates the feasibility of the proposed method.

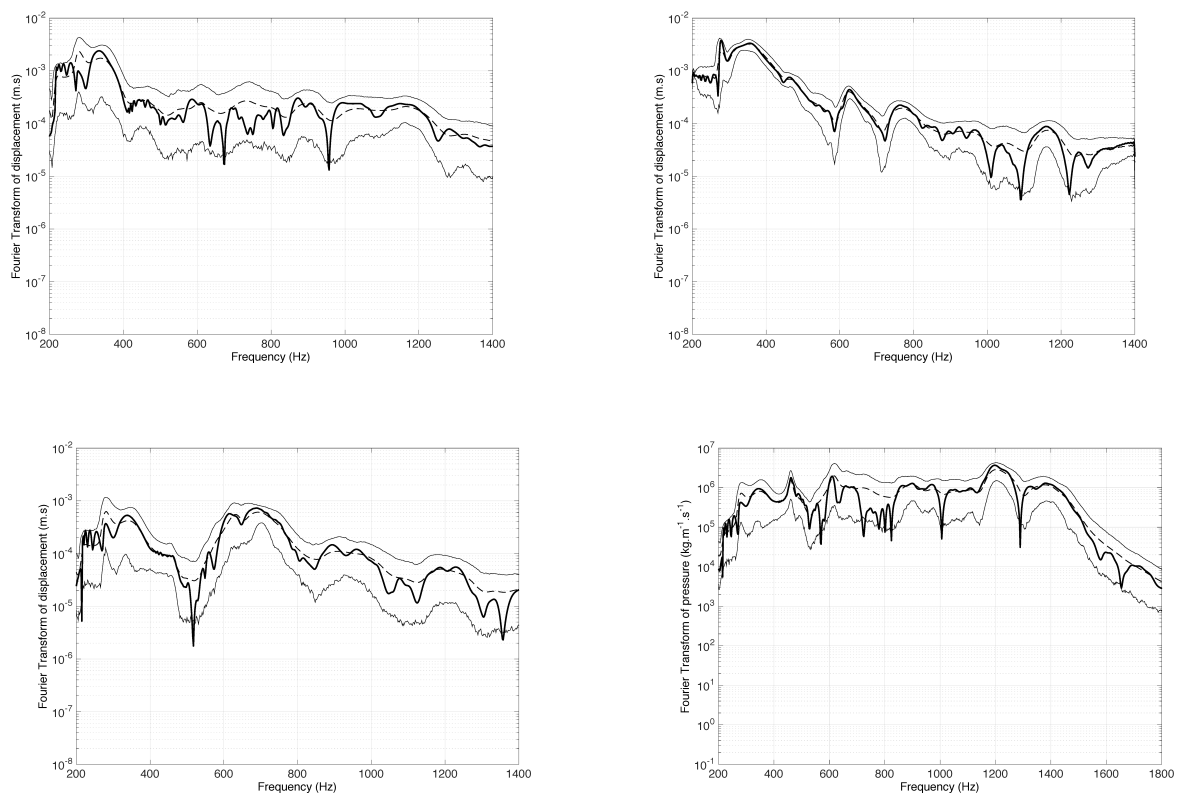


FIGURE 2 – For  $\delta_{K_S} = 0.2$ , graph of the deterministic response (thick line), the mean response (thin dashed line) and of the confidence region related to (a) displacement  $\hat{U}_x(\nu)$ , (b) displacement  $\hat{U}_y(\nu)$ , (c) displacement  $\hat{U}_z(\nu)$ , (d) pressure  $\hat{P}(\nu)$ .

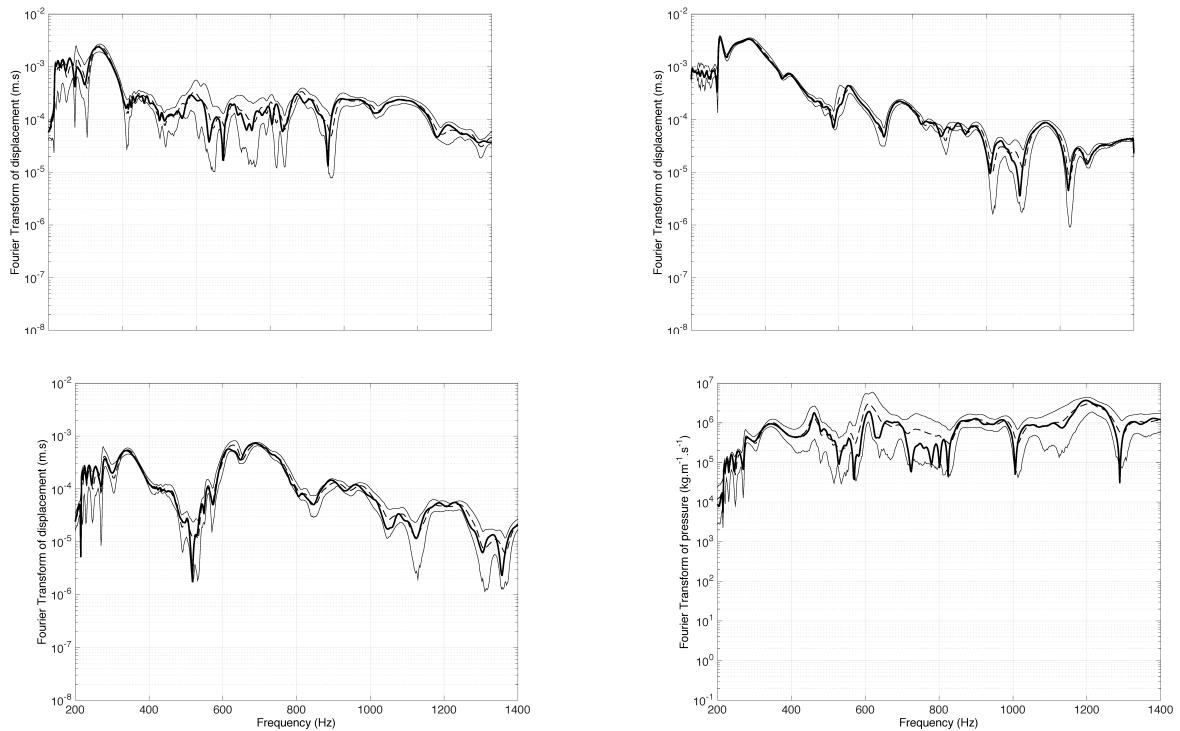


FIGURE 3 – For  $\delta_{K_F} = 0.2$ , graph of the deterministic response (thick line), the mean response (thin dashed line) and of the confidence region related to (a) displacement  $\hat{U}_x(\nu)$ , (b) displacement  $\hat{U}_y(\nu)$ , (c) displacement  $\hat{U}_z(\nu)$ , (d) pressure  $\hat{P}(\nu)$ .

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