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van Hao Can, Viet-Hung Pham

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MODERATE DEVIATIONS FOR CRITICAL CURIE-WEISS MODEL

VAN HAO CAN AND VIET-HUNG PHAM

ABSTRACT. In this paper we study the moderate deviations for the magnetization of critical Curie-Weiss model. Chen, Fang and Shao considered a similar problem for non-critical model by using Stein method. By direct and simple arguments based on Laplace method, we provide an explicit formula of the error and deduce a Cramér-type result.

1. INTRODUCTION

Moderate deviations have been investigated for a long time with numerous results. This kind of phenomenon was first introduced by Cramér as a refinement of the classic Central limit theorems for sum of i.i.d random variables. Let (X_i) be a sequence of i.i.d. random variables satisfying $\mathbb{E}X_i = 0$, $\text{Var}(X_i) = 1$ and $\mathbb{E}(e^{\alpha X_i}) < \infty$, for some $\alpha > 0$. Then the normalized sum $W_n = (X_1 + \dots + X_n)/\sqrt{n}$ satisfies

$$\frac{P(\sigma : W_n > x)}{1 - \Phi(x)} = 1 + O(1)(1 + x^3)/\sqrt{n}$$

for $0 \leq x \leq n^{1/6}$, with Φ the standard normal distribution function. It means that while the classic CLT ensures that for a fixed real number x , the tail distribution $P(\sigma : W_n > x)$ tends to $1 - \Phi(x)$, this Cramér-type moderate deviation result provides the rate of convergence of the ratio $P(\sigma : W_n > x)/(1 - \Phi(x))$ to 1 for a regime of variable x . For the proof of this result, we refer to the book [16].

Later on, many authors managed to generalize this result in many directions like as: sum of independent but not identically distributed [6], and then sum of dependent random variables such as stationary process [1, 20], linear process [15], and process with stationary increments [19].

In general, many authors considered the general objects such as: random sums [7], self-normalized sums [4, 18], functionals of Markov chain [11], functionals of geometric systems [8], determinant of random matrices [12], spectral measures of random matrices [10], L-statistics [13], ...; and also general corresponding limit distribution (maybe non-Gaussian): chi-squared distribution [14], sub-Gaussian or exponential distribution [2]. They also provided many powerful tools to study the problem: Laplace method [16], Lindeberg method [7] and Stein method [2, 3].

In applications, moderate deviations have been proved for various models, such as non-critical Curie-Weiss model, anti voter model, dynamical Curie Weiss model, see [3, 5].

In this paper, we are interested in the *Curie-Weiss model*. Let us first recall some definitions and existing results. For $n \in \mathbb{N}$, let $\Omega_n = \{\pm 1\}^n$ be the space of spin configurations.

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The configuration probability is given by Boltzman-Gibbs distribution: for any $\sigma \in \Omega_n$,

$$\mu_n(\sigma) = Z_n^{-1} \exp \left(\frac{\beta}{n} \sum_{1 \leq i < j \leq n} \sigma_i \sigma_j + \beta h \sum_{i=1}^n \sigma_i \right),$$

where Z_n is the normalizing factor, $\beta > 0$ and $h \in \mathbb{R}$ are inverse temperature and external field respectively. The asymptotic behavior of the total spin (also called magnetization) $S_n = \sigma_1 + \dots + \sigma_n$ is characterized by the solution of the following fixed-point equation

$$(1) \quad m = \tanh(\beta(m + h)).$$

Then the limit distribution of S_n is considered in three following cases. The first two ones are called *non-critical* and the last one is called *critical*.

Case 1. $0 < \beta < 1$, $h \in \mathbb{R}$ or $\beta \geq 1, h \neq 0$ (*non-critical case*). The equation (1) has a unique solution m_0 such that $m_0 h \geq 0$. In this case, S/n is concentrated around m_0 and has a Gaussian limit under proper standardization, see [9]. Moreover, in [3] the authors prove moderate deviations for the magnetization.

Theorem 1.1. [3, Proposition 4.3] *In case 1, let us define*

$$W_n = \frac{S_n - nm_0}{v_n},$$

where

$$v_n = \sqrt{\frac{n(1 - m_0^2)}{1 - (1 - m_0^2)\beta}}.$$

Then we have

$$\frac{\mu_n(\sigma : W_n > x)}{1 - \Phi(x)} = 1 + O(1)(1 + x^3)/\sqrt{n}$$

for $0 \leq x \leq n^{1/6}$.

Case 2. $\beta > 1, h = 0$ (*non-critical case*). The equation (1) has two nonzero solutions $m_1 < 0 < m_2$, where $m_1 = -m_2$. In this case, one has the conditional central limit theorems as follows: conditioned on $S_n < 0$ (resp. $S_n > 0$), S/n is concentrated around m_1 (resp. m_2) and has a Gaussian limit after proper scaling, see [9]. Similarly to case 1, a moderate deviation result has been also proved.

Theorem 1.2. [3, Proposition 4.4] *In case 2, define*

$$W_{1,n} = \frac{S_n - nm_1}{v_{1,n}} \quad \text{and} \quad W_{2,n} = \frac{S_n - nm_2}{v_{2,n}},$$

where

$$v_{1,n} = \sqrt{\frac{n(1 - m_1^2)}{1 - (1 - m_1^2)\beta}} \quad \text{and} \quad v_{2,n} = \sqrt{\frac{n(1 - m_2^2)}{1 - (1 - m_2^2)\beta}}.$$

Then we have

$$\frac{\mu_n(\sigma : W_{1,n} > x \mid S_n < 0)}{1 - \Phi(x)} = 1 + O(1)(1 + x^3)/\sqrt{n}$$

and

$$\frac{\mu_n(\sigma : W_{2,n} > x \mid S_n > 0)}{1 - \Phi(x)} = 1 + O(1)(1 + x^3)/\sqrt{n}$$

for $0 \leq x \leq n^{1/6}$.

Case 3. $\beta = 1$ and $h = 0$ (*critical case*). The equation (1) has a unique solution 0 and S/n is concentrated around 0. In this case, $S_n/n^{3/4}$ converges to a non-Gaussian distribution, see [9]. In [2], Chatterjee and Shao gave an upper bound for the rate of this convergence.

Theorem 1.3. [2, Theorem 2.1] *In the case 3, define*

$$W_n = \frac{S_n}{n^{3/4}}.$$

Then there exists a positive constant C , such that for all x

$$(2) \quad \limsup_{n \rightarrow \infty} \sqrt{n} \left| \mu_n(\sigma : W_n \leq x) - F(x) \right| \leq C,$$

where

$$F(x) = \frac{\int_{-\infty}^x e^{-t^4/12} dt}{\int_{-\infty}^{\infty} e^{-t^4/12} dt}.$$

However from Theorem 1.3, one could not say much about the ratio $\mu_n(\sigma : W_n > x) / (1 - F(x))$ as in Theorems 1.1 and 1.2.

In this paper, we will fill this gap by proving the moderate deviations for S_n in the critical case. Our main result is the following theorem.

Theorem 1.4. *For the critical case, when $\beta = 1$ and $h = 0$, define*

$$W_n = \frac{S_n}{n^{3/4}}.$$

Then there exists a positive constant C , such that for all n large enough and $0 \leq x \leq n^{1/12}$,

$$(3) \quad \left| \frac{\mu_n(\sigma : W_n > x)}{1 - F(x)} - 1 - \frac{G(x)}{\sqrt{n}} \right| \leq \frac{C(x^{12} + n^{1/3})}{n},$$

where

$$F(x) = \frac{\int_{-\infty}^x p_1(t) dt}{\int_{-\infty}^{\infty} p_1(t) dt},$$

and

$$G(x) = \left(\frac{\int_x^{\infty} p_2(t) dt}{\int_x^{\infty} p_1(t) dt} - \frac{\int_{-\infty}^{\infty} p_2(t) dt}{\int_{-\infty}^{\infty} p_1(t) dt} \right)$$

with

$$p_1(t) = e^{-\frac{t^4}{12}} \quad \text{and} \quad p_2(t) = \left(\frac{t^2}{2} - \frac{t^6}{30} \right) e^{-\frac{t^4}{12}}.$$

It is worth to mention that Theorem 1.4 gives exact formula of the error term of order $n^{-1/2}$, while moderate deviation results in Theorems 1.1 and 1.2 only give asymptotic estimates of the error terms. The range of estimate $n^{1/6}$ is replaced by $n^{1/12}$ due to the change of scaling and limiting distribution. The proof of Theorem 1.4 is simple and direct, based on Laplace method-like arguments.

We have a direct corollary.

Corollary 1.5. For $0 \leq x \leq n^{1/12}$, we have

$$\frac{\mu_n(\sigma : W_n > x)}{1 - F(x)} = 1 + O(1)(1 + x^6)/\sqrt{n}.$$

Moreover, for any fixed real number x ,

$$\lim_{n \rightarrow \infty} \sqrt{n} \left(\mu_n(\sigma : W_n \leq x) - F(x) \right) = (F(x) - 1)G(x).$$

The first part of this corollary is a Cramér-type moderate deviation result in classic form, whereas the second part is an improvement of Theorem 1.3.

We fix here some notation. If f and g are two real functions, we write $f = O(g)$ if there exists a constant $C > 0$, such that $f(x) \leq Cg(x)$ for all x ; $f = \Omega(g)$ if $g = O(f)$; and $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.

2. PROOF OF THEOREM 1.4

For any $\sigma \in \Omega_n$, we define

$$\sigma_+ = \{i : \sigma_i = 1\}.$$

Observe that if $|\sigma_+| = k$, then

$$\begin{aligned} \frac{1}{n} \sum_{i \leq j} \sigma_i \sigma_j &= 1 + \frac{1}{n} \sum_{i < j} \sigma_i \sigma_j = 1 + \frac{1}{2n} \left(\left(\sum_{1 \leq i \leq n} \sigma_i \right)^2 - n \right) \\ (4) \qquad \qquad \qquad &= \frac{(2k - n)^2}{2n} + \frac{1}{2}. \end{aligned}$$

Hence,

$$Z_n = \sum_{\sigma \in \Omega_n} \exp \left(\frac{1}{n} \sum_{i \leq j} \sigma_i \sigma_j \right) = \sum_{k=0}^n \sum_{\substack{\sigma \in \Omega_n \\ |\sigma_+|=k}} \exp \left(\frac{1}{n} \sum_{i \leq j} \sigma_i \sigma_j \right) = \sum_{k=0}^n \binom{n}{k} e^{\frac{(2k-n)^2}{2n} + \frac{1}{2}}.$$

Let us define

$$x_{k,n} = \binom{n}{k} e^{\frac{(2k-n)^2}{2n} + \frac{1}{2}}.$$

Then

$$(5) \qquad \qquad \qquad Z_n = \sum_{k=0}^n x_{k,n},$$

and

$$(6) \qquad \qquad \qquad \mu_n(\sigma : |\sigma_+| = k) = \frac{x_{k,n}}{Z_n}.$$

Let us define for $x \in \mathbb{R}$

$$F_n(x) = \mu_n(\sigma : W_n \leq x).$$

Then by (6),

$$\begin{aligned}
 1 - F_n(x) &= \mu_n(\sigma : \sigma_1 + \dots + \sigma_n > n^{3/4}x) \\
 &= \mu_n(\sigma : 2|\sigma_+| - n > n^{3/4}x) = \mu_n\left(\sigma : |\sigma_+| > \frac{n + n^{3/4}x}{2}\right) \\
 (7) \quad &= \frac{1}{Z_n} \sum_{k=0}^n x_{k,n} \mathbb{I}\left(k > \frac{n + n^{3/4}x}{2}\right),
 \end{aligned}$$

where $\mathbb{I}(\cdot)$ stands for the indicator function.

To understand $x_{k,n}$, we need to study the binomial coefficient. Let us recall a version of Stirling's approximation (see [17]): for all $n \geq 1$,

$$\log(\sqrt{2\pi n}) + n \log n - n + \frac{1}{12n+1} \leq \log(n!) \leq \log(\sqrt{2\pi n}) + n \log n - n + \frac{1}{12n}.$$

Using this approximation, we can easily prove that

$$(8) \quad \binom{n}{k} \leq e^{nI(k/n)}, \quad \text{for all } k = 0, \dots, n,$$

and

$$(9) \quad \binom{n}{k} = (1 - O(n^{-1})) \sqrt{\frac{n}{2\pi k(n-k)}} \times e^{nI(k/n)}, \quad \text{for } |k - (n/2)| < n/4,$$

where for $t \in (0, 1)$,

$$I(t) = (t-1) \log(1-t) - t \log t.$$

Using (8) and (9), we obtain

$$(10) \quad x_{k,n} \leq e^{nJ(k/n)+1/2}, \quad \text{for all } k = 0, \dots, n,$$

and

$$(11) \quad x_{k,n} = (1 - O(n^{-1})) \sqrt{\frac{n}{2\pi k(n-k)}} \times e^{nJ(k/n)+1/2}, \quad \text{for } |k - (n/2)| < n/4,$$

where

$$J(t) = I(t) + \frac{(2t-1)^2}{2}.$$

We have a lemma on the behavior of $J(t)$.

Lemma 2.1. *We have*

$$(i) \quad J'(1/2) = J''(1/2) = J'''(1/2) = J^{(5)}(1/2) = J^{(7)}(1/2) = 0, \text{ and for all } t \neq 1/2 \\ J''(t) < 0.$$

$$(ii) \quad J^{(4)}(1/2) = -32, \quad J^{(6)}(1/2) = -1536, \text{ and for all } 1/4 \leq t \leq 3/4 \\ -2^{25} < J^{(8)}(t) < 0.$$

Proof. We have

$$J'(t) = \log\left(\frac{1-t}{t}\right) + (4t-2).$$

Hence

$$\begin{aligned}
 J''(t) &= -[t^{-1} + (1-t)^{-1}] + 4, & J'''(t) &= [t^{-2} - (1-t)^{-2}], & J^{(4)}(t) &= -2[t^{-3} + (1-t)^{-3}], \\
 J^{(5)}(t) &= 6[t^{-4} - (1-t)^{-4}], & J^{(6)}(t) &= -24[t^{-5} + (1-t)^{-5}], & J^{(7)}(t) &= 120[t^{-6} - (1-t)^{-6}],
 \end{aligned}$$

and

$$J^{(8)}(t) = -720[t^{-7} + (1-t)^{-7}].$$

Using these equations, we can easily prove desired results. \square

By Lemma 2.1, we see that $J(t)$ attains the maximum at a unique point $1/2$. This fact suggest that the value of Z_n (the sum of $(x_{k,n})$) is concentrated at the middle terms. Let us define

$$y_n = \sqrt{\frac{2}{\pi n}} \times e^{nJ(1/2)+1/2},$$

which is asymptotic to $x_{[n/2],n}$. Using (10) and (11), we have the comparisons between $x_{k,n}$ and y_n

$$(12) \quad \frac{x_{k,n}}{y_n} \leq \sqrt{\frac{\pi n}{2}} \times e^{n[J(k/n)-J(1/2)]}, \quad \text{for all } k = 0, \dots, n,$$

and

$$(13) \quad \frac{x_{k,n}}{y_n} = (1 - O(n^{-1})) \sqrt{\frac{n^2}{4k(n-k)}} \times e^{n[J(k/n)-J(1/2)]} \quad \text{for } |k - (n/2)| < n/4.$$

Combining (5) and (7), we have

$$(14) \quad 1 - F_n(x) = \frac{1}{\sum_{k=0}^n y_{k,n}} \times \sum_{k=0}^n y_{k,n} \mathbb{I} \left(k > \frac{n + n^{3/4}x}{2} \right),$$

where

$$y_{k,n} = \frac{x_{k,n}}{y_n}.$$

In the sequel, we consider two cases: $x > 10$ and $x \leq 10$. The choice of number 10 is flexible. We just need the fact that the functions $p_1(\cdot), p_2(\cdot)$ and $r(\cdot)$ defined in (19) are decreasing on some intervals (c, ∞) for a positive constant c (see Lemma 2.3).

2.1. Case $x > 10$. We define

$$\begin{aligned} A_n &= \sum_{k=0}^n y_{k,n} \mathbb{I} \left(\left| k - \frac{n}{2} \right| \geq \frac{n}{4} \right), \quad B_n = \sum_{k=0}^n y_{k,n} \mathbb{I} \left(\left| k - \frac{n}{2} \right| < \frac{n}{4} \right) \\ \hat{A}_n &= \sum_{k=0}^n y_{k,n} \mathbb{I} \left(k - \frac{n}{2} \geq \frac{n}{4} \right), \quad B_{n,x} = \sum_{k=0}^n y_{k,n} \mathbb{I} \left(\frac{n}{4} > k - \frac{n}{2} > \frac{n^{3/4}x}{2} \right). \end{aligned}$$

Then

$$(15) \quad 1 - F_n(x) = \frac{\hat{A}_n + B_{n,x}}{A_n + B_n}.$$

2.1.1. Estimates of A_n and \hat{A}_n .

Lemma 2.2. *There exists a positive constant c , such that for n large enough,*

$$\hat{A}_n \leq A_n \leq e^{-cn}.$$

Proof. Since $J(t)$ attains the maximum at a unique point $1/2$,

$$\max_{|x-0.5|\geq 0.25} J(x) = \max\{J(0.75), J(0.25)\} = J(0.25).$$

Hence for all $|k - (n/2)| \geq n/4$,

$$J(k/n) - J(1/2) \leq J(0.25) - J(0.5) < -0.005.$$

Thus for all $|k - (n/2)| \geq n/4$,

$$(16) \quad n(J(k/n) - J(1/2)) < -0.005n.$$

It follows from (12) and (16) that for $|k - (n/2)| \geq n/4$,

$$y_{k,n} \leq \sqrt{2n} \exp(-0.005n).$$

Thus

$$\hat{A}_n \leq A_n \leq n\sqrt{2n} \exp(-0.005n) < \exp(-0.004n),$$

for all n large enough. \square

2.1.2. *Estimates of B_n and $B_{n,x}$.* By using Lemma 2.1 and Taylor expansion, we get

$$J\left(\frac{k}{n}\right) - J\left(\frac{1}{2}\right) = \frac{1}{4!}J^{(4)}(1/2)\left(\frac{k}{n} - \frac{1}{2}\right)^4 + \frac{1}{6!}J^{(6)}(1/2)\left(\frac{k}{n} - \frac{1}{2}\right)^6 + \frac{1}{8!}J^{(8)}(\xi_{k,n})\left(\frac{k}{n} - \frac{1}{2}\right)^8,$$

with some $\xi_{k,n}$ between k/n and $1/2$. Then using again Lemma 2.1,

$$J\left(\frac{k}{n}\right) - J\left(\frac{1}{2}\right) \leq -\frac{(2k-n)^4}{12n^4} - \frac{(2k-n)^6}{30n^6}.$$

and

$$J\left(\frac{k}{n}\right) - J\left(\frac{1}{2}\right) \geq -\frac{(2k-n)^4}{12n^4} - \frac{(2k-n)^6}{30n^6} - \frac{2^{17}(2k-n)^8}{n^8 8!}.$$

Thus

$$n(J(k/n) - J(1/2)) \leq -\frac{(2k-n)^4}{12n^3} - \frac{(2k-n)^6}{30n^5}$$

and

$$n(J(k/n) - J(1/2)) \geq -\frac{(2k-n)^4}{12n^3} - \frac{(2k-n)^6}{30n^5} - \frac{2^{17}(2k-n)^8}{n^7 8!}.$$

Hence, using the inequality that $1 - x \leq e^{-x} \leq 1 - x + x^2/2$ for all $x \geq 0$,

$$e^{n[J(k/n) - J(1/2)]} \leq \exp\left(\frac{-(2k-n)^4}{12n^3}\right) \left(1 - \frac{(2k-n)^6}{30n^5} + \frac{(2k-n)^{12}}{1800n^{10}}\right),$$

and

$$e^{n[J(k/n) - J(1/2)]} \geq \exp\left(\frac{-(2k-n)^4}{12n^3}\right) \left(1 - \frac{(2k-n)^6}{30n^5}\right) \left(1 - \frac{2^{17}(2k-n)^8}{n^7 8!}\right).$$

In conclusion,

$$(17) \quad e^{n[J(k/n) - J(1/2)]} = \exp\left(\frac{-(2k-n)^4}{12n^3}\right) \left(1 - \frac{(2k-n)^6}{30n^5} + O(1)X_{k,n}\right),$$

where

$$X_{k,n} = \frac{(2k-n)^8}{n^7} + \frac{(2k-n)^{12}}{n^{10}} + \frac{(2k-n)^{14}}{n^{12}}.$$

On the other hand for $|k - (n/2)| < n/4$,

$$(18) \quad \sqrt{\frac{n^2}{4k(n-k)}} = 1 + \frac{(2k-n)^2}{2n^2} + O\left(\frac{(2k-n)^4}{n^4}\right).$$

Combining (13), (17) and (18), we have for $|k - (n/2)| < n/4$,

$$y_{k,n} = \left(1 + \frac{(2k-n)^2}{2n^2} - \frac{(2k-n)^6}{30n^5} + O(1)R_{k,n}\right) \exp\left(-\frac{(2k-n)^4}{12n^3}\right),$$

where

$$R_{k,n} = \frac{1}{n} + \frac{(2k-n)^4}{n^4} + \frac{(2k-n)^8}{n^7} + \frac{(2k-n)^{12}}{n^{10}} + \frac{(2k-n)^{14}}{n^{12}}.$$

By letting $\ell = 2k - n$,

$$\begin{aligned} B_n &= \sum_{n/4 < k < 3n/4} y_{k,n} \\ &= \sum_{\substack{|\ell| < n/2 \\ 2|(\ell+n)}} e^{-\frac{\ell^4}{12n^3}} \left(1 + \frac{\ell^2}{2n^2} - \frac{\ell^6}{30n^5} + O\left(\frac{1}{n} + \frac{\ell^4}{n^4} + \frac{\ell^8}{n^7} + \frac{\ell^{12}}{n^{10}} + \frac{\ell^{14}}{n^{12}}\right)\right) \\ &= \sum_{\substack{|\ell| < n/2 \\ 2|(\ell+n)}} e^{-\left(\frac{\ell}{n^{3/4}}\right)^4/12} \left(1 + \frac{1}{\sqrt{n}} \left(\frac{1}{2} \left(\frac{\ell}{n^{3/4}}\right)^2 - \frac{1}{30} \left(\frac{\ell}{n^{3/4}}\right)^6\right) \right. \\ &\quad \left. + \frac{O(1)}{n} \left(1 + \left(\frac{\ell}{n^{3/4}}\right)^4 + \left(\frac{\ell}{n^{3/4}}\right)^8 + \left(\frac{\ell}{n^{3/4}}\right)^{12} + \frac{1}{\sqrt{n}} \left(\frac{\ell}{n^{3/4}}\right)^{14}\right)\right) \\ (19) &= \sum_{\substack{|\ell| < n/2 \\ 2|(\ell+n)}} p_1\left(\frac{\ell}{n^{3/4}}\right) + \frac{1}{\sqrt{n}} \sum_{\substack{|\ell| < n/2 \\ 2|(\ell+n)}} p_2\left(\frac{\ell}{n^{3/4}}\right) + \frac{O(1)}{n} \sum_{\substack{|\ell| < n/2 \\ 2|(\ell+n)}} r\left(\frac{\ell}{n^{3/4}}\right), \end{aligned}$$

where

$$\begin{aligned} p_1(t) &= e^{-\frac{t^4}{12}} \\ p_2(t) &= \left(\frac{t^2}{2} - \frac{t^6}{30}\right) e^{-\frac{t^4}{12}} \\ r(t) &= (1 + t^4 + t^8 + t^{12} + t^{14}/\sqrt{n}) e^{-\frac{t^4}{12}}. \end{aligned}$$

Similarly,

$$\begin{aligned} B_{n,x} &= \sum y_{k,n} 1\left(\frac{n}{4} > k - \frac{n}{2} > \frac{n^{3/4}x}{2}\right) \\ (20) &= \sum_{\substack{n^{3/4}x < \ell < n/2 \\ 2|(\ell+n)}} p_1\left(\frac{\ell}{n^{3/4}}\right) + \frac{1}{\sqrt{n}} \sum_{\substack{n^{3/4}x < \ell < n/2 \\ 2|(\ell+n)}} p_2\left(\frac{\ell}{n^{3/4}}\right) + \frac{O(1)}{n} \sum_{\substack{n^{3/4}x < \ell < n/2 \\ 2|(\ell+n)}} r\left(\frac{\ell}{n^{3/4}}\right), \end{aligned}$$

We have the following lemmas.

Lemma 2.3. *These functions $p_1(t)$, $p_2(t)$ and $r(t)$ are decreasing in $(9, \infty)$.*

Lemma 2.4. *Let m, q, p be positive real numbers. If $f(t)$ is a decreasing function in $[(m-1)/p, (q+1)/p]$, then*

$$\left| \sum_{\substack{m < \ell < n \\ 2|\ell}} f\left(\frac{\ell}{p}\right) - \frac{p}{2} \int_{m/p}^{q/p} f(t) dt \right| \leq \left| f\left(\frac{m}{p}\right) \right| + \left| f\left(\frac{q}{p}\right) \right|,$$

and

$$\left| \sum_{\substack{m < \ell < q \\ 2|\ell}} f\left(\frac{\ell}{p}\right) - \frac{p}{2} \int_{m/p}^{q/p} f(t) dt \right| \leq \left| f\left(\frac{m}{p}\right) \right| + \left| f\left(\frac{q}{p}\right) \right|.$$

The proof of Lemmas 2.3 and 2.4 is simple, so we safely leave it to the reader. Applying these lemmas to the sums in (20), we have

$$\begin{aligned} B_{n,x} &= \frac{n^{3/4}}{2} \int_x^{n^{1/4}/2} p_1(t) dt + \frac{n^{1/4}}{2} \int_x^{n^{1/4}/2} p_2(t) dt \\ &+ O(1) \left(\frac{1}{n^{1/4}} \int_x^{n^{1/4}/2} r(t) dt + p_1(x) + \frac{p_2(x)}{\sqrt{n}} + \frac{r(x)}{n} + p_1(n^{1/4}/2) + \frac{p_2(n^{1/4}/2)}{\sqrt{n}} + \frac{r(n^{1/4}/2)}{\sqrt{n}} \right). \end{aligned}$$

It is not difficult to see that

$$p_1(n^{1/4}/2), p_2(n^{1/4}/2), r(n^{1/4}/2), \int_{n^{1/4}/2}^{\infty} p_1(t) dt, \int_{n^{1/4}/2}^{\infty} p_2(t) dt, \int_{n^{1/4}/2}^{\infty} r(t) dt = e^{-\Omega(n)}.$$

Therefore

$$(21) \quad B_{n,x} = \frac{n^{3/4}}{2} \hat{P}_1(x) + \frac{n^{1/4}}{2} \hat{P}_2(x) + O(1) \left(\frac{\hat{R}(x)}{n^{1/4}} + p_1(x) + \frac{p_2(x)}{\sqrt{n}} + \frac{r(x)}{n} \right) + e^{-\Omega(n)},$$

where

$$\begin{aligned} \hat{P}_1(x) &= \int_x^{\infty} p_1(t) dt \\ \hat{P}_2(x) &= \int_x^{\infty} p_2(t) dt \\ \hat{R}(x) &= \int_x^{\infty} r(t) dt. \end{aligned}$$

Similarly, using (19) and Lemmas 2.3 and 2.4 we have

$$(22) \quad B_n = \frac{n^{3/4}}{2} \hat{P}_1(-\infty) + \frac{n^{1/4}}{2} \hat{P}_2(-\infty) + \frac{O(1)}{n^{1/4}} + e^{-\Omega(n)}.$$

2.1.3. *Conclusion.* Combining Lemma 2.2 and (15), (22), (21), we have

$$\begin{aligned} 1 - F_n(x) &= \frac{n^{3/4} \hat{P}_1(x) + n^{1/4} \hat{P}_2(x) + O(1) \left(\hat{R}(x) n^{-1/4} + p_1(x) + p_2(x) n^{-1/2} + r(x) n^{-1} \right)}{n^{3/4} \hat{P}_1(-\infty) + n^{1/4} \hat{P}_2(-\infty) + O(n^{-1/4})} \\ &= \frac{\hat{P}_1(x) + n^{-1/2} \hat{P}_2(x) + O(1) \left(\hat{R}(x) n^{-1} + p_1(x) n^{-3/4} + p_2(x) n^{-5/4} + r(x) n^{-7/4} \right)}{\hat{P}_1(-\infty) + n^{-1/2} \hat{P}_2(-\infty) + O(n^{-1})}. \end{aligned}$$

Notice that $1 - F(x) = \hat{P}_1(x)/\hat{P}_1(-\infty)$. Therefore

$$\begin{aligned} -1 + \frac{1 - F_n(x)}{1 - F(x)} &= -1 + (1 - F_n(x)) \frac{\hat{P}_1(-\infty)}{\hat{P}_1(x)} \\ &= -1 + \frac{\hat{P}_1(-\infty) + \frac{\hat{P}_1(-\infty)\hat{P}_2(x)}{\sqrt{n}} + O(1) \left(n^{-1} \frac{\hat{R}(x)}{\hat{P}_1(x)} + n^{-3/4} \frac{p_1(x)}{\hat{P}_1(x)} + n^{-5/4} \frac{p_2(x)}{\hat{P}_1(x)} + n^{-7/4} \frac{r(x)}{\hat{P}_1(x)} \right)}{\hat{P}_1(-\infty) + n^{-1/2} \hat{P}_2(-\infty) + O(n^{-1})} \\ &= \frac{1}{\sqrt{n}} \left(\frac{\hat{P}_2(x)}{\hat{P}_1(x)} - \frac{\hat{P}_2(-\infty)}{\hat{P}_1(-\infty)} \right) + O(1) \left(\frac{x^{12}}{n} + n^{-3/4} \right), \end{aligned}$$

where for the last line we have used that

$$\frac{\hat{R}(x)}{\hat{P}_1(x)} = \Theta(x^{12}), \quad \frac{p_1(x)}{\hat{P}_1(x)} = \Theta(1), \quad \frac{p_2(x)}{\hat{P}_1(x)} = \Theta(x^6), \quad \frac{r(x)}{\hat{P}_1(x)} = \Theta(x^{12}).$$

In conclusion,

$$\frac{1 - F_n(x)}{1 - F(x)} = 1 + \frac{G(x)}{\sqrt{n}} + O(1) \left(\frac{x^{12}}{n} + n^{-3/4} \right),$$

with $G(x)$ as in Theorem 1.4.

2.2. Case $x \leq 10$. We define

$$\begin{aligned} U_n &= \sum_{k=0}^n y_{k,n} \mathbb{I} \left(\left| k - \frac{n}{2} \right| \geq n^{5/6} \right), \quad V_n = \sum_{k=0}^n y_{k,n} \mathbb{I} \left(\left| k - \frac{n}{2} \right| < n^{5/6} \right) \\ \hat{U}_n &= \sum_{k=0}^n y_{k,n} \mathbb{I} \left(k - \frac{n}{2} \geq n^{5/6} \right), \quad V_{n,x} = \sum_{k=0}^n y_{k,n} \mathbb{I} \left(n^{5/6} > k - \frac{n}{2} > \frac{n^{3/4}x}{2} \right). \end{aligned}$$

Then

$$(23) \quad 1 - F_n(x) = \frac{\hat{U}_n + V_{n,x}}{U_n + V_n}.$$

2.2.1. Estimates of U_n and \hat{U}_n .

Lemma 2.5. *We have*

$$\hat{U}_n \leq U_n \leq n^{-4}.$$

Proof. Since $J(t)$ attains the maximum at a unique point $1/2$,

$$\max_{|x-0.5| \geq n^{-1/6}} J(x) = \max\{J(0.5 \pm n^{-1/6})\}.$$

Using Lemma 2.1 and Taylor expansion,

$$J(0.5 \pm n^{-1/6}) - J(0.5) = \frac{-4n^{-2/3}}{3} + o(n^{-2/3}).$$

Hence for all n large enough and $|k - (n/2)| \geq n^{5/6}$,

$$(24) \quad n(J(k/n) - J(1/2)) < -n^{1/3}.$$

It follows from (12) and (24) that for $|k - (n/2)| \geq n^{5/6}$,

$$y_{k,n} \leq \sqrt{2n} \exp(-n^{1/3}).$$

Thus

$$\hat{U}_n \leq U_n \leq n\sqrt{2n} \exp(-n^{1/3}) < n^{-4},$$

for all n large enough. \square

2.2.2. *Estimates of V_n and $V_{n,x}$.* We can easily prove that there exists a positive constant K , such that

$$(25) \quad \sup_{t \in \mathbb{R}} |p_1(t)| + |p_2(t)| + |r(t)| + |p'_1(t)| + |p'_2(t)| + |r'(t)| \leq K.$$

Lemma 2.6. *Let K be a positive constant, and f be a differentiable function satisfying $|f(t)| + |f'(t)| \leq K$ for all $t \geq 0$. Let m, q, p be positive real numbers. Then*

$$\left| \sum_{\substack{m < \ell < q \\ 2^{\ell}}} f\left(\frac{\ell}{p}\right) - \frac{p}{2} \int_{m/p}^{q/p} f(t) dt \right| \leq \frac{K(q-m)}{p} + 2K,$$

and

$$\left| \sum_{\substack{m < \ell < q \\ 2^{\ell}}} f\left(\frac{\ell}{p}\right) - \frac{p}{2} \int_{m/p}^{q/p} f(t) dt \right| \leq \frac{K(q-m)}{p} + 2K.$$

Proof. For any ℓ , by using the mean value theorem,

$$\left| f\left(\frac{\ell}{p}\right) - \frac{p}{2} \int_{\frac{\ell}{p}}^{\frac{\ell+2}{p}} f(t) dt \right| \leq \frac{pK}{2} \int_{\frac{\ell}{p}}^{\frac{\ell+2}{p}} \left(t - \frac{\ell}{p}\right) dt = \frac{K}{p}.$$

Therefore, by summing over ℓ we get desired results. \square

Using the same arguments for (21) and (22) (here (25) and Lemma 2.6 play the same role of Lemmas 2.3 and 2.4), we can prove that

$$(26) \quad B_{n,x} = \frac{n^{3/4}}{2} \hat{P}_1(x) + \frac{n^{1/4}}{2} \hat{P}_2(x) + O(n^{1/12}),$$

and

$$(27) \quad B_n = \frac{n^{3/4}}{2} \hat{P}_1(-\infty) + \frac{n^{1/4}}{2} \hat{P}_2(-\infty) + O(n^{1/12}).$$

Notice that the term $O(n^{1/12})$ comes from the estimate of Lemma 2.6 with $(q-m) = 2n^{5/6}$ and $p = n^{3/4}$.

2.2.3. *Conclusion.* Now, using Lemma 2.5 and (23), (26), (27), and similar arguments as in the case $x > 10$, we can prove that

$$\frac{1 - F_n(x)}{1 - F(x)} = 1 + \frac{G(x)}{\sqrt{n}} + O(n^{-2/3}),$$

here the term $O(n^{-2/3})$ comes from the quotient $O(n^{1/12})/n^{3/4}$.

We remark that in the proof of case $x \leq 10$, we can change the exponent $5/6$ by any constant in $(3/4, 1)$. Thus the error term $n^{-2/3}$ can be improved to $n^{-3/4+\varepsilon}$ for any $\varepsilon > 0$.

REFERENCES

- [1] T. Çağın, P. E. Oliveiraa, N. Torradoa. *A moderate deviation for associated random variables*. J. Korean Statist. Soc. 45 (2016), no. 2, 285–294.
- [2] S. Chatterjee, Q. Shao. *Nonnormal approximation by Stein’s method of exchangeable pairs with application to the Curie-Weiss model*. Ann. Appl. Probab. 21, 464–483 (2011).
- [3] L. Chen, X. Fang, Q. Shao. *From Stein identities to moderate deviations*. Ann. Probab. 41, 262–293 (2013).
- [4] X. Chen, Q.-M. Shao, W. B. Wu, L. Xu. *Self-normalized Cramér-type moderate deviations under dependence*. Ann. Statist. 44 (2016), no. 4, 1593–1617.
- [5] F. Collet, R. C. Kraaij. *Dynamical moderate deviations for the Curie-Weiss model*. Stochastic Process. Appl. In press, doi:j.spa.2017.01.002.
- [6] P. Eichelsbacher, M. Lowe. *Moderate deviations for i.i.d. random variables*. ESAIM Probab. Stat. 7 (2003), 209–218.
- [7] P. Eichelsbacher, M. Lowe. *Lindeberg’s method for moderate deviations and random summation*. Preprint, arXiv:1705.03837.
- [8] P. Eichelsbacher, M. Raic, T. Schreiber. *Moderate deviations for stabilizing functionals in geometric probability*. Ann. Inst. Henri Poincaré Probab. Stat. 51 (2015), no. 1, 89–128.
- [9] R. Ellis. *Entropy, large deviations, and statistical mechanics*. Grundlehren der Mathematischen Wissenschaften, 271. Springer-Verlag, New York, 1985.
- [10] A. Dembo, A. Guionnet, O. Zeitouni. *Moderate deviations for the spectral measure of certain random matrices*. Ann. Inst. H. Poincaré Probab. Statist. 39 (2003), no. 6, 1013–1042.
- [11] H. Djellout, A. Guillin. *Moderate deviations for Markov chains with atom*. Stochastic Process. Appl. 95 (2001), no. 2, 203–217.
- [12] H. Doring, P. Eichelsbacher. *Moderate deviations for the determinant of Wigner matrices*. Limit theorems in probability, statistics and number theory, 253–275, Springer Proc. Math. Stat., 42, Springer, Heidelberg, 2013.
- [13] N. Gribkova. *Cramér type moderate deviations for trimmed L-statistics*. Math. Methods Statist. 25 (2016), no. 4, 313–322.
- [14] W. Liu, Q.-M. Shao. *A Carmér moderate deviation theorem for Hotelling’s T^2 -statistic with applications to global tests*. Ann. Statist. 41 (2013), no. 1, 296–322.
- [15] M. Peligrad, H. Sang, Y. Zhong, W. B. Wu. *Exact moderate and large deviations for linear processes*. Statist. Sinica 24 (2014), no. 2, 957–969.
- [16] V. V. Petrov. *Sums of Independent Random Variables*. Springer, New York, 1975.
- [17] H. Robbins. *A Remark of Stirling’s Formula*. Amer. Math. Monthly 62, 26–29 (1955).
- [18] Q.-M. Shao, Q. Wang. *Self-normalized limit theorems: a survey*. Probab. Surv. 10 (2013), 69–93.
- [19] J. Tiefeng, M. B. Rao, W. Xiangchen, L. Deli. *Laws of large numbers and moderate deviations for stochastic processes with stationary and independent increments*. Stochastic Process. Appl. 44 (1993), no. 2, 205–219.
- [20] W. B. Wu, Z. Zhao. *Moderate deviations for stationary processes*. Statist. Sinica 18 (2008), no. 2, 769–782.

INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY, 18 HOANG QUOC VIET STREET, 10307 HANOI, VIETNAM

E-mail address: cvhao89@gmail.com

E-mail address: pgviethung@gmail.com