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Padé type approximants of Hurwitz zeta function

$\zeta(4, x)$

Tanguy Rivoal

January 15, 2018

1 Introduction

The Hurwitz zeta function is defined by

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

for any s, x such that $\operatorname{Re}(s) > 1$ and $x \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$. For any fixed integer s , $\zeta(s, x)$ is meromorphic in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, with poles of order s at each non-negative integer. For any fixed $\varepsilon > 0$, $\zeta(s, x)$ has an asymptotic expansion when $x \rightarrow \infty$ in the angular sector $|\arg(x)| < \pi - \varepsilon$:

$$\zeta(s, x) \sim \frac{x^{1-s}}{s-1} + \sum_{k=1}^{\infty} \frac{(s)_{k-1}}{k!} \frac{B_k}{x^{k+s-1}},$$

where $(B_k)_{k \geq 0}$ is the sequence of Bernoulli numbers.

In this paper, we address the following problem: *Given two integers $m, n \geq 0$, determine two polynomials $A(x)$ and $B(x) \in \mathbb{Q}[x]$, of degree $\leq n$, such that*

$$A(x)\zeta(4, x) + B(x) = \mathcal{O}\left(\frac{1}{x^{m+1}}\right) \quad (1)$$

as $x \rightarrow \infty$ in any angular sector $|\arg(x)| < \pi - \varepsilon$.

Stated in this form, this is an analytic problem. However, using the asymptotic expansion of $\zeta(s, x)$ at $x = \infty$, Eq. (1) can also be interpreted as a Padé type problem at $x = \infty$ for the formal series

$$2x^{-3} + \sum_{k=0}^{\infty} (k+2)(k+3)B_{k+1}x^{-k-4}.$$

See [10, Sec. 2] for details. This Padé problem amounts to solving a linear system with $2n+2$ indeterminates (the polynomial coefficients) and $m+n+1$ equations (the vanishing conditions): provided $m \leq n$, this system has at least one non-zero solution. The case

$m = n$ corresponds to the usual diagonal Padé approximation. The explicit polynomials obtained below are automatically solutions of the associated Padé type problem. Our main result is the explicit determination of a non-zero analytic solution of (1) when $m \leq n/2$ (essentially). Unicity of the solution is obviously not guaranteed.

Theorem 1. *For any integer $n \geq 0$, consider the following Padé type problem: determine two polynomials $Q_{0,n}(x)$ and $Q_{2,n}(x) \in \mathbb{Q}[x]$, of degree $\leq 4n$, such that*

$$S_n(x) := 3Q_{0,n}(x)\zeta(4, x) + Q_{2,n}(x) = \mathcal{O}\left(\frac{1}{x^{2n+3}}\right) \quad (2)$$

Problem (2) admits the following solution:

$$S_n(x) = - \sum_{k=0}^{\infty} \frac{\partial}{\partial k} \left(\left(k + x + \frac{n}{2} \right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} \right). \quad (3)$$

The series converges for all $x \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$.

Moreover, for the series on the right-hand side of (3), we have

$$Q_{0,n}(x) = \sum_{j=0}^n \frac{\partial}{\partial \varepsilon} \left(\left(\frac{n}{2} - j + \varepsilon \right) \frac{(x+j-n-\varepsilon)_n^2 (x-j+\varepsilon)_n^2}{(1-\varepsilon)_j^4 (1+\varepsilon)_{n-j}^4} \right) \Big|_{\varepsilon=0}$$

and

$$Q_{2,n}(x) = - \frac{1}{6} \sum_{j=0}^n \left(\frac{\partial}{\partial \varepsilon} \right)^3 \left(\left(\frac{n}{2} - j + \varepsilon \right) \frac{(x+j-n-\varepsilon)_n^2 (x-j+\varepsilon)_n^2}{(1-\varepsilon)_j^4 (1+\varepsilon)_{n-j}^4} \sum_{k=0}^{j-1} \frac{1}{(x+k-\varepsilon)^2} \right) \Big|_{\varepsilon=0}.$$

Diagonal Padé approximants are known for $\zeta(2, x)$ and $\zeta(3, x)$: the formulas are given in [8] and [10, Theorem 2]. However diagonal Padé approximants are not explicitly known for $\zeta(4, x)$ and Theorem 1 offers a weaker alternative. The polynomial $Q_{0,n}(x)$ can also be written more symbolically in the form

$$Q_{0,n}(x) = \sum_{j=0}^n \frac{\partial}{\partial j} \left(\left(\frac{n}{2} - j \right) \binom{n}{j}^4 \binom{x+j-1}{n}^2 \binom{x+n-j-1}{n}^2 \right).$$

We now let

$$Q_{1,n}(x) = - \frac{1}{6} \sum_{j=0}^n \left(\frac{\partial}{\partial \varepsilon} \right)^3 \left(\left(\frac{n}{2} - j + \varepsilon \right) \frac{(x+j-n-\varepsilon)_n^2 (x-j+\varepsilon)_n^2}{(1-\varepsilon)_j^4 (1+\varepsilon)_{n-j}^4} \sum_{k=0}^{j-1} \frac{1}{x+k-\varepsilon} \right) \Big|_{\varepsilon=0}$$

which is also a polynomial of degree $\leq 4n$. Then, our proof will also show that

$$R_n(x) := Q_{0,n}(x)\zeta(3, x) + Q_{1,n}(x) = \sum_{k=0}^{\infty} \left(k + x + \frac{n}{2}\right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4}.$$

As $x \rightarrow \infty$,

$$\left(k + x + \frac{n}{2}\right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} \sim \frac{4^n (k+1)_n^2}{x^{2n+3}} \quad (4)$$

and this suggests that $R_n(x) = \mathcal{O}(\frac{1}{x^{2n+3}})$ as $S_n(x)$. However, this is false and in fact one can not prove anything better than $R_n(x) = \mathcal{O}(\frac{1}{x^2})$. Therefore, the property $S_n(x) = \mathcal{O}(\frac{1}{x^{2n+3}})$ is a non-trivial property, which is not a simple consequence of (4).

We give the proof of Theorem 1 in Section 2 while we make connections with other results in the literature in Section 3.

I warmly thank Pierre Bel for his careful reading of a previous version of this paper, and for pointing out that $R_n(x)$ is only $\mathcal{O}(\frac{1}{x^2})$ and not $\mathcal{O}(\frac{1}{x^{2n+3}})$.

2 Proof of Theorem 1

We follow the method used in [10] and split the proof in three parts. We also include the case of $\zeta(3, x)$ in the first part of the proof.

2.1 Linear forms in 1, $\zeta(3, x)$, respectively 1, $\zeta(4, x)$

We define the rational function

$$\rho(t) = \left(t + x + \frac{n}{2}\right) \frac{(t+1)_n^2 (t+2x)_n^2}{(t+x)_{n+1}^4}.$$

By partial fraction expansion, we have

$$\rho(t) = \sum_{s=1}^4 \sum_{j=0}^n \frac{E_{j,n,s}(x)}{(t+x+j)^s}$$

with

$$E_{j,n,s}(x) = \frac{1}{(4-s)!} \left(\frac{\partial}{\partial t}\right)^{4-s} (\rho(t) (t+x+j)^4) \Big|_{k=-j-x}.$$

Exchanging summations, we thus get

$$\begin{aligned} & \sum_{k=0}^{\infty} \left(k + x + \frac{n}{2}\right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} \\ &= \sum_{s=2}^4 \left(\sum_{j=0}^n E_{j,n,s}(x) \right) \zeta(s, x) - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{E_{j,n,s}(x)}{(k+x)^s}. \end{aligned} \quad (5)$$

Here we must explain how we have disposed of the divergent series $\sum_{k=0}^{\infty} \frac{1}{k+x}$ in (5), i.e. why the first sum over s does not start at $s = 1$. The series on the left-hand side of (5) being convergent, this forces to assign the value $-\sum_{j=0}^n E_{j,n,1}(x) \sum_{k=0}^{j-1} \frac{1}{k+x}$ to the divergent expression $(\sum_{j=0}^n E_{j,n,1}(x)) \sum_{k=0}^{\infty} \frac{1}{k+x+j}$. Indeed we have $\sum_{j=0}^n E_{j,n,1}(x) = 0$ because this is the sum over the residues at all the finite poles of $\rho(k)$, hence also equal to minus its residue at infinity, which is zero. Formally, one should introduce the Lerch series $\sum_{k=0}^{\infty} \frac{z^k}{(k+x)^s}$ with $|z| < 1$ and eventually to let $z \rightarrow 1$; see [10, Sec. 3.2] for details.

We now observe that $\rho(k) = -\rho(-k - 2x - n)$. Since

$$\rho(-k - 2x - n) = \sum_{s=1}^4 \sum_{j=0}^n \frac{E_{j,n,s}(x)}{(-k - x - n + j)^s} = \sum_{s=1}^4 \sum_{j=0}^n (-1)^s \frac{E_{n-j,n,s}(x)}{(k+x+j)^s},$$

we then deduce that $E_{n-j,n,s}(x) = (-1)^{s+1} E_{j,n,s}(x)$. Therefore

$$\sum_{j=0}^n E_{j,n,s}(x) = (-1)^{s+1} \sum_{j=0}^n E_{j,n,s}(x)$$

which is thus equal to 0 for $s = 2$ and $s = 4$, and consequently, the first sum in (5) is for $s = 3$ only:

$$\sum_{k=0}^{\infty} \left(k+x+\frac{n}{2}\right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} = \left(\sum_{j=0}^n E_{j,n,3}(x)\right) \zeta(3, x) - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{E_{j,n,s}(x)}{(k+x)^s}. \quad (6)$$

Similarly,

$$\begin{aligned} -\sum_{k=0}^{\infty} \left(\frac{\partial}{\partial k} \left(k+x+\frac{n}{2}\right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4}\right) &= -\sum_{s=1}^4 \sum_{j=0}^n E_{j,n,s}(x) \sum_{k=0}^{\infty} \frac{\partial}{\partial k} \frac{1}{(k+x+j)^s} \\ &= \sum_{s=1}^4 \left(\sum_{j=0}^n E_{j,n,s}(x)\right) s \zeta(s+1, x) - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{s E_{j,n,s}(x)}{(k+x)^{s+1}} \end{aligned}$$

so that

$$\begin{aligned} -\sum_{k=0}^{\infty} \left(\frac{\partial}{\partial k} \left(k+x+\frac{n}{2}\right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4}\right) \\ = \left(\sum_{j=0}^n E_{j,n,3}(x)\right) 3 \zeta(4, x) - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{s E_{j,n,s}(x)}{(k+x)^{s+1}}. \quad (7) \end{aligned}$$

2.2 The coefficients are polynomials of degree $\leq 4n$

We set $Q_{0,n}(x) := \sum_{j=0}^n E_{j,n,3}(x)$ and

$$Q_{1,n}(x) := - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{E_{j,n,s}(x)}{(k+x)^s}, \quad Q_{2,n}(x) := - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{sE_{j,n,s}(x)}{(k+x)^{s+1}}$$

so that the right-hand sides of (6) and (7) are respectively equal to

$$Q_{0,n}(x)\zeta(3, x) + Q_{1,n}(x) \quad \text{and} \quad 3Q_{0,n}(x)\zeta(4, x) + Q_{2,n}(x).$$

Let us prove that for $s \in \{0, 1, 2\}$, the $Q_{s,n}(x)$ are in $\mathbb{Q}[x]$ and of degree $\leq 4n$. We have

$$\begin{aligned} Q_{0,n}(x) &= \sum_{j=0}^n \frac{\partial}{\partial k} (\rho(k) (k+x+j)^4)_{|k=-j-x} = \sum_{j=0}^n \frac{\partial}{\partial \ell} (\ell^4 \rho(\ell-j-x))_{|\ell=0} \\ &= \sum_{j=0}^n \frac{\partial}{\partial \ell} \left(\left(\frac{n}{2} - j + \ell \right) \frac{(x+j-n-\ell)_n^2 (x-j+\ell)_n^2}{(1-\ell)_j^4 (1+\ell)_{n-j}^4} \right)_{|\ell=0}. \end{aligned} \quad (8)$$

Eq. (8) shows that $Q_{0,n}(x) \in \mathbb{Q}[x]$ and $\deg(Q_{0,n}) \leq 4n$. Furthermore, by Leibniz' formula

$$\begin{aligned} Q_{1,n}(x) &= - \sum_{s=1}^4 \sum_{j=0}^n \left[\frac{1}{(4-s)!} \left(\frac{\partial}{\partial \ell} \right)^{4-s} (\ell^4 \rho(\ell-j-x)) \right. \\ &\quad \left. \times \frac{1}{(s-1)!} \left(\frac{\partial}{\partial \ell} \right)^{s-1} \left(\sum_{k=0}^{j-1} \frac{1}{k-\ell+x} \right) \right]_{|\ell=0} \\ &= -\frac{1}{6} \sum_{j=0}^n \left(\frac{\partial}{\partial \ell} \right)^3 (\ell^4 \rho(\ell-j-x) \sum_{k=0}^{j-1} \frac{1}{k-\ell+x})_{|\ell=0} \\ &= -\frac{1}{6} \sum_{j=0}^n \left(\frac{\partial}{\partial \ell} \right)^3 \left(\left(\frac{n}{2} - j + \ell \right) \frac{(x+j-n-\ell)_n^2 (x-j+\ell)_n^2}{(1-\ell)_j^4 (\ell+1)_{n-j}^4} \sum_{k=0}^{j-1} \frac{1}{k-\ell+x} \right)_{|\ell=0} \end{aligned}$$

and similarly,

$$\begin{aligned} Q_{2,n}(x) &= - \sum_{s=1}^4 \sum_{j=0}^n \left[\frac{(-1)^{4-s}}{(4-s)!} \left(\frac{\partial}{\partial \ell} \right)^{4-s} (\ell^4 R(\ell-j-x)) \right. \\ &\quad \left. \times \frac{1}{(s-1)!} \left(\frac{\partial}{\partial \ell} \right)^{s-1} \left(\sum_{k=0}^{j-1} \frac{1}{(x+k-\ell)^2} \right) \right]_{|\ell=0} \\ &= -\frac{1}{6} \sum_{j=0}^n \left(\frac{\partial}{\partial \ell} \right)^3 (\ell^4 R(\ell-j-x) \sum_{k=0}^{j-1} \frac{1}{(k-\ell+x)^2})_{|\ell=0} \\ &= -\frac{1}{6} \sum_{j=0}^n \left(\frac{\partial}{\partial \ell} \right)^3 \left(\left(\frac{n}{2} - j + \ell \right) \frac{(x+j-n-\ell)_n^2 (x-j+\ell)_n^2}{(1-\ell)_j^4 (\ell+1)_{n-j}^4} \sum_{k=0}^{j-1} \frac{1}{(x+k-\ell)^2} \right)_{|\ell=0}. \end{aligned}$$

It follows that $Q_{1,n}(x)$ and $Q_{2,n}(x)$ are in $\mathbb{Q}[x]$ and of degree $\leq 4n$, because for all $j \in \{0, \dots, n\}$, $k \in \{0, \dots, j-1\}$ and any ℓ , we have that

$$\frac{(x+j-n-\ell)_n}{x+k-\ell} \in \mathbb{Q}[x].$$

2.3 Proof that $S_n(x) = \mathcal{O}\left(\frac{1}{x^{2n+3}}\right)$

We shall prove that $S_n(x) = \mathcal{O}\left(\frac{1}{x^{2n+3}}\right)$ as $x \rightarrow \infty$ in any open angular sector that does not contain the negative real axis. The methods of [10, Sec. 3.1] can not be used here because they lead to divergent series.

We first observe that it is enough to consider the case $x \rightarrow +\infty$ on the real axis. Indeed, $S_n(x) = 3Q_{0,n}(x)\zeta(4, x) + Q_{2,n}(x)$ so that we know a priori that $S_n(x)$ has an asymptotic expansion in any angular sector that does not contain the negative real axis. Thus the leading power of this expansion can be determined by letting $x \rightarrow +\infty$ on the real positive axis.

Let $N \geq 0$ be an integer. We assume that $x \geq 1$. Consider the positively oriented square C_N with sides $[-\frac{1}{2} - iN, N + \frac{1}{2} - iN]$, $[N + \frac{1}{2} - iN, N + \frac{1}{2} + iN]$, $[N + \frac{1}{2} + iN, -\frac{1}{2} + iN]$, $[-\frac{1}{2} + iN, -\frac{1}{2} - iN]$. Then by the residues theorem,

$$\frac{1}{2i\pi} \int_{C_N} \frac{\pi^2}{\sin(\pi t)^2} \rho(t) dt = \sum_{k=0}^N \frac{\partial}{\partial k} \left((k+x+\frac{n}{2}) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} \right).$$

(The only poles of the integrand inside C_N are those of $\frac{\pi^2}{\sin(\pi t)^2}$ because $x \geq 1$.)

For any fixed real number u , $\frac{\pi^2}{\sin(\pi(u+iv))^2} = \mathcal{O}(e^{-2\pi|v|})$ as the real number $v \rightarrow \pm\infty$, and $\rho(t) = \mathcal{O}(1/t^3)$ as $t \rightarrow \infty$. Letting $N \rightarrow +\infty$, it follows that

$$\frac{1}{2i\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi^2}{\sin(\pi t)^2} \rho(t) dt = - \sum_{k=0}^{\infty} \frac{\partial}{\partial k} \left((k+x+\frac{n}{2}) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} \right) = S_n(x).$$

Then for any $t \in [-\frac{1}{2} - i\infty, -\frac{1}{2} + i\infty]$ and any $x \geq 1$, we have $|x^{2n+3}\rho(t)| \leq c_n |(t+1)_n^2|$ for some constant $c_n > 0$ independent of x and t . Since $\int_{-1/2+i\mathbb{R}} |\frac{\pi^2}{\sin(\pi t)^2} (t+1)_n^2| |dt| < \infty$ for any n , and $\lim_{x \rightarrow +\infty} x^{2n+3}\rho(t) = 4^n (t+1)_n^2$, it follows by the dominated convergence theorem that

$$\lim_{x \rightarrow +\infty} x^{2n+3} S_n(x) = \frac{4^n}{2i\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi^2}{\sin(\pi t)^2} (t+1)_n^2 dt.$$

Similarly, we can prove that

$$\frac{1}{2i\pi} \int_{-\frac{1}{2}+i\infty}^{-\frac{1}{2}-i\infty} \pi \cot(\pi t) \rho(t) dt = \sum_{k=0}^{\infty} (k+x+\frac{n}{2}) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} = R_n(x).$$

However, we can not deduce from this representation that $\lim_{x \rightarrow +\infty} x^{2n+3} R_n(x)$ is finite by the method above, because $\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \pi \cot(\pi t) (t+1)_n^2 dt$ is divergent. In fact, it turns out that $\lim_{x \rightarrow +\infty} x^{2n+3} R_n(x)$ is not finite when $n \geq 1$, because it can be proved that $\lim_{x \rightarrow +\infty} x^2 R_n(x)$ is finite and non-zero.

3 Connections with other works

3.1 Cohen's continued fraction for $\zeta(4, x)$

In [6], Cohen presented certain continued fractions for values of the Riemann zeta function and the Gamma function. In particular he stated the following one (in his notations):

$$\zeta(4, x+1) \approx \left| \frac{(2x+1)/3}{1P_x(1)} \right| + \left| \frac{1^8 2x(2x+2)}{3P_x(2)} \right| + \left| \frac{2^8(2x-1)(2x+3)}{5P_x(3)} \right| + \dots \quad (9)$$

where

$$P_x(\ell) = 2x^4 + 4x^3 + (2\ell^2 - 2\ell + 4)x^2 + (2\ell^2 - 2\ell + 2)x - \ell(\ell-1)(\ell^2 - \ell + 1).$$

He wrote that \approx means ‘‘asymptotic expansion as the integer $x \rightarrow \infty$ ’’, and that it is not an equality.

Maple implementation of Zeilberger's algorithm shows that our sequences $(S_n(x+1))_{n \geq 0}$, $(Q_{0,n}(x+1))_{n \geq 0}$ and $(Q_{2,n}(x+1))_{n \geq 0}$ are solutions of the linear recurrence

$$n^5 U_n + (2n-1)P_x(n)U_{n-1} + (n+1)^3(n+2x)(n-2-2x)U_{n-2} = 0.$$

It is then not difficult to prove that $\frac{Q_{2,n}(x+1)}{3Q_{0,n}(x+1)}$ are the convergents of Cohen's continued fraction (9). See also Lange's paper [7] for many continued fractions related to Hurwitz zeta function, though (9) does not seem to be listed.

Cohen then mentioned that Apéry's ‘‘continued fraction acceleration’’ method shows

$$\zeta(4) = \left| \frac{13}{C(1)} \right| + \left| \frac{2 \cdot 3 \cdot 4 \cdot 1^7}{C(2)} \right| + \left| \frac{5 \cdot 6 \cdot 7 \cdot 2^7}{C(3)} \right| + \dots \quad (10)$$

where

$$C(n) = 3(2n-1)(45n^4 - 90n^3 + 72n^2 - 27n + 4). \quad (11)$$

He also wrote that the convergents $\frac{a_n}{b_n}$ of the continued fraction (10) are such that

$$\zeta(4) - \frac{a_n}{b_n} \approx \frac{c(-1)^n}{(2 + \sqrt{3})^{6n}}$$

for some constant $c \neq 0$, which is not enough to prove the irrationality of $\zeta(4)$. The continued fraction (10) had been announced before in [4], with details given in [5].

To conclude this section, we remark that the polynomials $Q_{j,n,r}(x)$ and series stated after Theorem 1 do not seem to satisfy a linear recurrence of order ≤ 2 when $r < n$. The case $r = n$ is thus very remarkable.

3.2 Zudilin's approximations to $\zeta(4)$

In [12, Section 2], Zudilin showed that for any integer $n \geq 0$

$$Z_n := - \sum_{k=0}^{\infty} \frac{\partial}{\partial k} \left(\left(k + \frac{n}{2} \right) \frac{(k-n)_n^2 (k+n+1)_n^2}{(k)_{n+1}^4} \right) = u_n 3\zeta(4) + v_n$$

where u_n and v_n are rational numbers. In particular,

$$u_n = \sum_{j=0}^n \frac{\partial}{\partial j} \left(\left(\frac{n}{2} - j \right) \binom{n}{j}^4 \binom{n+j}{n}^2 \binom{2n-j}{n}^2 \right)$$

and the expression for v_n is more complicated. He also proved that $(Z_n)_n$, $(u_n)_n$ and $(v_n)_n$ are solutions of the linear recurrence

$$n^5 U_n + C(n)U_{n-1} - 3(3n-2)(3n-4)(n-1)^3 U_{n-2} = 0$$

where $Q(n)$ is Cohen's polynomial (11). It can be verified that $\frac{u_n}{v_n}$ coincide with the convergents $\frac{a_n}{b_n}$ of (10); see [12, Section 2, Theorem 2].

We observe that $Z_n = S_n(n+1)$ and $u_n = Q_{0,n}(n+1)$. Since $\zeta(4, n+1) = \zeta(4) - \sum_{j=1}^n \frac{1}{j^4}$, the specialization of Theorem 1 at $x = n+1$:

$$S_n(n+1) = 3Q_{0,n}(n+1)\zeta(4, n+1) + Q_{2,n}(n+1)$$

becomes

$$Z_n = u_n 3\zeta(4) + Q_{2,n}(n+1) - 3u_n \sum_{j=1}^n \frac{1}{j^4}$$

and thus we recover Zudilin's sequence $(v_n)_n$ by the identity

$$v_n = Q_{2,n}(n+1) - 3u_n \sum_{j=1}^n \frac{1}{j^4}.$$

3.3 Prévost's remainder Padé approximants for $\zeta(s, x)$

In [8], Prévost showed a very original method to prove the irrationality of $\zeta(2)$ and $\zeta(3)$. We present the slightly modified approach he recently presented in [9]. For any integer $x \geq 1$, we have

$$\zeta(s) = \sum_{k=1}^{x-1} \frac{1}{k^s} + \zeta(s, x).$$

He then computed explicitly the Padé approximants $[n+1/n](x)$ at $x = \infty$ of $\zeta(2, x)$, respectively the Padé approximants $[n+2/n](x)$ at $x = \infty$ of $\zeta(3, x)$. After taking $x = n+1$, he obtained Apéry's famous sequences for $\zeta(2)$ and $\zeta(3)$.

For $s = 2$, the denominators of $[n + 1/n](x)$ are

$$P_n(x) = \sum_{j=0}^n \binom{n+1}{j+1} \binom{n+j+2}{j+1} \binom{x-1}{j}, \quad n \geq 0,$$

and they satisfy the orthogonality relation

$$\int_{i\mathbb{R}} P_n(x) P_m(x) \frac{x^2}{\sin(\pi x)^2} dx = 0, \quad n \neq m.$$

For $s = 3$, the denominators $Q_n(x)$ of $[n + 2/n](x)$ are such that

$$Q_n(x^2) = \sum_{j=0}^n \frac{1}{j+1} \binom{n+1}{j+1} \binom{n+j+2}{j+1} \binom{x-1}{j} \binom{x+1}{j}, \quad n \geq 0,$$

and they satisfy the orthogonality relation

$$\int_{i\mathbb{R}} Q_n(x) Q_m(x) \frac{x^5 \cos(\pi x)}{\sin(\pi x)^3} dx = 0, \quad n \neq m.$$

The two families of orthogonal polynomials $(P_n)_n$ and $(Q_n)_n$ are specializations of Wilson's orthogonal polynomials [11].

Recently, Prévost [9] proved that the Padé approximants $[n + 1/n](x)$ of $\zeta(s, x)$ at $x = \infty$ converge to $\zeta(s, x)$ for any fixed real number $s > 1$, but convergence is still an open problem when s is a complex number. Moreover, except for $s = 2, 3$, no expression of these approximants is known, even for $s = 4$. In this case, the problem is to find explicit expressions for polynomials $A_n(x)$ (of degree n) such that

$$\int_{i\mathbb{R}} A_n(x) A_m(x) \frac{x^8 (2 + \cos(2\pi x))}{\sin(\pi x)^4} dx = 0, \quad n \neq m.$$

Unfortunately, the weight function $\frac{x^8 (2 + \cos(2\pi x))}{\sin(\pi x)^4}$ is not of the form studied by Wilson. The sequence $(Q_{0,n}(x))_n$ is not orthogonal for this weight, but it is bi-orthogonal in the following sense: for any n and m such that $0 \leq m \leq 2n - 1$, we have

$$\int_{i\mathbb{R}} x^{m+5} Q_{0,n}(x) \frac{\cos(\pi x)}{\sin(\pi x)^3} dx = 0 = \int_{i\mathbb{R}} x^{m+8} Q_{0,n}(x) \frac{2 + \cos(2\pi x)}{\sin(\pi x)^4} dx.$$

3.4 Beukers and Bel's p -adic irrationality proofs

In [3], Calegari proved the irrationality of the 2-adic numbers $\zeta_2(2)$ and $\zeta_2(3)$, as well as of the 3-adic numbers $\zeta_3(3)$. His proof used overconvergent p -adic modular forms. Later, Beukers [2] obtained another proof of these facts, of a more classical flavor. In fact, he essentially used Prévost's Padé approximants for $\zeta(2, x)$ and $\zeta(3, x)$, though his formulas

are written differently. The Padé type approximants constructed in [10] for $\zeta(s, x)$ contain as initial cases Beukers and Prévost approximants; Bel [1] used them to prove certain linear independence results for values of p -adic Hurwitz zeta functions. It would be interesting to know if Theorem 1 or its generalization could be used to prove the irrationality of the numbers $\zeta_p(4)$ for some p . The arithmetic and asymptotic properties of Zudilin's series Z_n are not good enough to imply the irrationality of $\zeta(4)$, but a modification of Z_n conjecturally proves that $\zeta(4) \notin \mathbb{Q}$.

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