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Simultaneous Padé approximants of Hurwitz zeta functions $\zeta(3, x)$ and $\zeta(4, x)$

Tanguy Rivoal

September 9, 2017

1 Introduction

The Hurwitz zeta function is defined by

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

for any s, x such that $\operatorname{Re}(s) > 1$ and $x \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$. For any fixed integer s , $\zeta(s, x)$ is meromorphic in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, with poles of order s at each non-negative integer. For any fixed $\varepsilon > 0$, $\zeta(s, x)$ has an asymptotic expansion when $x \rightarrow \infty$ in the angular sector $|\arg(x)| < \pi - \varepsilon$:

$$\zeta(s, x) \sim \frac{x^{1-s}}{s-1} + \sum_{k=1}^{\infty} \frac{(s)_{k-1}}{k!} \frac{B_k}{x^{k+s-1}},$$

where $(B_k)_{k \geq 0}$ is the sequence of Bernoulli numbers.

The goal of this paper is to solve the following problem: *Given any integer $n \geq 0$, determine some polynomials $Q_{0,n}(x), Q_{1,n}(x), Q_{2,n}(x) \in \mathbb{Q}[x]$, all of degree $\leq 4n$, such that*

$$\begin{cases} R_n(x) := Q_{0,n}(x)\zeta(3, x) + Q_{1,n}(x) = \mathcal{O}\left(\frac{1}{x^{2n+1}}\right) \\ S_n(x) := 3Q_{0,n}(x)\zeta(4, x) + Q_{2,n}(x) = \mathcal{O}\left(\frac{1}{x^{2n+1}}\right) \end{cases} \quad (1)$$

as $x \rightarrow \infty$ in any angular sector $|\arg(x)| < \pi - \varepsilon$.

Stated in this form, this is an analytic problem. However, using the asymptotic expansion of $\zeta(s, x)$ at $x = \infty$, Eqs. (1) can also be interpreted as an ordinary simultaneous type II Padé problem at $x = \infty$ for the two formal series

$$x^{-2} + \sum_{k=0}^{\infty} (k+2)B_{k+1}x^{-k-3} \quad \text{and} \quad 2x^{-3} + \sum_{k=0}^{\infty} (k+2)(k+3)B_{k+1}x^{-k-4}.$$

This Padé problem amounts to solving a linear system with $4n + 3$ unknowns (the polynomial coefficients) and $4n + 2$ equations (the vanishing conditions): this system has at least one non-zero solution. Our main result is the explicit determination of a non-zero analytic solution of (1). As explained in [10, Sec. 2], the explicit polynomials obtained are automatically solutions of the above mentioned Padé problem.

Theorem 1. *For any integer $n \geq 0$, Problem (1) admits the following solution:*

$$\begin{cases} R_n(x) = \sum_{k=0}^{\infty} \left(k + x + \frac{n}{2}\right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4}, \\ S_n(x) = - \sum_{k=0}^{\infty} \frac{\partial}{\partial k} \left(\left(k + x + \frac{n}{2}\right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} \right). \end{cases} \quad (2)$$

Both series converge for all $x \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$.

Moreover, associated to these series, we have

$$Q_{0,n}(x) = \sum_{j=0}^n \frac{\partial}{\partial j} \left(\left(\frac{n}{2} - j + \varepsilon\right) \frac{(x+j-n-\varepsilon)_n^2 (x-j+\varepsilon)_n^2}{(1-\varepsilon)_j^4 (1+\varepsilon)_{n-j}^4} \right) \Big|_{\varepsilon=0}$$

and, for $s = 1$ or 2 ,

$$Q_{s,n}(x) = -\frac{1}{6} \sum_{j=0}^n \left(\frac{\partial}{\partial \varepsilon} \right)^3 \left(\left(\frac{n}{2} - j + \varepsilon\right) \frac{(x+j-n-\varepsilon)_n^2 (x-j+\varepsilon)_n^2}{(1-\varepsilon)_j^4 (1+\varepsilon)_{n-j}^4} \sum_{k=0}^{j-1} \frac{1}{(x+k-\varepsilon)^s} \right) \Big|_{\varepsilon=0}.$$

Diagonal Padé approximants are known for $\zeta(2, x)$ and $\zeta(3, x)$: the formulas are given in [8] and [10, Theorem 2]. However diagonal Padé approximants are not explicitly known for $\zeta(4, x)$ and Theorem 1 offers an alternative. The polynomial $Q_{0,n}(x)$ can also be written more symbolically in the form

$$Q_{0,n}(x) = \sum_{j=0}^n \frac{\partial}{\partial j} \left(\left(\frac{n}{2} - j\right) \binom{n}{j}^4 \binom{x+j-1}{n}^2 \binom{x+n-j-1}{n}^2 \right).$$

Let us mention the following generalization, which can be proved along the same lines as Theorem 1, mutatis mutandis: *For any integers $0 \leq r \leq n$, there exist some polynomials $Q_{0,n,r}(x) \in \mathbb{Q}[x]$ of degree $\leq 4r$ and $Q_{1,n,r}(x), Q_{2,n,r}(x) \in \mathbb{Q}[x]$ of degree $\leq 4n$, such that*

$$\begin{aligned} \sum_{k=0}^{\infty} \left(k + x + \frac{n}{2}\right) \frac{(k+1)_r^2 (k+2x+n-r)_r^2}{(k+x)_{n+1}^4} \\ = Q_{0,n,r}(x) \zeta(3, x) + \frac{Q_{1,n,r}(x)}{(x)_{n-r}^4} = \mathcal{O}\left(\frac{1}{x^{4n-2r+1}}\right) \end{aligned}$$

$$\begin{aligned}
& - \sum_{k=0}^{\infty} \frac{\partial}{\partial k} \left(\left(k + x + \frac{n}{2} \right) \frac{(k+1)_r^2 (k+2x+n-r)_r^2}{(k+x)_{n+1}^4} \right) \\
& \qquad \qquad \qquad = 3Q_{0,n,r}(x)\zeta(4, x) + \frac{Q_{2,n,r}(x)}{(x)_{n-r}^5} = \mathcal{O}\left(\frac{1}{x^{4n-2r+1}}\right)
\end{aligned}$$

as $x \rightarrow \infty$ in any angular sector $|\arg(x)| < \pi - \varepsilon$.

Theorem 1 corresponds to the case $r = n$. Explicit expressions for the polynomials $Q_{j,n,r}(x)$ could be given as well but an interpretation in terms of Padé approximants is not known in general.

We give the proof of Theorem 1 in Section 2 while we make connections with other results in the literature in Section 3.

2 Proof of Theorem 1

We follow the method used in [10] and split the proof in two parts.

2.1 Linear forms in $1, \zeta(3, x)$, respectively $1, \zeta(4, x)$

We set

$$R(k) = \left(k + x + \frac{n}{2} \right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4}.$$

By partial fraction expansion, we have

$$R(k) = \sum_{s=1}^4 \sum_{j=0}^n \frac{E_{j,n,s}(x)}{(k+x+j)^s}$$

with

$$E_{j,n,s}(x) = \frac{1}{(4-s)!} \left(\frac{\partial}{\partial k} \right)^{4-s} (R(k) (k+x+j)^4) \Big|_{k=-j-x}.$$

Exchanging summations, we get

$$\begin{aligned}
& \sum_{k=0}^{\infty} \left(k + x + \frac{n}{2} \right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} \\
& \qquad \qquad \qquad = \sum_{s=2}^4 \left(\sum_{j=0}^n E_{j,n,s}(x) \right) \zeta(s, x) - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{E_{j,n,s}(x)}{(k+x)^s}. \tag{3}
\end{aligned}$$

Here we must explain how we have disposed of the divergent series $\sum_{k=0}^{\infty} \frac{1}{k+x}$ in (3), i.e. why the first sum over s does not start at $s = 1$. The series on the left-hand side of (3) being convergent, this forces to assign the value $-\sum_{j=0}^n E_{j,n,1}(x) \sum_{k=0}^{j-1} \frac{1}{k+x}$ to the divergent expression $(\sum_{j=0}^n E_{j,n,1}(x)) \sum_{k=0}^{\infty} \frac{1}{k+x+j}$. Indeed we have $\sum_{j=0}^n E_{j,n,1}(x) = 0$ because this is

the sum over the residues at all the finite poles of $R(k)$, hence also equal to minus its residue at infinity, which is zero. Formally, one should introduce the Lerch series $\sum_{k=0}^{\infty} \frac{z^k}{(k+x)^s}$ with $|z| < 1$ and eventually to let $z \rightarrow 1$; see [10, Sec. 3.2] for details.

We now observe that $R(k) = -R(-k - 2x - n)$. Since

$$R(-k - 2x - n) = \sum_{s=1}^4 \sum_{j=0}^n \frac{E_{j,n,s}(x)}{(-k - x - n + j)^s} = \sum_{s=1}^4 \sum_{j=0}^n (-1)^s \frac{E_{n-j,n,s}(x)}{(k+x+j)^s},$$

we then deduce that

$$E_{n-j,n,s}(x) = (-1)^{s+1} E_{j,n,s}(x).$$

Therefore

$$\sum_{j=0}^n E_{j,n,s}(x) = (-1)^{s+1} \sum_{j=0}^n E_{j,n,s}(x)$$

which is thus equal to 0 for $s = 2$ and $s = 4$, and consequently, the first sum in (3) is for $s = 3$ only:

$$\sum_{k=0}^{\infty} \left(k+x+\frac{n}{2}\right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4} = \left(\sum_{j=0}^n E_{j,n,3}(x)\right) \zeta(3, x) - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{E_{j,n,s}(x)}{(k+x)^s}. \quad (4)$$

Similarly,

$$\begin{aligned} -\sum_{k=0}^{\infty} \left(\frac{\partial}{\partial k} \left(k+x+\frac{n}{2}\right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4}\right) &= -\sum_{s=1}^4 \sum_{j=0}^n E_{j,n,s}(x) \sum_{k=0}^{\infty} \frac{\partial}{\partial k} \frac{1}{(k+x+j)^s} \\ &= \sum_{s=1}^4 \left(\sum_{j=0}^n E_{j,n,s}(x)\right) s \zeta(s+1, x) - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{s E_{j,n,s}(x)}{(k+x)^{s+1}} \end{aligned}$$

so that

$$\begin{aligned} -\sum_{k=0}^{\infty} \left(\frac{\partial}{\partial k} \left(k+x+\frac{n}{2}\right) \frac{(k+1)_n^2 (k+2x)_n^2}{(k+x)_{n+1}^4}\right) \\ = \left(\sum_{j=0}^n E_{j,n,3}(x)\right) 3 \zeta(4, x) - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{s E_{j,n,s}(x)}{(k+x)^{s+1}}. \quad (5) \end{aligned}$$

Finally, computations similar to those made in [10, Sec. 3.1] show that the two series on the left-hand sides of (4) and (5) are $\mathcal{O}\left(\frac{1}{x^{2n+1}}\right)$ as $x \rightarrow \infty$ in any open angular sector that does not contain the negative real axis.

2.2 The coefficients are polynomials of degree $\leq 4n$

We set $Q_{0,n}(x) := \sum_{j=0}^n E_{j,n,3}(x)$ and

$$Q_{1,n}(x) := - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{E_{j,n,s}(x)}{(k+x)^s}, \quad Q_{2,n}(x) := - \sum_{s=1}^4 \sum_{j=0}^n \sum_{k=0}^{j-1} \frac{sE_{j,n,s}(x)}{(k+x)^{s+1}}$$

so that the right-hand sides of (4) and (5) are respectively equal to

$$Q_{0,n}(x)\zeta(3, x) + Q_{1,n}(x) \quad \text{and} \quad 3Q_{0,n}(x)\zeta(4, x) + Q_{2,n}(x).$$

We shall now prove that for $s \in \{0, 1, 2\}$, the $Q_{s,n}(x)$ are in $\mathbb{Q}[x]$ and of degree $\leq 4n$. We have

$$\begin{aligned} Q_{0,n}(x) &= \sum_{j=0}^n \frac{\partial}{\partial k} (R(k) (k+x+j)^4)_{|k=-j-x} = \sum_{j=0}^n \frac{\partial}{\partial \ell} (\ell^4 R(\ell-j-x))_{|\ell=0} \\ &= \sum_{j=0}^n \frac{\partial}{\partial \ell} \left(\binom{n}{2} - j + \ell \right) \frac{(x+j-n-\ell)_n^2 (x-j+\ell)_n^2}{(1-\ell)_j^4 (1+\ell)_{n-j}^4} \Big|_{\ell=0}. \end{aligned} \quad (6)$$

Eq. (6) shows that $Q_{0,n}(x) \in \mathbb{Q}[x]$ and $\deg(Q_{0,n}) \leq 4n$. Furthermore,

$$\begin{aligned} Q_{1,n}(x) &= - \sum_{s=1}^4 \sum_{j=0}^n \left[\frac{1}{(4-s)!} \left(\frac{\partial}{\partial \ell} \right)^{4-s} (\ell^4 R(\ell-j-x)) \right. \\ &\quad \left. \times \frac{1}{(s-1)!} \left(\frac{\partial}{\partial \ell} \right)^{s-1} \left(\sum_{k=0}^{j-1} \frac{1}{k-\ell+x} \right) \right]_{|\ell=0} \\ &= -\frac{1}{6} \sum_{j=0}^n \left(\frac{\partial}{\partial \ell} \right)^3 (\ell^4 R(\ell-j-x) \sum_{k=0}^{j-1} \frac{1}{k-\ell+x})_{|\ell=0} \\ &= -\frac{1}{6} \sum_{j=0}^n \left(\frac{\partial}{\partial \ell} \right)^3 \left(\binom{n}{2} - j + \ell \right) \frac{(x+j-n-\ell)_n^2 (x-j+\ell)_n^2}{(1-\ell)_j^4 (\ell+1)_{n-j}^4} \sum_{k=0}^{j-1} \frac{1}{k-\ell+x} \Big|_{\ell=0} \end{aligned}$$

and similarly,

$$\begin{aligned}
Q_{2,n}(x) &= - \sum_{s=1}^4 \sum_{j=0}^n \left[\frac{(-1)^{4-s}}{(4-s)!} \left(\frac{\partial}{\partial \ell} \right)^{4-s} (\ell^4 R(\ell - j - x)) \right. \\
&\quad \left. \times \frac{1}{(s-1)!} \left(\frac{\partial}{\partial \ell} \right)^{s-1} \left(\sum_{k=0}^{j-1} \frac{1}{(x+k-\ell)^2} \right) \right]_{|\ell=0} \\
&= -\frac{1}{6} \sum_{j=0}^n \left(\frac{\partial}{\partial \ell} \right)^3 \left(\ell^4 R(\ell - j - x) \sum_{k=0}^{j-1} \frac{1}{(k-\ell+x)^2} \right)_{|\ell=0} \\
&= -\frac{1}{6} \sum_{j=0}^n \left(\frac{\partial}{\partial \ell} \right)^3 \left(\left(\frac{n}{2} - j + \ell \right) \frac{(x+j-n-\ell)_n^2 (x-j+\ell)_n^2}{(1-\ell)_j^4 (\ell+1)_{n-j}^4} \sum_{k=0}^{j-1} \frac{1}{(x+k-\ell)^2} \right)_{|\ell=0}.
\end{aligned}$$

It follows that $Q_{1,n}(x)$ and $Q_{2,n}(x)$ are in $\mathbb{Q}[x]$ and of degree $\leq 4n$, because for all $j \in \{0, \dots, n\}$, $k \in \{0, \dots, j-1\}$ and any ℓ , we have that

$$\frac{(x+j-n-\ell)_n}{x+k-\ell} \in \mathbb{Q}[x].$$

3 Connections with other works

3.1 Cohen's continued fraction for $\zeta(4, x)$

In [6], Cohen presented certain continued fractions for values of the Riemann zeta function and the Gamma function. In particular he stated the following one (in his notations):

$$\zeta(4, x+1) \approx \left| \frac{(2x+1)/3}{1P_x(1)} \right| + \left| \frac{1^8 2x(2x+2)}{3P_x(2)} \right| + \left| \frac{2^8 (2x-1)(2x+3)}{5P_x(3)} \right| + \dots \quad (7)$$

where

$$P_x(\ell) = 2x^4 + 4x^3 + (2\ell^2 - 2\ell + 4)x^2 + (2\ell^2 - 2\ell + 2)x - \ell(\ell-1)(\ell^2 - \ell + 1).$$

He wrote that \approx means ‘‘asymptotic expansion as the integer $x \rightarrow \infty$ ’’, and that it is not an equality.

Maple implementation of Zeilberger's algorithm shows that our sequences $(S_n(x+1))_{n \geq 0}$, $(Q_{0,n}(x+1))_{n \geq 0}$ and $(Q_{2,n}(x+1))_{n \geq 0}$ are solutions of the linear recurrence

$$n^5 U_n + (2n-1)P_x(n)U_{n-1} + (n+1)^3(n+2x)(n-2-2x)U_{n-2} = 0.$$

It is then not difficult to prove that $\frac{Q_{2,n}(x+1)}{3Q_{0,n}(x+1)}$ are the convergents of Cohen's continued fraction (7). See also Lange's paper [7] for many continued fractions related to Hurwitz zeta function, though (7) does not seem to be listed.

Cohen then mentioned that Apéry's "continued fraction acceleration" method shows

$$\zeta(4) = \frac{13}{C(1)} + \frac{2 \cdot 3 \cdot 4 \cdot 1^7}{C(2)} + \frac{5 \cdot 6 \cdot 7 \cdot 2^7}{C(3)} + \dots \quad (8)$$

where

$$C(n) = 3(2n-1)(45n^4 - 90n^3 + 72n^2 - 27n + 4). \quad (9)$$

He also wrote that the convergents $\frac{a_n}{b_n}$ of the continued fraction (8) are such that

$$\zeta(4) - \frac{a_n}{b_n} \approx \frac{c(-1)^n}{(2 + \sqrt{3})^{6n}}$$

for some constant $c \neq 0$, which is not enough to prove the irrationality of $\zeta(4)$. The continued fraction (8) had been announced before in [4], with details given in [5].

To conclude this section, we remark that the polynomials $Q_{j,n,r}(x)$ and series stated after Theorem 1 do not seem to satisfy a linear recurrence of order ≤ 2 when $r < n$. The case $r = n$ is thus very remarkable.

3.2 Zudilin's approximations to $\zeta(4)$

In [12, Section 2], Zudilin showed that for any integer $n \geq 0$

$$Z_n := - \sum_{k=0}^{\infty} \frac{\partial}{\partial k} \left(\left(k + \frac{n}{2} \right) \frac{(k-n)_n^2 (k+n+1)_n^2}{(k)_{n+1}^4} \right) = u_n 3\zeta(4) + v_n$$

where u_n and v_n are rational numbers. In particular,

$$u_n = \sum_{j=0}^n \frac{\partial}{\partial j} \left(\left(\frac{n}{2} - j \right) \binom{n}{j}^4 \binom{n+j}{n}^2 \binom{2n-j}{n}^2 \right)$$

and the expression for v_n is more complicated. He also proved that $(Z_n)_n$, $(u_n)_n$ and $(v_n)_n$ are solutions of the linear recurrence

$$n^5 U_n + C(n) U_{n-1} - 3(3n-2)(3n-4)(n-1)^3 U_{n-2} = 0$$

where $Q(n)$ is Cohen's polynomial (9). It can be verified that $\frac{u_n}{v_n}$ coincide with the convergents $\frac{a_n}{b_n}$ of (8); see [12, Section 2, Theorem 2].

We observe that $Z_n = S_n(n+1)$ and $u_n = Q_{0,n}(n+1)$. Since $\zeta(4, n+1) = \zeta(4) - \sum_{j=1}^n \frac{1}{j^4}$, the specialization of Theorem 1 at $x = n+1$:

$$S_n(n+1) = 3Q_{0,n}(n+1)\zeta(4, n+1) + Q_{2,n}(n+1)$$

becomes

$$Z_n = u_n 3\zeta(4) + Q_{2,n}(n+1) - 3u_n \sum_{j=1}^n \frac{1}{j^4}$$

and thus we recover Zudilin's sequence $(v_n)_n$ by the identity

$$v_n = Q_{2,n}(n+1) - 3u_n \sum_{j=1}^n \frac{1}{j^4}.$$

3.3 Prévost's remainder Padé approximants for $\zeta(s, x)$

In [8], Prévost showed a very original method to prove the irrationality of $\zeta(2)$ and $\zeta(3)$. We present the slightly modified approach he recently presented in [9]. For any integer $x \geq 1$, we have

$$\zeta(s) = \sum_{k=1}^{x-1} \frac{1}{k^s} + \zeta(s, x).$$

He then computed explicitly the Padé approximants $[n+1/n](x)$ at $x = \infty$ of $\zeta(2, x)$, respectively the Padé approximants $[n+2/n](x)$ at $x = \infty$ of $\zeta(3, x)$. After taking $x = n+1$, he obtained Apéry's famous sequences for $\zeta(2)$ and $\zeta(3)$.

For $s = 2$, the denominators of $[n+1/n](x)$ are

$$P_n(x) = \sum_{j=0}^n \binom{n+1}{j+1} \binom{n+j+2}{j+1} \binom{x-1}{j}, \quad n \geq 0,$$

and they satisfy the orthogonality relation

$$\int_{i\mathbb{R}} P_n(x) P_m(x) \frac{x^2}{\sin(\pi x)^2} dx = 0, \quad n \neq m.$$

For $s = 3$, the denominators $Q_n(x)$ of $[n+2/n](x)$ are such that

$$Q_n(x^2) = \sum_{j=0}^n \frac{1}{j+1} \binom{n+1}{j+1} \binom{n+j+2}{j+1} \binom{x-1}{j} \binom{x+1}{j}, \quad n \geq 0,$$

and they satisfy the orthogonality relation

$$\int_{i\mathbb{R}} Q_n(x) Q_m(x) \frac{x^5 \cos(\pi x)}{\sin(\pi x)^3} dx = 0, \quad n \neq m.$$

The two families of orthogonal polynomials $(P_n)_n$ and $(Q_n)_n$ are specializations of Wilson's orthogonal polynomials [11].

Recently, Prévost [9] proved that the Padé approximants $[n+1/n](x)$ of $\zeta(s, x)$ at $x = \infty$ converge to $\zeta(s, x)$ for any fixed real number $s > 1$, but convergence is still an open problem when s is a complex number. Moreover, except for $s = 2, 3$, no expression of these approximants is known, even for $s = 4$. In this case, the problem is to find explicit expressions for polynomials $A_n(x)$ (of degree n) such that

$$\int_{i\mathbb{R}} A_n(x) A_m(x) \frac{x^8 (2 + \cos(2\pi x))}{\sin(\pi x)^4} dx = 0, \quad n \neq m.$$

Unfortunately, the weight function $\frac{x^8(2+\cos(2\pi x))}{\sin(\pi x)^4}$ is not of the form studied by Wilson. The sequence $(Q_{0,n}(x))_n$ is not orthogonal for this weight, but it is bi-orthogonal in the following sense: for any n and m such that $0 \leq m \leq 2n - 1$, we have

$$\int_{i\mathbb{R}} x^{m+5} Q_{0,n}(x) \frac{\cos(\pi x)}{\sin(\pi x)^3} dx = 0 = \int_{i\mathbb{R}} x^{m+8} Q_{0,n}(x) \frac{2 + \cos(2\pi x)}{\sin(\pi x)^4} dx.$$

3.4 Beukers and Bel's p -adic irrationality proofs

In [3], Calegari proved the irrationality of the 2-adic numbers $\zeta_2(2)$ and $\zeta_2(3)$, as well as of the 3-adic numbers $\zeta_2(3)$. His proof used overconvergent p -adic modular forms. Later, Beukers [2] obtained another proof of these facts, of a more classical flavor. In fact, he essentially used Prévost's Padé approximants for $\zeta(2, x)$ and $\zeta(3, x)$, though his formulas are written differently. The Padé type approximants constructed in [10] for $\zeta(s, x)$ contain as initial cases Beukers and Prévost approximants; Bel [1] used them to prove certain linear independence results for values of p -adic Hurwitz zeta functions. It would be interesting to know if Theorem 1 or its generalization could be used to prove the irrationality of the numbers $\zeta_p(4)$ for some p . The arithmetic and asymptotic properties of Zudilin's series Z_n are not good enough to imply the irrationality of $\zeta(4)$, but a modification of Z_n conjecturally proves that $\zeta(4) \notin \mathbb{Q}$.

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