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# NUMERICAL MODELLING OF TURBULENT TWO PHASE FLOWS USING THE TWO FLUID APPROACH

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## ABSTRACT

This paper addresses the problem of turbulence modelling when focusing on two-fluid models with distinct pressures in each phase. This work represents the sequel of a previous contribution which was devoted to laminar flows. The turbulence effects are accounted for through a very simple model which only requires computing the so-called turbulent kinetic energy within each phase. Dissipation mechanism is represented by an algebraic relation. One shows that one may obtain a physically relevant entropy inequality while writing both interface velocity and interface pressure in terms of local unknowns. The whole convective set is a non strictly hyperbolic system, and involves three contact discontinuities, which require an accurate numerical treatment. Unlike in the pure laminar case, one cannot recast the whole convective set in a conservative form. This is simply due to the occurrence of turbulent kinetic energies. Approximate jump conditions are introduced to close the true non conservative products arising in the governing equations of turbulent kinetic energies. Investigation of the field-by-field structure enables to establish that

expected positive quantities should remain so. A few numerical experiments involving the standard two phase flow shock tube apparatus complete the whole.

## INTRODUCTION

This paper focuses on the physical and numerical modelling of turbulent two phase flows, when one wishes to retain the two fluid approach without assuming local instantaneous pressure equilibrium. For a general presentation of the two fluid approach and the basics of ensemble averaging, the reader is referred to<sup>26, 3, 25, 10, 9, 39, 21</sup> and<sup>34</sup> for instance. An overwhelming number of specific papers refers to the so-called single pressure two fluid model. This has motivated a large debate over the past twenty years, which mainly relies on the problem of the lack of hyperbolicity of the convective process in these sets of PDE. A partial review of potential problems and cures was recently described in<sup>38</sup> in the framework of the ASTAR project, and one may also refer to<sup>6</sup> which discusses the problem of the maximum principle for the void fraction in this kind of models. Some ways to overcome the potential loss of hyperbolicity on coarse meshes have been suggested recently in<sup>7</sup> and<sup>28</sup> among others. In the present approach, one no longer assumes the crude hypothesis of local pressure equilibrium between phases, and we follow the pioneering

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work of<sup>29, 1, 24, 27, 23</sup> and other workers, though using different closures for the specific terms occurring, namely the interfacial pressure and the interfacial velocity.

This work is a sequel of a previous work described in<sup>15</sup> and<sup>8</sup>.

## GOVERNING EQUATIONS

The governing set of equations, which involves convective terms, source terms  $S(W)$  (to account for the mass transfer, drag effects, energy exchange and dissipation mechanisms), may be written as (neglecting diffusive terms):

$$\begin{cases} A(W) \frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} + C(W) \frac{\partial G(W)}{\partial x} = S(W) \\ W(x, 0) = W_0(x) \end{cases}$$

with  $W$ ,  $F(W)$ ,  $G(W)$  and  $S(W)$  in  $\mathbb{R}^9$ , and  $C(W), A(W)$  in  $\mathbb{R}^{9 \times 9}$ . The variable  $W$  is :

$$W = \begin{pmatrix} \alpha_2 \\ \alpha_2 \rho_2 \\ \alpha_2 \rho_2 U_2 \\ \alpha_2 E_2 \\ \alpha_2 K_2 \\ \alpha_1 \rho_1 \\ \alpha_1 \rho_1 U_1 \\ \alpha_1 E_1 \\ \alpha_1 K_1 \end{pmatrix}$$

All over the paper,  $\alpha_2$  stands for the void fraction of phase referenced 2, and  $\alpha_1 = 1 - \alpha_2$ . The mean density, the mean velocity, the turbulent kinetic energy, and the total mean energy within phase  $k$  are denoted  $\rho_k$ ,  $U_k$ ,  $K_k$  and  $E_k$  respectively. Viscous terms are included in the contribution  $E(W)$ . The conservative convective flux  $F(W)$  reads :

$$F(W) = \begin{pmatrix} 0 \\ \alpha_2 \rho_2 U_2 \\ \alpha_2 (\rho_2 U_2^2 + P_2 + \frac{2K_2}{3}) \\ \alpha_2 U_2 (E_2 + P_2 + \frac{2K_2}{3}) \\ \alpha_2 K_2 U_2 \\ \alpha_1 \rho_1 U_1 \\ \alpha_1 (\rho_1 U_1^2 + P_1 + \frac{2K_1}{3}) \\ \alpha_1 U_1 (E_1 + P_1 + \frac{2K_1}{3}) \\ \alpha_1 K_1 U_1 \end{pmatrix}$$

$$(A(W) - I) \frac{\partial W}{\partial t} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ P_{int}^2 \frac{\partial \alpha_2}{\partial t} \\ 0 \\ 0 \\ 0 \\ P_{int}^1 \frac{\partial \alpha_1}{\partial t} \\ 0 \end{pmatrix}$$

and :

$$C(W) \frac{\partial G(W)}{\partial x} = \begin{pmatrix} V_i \frac{\partial \alpha_2}{\partial x} \\ 0 \\ -P_{int}^2 \frac{\partial \alpha_2}{\partial x} \\ 0 \\ \frac{2\alpha_2 K_2}{3} \frac{\partial U_2}{\partial x} \\ 0 \\ -P_{int}^1 \frac{\partial \alpha_1}{\partial x} \\ 0 \\ \frac{2\alpha_1 K_1}{3} \frac{\partial U_1}{\partial x} \end{pmatrix}$$

The internal energy  $e_k$  and the turbulent mean total energy  $E_k$  are given through:

$$\begin{cases} e_k = e_k(P_k, \rho_k) \\ E_k = 0.5 \rho_k U_k U_k + K_k + \rho_k e_k \end{cases}$$

First, we emphasize that the list of arguments for the mean internal energy imply rather crude assumptions on the statistical thermodynamics, unless one focuses on simple instantaneous EOS within each phase (such as perfect gas EOS or Tamman EOS for instance). Second, we also immediately assume the following closure law on interfacial pressures  $P_{int}^k$  :

$$P_{int}^1 - P_{int}^2 = 0$$

which enables to ensure the conservation principle of both the total momentum equation and total energy equation for the mixture. Extensions to the framework of situations where surface tensions cannot be neglected are not be discussed herein. The drag term, turbulent dissipation, and the pressure relaxation term require giving associated time scales  $\tau_\alpha, \tau_U, \tau_{K_k}$  in:

$$S(W) = \begin{pmatrix} -\alpha_2(1-\alpha_2)\frac{P_1-P_2}{\tau_\alpha(P_1+P_2)} \\ 0 \\ \alpha_2(1-\alpha_2)(\rho_1\rho_2)^{1/2}\frac{U_1-U_2}{\tau_U} \\ V_i\alpha_2(1-\alpha_2)(\rho_1\rho_2)^{1/2}\frac{\tau_U}{\tau_U} \\ -\frac{\alpha_2 K_2}{\tau_{K_2}} \\ 0 \\ \alpha_2(1-\alpha_2)(\rho_1\rho_2)^{1/2}\frac{U_2-U_1}{\tau_U} \\ V_i\alpha_2(1-\alpha_2)(\rho_1\rho_2)^{1/2}\frac{\tau_U}{\tau_U} \\ -\frac{\alpha_1 K_1}{\tau_{K_1}} \end{pmatrix}$$

We choose data for  $\tau_{K_k}$  in agreement with single phase turbulent literature (see for instance the work of Spalart<sup>37</sup> or Baldwin-Lomax.<sup>2</sup>) Standard drag correlations (<sup>21, 34, 35</sup>) implicitly provide  $\tau_U$ , and may possibly account for interaction between the turbulent fields. The form of  $\tau_\alpha$  is chosen in agreement with those used in the laminar framework (see<sup>31, 15, 8</sup>). In order to detail both interfacial pressure and velocity fields  $V_i$  and  $P_i$ , we from now set (these notations will be used throughout the paper):

$$m_k = \alpha_k \rho_k \quad (1)$$

and

$$\beta_k = \frac{m_k}{m_1 + m_2} \quad (2)$$

$$\begin{cases} P_i = \mu_1(W)P_1 + (1-\mu_1(W))P_2 \\ + \mu_2(W)(m_1 m_2)^{1/2}(U_1 - U_2)^2 \\ V_i = \beta(W)U_1 + (1-\beta(W))U_2 \end{cases}$$

The basic principle which is involved is that closures should obey the local objective rule: ( $U_1(x, t) = U_2(x, t)$  and  $P_1(x, t) = P_2(x, t)$ ) imply:  $V_i(x, t) = U_1(x, t) = U_2(x, t)$  and  $P_i(x, t) = P_1(x, t) = P_2(x, t)$ . Scalar functions  $\mu_1(W)(1-\mu_1(W))$ ,  $\mu_2(W)$  and  $\beta(W)(1-\beta(W))$  are assumed non negative. Following a previous work (<sup>15, 8</sup>), we will use the specific choice:

$$\begin{cases} \mu_1(W) = \frac{a_1 \beta_2}{a_1 \beta_2 + a_2 \beta_1} \\ \mu_2(W) = 0 \\ \beta(W) = \beta_1 \end{cases}$$

using the following definitions:

$$a_k = \frac{1}{s_k} \frac{\frac{\partial s_k(P_k, \rho_k)}{\partial P_k}}{\frac{\partial e_k(P_k, \rho_k)}{\partial P_k}}$$

where the specific entropy  $s_k$  and the speed of density waves  $c_k$  comply with:

$$\begin{cases} (c_k)^2 = \frac{\frac{P_k}{(\rho_k)^2} - \frac{\partial e_k(P_k, \rho_k)}{\partial \rho_k}}{\frac{\partial e_k(P_k, \rho_k)}{\partial P_k}} = \frac{\hat{\gamma}_k P_k}{\rho_k} \\ (c_k)^2 \frac{\partial s_k(P_k, \rho_k)}{\partial P_k} + \frac{\partial s_k(P_k, \rho_k)}{\partial \rho_k} = 0 \end{cases}$$

The reader is referred to the papers by Glimm and co authors<sup>22, 23</sup> and also to the references<sup>24, 27</sup>. In the latter work in particular, the choice of the couple  $P_i, V_i$  is precisely  $P_1, U_2$  which is actually one particular example in the formulation above. In another recent work<sup>20</sup>, a similar formulation is used, setting :  $V_i = U_1$  and :  $P_i = P_2 + \psi(\alpha_2)$ . Some comments pertaining to the influence of the interface velocity will follow in a next section.

### MAIN PROPERTIES

We provide below some of the main properties of the whole closed set of equations.

#### Property 1.a

The homogeneous convective subset :

$$(Id + D(W)) \frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} + C(W) \frac{\partial G(W)}{\partial x} = 0$$

is hyperbolic, since it has real eigenvalues  $\lambda_k$ , and the associated right eigenvectors span the whole space  $\mathbb{R}^9$  -unless some resonance occurs-:

$$\begin{cases} \lambda_1 = V_i \\ \lambda_2 = U_2 - \hat{c}_2 \\ \lambda_{3,4} = U_2 \\ \lambda_5 = U_2 + \hat{c}_2 \\ \lambda_6 = U_1 - \hat{c}_1 \\ \lambda_{7,8} = U_1 \\ \lambda_9 = U_1 + \hat{c}_1 \end{cases}$$

Problems arising with resonant waves are disregarded here, though of great importance<sup>13</sup>. The reader may have a look at<sup>33</sup> and many references therein for a first approach. The exact expression of celerities shows the influence of the turbulent kinetic energy :

$$\rho_k (\hat{c}_k)^2 = \hat{\gamma}_k P_k + \frac{10K_k}{9}$$

which is the straightforward counterpart of what happens in single phase turbulent models. The physical significance of contribution proportional to  $K_k$  may be easily understood by investigating

the linearised system, and focusing on density perturbations, with given entropy rate  $P_k(\rho_k)^{-\gamma_k}$  and turbulent entropy rate  $K_k(\rho_k)^{-5/3}$  (assuming perfect gas EOS within each phase).

*Property 1.b*

Whatever the closure laws for  $P_i$  and  $V_i$  are, the 3, 4, 7, 8 fields are linearly degenerated, and the 2, 5, 6, 9 waves are genuinely non linear fields.

These results were expected owing to the form of the eigenstructure of the two-phase flow model in the laminar framework <sup>(8)</sup>, and in the single phase turbulent frame <sup>(11)</sup>. Note that these results do not depend on the specific choice of closure laws for  $V_i$  and  $P_i$ . On the contrary, the following properties are tightly bound with the latter:

*Property 2*

Defining the entropy-entropy flux pair  $(\eta, F_\eta)$ :

$$\begin{cases} \eta_k = \text{Log}(s_k) + \psi(\alpha_2) \\ \eta = -\alpha_1 \rho_1 \eta_1 - \alpha_2 \rho_2 \eta_2 \\ F_\eta = -\alpha_1 \rho_1 \eta_1 U_1 - \alpha_2 \rho_2 \eta_2 U_2 \end{cases}$$

the following entropy inequality holds for regular solutions of the above mentioned system :

$$\frac{\partial \eta}{\partial t} + \frac{\partial F_\eta}{\partial x} < 0.$$

This entropy inequality enables to select the admissible shock waves. The closure law for the interface velocity ensures that:

*Property 3*

We still set  $\beta(W) = \beta_1$ . The 1- field is linearly degenerated since it agrees with :

$$\nabla_W V_i(W) \cdot r_1(W) = 0 \quad (3)$$

where  $r_1(W)$  stands for the right eigenvector associated with the first eigenvalue  $\lambda_1(W)$ .

Other possible choices correspond to the extreme values

$$\beta(W)(1 - \beta(W)) = 0. \quad (4)$$

In that case, it must be emphasized that some specific behaviour arises, since the choice  $V_i = U_1$  induces  $P_i = P_2$ , and thus the governing equations for variables  $s_1, s_2$  become:

$$\frac{\partial s_k}{\partial t} + U_k \frac{\partial s_k}{\partial x} = 0.$$

An amazing point is that this clearly corresponds to one of the assumptions in the approach recently proposed by Gavriluyuk (see<sup>20, 32</sup>) This model is clearly no longer a mixture model but a true interface model. It also has some numerical consequences when using the rough Godunov schemes described in the next section. We also emphasize that an alternative way to obtain relevant closures for the interfacial pressure  $P_i$  was proposed in<sup>8</sup>. Though it seems another interesting way to improve closure laws, this will not be discussed herein. We turn now to the description of the structure of waves.

*Property 4*

If  $I_k$  refers to the list of Riemann invariants associated with the  $k$ -wave, one has the following result:

$$\begin{aligned} I_1 &= V_i, \frac{m_1 m_2}{m_1 + m_2} (U_1 - U_2), \frac{s_2}{s_1}, \hat{s}_1, \hat{s}_2, \\ e_1 + \frac{\pi_1 + K_1}{\rho_1} + \frac{(U_1 - V_i)^2}{2}, e_2 + \frac{\pi_2 + K_2}{\rho_2} + \frac{(U_2 - V_i)^2}{2}, \\ \frac{m_1 m_2}{m_1 + m_2} (U_1 - U_2)^2 + \alpha_1 \pi_1 + \alpha_2 \pi_2 \\ I_2 &= \alpha_2, U_2 + \psi_2, s_2, \hat{s}_2, \pi_1, U_1, s_1, \hat{s}_1 \\ I_{3-4} &= \alpha_2, U_2, \pi_2, \pi_1, U_1, s_1, \hat{s}_1 \\ I_5 &= \alpha_2, U_2 - \psi_2, s_2, \hat{s}_2, \pi_1, U_1, s_1, \hat{s}_1 \\ I_6 &= \alpha_2, U_1 + \psi_1, s_1, \hat{s}_1, \pi_2, U_2, s_2, \hat{s}_2 \\ I_{7-8} &= \alpha_2, U_1, \pi_1, \pi_2, U_2, s_2, \hat{s}_2 \\ I_9 &= \alpha_2, U_1 - \psi_1, s_1, \hat{s}_1, \pi_2, U_2, s_2, \hat{s}_2 \end{aligned}$$

noting :

$$\begin{aligned} \psi_k &= \int_0^{\rho_k} (\frac{\hat{c}_k(\rho_k, s_k, \hat{s}_k)}{\rho_k}) d\rho_k \\ \pi_k &= P_k + \frac{2K_k}{3} \\ \hat{s}_k &= \frac{K_k}{\rho_k (m_k)^{2/3}} \end{aligned}$$

The Riemann invariants in the 1 - wave will be respectively noted  $I_1^k$  afterwards for  $k = 1, 8$ . Obviously, inside the  $k$ - wave ( $2 \leq k \leq 8$ ), the explicit form of Riemann invariants ensures that all expected positive quantities remain positive in genuinely non linear fields, provided that no vacuum occurs in the solution in any phase. This result is equivalent of what happens in the laminar frame <sup>(15)</sup>, for regular rarefaction waves. The specific case of the 1 wave will be discussed later on, since it is a Linearly Degenerate field.

Turning then to discontinuities, we denote as usual  $[\phi] = \phi_r - \phi_l$ , the jump between the two left and right states (labeled  $l, r$ ) on each side of the discontinuity travelling at speed  $\sigma$ . We also define the mean  $\bar{\phi} = (\phi_r + \phi_l)/2$ . For weak shocks, jump

conditions may be written :

$$\begin{aligned}
(\bar{V}_i - \sigma)\Delta(\alpha_k) &= 0 \\
\Delta(m_k(U_k - \sigma)) &= 0 \\
\Delta(m_k U_k(U_k - \sigma) + \alpha_k \pi_k) - \hat{P}_i \Delta(\alpha_k) &= 0 \\
\Delta(\alpha_k E_k(U_k - \sigma) + \alpha_k \pi_k U_k) \\
- \hat{P}_i \bar{V}_i \Delta(\alpha_k) &= 0 \\
\Delta(m_k \hat{s}_k(U_k - \sigma)) &= 0.
\end{aligned}$$

The latter are valid in the pure laminar frame ( $K_1(x, t) = K_2(x, t) = 0$ ), and at least for weak shocks in turbulent single phase flows. On both sides of the 1-contact discontinuity, these make sense since  $\Delta(\alpha_k) = 0$ , which results in the following jump relations within each phase:

$$\begin{aligned}
\Delta(\alpha_k) &= 0 \\
\Delta(\rho_k(U_k - \sigma)) &= 0 \\
\Delta(\rho_k U_k(U_k - \sigma) + \pi_k) &= 0 \\
\Delta(E_k(U_k - \sigma) + \pi_k U_k) &= 0 \\
\Delta(\rho_k \hat{s}_k(U_k - \sigma)) &= 0
\end{aligned}$$

#### Property 5

Assume that the perfect gas EOS holds within each phase, that is:  $P_k = (\gamma_k - 1)\rho_k e_k$ . Then above jump conditions ensure that through a shock wave, the right state is relevant, which means that:

$$(\alpha_k)_r > 0, (\rho_k)_r > 0, (P_k)_r > 0, (K_k)_r > 0$$

provided that no vacuum occurs in the solution and that the left state agrees with :  $(\alpha_k)_l > 0$ ,  $(\rho_k)_l > 0$ ,  $(P_k)_l > 0$ ,  $(K_k)_l > 0$ . More over, the ratio of densities is bounded :

$$(b_k)^{-1} < \frac{(\rho_k)_r}{(\rho_k)_l} < b_k$$

and the rate of turbulence has a lower and an upper bound:

$$1 \leq \sup\left(\frac{(K_k)_r}{(K_k)_l}, \frac{(K_k)_l}{(K_k)_r}\right) \leq (b_k)^{5/3}$$

with  $b_k = \frac{\gamma_k + 1}{\gamma_k - 1}$ . The pressure ratio  $\sup\left(\frac{(P_k)_r}{(P_k)_l}, \frac{(P_k)_l}{(P_k)_r}\right)$  has no upper bound.

We note that conditions on the ratio of the mean densities through a shock wave correspond to the conditions arising in the pure laminar framework (see for instance Smoller<sup>36</sup>). This is expected since the instantaneous inequality  $a\phi_l < \phi_r < b\phi_l$  obviously provides the following relation between the mean quantities:  $a < \phi_l \leq \phi_r \leq b < \phi_l$ . We also recall

that approximate jump conditions proposed in<sup>11</sup> induce  $1/4 < \frac{(\rho_k)_r}{(\rho_k)_l} < 4$ , which is indeed more restrictive than the latter, since the usual condition  $1 < \gamma_k < 5/3$  implies  $4 < b_k$ . Moreover<sup>11</sup> also implies that the ratio of turbulent kinetic energies is unbounded, unlike in the present case.

#### Proof

The proof is straightforward. Setting:  $v_k = U_k - \sigma$ , above jump conditions immediately provide the following relations between states through genuinely non linear simple fields associated with  $\lambda_2$  or  $\lambda_6$  :

$$\begin{aligned}
(\alpha_k)_r &= (\alpha_k)_l \\
(\rho_k v_k)_r &= (\rho_k v_k)_l \\
(\rho_k)_r &= z_k (\rho_k)_l \\
(U_k)_r &= (U_k)_l - \left(\frac{(\pi_k)_l}{(\rho_k)_l} \left(\frac{(\pi_k)_r}{(\pi_k)_l} - 1\right) \left(\frac{z_k - 1}{z_k}\right)\right)^{1/2} \\
(K_k)_r &= (z_k)^{5/3} (K_k)_l \\
(P_k)_r &= (P_k)_l \frac{b_k z_k - 1}{b_k - z_k} \\
&+ \frac{2(K_k)_l}{3(b_k - z_k)} (-4((z_k)^{5/3} - z_k) + ((z_k)^{8/3} - 1))
\end{aligned}$$

with  $z_k > 1$ . The connection of both the pressure and the turbulent kinetic energy on each side of the shock is slightly distinct from the one introduced in<sup>11</sup>, at least for shocks which are not too weak. Nonetheless, the true non-conservative products theoretically inhibit investigating such a situation. A similar proof holds through simple waves connected with eigenvalues  $\lambda_5$  or  $\lambda_9$  (with  $z_k < 1$ ).

When focusing on simple EOS such as those considered in our work (perfect gas EOS or alternatively Tamman EOS within each phase), these approximate jump conditions also preserve the positivity constraints through the 1-wave:

#### Property 6

We still assume the form of  $V_i$  with  $\beta(W) = \beta_1$ . We also still restrict to the framework :

$$P_k = (\gamma_k - 1)\rho_k e_k$$

with  $1 < \gamma_k < 5/3$ . Then , the connection of states through the 1-wave ensures that  $\alpha_k > 0$  but also:

$$m_k > 0, P_k > 0, K_k > 0$$

#### Proof

Obviously, the only wave through which the void fraction varies is the 1-wave, thus:  $\alpha_l = \alpha_L$  and also:  $\alpha_r = \alpha_R$ . Through the 1-contact discontinuity

nuity, jump conditions yield:

$$\begin{aligned}
\sigma &= \overline{V}_i = (V_i)_l = (V_i)_r \\
\Delta(m_k(U_k - \sigma)) &= 0 \\
\Delta(m_k \hat{s}_k(U_k - \sigma)) &= 0 \\
\Delta(\alpha_k E_k(U_k - \sigma) + \alpha_k \pi_k U_k) \\
-\overline{V}_i \Delta(m_k U_k(U_k - \sigma) + \alpha_k \pi_k) &= 0 \\
\Delta(m_1 U_1(U_1 - \sigma) + \alpha_1 \pi_1) \\
+\Delta(m_2 U_2(U_2 - \sigma) + \alpha_2 \pi_2) &= 0 \\
\Delta\left(\frac{\hat{s}_2}{\hat{s}_1}\right) &= 0
\end{aligned}$$

We still denote with subscript  $l$  quantities on the left side on the 1- wave. We recall that connecting states through this wave with help of Riemann invariants or jump relations is perfectly identical and leads to:

$$(\gamma_k e_k + \frac{5(\hat{s}_k)_l m_k^{2/3}}{3} + \frac{Q_l^2}{2m_k^2})_r = (C_k)_l$$

setting :

$$\begin{aligned}
Q_l &= I_1^2(W_l) = \left(\frac{m_1 m_2}{m_1 + m_2}(U_1 - U_2)\right)_l \\
(C_1)_l &= I_1^6(W_l) = \left(e_1 + \frac{\pi_1 + K_1}{\rho_1} + \frac{(U_1 - V_i)^2}{2}\right)_l \\
(C_2)_l &= I_1^7(W_l) = \left(e_2 + \frac{\pi_2 + K_2}{\rho_2} + \frac{(U_2 - V_i)^2}{2}\right)_l \\
E_l &= I_1^8(W_l)
\end{aligned}$$

The main two equations correspond to the conservation of the total energy for both phases:

$$\begin{aligned}
&Q_l^2 \left(\frac{\gamma_1 + 1}{2\gamma_1 m_1} + \frac{\gamma_2 + 1}{2\gamma_2 m_2}\right)_r \\
&+ (\frac{\gamma_1 - 1}{\gamma_1} (C_1)_l m_1 + \frac{\gamma_2 - 1}{\gamma_2} (C_2)_l m_2)_r \\
&+ \left(\frac{5-3\gamma_1}{3\gamma_1} (\hat{s}_1)_l (m_1)^{5/3}\right)_r \\
&+ \frac{5-3\gamma_2}{3\gamma_2} (\hat{s}_2)_l (m_2)^{5/3})_r = E_l
\end{aligned}$$

and the constant ratio of phase entropies:

$$\begin{aligned}
&(\alpha_1)_r^{\gamma_1 - 1} \frac{\gamma_1 - 1}{\gamma_1} (m_1)_r^{1 - \gamma_1} \\
&((C_1)_l - \frac{Q_l^2}{2(m_1)_r^2} - \frac{5}{3}(\hat{s}_1)_l (m_1)_r^{2/3}) \\
&= \left(\frac{\hat{s}_2}{\hat{s}_1}\right)_l (\alpha_2)_r^{\gamma_2 - 1} \frac{\gamma_2 - 1}{\gamma_2} (m_2)_r^{1 - \gamma_2} \\
&((C_2)_l - \frac{Q_l^2}{2(m_2)_r^2} - \frac{5}{3}(\hat{s}_2)_l (m_2)_r^{2/3})
\end{aligned}$$

This provides a coupled set of two non-linear equations with two unknowns  $(m_1)_r$  and  $(m_2)_r$ . Performing implicit elimination of  $(m_1)_r$  through the last equation and introducing in the former enables to compute solution in terms of only one unknown  $(m_2)_r$ . One may then check that the resulting equation has either zero, one or two positive solutions. As a consequence, the partial mass  $(m_1)_r$  is also positive. Though not totally straightforward, a similar remark holds for internal energies  $e_k$  and obviously for turbulent kinetic energies  $K_k$ , owing to the invariance of

turbulent kinetic entropies through the one wave:  $\hat{s}_{kr} = \hat{s}_{kl}$ .

On the whole, it clearly appears that there is no major difference arising in the turbulent framework, when compared with the laminar approach<sup>8, 15</sup>.

## NUMERICAL METHOD

We only provide here the main guidelines of the numerical method and a few numerical results. Actually, we use the same fractional step approach as in<sup>15</sup> to compute the whole set. Thus we first compute an approximate solution of the homogeneous hyperbolic system :

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} + C'(W) \frac{\partial G(W)}{\partial x} = 0 \\ W(x, 0) = \tilde{W}_n(x) \end{cases}$$

over  $[t, t + \delta t]$ , with given adequate boundary conditions. We then look for solutions of the ODE:

$$\begin{cases} \frac{\partial W}{\partial t} = S(W) \\ W(x, 0) = W^*(x) \end{cases}$$

provided initial data  $W^*(x)$  given by an approximation of the solution of the first step at time  $t + \delta t$ . The second step is once again split in two, so that one first computes the ODE:

$$\frac{\partial W}{\partial t} = S_1(W) = \begin{pmatrix} 0 \\ 0 \\ \frac{\alpha_1 \alpha_2 (\rho_1 \rho_2)^{1/2}}{\tau_U} (U_1 - U_2) \\ \frac{\alpha_1 \alpha_2 (\rho_1 \rho_2)^{1/2}}{\tau_U} V_i (U_1 - U_2) \\ -\frac{\alpha_2 K_2}{\tau_{K_2}} \\ 0 \\ -\frac{\alpha_1 \alpha_2 (\rho_1 \rho_2)^{1/2}}{\tau_U} (U_1 - U_2) \\ -\frac{\alpha_1 \alpha_2 (\rho_1 \rho_2)^{1/2}}{\tau_U} V_i (U_1 - U_2) \\ -\frac{\alpha_1 K_1}{\tau_{K_1}} \end{pmatrix}$$

Through the latter the following behaviour is ensured:

$$\begin{aligned}
m_1 \frac{\partial e_1}{\partial P_1} \frac{\partial P_1}{\partial t} &= \frac{M}{\tau_U} (1 - \beta_1) (U_1 - U_2)^2 + \frac{\alpha_1 K_1}{\tau_{K_1}} \\
m_2 \frac{\partial e_2}{\partial P_2} \frac{\partial P_2}{\partial t} &= \frac{M}{\tau_U} \beta_1 (U_1 - U_2)^2 + \frac{\alpha_2 K_2}{\tau_{K_2}} \\
M &= \alpha_1 \alpha_2 (\rho_1 \rho_2)^{1/2}
\end{aligned}$$

Hence, as soon as  $0 \leq \frac{\partial e_k}{\partial P_k}$ , both pressures are increasing, which guarantees positive values of  $P_1$  and  $P_2$  during this step. Moreover, velocities follow the rule  $\frac{\partial(U_1 - U_2)^2}{\partial t} \leq 0$ . Since

$\frac{\partial \alpha_2}{\partial t} = \frac{\partial m_k}{\partial t} = 0$ , the void fraction and the densities  $\rho_k$  remain constant wrt time.

Then, one needs to update values of pressure, density and void fractions through:

$$\frac{\partial W}{\partial t} = S(W) - S_1(W)$$

(obviously here, both  $\frac{\partial U_k}{\partial t}$  and  $\frac{\partial K_k}{\partial t}$  are null). In this latter step, both pressures relax towards an equilibrium state and remain positive meanwhile. Even more, the void fraction remains in its bounds, if:  $0 \leq \hat{\gamma}_k + (P_i - P_k)(\rho_k P_k \frac{\partial e_k}{\partial P_k})^{-1}$  (the proof is exactly the same as the one given in<sup>15</sup>). Actually, the computation of the second step involving ODE is not that obvious, and the reader is referred to<sup>15</sup> for details. As it was expected, but surprisingly not clearly explained in the literature, this second step indeed ensures the relaxation of pressures  $P_1$  and  $P_2$  towards an equilibrium state.

On the whole, the first, second and third step preserve the maximum principle for the void fraction. More over, for finite time interval  $[0, T]$ , the turbulent kinetic energy remains positive within each phase, assuming of course positive inlet boundary conditions, positive initial conditions, and also assuming that  $U_k$ ,  $\frac{\partial U_k}{\partial x}$  and  $\frac{1}{\tau_{K_k}}$  remain bounded. This result is only valid for regular solutions.

Approximate solutions of the latter first-order homogeneous system may be obtained applying for an extension of Rusanov scheme<sup>30</sup>, or the approximate Godunov scheme called VFROencv. Details on the latter, which is based on the early proposal<sup>19</sup>, can be found for instance in<sup>5, 4</sup>. More details on the speed of convergence of this family of schemes may be found in<sup>16</sup> and<sup>14</sup>. Extensions of these schemes to the frame of mixed type equations is also described in<sup>18</sup>, which is based on the pioneering work of A.Y. LeRoux and co workers. An important point to note is that an equivalent form of the latter homogeneous system may be written (the explicit form of the matrix  $C(Z)$  is given in appendix A):

$$\frac{\partial Z}{\partial t} + C(Z) \frac{\partial Z}{\partial x} = 0$$

when restricting to regular solutions, and noting:

$$Z = (\alpha_2, s_2, \hat{s}_2, U_2, \pi_2, s_1, \hat{s}_1, U_1, \pi_1)$$

$$A_k = \frac{\partial s_k}{\partial P_k} (P_i - P_k) (m_k \frac{\partial e_k}{\partial P_k})^{-1}$$

$$B_k = \frac{2K_k}{3\alpha_k} + \frac{\hat{\gamma}_k P_k}{\alpha_k} + (P_i - P_k) (m_k \frac{\partial e_k}{\partial P_k})^{-1}$$

It may prevent the occurrence of negative interface values of expected discrete quantities, with reasonable CFL constraint on the time stepping, in single phase areas<sup>17</sup>. This enables to develop a rather efficient solver, following the ideas developed in<sup>15</sup>. Eventually, it is emphasized that we compute here approximate solutions of the non-conservative equation of  $K_k$ .

The first series of figures (Fig. 1 – 4) displays the behaviour of the approximate solutions obtained when computing an unsteady shock tube in the turbulent framework, using perfect gas EOS within each phase ( $\gamma_1 = 1.4$ ,  $\gamma_2 = 1.2$ ). The mesh contains 10000 regular cells. The CFL number is set to 1/2. The initial data is :  $(U_1)_L = (U_2)_L = 0$ ,  $(P_1)_L = (P_2)_L = 10^5$ ,  $(K_1)_L = (K_2)_L = 10^3$ ,  $(\rho_1)_L = (\rho_2)_L = 1$ ,  $(\alpha_1)_L = 0.6$ , and :  $(U_1)_R = (U_2)_R = 0$ ,  $(P_1)_R = (P_2)_R = 10^4$ ,  $(K_1)_R = (K_2)_R = 10^2$ ,  $(\rho_1)_R = (\rho_2)_R = 0.125$ ,  $(\alpha_1)_R = 0.5$ , where subscripts  $L$  and  $R$  refer to  $x < 0.5$  and  $x > 0.5$  respectively. The pressure  $P_k$  varies through the three contact discontinuities associated with  $U_1$ ,  $U_2$  and  $V_i$ , which are close one to another in this example. The turbulent kinetic energies also vary through the latter. However,  $\pi_k$  remains constant through both fields associated with:  $\lambda = U_1$  and  $\lambda = U_2$ . The void fraction, which is governed by a LD field, converges very slowly (as  $h^{1/2}$ ), and this renders accurate computations rather uneasy. Extension to so-called 'second-order' is thus mandatory. The accuracy may be expected even worse on 2D or 3D meshes.

The last figures (Fig. 5 to 7) compare results obtained with the same initial condition, when using  $V_i = \beta_1 U_1 + (1 - \beta_1) U_2$  and  $V'_i = U_2$  alternatively.

## CONCLUSION

As happens when computing laminar flows with help of the two fluid two pressure model, fine meshes are required in order to get a sufficiently accurate approximation of solutions of the whole set of PDE. One of the major reasons is that three distinct contact discontinuities are involved herein - which in addition are close to one another-, and this implies that very high order



schemes or alternatively very fine meshes with low order schemes are used for predictions. This has motivated the building of new schemes<sup>14, 12</sup> (that is mixed conservative - non conservative schemes) since conservative upwind schemes only ensure a small rate of convergence ( $h^{\frac{2}{p+1}}$ ) for expected schemes of order  $p$ , when LD fields are present in the eigenstructure. This represents a great challenge for future work.

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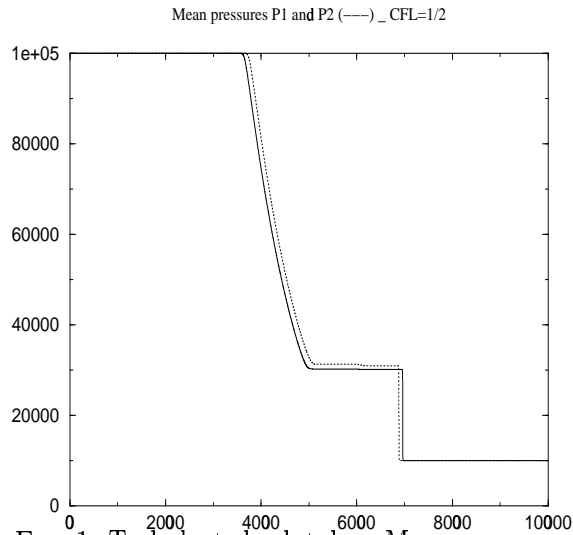


FIG. 1: Turbulent shock tube - Mean pressures

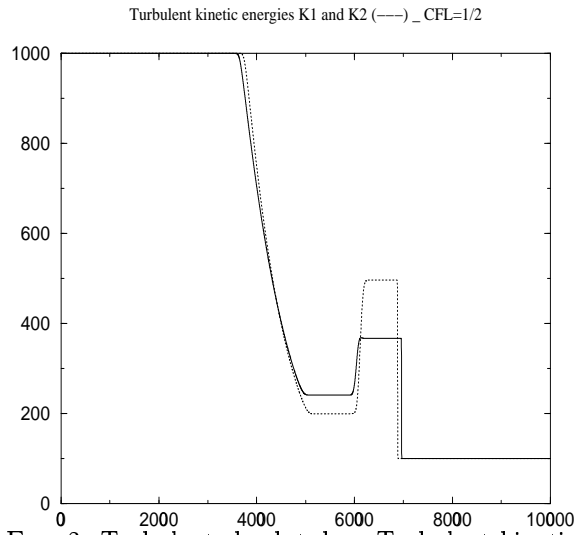


FIG. 2: Turbulent shock tube - Turbulent kinetic energies

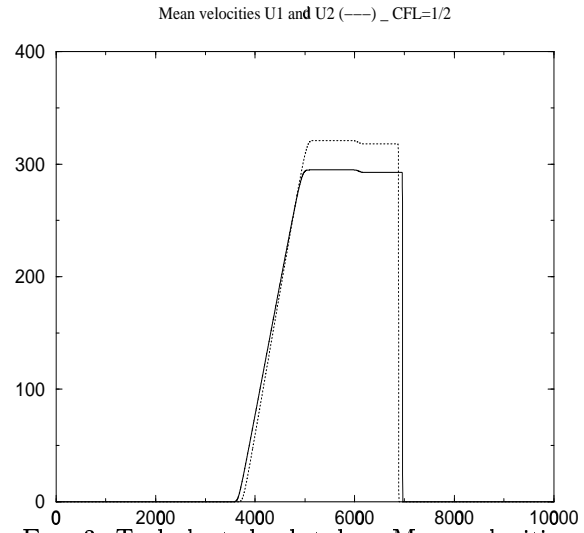


FIG. 3: Turbulent shock tube - Mean velocities

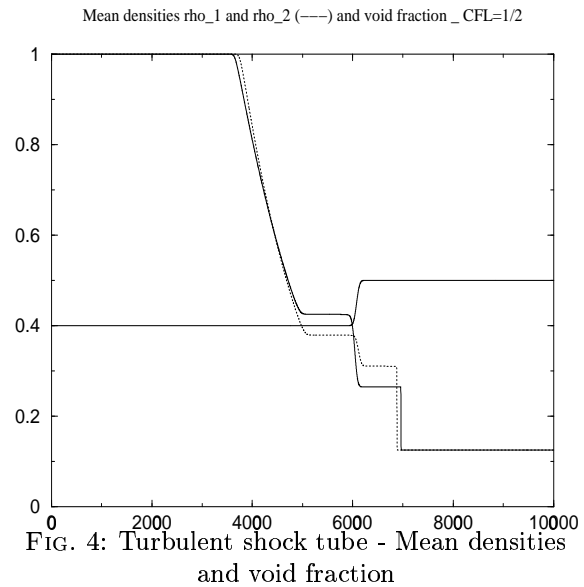


FIG. 4: Turbulent shock tube - Mean densities and void fraction

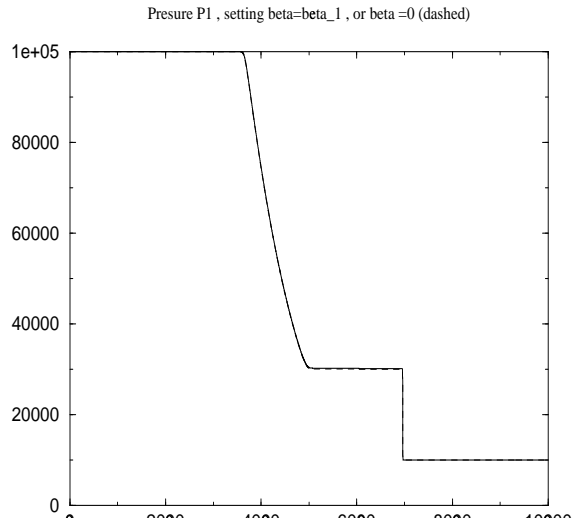


FIG. 5: Turbulent shock tube - Mean pressure P1 for two distinct values of  $V_i$

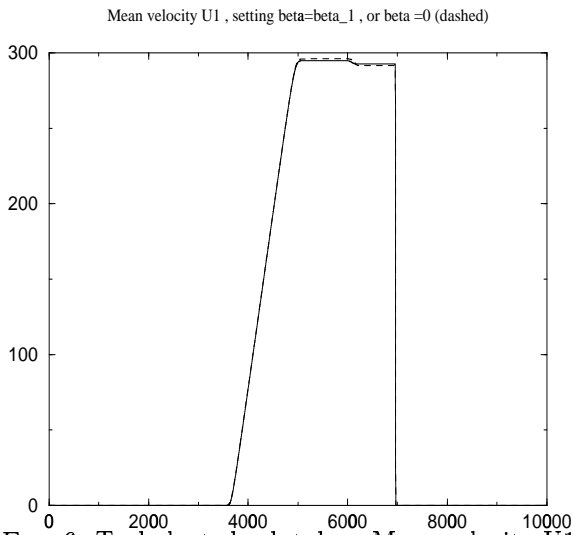


FIG. 6: Turbulent shock tube - Mean velocity U1 for two distinct values of  $V_i$

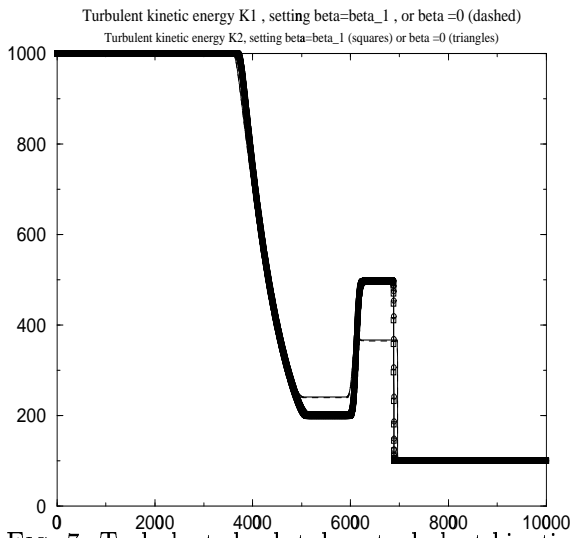


FIG. 7: Turbulent shock tube - turbulent kinetic energies for two distinct values of  $V_i$

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## Appendix A

We give below the form of the matrix of convective terms.

$$C(Z) = \begin{pmatrix} V_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_2(U_2 - V_i) & U_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\pi_2 - P_i)/m_2 & 0 & 0 & U_2 & \tau_2 & 0 & 0 & 0 & 0 \\ B_2(U_2 - V_i) & 0 & 0 & \gamma_2 P_2 + 10K_2/9 & U_2 & 0 & 0 & 0 & 0 \\ -A_1(U_1 - V_i) & 0 & 0 & 0 & 0 & U_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U_1 & 0 & 0 \\ -(\pi_1 - P_i)/m_1 & 0 & 0 & 0 & 0 & 0 & 0 & U_1 & \tau_1 \\ -B_1(U_1 - V_i) & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 P_1 + 10K_1/9 & U_1 \end{pmatrix}.$$