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An example of non-uniqueness for the weighted Radon transforms along hyperplanes in multidimensions

F.O. Goncharov* R. G. Novikov*†

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Abstract

We consider the weighted Radon transforms R_W along hyperplanes in \mathbb{R}^d , $d \geq 3$, with strictly positive weights $W = W(x, \theta)$, $x \in \mathbb{R}^d$, $\theta \in \mathbb{S}^{d-1}$. We construct an example of such a transform with non-trivial kernel in the space of infinitely smooth compactly supported functions. In addition, the related weight W is infinitely smooth almost everywhere and is bounded. Our construction is based on the famous example of non-uniqueness of J. Boman (1993) for the weighted Radon transforms in \mathbb{R}^2 and on a recent result of F. Goncharov and R. Novikov (2016).

Keywords: weighted Radon transforms, injectivity, non-injectivity

AMS Mathematics Subject Classification: 44A12, 65R32

1 Introduction

We consider the weighted Radon transforms R_W , defined by the formulas:

$$R_W f(s, \theta) = \int_{x\theta=s} W(x, \theta) f(x) dx, \quad (s, \theta) \in \mathbb{R} \times \mathbb{S}^{d-1}, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad (1.1)$$

where $W = W(x, \theta)$ is the weight, $f = f(x)$ is a test function on \mathbb{R}^d .

We assume that W is real valued, bounded and strictly positive, i.e.:

$$W = \overline{W} \geq c > 0, \quad W \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1}), \quad (1.2)$$

where \overline{W} denotes the complex conjugate of W , c is a constant.

If $W \equiv 1$, then R_W is reduced to the classical Radon transform R along hyperplanes in \mathbb{R}^d . This transform is invertible by the classical Radon inversion formulas; see [Rad17].

If W is strictly positive, $W \in C^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$ and $f \in C_0^\infty(\mathbb{R}^d)$, then in [Bey84] the inversion of R_W is reduced to solving a Fredholm type linear integral equation. Besides, in [BQ87] it was proved that R_W is injective (for example, in $L_0^2(\mathbb{R}^d)$) if W is real-analytic and strictly positive. In addition, an example of R_W in \mathbb{R}^2 with infinitely smooth strictly positive W and with non-trivial kernel $\text{Ker} R_W$ in $C_0^\infty(\mathbb{R}^2)$ was constructed in [Bom93]. Here C_0^∞ , L_0^2 denote the spaces of functions from C^∞ , L^2 with compact support, respectively.

In connection with the most recent progress in inversion methods for weighted Radon transforms R_W , see [Gon17].

We recall also that inversion methods for R_W in \mathbb{R}^3 admit applications in the framework of emission tomographies (see [GN16]).

In the present work we construct an example of R_W in \mathbb{R}^d , $d \geq 3$, with non-trivial kernel $\text{Ker} R_W$ in $C_0^\infty(\mathbb{R}^d)$. The related W satisfies (1.2). In addition, our weight W is infinitely smooth almost everywhere on $\mathbb{R}^d \times \mathbb{S}^{d-1}$.

In our construction we proceed from results of [Bom93] and [GN16].

In Section 2, in particular, we recall the result of [GN16].

In Section 3 we recall the result of [Bom93].

In Section 4 we obtain the main result of the present work.

*CMAP, Ecole Polytechnique, CNRS, Université Paris-Saclay, 91128 Palaiseau, France;
email: fedor.goncharov.ol@gmail.com

†IEPT RAS, 117997 Moscow, Russia;
email: roman.novikov@polytechnique.edu

2 Relations between the Radon and the ray transforms

We consider also the weighted ray transforms P_w in \mathbb{R}^d , defined by the formulas:

$$P_w f(x, \theta) = \int_{\mathbb{R}} w(x + t\theta, \theta) f(x + t\theta) dt, \quad (x, \theta) \in T\mathbb{S}^{d-1}, \quad (2.1)$$

$$T\mathbb{S}^{d-1} = \{(x, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1} : x\theta = 0\}, \quad d \geq 2, \quad (2.2)$$

where $w = w(x, \theta)$ is the weight, $f = f(x)$ is a test-function on \mathbb{R}^d .

We assume that w is real valued, bounded and strictly positive, i.e.:

$$w = \bar{w} \geq c > 0, \quad w \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1}). \quad (2.3)$$

We recall that $T\mathbb{S}^{d-1}$ can be interpreted as the set of all oriented rays in \mathbb{R}^d . In particular, if $\gamma = (x, \theta) \in T\mathbb{S}^{d-1}$, then

$$\gamma = \{y \in \mathbb{R}^d : y = x + t\theta, t \in \mathbb{R}\}, \quad (2.4)$$

where θ gives the orientation of γ .

We recall that for $d = 2$, transforms P_w and R_W are equivalent up to the following change of variables:

$$R_W f(s, \theta) = P_w f(s\theta, \theta^\perp), \quad s \in \mathbb{R}, \quad \theta \in \mathbb{S}^1, \quad (2.5)$$

$$W(x, \theta) = w(x, \theta^\perp), \quad x \in \mathbb{R}^2, \quad \theta \in \mathbb{S}^1, \quad (2.6)$$

$$\theta^\perp = (-\sin \phi, \cos \phi) \text{ for } \theta = (\cos \phi, \sin \phi), \quad \phi \in [0, 2\pi),$$

where f is a test-function on \mathbb{R}^2 .

For $d = 3$, the transforms R_W and P_w are related by the following formulas (see [GN16]):

$$R_W f(s, \theta) = \int_{\mathbb{R}} P_w f(s\theta + \tau[\theta, \alpha(\theta)], \alpha(\theta)) d\tau, \quad (s, \theta) \in \mathbb{R} \times \mathbb{S}^2, \quad (2.7)$$

$$W(x, \theta) = w(x, \alpha(\theta)), \quad x \in \mathbb{R}^3, \quad \theta \in \mathbb{S}^2, \quad (2.8)$$

$$\alpha(\theta) = \begin{cases} \frac{[\eta, \theta]}{|[\eta, \theta]|}, & \text{if } \theta \neq \pm\eta, \\ \text{any vector } e \in \mathbb{S}^2, \text{ such that } e \perp \theta, & \text{if } \theta = \pm\eta, \end{cases} \quad (2.9)$$

where η is some fixed vector from \mathbb{S}^2 , $[\cdot, \cdot]$ denotes the standard vector product in \mathbb{R}^3 , \perp denotes the orthogonality of vectors. Actually, formula (2.7) gives an expression for $R_W f$ on $\mathbb{R} \times \mathbb{S}^2$ in terms of $P_w f$ restricted to the rays $\gamma = \gamma(x, \theta)$, such that $\theta \perp \eta$, where W and w are related by (2.8).

Below we present analogs of (2.7)-(2.8) for $d > 3$.

Let

$$\Sigma(s, \theta) = \{x \in \mathbb{R}^d : x\theta = s\}, \quad s \in \mathbb{R}, \quad \theta \in \mathbb{S}^{d-1}, \quad (2.10)$$

$$\Xi(v_1, \dots, v_k) = \text{Span}\{v_1, \dots, v_k\}, \quad v_i \in \mathbb{R}^d, \quad i = \overline{1, k}, \quad 1 \leq k \leq d, \quad (2.11)$$

$$\Theta(v_1, v_2) = \{\theta \in \mathbb{S}^{d-1} : \theta \perp v_1, \theta \perp v_2\} \simeq \mathbb{S}^{d-3}, \quad v_1, v_2 \in \mathbb{R}^d, \quad v_1 \perp v_2, \quad (2.12)$$

$$(e_1, e_2, e_3, \dots, e_d) \text{ - be some fixed orthonormal, positively oriented basis in } \mathbb{R}^d. \quad (2.13)$$

If (e_1, \dots, e_d) is not specified otherwise, it is assumed that (e_1, \dots, e_d) is the standard basis in \mathbb{R}^d .

For $d > 3$, the transforms R_W and P_w are related by the following formulas:

$$R_W f(s, \theta) = \int_{\mathbb{R}^{d-2}} P_w f(s\theta + \sum_{i=1}^{d-2} \tau_i \beta_i(\theta), \alpha(\theta)) d\tau_1 \dots d\tau_{d-2}, \quad (s, \theta) \in \mathbb{R} \times \mathbb{S}^{d-1}, \quad (2.14)$$

$$W(x, \theta) = w(x, \alpha(\theta)), \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \quad (2.15)$$

where $\alpha(\theta)$, $\beta_i(\theta)$, $i = \overline{1, d-2}$, are defined as follows:

$$\alpha(\theta) = \begin{cases} \text{direction of one-dimensional intersection } \Sigma(s, \theta) \cap \Xi(e_1, e_2), & \text{where} \\ \text{the orientation of } \alpha(\theta) \text{ is chosen such that } \det(\alpha(\theta), \theta, e_3, \dots, e_d) > 0, & \text{if } \theta \notin \Theta(e_1, e_2), \\ \text{any vector } e \in \mathbb{S}^{d-1} \cap \Xi(e_1, e_2), & \text{if } \theta \in \Theta(e_1, e_2), \end{cases} \quad (2.16)$$

$$(\alpha(\theta), \beta_1(\theta), \dots, \beta_{d-2}(\theta)) \text{ is an orthonormal basis on } \Sigma(s, \theta), \quad (2.17)$$

and $\Sigma(s, \theta)$, $\Theta(e_1, e_2)$ are given by (2.10), (2.12), respectively. Here, due to the condition $\theta \notin \Theta(e_1, e_2)$:

$$\dim(\Sigma(s, \theta) \cap \Xi(e_1, e_2)) = 1. \quad (2.18)$$

Formula (2.18) is proved in Section 5.

Note that formulas (2.14)-(2.18) are also valid for $d = 3$. In this case these formulas are reduced to (2.7)-(2.9), where $e_3 = -\eta$.

Note that, formula (2.14) gives an expression for $R_W f$ on $\mathbb{R} \times \mathbb{S}^{d-1}$ in terms of $P_w f$ restricted to the rays $\gamma = (x, \alpha)$, such that $\alpha \in \mathbb{S}^{d-1} \cap \Xi(e_1, e_2)$.

Remark 1. In (2.16) one can also write:

$$\alpha(\theta) = (-1)^{d-1} \star (\theta \wedge e_3 \wedge \cdots \wedge e_d), \text{ if } \theta \notin \Theta(e_1, e_2), \quad (2.19)$$

where \star -denotes the Hodge star, \wedge - is the exterior product in $\Lambda^* \mathbb{R}^d$ (exterior algebra on \mathbb{R}^d); see, for example, Chapters 2.1.c, 4.1.c of [Mor01].

Note that the value of the integral in the right hand-side of (2.14) does not depend on the particular choice of $(\beta_1(\theta), \dots, \beta_{d-2}(\theta))$ of (2.17).

Note also that, due to (2.8), (2.9), (2.15), (2.16), the weight W is defined everywhere on $\mathbb{R}^d \times \mathbb{S}^{d-1}$, $d \geq 3$. In addition, this W has the same smoothness as w in x on \mathbb{R}^d and in θ on $\mathbb{S}^{d-1} \setminus \Theta(e_1, e_2)$, where $\Theta(e_1, e_2)$ is defined in (2.12) and has zero Lebesgue measure on \mathbb{S}^{d-1} .

3 Boman's example

For $d = 2$, in [Bom93] there were constructed a weight W and a function f , such that:

$$R_W f \equiv 0 \text{ on } \mathbb{R} \times \mathbb{S}^1, \quad (3.1)$$

$$1/2 \leq W \leq 1, W \in C^\infty(\mathbb{R}^2 \times \mathbb{S}^1), \quad (3.2)$$

$$f \in C_0^\infty(\mathbb{R}^2), f \not\equiv 0, \text{ supp } f \subset \overline{B^2} = \{x \in \mathbb{R}^2 : |x| \leq 1\}. \quad (3.3)$$

In addition, as a corollary of (2.5), (2.6), (3.1)-(3.3), we have that

$$P_{w_0} f_0 \equiv 0 \text{ on } T\mathbb{S}^1, \quad (3.4)$$

$$1/2 \leq w_0 \leq 1, w_0 \in C^\infty(\mathbb{R}^2 \times \mathbb{S}^1), \quad (3.5)$$

$$f_0 \in C_0^\infty(\mathbb{R}^2), f_0 \not\equiv 0, \text{ supp } f_0 \subset \overline{B^2} = \{x \in \mathbb{R}^2 : |x| \leq 1\}, \quad (3.6)$$

where

$$w_0(x, \theta) = W(x, -\theta^\perp), x \in \mathbb{R}^2, \theta \in \mathbb{S}^1, \quad (3.7)$$

$$f_0 \equiv f. \quad (3.8)$$

4 Main results

Let

$$B^d = \{x \in \mathbb{R}^d : |x| < 1\}, \quad (4.1)$$

$$\overline{B^d} = \{x \in \mathbb{R}^d : |x| \leq 1\}, \quad (4.2)$$

$$(e_1, \dots, e_d) \text{ - be the canonical basis in } \mathbb{R}^d. \quad (4.3)$$

Theorem 1. *There are W and f , such that*

$$R_W f \equiv 0 \text{ on } \mathbb{R} \times \mathbb{S}^{d-1}, \quad (4.4)$$

$$W \text{ satisfies (1.2), } f \in C_0^\infty(\mathbb{R}^d), f \not\equiv 0, \quad (4.5)$$

where R_W is defined by (1.1). In addition,

$$1/2 \leq W \leq 1, W \text{ is } C^\infty\text{-smooth on } \mathbb{R}^d \times (\mathbb{S}^{d-1} \setminus \Theta(e_1, e_2)), \quad (4.6)$$

where $\Theta(e_1, e_2)$ is defined by (2.12). Moreover, weight W and function f are given by formulas (2.15), (4.8)-(4.10) in terms of the J . Boman's weight w_0 and function f_0 of (3.7), (3.8).

Remark 2. According to (2.15), (2.16), $W(x, \theta)$ for $\theta \in \Theta(e_1, e_2)$ can be specified as follows:

$$W(x, \theta) = W(x_1, \dots, x_d, \theta) \stackrel{def}{=} w_0(x_1, x_2, e_1), \theta \in \Theta(e_1, e_2), x \in \mathbb{R}^d. \quad (4.7)$$

Proof of Theorem 1. We define

$$w(x, \alpha) = w(x_1, \dots, x_d, \alpha) \stackrel{def}{=} w_0(x_1, x_2, \alpha_1, \alpha_2), \quad (4.8)$$

$$f(x) = f(x_1, \dots, x_d) \stackrel{def}{=} \psi(x_3, \dots, x_d) f_0(x_1, x_2), \quad (4.9)$$

for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\alpha = (\alpha_1, \alpha_2, 0, \dots, 0) \in \mathbb{S}^{d-1} \cap \Xi(e_1, e_2) \simeq \mathbb{S}^1$,

where

$$\psi \in C_0^\infty(\mathbb{R}^{d-2}), \text{ supp } \psi = \overline{B^{d-2}} \text{ and } \psi(x) > 0 \text{ for } x \in B^{d-2}. \quad (4.10)$$

From (2.1), (3.4), (4.8)-(4.10) it follows that:

$$\begin{aligned} P_w f(x, \alpha) &= \int_{\mathbb{R}} w(x_1 + t\alpha_1, x_2 + t\alpha_2, x_3, \dots, x_d, \alpha) f(x_1 + t\alpha_1, x_2 + t\alpha_2, x_3, \dots, x_d) dt \\ &= \psi(x_3, \dots, x_d) \int_{\mathbb{R}} w_0(x_1 + t\alpha_1, x_2 + t\alpha_2, \alpha_1, \alpha_2) f_0(x_1 + t\alpha_1, x_2 + t\alpha_2) dt \\ &= \psi(x_3, \dots, x_d) P_{w_0} f_0(x_1, x_2, \alpha_1, \alpha_2) = 0 \text{ for any } \alpha = (\alpha_1, \alpha_2, 0, \dots, 0) \in \Xi(e_1, e_2) \cap \mathbb{S}^{d-1} \simeq \mathbb{S}^1. \end{aligned} \quad (4.11)$$

Properties (4.4)-(4.6) follow from (2.15)-(2.17), (2.19), (3.2), (3.3), (4.7), (4.8).

Theorem 1 is proved. \square

5 Proof of formula (2.18)

Note that

$$\dim(\Xi(e_1, e_2)) + \dim(\Sigma(s, \theta)) = d + 1 > d, \quad (5.1)$$

which implies that the intersection $\Sigma(s, \theta) \cap \Xi(e_1, e_2)$ is one of the following:

1. The intersection is the one dimensional line $l = l(s, \theta)$:

$$l(s, \theta) = \{x \in \mathbb{R}^d : x = x_0(s, \theta) + \alpha(\theta)t, t \in \mathbb{R}\}, \alpha(\theta) \in \mathbb{S}^2, \quad (5.2)$$

where $x_0(s, \theta)$ is an arbitrary point of $\Sigma(s, \theta) \cap \Xi(e_1, e_2)$, the orientation of $\alpha(\theta)$ is chosen such that:

$$\det(\alpha(\theta), \theta, e_3, \dots, e_d) > 0. \quad (5.3)$$

Condition (5.3) fixes uniquely the direction of $\alpha(\theta)$ of (5.2).

Formulas (2.10), (2.11), (2.12) imply that (5.3) can hold if and only if $\theta \notin \Theta(e_1, e_2)$.

2. The intersection is the two-dimensional plane $\Xi(e_1, e_2)$. Formulas (2.10), (2.11) imply that it is the case if and only if

$$s = 0, \theta \perp e_1, \theta \perp e_2. \quad (5.4)$$

3. The intersection is an empty set. Formulas (2.10), (2.11) imply that it is the case if and only if

$$s \neq 0, \theta \perp e_1, \theta \perp e_2. \quad (5.5)$$

Note that

$$\text{cases 2 and 3 occur if and only if } \theta \perp e_1, \theta \perp e_2, \text{ i.e., } \theta \in \Theta(e_1, e_2). \quad (5.6)$$

This completes the proof of formula (2.18).

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