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## PHASE SEPARATION FOR THE LONG RANGE ONE-DIMENSIONAL ISING MODEL \*

Marzio Cassandro <sup>1</sup>, Immacolata Merola <sup>2</sup>, Pierre Picco <sup>3</sup>

Dedicated to the memory of Enza Orlandi

**Abstract** We consider the phase separation problem for the one-dimensional ferromagnetic Ising model with long-range two-body interaction,  $J(n) = n^{-2+\alpha}$  where  $n \in \mathbb{N}$  denotes the distance of the two spins and  $\alpha \in ]0, \alpha_+[$  with  $\alpha_+ = (\log 3)/(\log 2) - 1$ . We prove that given  $m \in ]-1, +1[$ , if the temperature is small enough, then typical configuration for the  $\mu^+$  Gibbs measure conditionally to have an empirical magnetization of the order  $m$  are made of a single interval that occupies almost a proportion  $\frac{1}{2}(1 - \frac{m}{m_\beta})$  of the volume with the minus phase inside and the rest of the volume is the plus phase, here  $m_\beta > 0$  is the spontaneous magnetization.

## 1 Introduction and main results

We consider a one-dimensional ferromagnetic Ising model with a two body interaction  $J(n) = n^{-2+\alpha}$  where  $n$  denotes the distance of the two spins and  $\alpha$  tunes the decay of the interaction.

A systematic and successful analysis of these one dimensional models started more than forty years ago. [23,37,10,11,12] proved existence, uniqueness of the Gibbs states and analyticity in  $\beta$  of the free energy for  $\alpha < 0$  and [16,17,18] proved the occurrence of a phase transition for  $\alpha > 0$ .

The borderline case  $\alpha = 0$  was already distinguished by a number of unusual features in the early seventies [38, 18]. It took more than a decade to prove Dyson's conjecture [16] about the existence of a spontaneous magnetization at low temperature. This result was proved by Fröhlich & Spencer [22] by introducing a suitable notion of contours. Precise estimates on the low-temperature decay of the truncated correlations were given by Imbrie [26]; the existence of a Thouless effect [38], that is a discontinuity of the magnetization at the critical temperature was proved by Aizenman, Chayes, Chayes & Newman [1] and all these works culminate in the proof of the existence of an intermediate phase similar to a Kosterlitz-Thouless phase with a variable exponent power law decay for correlation functions given by Imbrie & Newman [27].

In one-dimensional systems it was proved that Fannes, Vanheuverzwijn & Verbeure [20] and Burkov & Sinai [4] that used an energy argument that comes from Bricmont, Lebowitz & Pfister [2] there are no non-translation invariant extremal Gibbs states. This argument was used in particular in [2] to prove uniqueness of Gibbs states for one-dimensional Ising systems under the Ruelle's condition [37].

In a more recent work, [5] revisited [22] to extend Peierls argument to the  $0 < \alpha < 1$  case. This already allowed to study the behaviour of these systems when an external stochastic field is added [8, 9] and also to

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study the localization of the interface when taking, say  $-$  boundary conditions on the left of an interval and  $+$  boundary on its right [7].

Another important fact currently observed in everyday life, therefore macroscopic, is the phase separation or phase segregation phenomena. To deduce it from Statistical Mechanics was first considered in a remarkable paper on the short range Ising model by Minlos & Sinai in 1967, see [31], we share with Pfister, see [33], that "many important ideas, which were later on developed in Statistical Mechanics were in germs in it". Then in 1988, Dobrushin, Kotecky & Shlosman, see [14], derived from statistical mechanics the phenomenological macroscopic theory of Wulff, that gives the shape of the spatial region occupied by one phase immersed in the other one. Later this was called the DKS theory. See also Pfister [33] for recent version of Minlos & Sinai work and of DKS theory, also Pfister & Velenik, see [34] for large deviations and continuum limit and [28, 29] for extensions of the DKS theory for all temperature below the critical one.

In this paper, we address the problem of phase separation or phase segregation where the empirical magnetization is fixed in the interval  $] -m_\beta, +m_\beta[$ , where  $m_\beta$  is the spontaneous magnetization. We assume that  $\beta$  is large enough to have  $m_\beta > 0$ , a sufficient condition on  $\beta$  was given in [5].

To be more precise, we consider in the finite interval  $\Lambda = [-L, +L] \cap \mathbb{Z}$ , the system of Ising spin configurations  $\underline{\sigma}_\Lambda = (\sigma_i, i \in \Lambda)$  described by the Hamiltonian with  $+$  boundary conditions

$$h^{++}(\sigma_\Lambda) = \frac{1}{2} \sum_{(i,j) \in \Lambda \times \Lambda} J(i-j) \frac{(1 - \sigma_i \sigma_j)}{2} + \sum_{i \in \Lambda} \sum_{j \in \Lambda^c} J(i-j) \frac{(1 - \sigma_i)}{2}. \quad (1.1)$$

where the pair interaction  $J(i-j)$  is defined by

$$J(n) = \begin{cases} 0 & \text{if } n = 0; \\ J+1 \gg 1 & \text{if } n = 1; \\ \frac{1}{|n|^{2-\alpha}} & \text{if } n \neq 1. \end{cases} \quad (1.2)$$

and  $\alpha \in [0, \alpha_+)$  with  $\alpha_+ = (\log 3)/(\log 2) - 1$ . In (1.2), the nearest neighbour interaction  $J+1$  is chosen large enough as in [5] appendix A, however new conditions will be imposed here. As proved in [5], or in [16] for such system, when  $\beta \geq \beta_0(\alpha)$  for some  $\beta_0(\alpha)$  that comes from an energy-entropy argument within a Peierls argument, there exist at least two different extremal Gibbs states  $\mu_\beta^+$  and  $\mu_\beta^-$  that are limit when  $|\Lambda| \uparrow \infty$  of the finite volume Gibbs measure with  $+$ , respectively  $-$  boundary conditions. Then the spontaneous magnetization is

$$m_\beta = \mu_\beta^+[\sigma_0] = -\mu_\beta^-[\sigma_0] > 0. \quad (1.3)$$

Let us take a  $\beta \geq \beta_1(\alpha) (\geq \beta_0(\alpha))$  as in [7] to have convergence of the cluster expansion then it follows from theorem 2.5 in [7] that

$$\mu_\beta^+[\sigma_0] = 1 - [2\xi^{++}(\beta) (1 + \mathcal{B}(x, ++))] \quad (1.4)$$

where

$$\xi^{++}(\beta) = e^{-2\beta(\zeta(2-\alpha)+J)} \quad (1.5)$$

with  $\zeta(2-\alpha)$  the Riemann zeta function, and  $\mathcal{B}(x, ++)$  is an absolutely convergent series that satisfies

$$|\mathcal{B}(x, ++)| \leq e^{-\frac{\beta}{32}(\frac{\zeta_\alpha}{\alpha(1-\alpha)} - 3\delta)} \quad (1.6)$$

where  $\delta$  is given in (6.25) and  $\zeta_\alpha = 1 - 2(2^\alpha - 1)$ .

Let

$$\epsilon_0 = |\Lambda|^{-a}, \quad a > 0 \quad (1.7)$$

and given  $m \in ]-1, +1[$ , let

$$\tau = \frac{1 - |m|}{2} \quad (1.8)$$

Let  $\beta^* = \beta^*(|m|)$  such that  $m_{\beta^*} = |m| + \tau = (1 + |m|)/2$ . Note that  $|m| + \tau < 1$  and therefore  $\beta^* < \infty$ . By GKS inequality we have : for all  $\beta > \beta^*$

$$m_\beta - |m| \geq m_{\beta^*} - |m| = \tau \quad (1.9)$$

we assume that  $|\Lambda|$  is large enough to have

$$\tau > \epsilon_0 \quad (1.10)$$

therefore we get

$$\frac{m_\beta - |m|}{m_\beta} \geq \frac{\tau}{m_\beta} > \frac{\epsilon_0}{m_\beta} > \epsilon_0. \quad (1.11)$$

Let

$$m_\Lambda(\underline{\sigma}_\Lambda) = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \sigma_i \quad (1.12)$$

be the empirical magnetisation. We consider the system under the constraint that

$$|m_\Lambda(\underline{\sigma}_\Lambda) - m| \leq \epsilon_0 m_\beta. \quad (1.13)$$

Note that  $m_\Lambda(\underline{\sigma}_\Lambda) \in ]-m_\beta, +m_\beta[$ , in fact it follows from (1.13) and (1.10) that for all  $\beta > \beta^*$ , we have

$$m_\beta - |m_\Lambda(\underline{\sigma}_\Lambda)| > \tau - \epsilon_0 m_\beta > \tau - \epsilon_0 > 0 \quad (1.14)$$

*i.e.*  $m_\Lambda(\underline{\sigma}_\Lambda)$  is well separated from  $m_\beta$  and  $-m_\beta$  as it should be.

Since the interaction is ferromagnetic, under the constraint (1.13) with  $\epsilon_0 = 0$  and  $m$  is a rational number of the form  $k/|\Lambda|$  for some positive odd integer  $k \leq |\Lambda|$ , say  $k = 2q + 1$ , a minimum of (1.1) is reached by a configuration made of a single run of  $-1$  of length  $L - q$ . In other word a ground state contains a single interval of  $-1$  with the correct length to satisfies (1.13) with  $\epsilon_0 = 0$ . However it can be located anywhere in  $\Lambda$ , therefore the ground state is  $L + 1 + q$  times degenerated. The main problem is therefore to understand what remains of this  $\beta = \infty$  picture for the configurations that are typical for the Gibbs measure  $\mu_\beta^+$  conditioned by (1.13) when we take  $\beta$  large enough but finite.

Roughly speaking, we show that for  $|\Lambda|$  very large, the configurations that are typical for the finite volume conditional  $\mu_\Lambda^{++}$  measure, given (1.13), are as follows:

**1** There exists in  $\Lambda$  an interval  $\Lambda'$  with  $|\Lambda'| \approx (|\Lambda|)^{\frac{1}{2}}(1 - \frac{m}{m_\beta})$  where

$$\frac{1}{|\Lambda'|} \sum_{i \in \Lambda'} \sigma_i \approx -m_\beta; \quad (1.15)$$

while

$$\frac{1}{|\Lambda \setminus \Lambda'|} \sum_{i \in \Lambda \setminus \Lambda'} \sigma_i \approx m_\beta. \quad (1.16)$$

**2** The statistics of the spin configurations, in the limit  $|\Lambda| \uparrow \infty$ , in the interval  $\Lambda'$  are similar to the one of the  $-$  phase, while in the region  $\Lambda \setminus \Lambda'$  they are similar the one of the  $+$  phase.

In this paper we deal only with the case  $\alpha > 0$ . For  $\alpha = 0$  the argument goes along the same lines but now , since many of the bounds we use are no more exponential, the proofs require substantial modifications.

These proofs will be presented in our next paper where we study the fluctuations of the interval where , with plus boundary conditions, the phase is negative. In this case the value of alfa should play a relevant role. In fact from our previous work [7] , where we study the separation point when the right and left boundary conditions are different (+, -) we have: 1) for  $\alpha > 0$  the density distribution of the fluctuations of the transition point ,suitably normalized, converges to a Gaussian , 2) for  $\alpha = 0$ ,rescaling the volume  $[-L, L]$  to  $[-1, 1]$ , the distribution converges to a non degenerate distribution with an explicit density in the interval  $[-1, 1]$ .

In section 2, we define the model, state the main theorem and two propositions that will imply the theorem.

In section 3, we state and prove two lemmata that will be extendedly used.

In section 4 and 5 we give the proofs of the two propositions of section 2.

All the proofs in these sections are based on the geometrical description of spin configurations introduced in [5] and use the cluster expansion developed in [7]. The appendix contains a resumé of the results obtained in [5], [7].

## 2 Definitions and main result

Let  $\Lambda = [-L, +L] \cap \mathbb{Z}$  and  $\mathcal{S}_\Lambda = \{-1, +1\}^\Lambda$  be the set of spin configurations in  $\Lambda$ . We denote by  $\underline{\sigma}_\Lambda \equiv (\sigma_i, i \in \Lambda) \in \mathcal{S}_\Lambda$  a configuration. For any subset  $A \subset \Lambda$ , we denote by  $\underline{\sigma}_A \equiv (\sigma_i, i \in A)$  the restriction of the configuration  $\underline{\sigma}_\Lambda$  to the subset  $A$ . For  $f$  a cylindrical bounded function with cylinder basis that is a subset of  $\Lambda$ , the finite volume Gibbs measure with + boundary conditions is given by

$$\mu_\Lambda^+[f] = \frac{\sum_{\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda} f(\underline{\sigma}_\Lambda) e^{-\beta h^{++}(\underline{\sigma}_\Lambda)}}{Z_\Lambda^{++}(\beta)} \quad (2.1)$$

The infinite volume limit,  $\lim_{\Lambda \uparrow \mathbb{Z}} \mu_\Lambda^+(f)$  exists by FKG inequalities, see [19], [21] or [32] and is translation invariant as all extremal Gibbs state are by [4] and [20].

**Definition 2.1** For  $\epsilon_0 = |\Lambda|^{-a}$ , with  $0 < a < 1$  to be chosen later, assuming that  $|\Lambda|$  is large enough to have (1.10), and for  $m \in [-1, +1]$ , let

$$\mathcal{S}_\Lambda(m, \epsilon_0) = \{\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda : |m_\Lambda(\underline{\sigma}_\Lambda) - m| < \epsilon_0 m_\beta\}. \quad (2.2)$$

Since the boundary condition is fixed, there is a bijection between spin configurations and spin-flip configurations, a spin-flip being a pair of consecutive sites  $(i, i + 1)$  with  $\sigma_i \sigma_{i+1} = -1$ . Triangles are a graphical representation of pairing together spin-flips, say  $(i, i + 1)$  and  $(j, j + 1)$  where  $i < j$ , with the property that  $\sigma_{i+1} \sigma_j = +1$ . It is obtained by an algorithm described in [5], see also section 6. In particular two triangles are either disjoint or one inside the other.

The mass of a triangle  $T$  will be denoted by  $|T|$  and is just the number of sites of  $\mathbb{Z}$  in the base of the triangle, i.e if  $T$  is associated to the two spin-flips  $(i, i + 1)$  and  $(j, j + 1)$  with  $i < j$  then  $|T| = j - i$ .

We say that a family of triangles  $\underline{T}$  is compatible, if there exists a spin configuration  $\underline{\sigma}_\Lambda$  such that  $\underline{T} = \underline{T}(\underline{\sigma}_\Lambda)$ , this spin configuration will be denoted by  $\underline{\sigma}_\Lambda(\underline{T})$ . The set of compatible configurations of triangles will be denoted by  $\mathcal{T}_\Lambda$ .

**Definition 2.2** For  $\epsilon_s = |\Lambda|^{-\gamma}$  with  $0 < \gamma < 1$  to be chosen later, let

$$\mathcal{T}_\Lambda^{small}(\epsilon_s) = \{\underline{T} \in \mathcal{T}_\Lambda : \forall T \in \underline{T}, |T| \leq \epsilon_s |\Lambda|\} \quad (2.3)$$

be the set of compatible configurations of small triangles and

$$\mathcal{S}_\Lambda^{small}(\epsilon_s) = \{\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda : \underline{T}(\underline{\sigma}_\Lambda) \in \mathcal{T}_\Lambda^{small}(\epsilon_s)\} \quad (2.4)$$

be the set of spin configurations such that the associated family of triangles is made of small triangles.

**Remark 2.3 .** It follows from [7] that for  $\epsilon_s = |\Lambda|^{-\gamma}$ , there exists a  $\beta_0(\alpha)$  such that for all  $\beta \geq \beta_0(\alpha)$ , the typical configurations of triangles for the measure  $\mu_\Lambda^+$  are within  $\mathcal{S}_\Lambda^{small}(\epsilon_s)$ , in the sense that  $\lim_{\Lambda \uparrow \mathbb{Z}} \mu_\Lambda^+[\mathcal{S}_\Lambda^{small}(\epsilon_s)] = 1$ . The set  $\mathcal{S}_\Lambda^{small}(\epsilon_s)$  plays the rôle of the phase of small contours as in [33].

**Definition 2.4** For a given family of compatible triangles  $\underline{T} \in \mathcal{T}_\Lambda$ , we say that a triangle  $T \in \underline{T}$  is external with respect to  $\underline{T}$  or more simply, external, if it is not contained in any other triangle of  $\underline{T}$ . We say that a family of triangles  $\underline{T}$  is made of mutually external triangles if each triangle  $T \in \underline{T}$  is external with respect to any other triangle of  $\underline{T}$ .

**Definition 2.5** Given a configuration  $\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda$ , we denote by  $\underline{T}^E(\underline{\sigma}_\Lambda)$  the subfamily of all triangles in  $\underline{T}(\underline{\sigma}_\Lambda)$  that are external with respect to  $\underline{T}(\underline{\sigma}_\Lambda)$  and not small. i.e.

$$\underline{T}^E(\underline{\sigma}_\Lambda) = \{T \in \underline{T}(\underline{\sigma}_\Lambda) : T \text{ is external, } |T| > \epsilon_s |\Lambda|\} \quad (2.5)$$

On the other hand, given a family  $\underline{T}^E$  of mutually external triangles that satisfies  $\forall T \in \underline{T}^E, |T| > \epsilon_s |\Lambda|$  we say that it is a compatible family if there exists a configuration  $\underline{\sigma}_\Lambda$  such that  $\underline{T}^E = \underline{T}^E(\underline{\sigma}_\Lambda)$ .

**Definition 2.6** Given  $\epsilon_s > 0$  let

$$\mathcal{T}_\Lambda^E = \left\{ \underline{T} \in \mathcal{T}_\Lambda : \underline{T} \text{ are mutually external, } \forall \tilde{T} \in \underline{T}, |\tilde{T}| > \epsilon_s |\Lambda| \right\}. \quad (2.6)$$

If  $\underline{T}^E \in \mathcal{T}_\Lambda^E$  we denote

$$|\underline{T}^E| = \sum_{T \in \underline{T}^E} |T|. \quad (2.7)$$

Given  $\underline{T}^E \in \mathcal{T}_\Lambda^E$ , there is a specific spin configuration, say  $\bar{\sigma}(\underline{T}^E)$ , defined by

$$\bar{\sigma}_i(\underline{T}^E) = -\mathbb{I}_{\{i \in \Delta(\underline{T}^E)\}} + \mathbb{I}_{\{i \in \Lambda \setminus \Delta(\underline{T}^E)\}} \quad (2.8)$$

where  $\Delta(\underline{T}^E)$  is just the union of the bases over all the triangles, large and external, that define  $\underline{T}^E$ .

Let us define an equivalence relation on  $\mathcal{S}_\Lambda$  by  $\underline{\sigma}_\Lambda \sim \underline{\sigma}'_\Lambda$  if and only if  $\underline{T}^E(\underline{\sigma}_\Lambda) = \underline{T}^E(\underline{\sigma}'_\Lambda)$ .

**Definition 2.7** Given a  $\underline{T}^E \in \mathcal{T}_\Lambda^E$ , let  $S_{\underline{T}^E}$  be the equivalent class of spin configurations corresponding to the representative  $\underline{T}^E$ :

$$S_{\underline{T}^E} \equiv \{\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda : \underline{T}^E(\underline{\sigma}_\Lambda) = \underline{T}^E\}. \quad (2.9)$$

Then we have  $S_{\underline{T}_1^E} \neq S_{\underline{T}_2^E}$  if  $\underline{T}_1^E \neq \underline{T}_2^E$  and therefore, recalling (2.4) the partition

$$\mathcal{S}_\Lambda = \mathcal{S}_\Lambda^{small}(\epsilon_s) \bigcup_{\underline{T}^E \in \mathcal{T}_\Lambda^E} S_{\underline{T}^E}. \quad (2.10)$$

Using (2.8), the point is that  $\forall \underline{\sigma}_\Lambda \in S_{\underline{T}^E}$  with  $\underline{\sigma}_\Lambda \neq \bar{\sigma}(\underline{T}^E)$  we have

$$h^{++}(\underline{\sigma}_\Lambda) > h^{++}(\bar{\sigma}(\underline{T}^E)). \quad (2.11)$$

Notice that given  $\underline{\sigma}_\Lambda \in S_{\underline{T}^E}$ , all triangles  $\tilde{T} \in \underline{T}(\underline{\sigma}_\Lambda) \setminus \underline{T}^E(\underline{\sigma}_\Lambda)$  describe fluctuations with respect to the fundamental state  $\underline{\sigma}(\underline{T}^E)$ .

**Definition 2.8** Given  $0 < \rho \leq 1$ , for  $\epsilon_s = |\Lambda|^{-\gamma}$  with  $0 < \gamma < 1$ , and assume that  $|\Lambda|$  is large enough to have  $\rho \geq \epsilon_s$ , let

$$\mathcal{T}_\Lambda^E(\rho) = \{\underline{T}^E \in \mathcal{T}_\Lambda^E : |\underline{T}^E| = \rho|\Lambda|\} \quad (2.12)$$

and

$$\mathcal{S}_\Lambda^1(\rho) = \{\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda : \underline{T}^E(\underline{\sigma}_\Lambda) \in \mathcal{T}_\Lambda^E(\rho)\}. \quad (2.13)$$

**Remark 2.9** . In this article the letter  $\mathcal{T}$  always denotes a set of compatible configuration of triangles as for  $\mathcal{T}_\Lambda^E(\rho)$  in (2.12) while the letter  $\mathcal{S}$  always denotes a set of spin configurations as for  $\mathcal{S}_\Lambda^1(\rho)$  in (2.13).

Note that  $\mathcal{S}_\Lambda^1(\rho)$  depends on  $\epsilon_s$  since  $\epsilon_s$  appears in the definition of  $\mathcal{T}^E$ , however for simplicity we do not write this dependence. Moreover if  $0 < \rho < 1$  is not a rational number that can be written as  $k/|\Lambda|$  for some integer  $k$ , the set  $\mathcal{S}_\Lambda^1(\rho)$  is empty. For future use let us denote

$$Q_\Lambda = \{\rho \in [0, 1] : \exists k \in \{0, \dots, |\Lambda|\}, \rho = k/|\Lambda|\}. \quad (2.14)$$

**Lemma 2.10** Given  $0 < \rho \leq 1$ , and  $\epsilon_s = |\Lambda|^{-\gamma}$ , the number of configurations of external triangles in  $\mathcal{T}_\Lambda^E(\rho)$ , say  $\#\mathcal{T}_\Lambda^E(\rho)$ , satisfies

$$\#\mathcal{T}_\Lambda^E(\rho) \leq e^{(2-\gamma)|\Lambda|^\gamma \log |\Lambda|}. \quad (2.15)$$

The proof is done in section 6.

For  $\epsilon_c = |\Lambda|^{-\nu}$ ,  $0 < \nu < 1$ , we define also

$$\mathcal{S}_\Lambda^1(\rho, \epsilon_c) = \bigcup_{\rho - \epsilon_c \leq \rho' \leq \rho + \epsilon_c} \mathcal{S}_\Lambda^1(\rho'). \quad (2.16)$$

Note that the previous union is merely over the set  $\{(\rho - \epsilon_c) \vee \epsilon_s \leq \rho' \leq (\rho + \epsilon_c) \wedge 1\} \cap Q_\Lambda$  whose cardinality is less than  $|\Lambda|$ .

**Definition 2.11** For all  $\rho \in (0, 1) \cap Q_\Lambda$  and for  $\epsilon_c = |\Lambda|^{-\nu}$  and  $|\Lambda|$  large enough to have  $8\epsilon_c < \rho$ , for all  $\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho)$  let

$$n_0[\underline{T}^E] = \sum_{T^E \in \underline{T}^E} \mathbb{1}_{\{|T^E| \geq |\underline{T}^E| - 6\epsilon_c|\Lambda|\}} \quad (2.17)$$

be the number of triangles in  $\underline{T}^E$  with mass larger than  $|\underline{T}^E| - 6\epsilon_c|\Lambda|$ , a number larger than  $2\epsilon_c|\Lambda|$ . Moreover let

$$\mathcal{T}_\Lambda^B(\rho) = \{\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho) : n_0[\underline{T}^E] = 1\} \quad (2.18)$$

and

$$\mathcal{S}_\Lambda^B(\rho) = \{\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda^1(\rho) : \underline{T}^E(\underline{\sigma}_\Lambda) \in \mathcal{T}_\Lambda^B(\rho)\} \quad (2.19)$$

be the set of spin configurations that give rise to a family of external triangles of total mass  $|\underline{T}^E| = \rho|\Lambda|$  that contains a single external triangle, say  $T_1$ , that satisfies  $|T_1| \geq (\rho - 6\epsilon_c)|\Lambda|$ .

**Remark 2.12** . Notice that the total mass of the other triangles of  $\underline{T}^E$ , that are mutually external and large by definition, fits the rest of the volume i.e.

$$\sum_{T \in \underline{T}^E, T \neq T_1} |T| = \rho|\Lambda| - |T_1| \leq 6\epsilon_c|\Lambda|. \quad (2.20)$$

In a way similar to (2.16) and with the same convention, we define

$$\mathcal{S}_\Lambda^B(\rho, \epsilon_c) = \bigcup_{\rho - \epsilon_c \leq \rho' \leq \rho + \epsilon_c} \mathcal{S}_\Lambda^B(\rho'). \quad (2.21)$$

### Choice of the parameters and some rounding conditions

Given  $m \in ]-1, 1[$ , and assume as in the introduction that  $\beta > \beta^*$  to have (1.11), let us define

$$\hat{\rho}(m) = \frac{1}{2} \left( 1 - \frac{m}{m_\beta} \right). \quad (2.22)$$

The point is that we have the following

**Lemma 2.13** *If  $\beta > \beta_1(\alpha)$ , for all  $\tilde{m} \in ]-1, 1[$ , for all  $\underline{\mathcal{I}}^E \in \mathcal{T}_\Lambda^E$  such that  $|\underline{\mathcal{I}}^E| = \hat{\rho}(m)|\Lambda|$  then*

$$\mu_\Lambda[m_\Lambda(\underline{\sigma}_\Lambda)|\mathcal{S}_{\underline{\mathcal{I}}^E}] = m \pm \frac{10\xi^{++}(\beta)}{\alpha(1-\alpha)} \frac{1}{|\Lambda|^{1-\alpha}} \quad (2.23)$$

where  $\xi^{++}(\beta)$  is defined in (1.5). On the other hand, for all  $\rho \in ]0, 1[$ , for all all  $\underline{\mathcal{I}}^E \in \mathcal{T}_\Lambda^E$  such that  $|\underline{\mathcal{I}}^E| = \rho|\Lambda|$  then

$$\mu_\Lambda[m_\Lambda(\underline{\sigma}_\Lambda)|\mathcal{S}_{\underline{\mathcal{I}}^E}] = (1-2\rho)m_\beta \pm \frac{10\xi^{++}(\beta)}{\alpha(1-\alpha)} \frac{1}{|\Lambda|^{1-\alpha}} \quad (2.24)$$

The proof which is a consequence of the cluster expansion will be done in section 6.

Note that it follows from (1.11) that

$$\frac{\tau}{2m_\beta} \leq \hat{\rho}(m) \leq 1 - \frac{\tau}{2m_\beta}. \quad (2.25)$$

To avoid rounding problems, let us define

$$\rho(m) \equiv \rho_\Lambda(m) = \sup\{\rho \in Q_\Lambda : \rho \leq \hat{\rho}(m)\} \quad (2.26)$$

where  $Q_\Lambda$  is defined in (2.14). Notice that  $\rho(m) \leq \hat{\rho}(m)$  and  $\hat{\rho}(m) - \rho(m) \leq |\Lambda|^{-1}$ .

Now we collect conditions on the parameters introduced above. We always assume that

$$\epsilon_s = |\Lambda|^{-\gamma}, \quad \epsilon_0 = |\Lambda|^{-a}, \quad \text{and} \quad \epsilon_c = |\Lambda|^{-\nu}. \quad (2.27)$$

with

$$0 < \gamma < \min\{\alpha - \nu, 2/3\}; \quad \frac{\gamma + \nu\alpha}{1 - \alpha} \leq (1 - \nu)\alpha; \quad \nu < a \quad \text{and} \quad \nu < \gamma(1 - \alpha). \quad (2.28)$$

**Remark 2.14 .** *The first condition  $0 < \gamma < \alpha - \nu$  comes from (5.25), the condition  $0 < \gamma < 2/3$  is stated before (5.44). The condition  $\frac{\gamma + \nu\alpha}{1 - \alpha} \leq (1 - \nu)\alpha$  comes from (4.34), to be able to find an  $\eta$  in between  $\frac{\gamma + \nu\alpha}{1 - \alpha}$  and  $(1 - \nu)\alpha$ ,  $\nu < a$  is for (2.31) where  $2\epsilon_c - \epsilon_0$  is present and the last  $\nu < \gamma(1 - \alpha)$  condition is (5.32) i.e.  $\alpha + \gamma(1 - \alpha) - \nu > \alpha$ ,*

**Remark 2.15 .** *It is easy to check that a possible choice is*

$$\nu = \frac{\alpha(1 - \alpha)}{4}; \quad \gamma = \frac{\alpha}{4}; \quad a = \frac{\alpha(1 - \alpha)}{2}. \quad (2.29)$$

The following theorem shows how the phenomenon of phase separation holds for long range Ising model in one dimension at low temperature. It is the analogue of the Minlos & Sinai theorem and its extension by Dobrushin, Kotecky & Shlosman that hold for the two dimensional short range Ising model.

**Theorem 2.16** *For all  $0 < \alpha < \alpha_+$ , for  $m \in ]-1, 1[$ , for  $(\epsilon_0, \epsilon_s, \epsilon_c)$  that satisfy (2.27) and (2.28) there exists a  $\beta_3(\alpha, m)$  such that for all  $\beta \geq \beta_3(\alpha, m)$  we have*

$$\lim_{\Lambda \uparrow \mathbb{Z}} \mu_\Lambda^+ [\mathcal{S}_\Lambda^B(\rho(m), \epsilon_c)|\mathcal{S}_\Lambda(m, \epsilon_0)] = 1. \quad (2.30)$$



The proof of the theorem 2.16 is a direct consequence of the two following propositions that give more precise estimates for the involved probabilities.

**Proposition 2.17** *For all  $0 < \alpha < \alpha_+$ , for  $m \in ]-1, 1[$ , for  $(\epsilon_0, \epsilon_s, \epsilon_c)$  that satisfy (2.27) and (2.28) there exists a  $\beta_4(\alpha, m)$  such that for all  $\beta \geq \beta_4(\alpha, m)$ , if  $|\Lambda|$  is large enough, we have*

$$\mu_\Lambda^+ [\mathcal{S}_\Lambda^1(\rho(m), \epsilon_c) | S_\Lambda(m, \epsilon_0)] \geq 1 - \left[ e^{-\frac{\zeta_\alpha}{16} \beta (2\epsilon_c - \epsilon_0) m \beta |\Lambda|^{\alpha + \gamma(1-\alpha)}} + e^{-\frac{\beta(\rho(m)|\Lambda|)^\alpha}{2\alpha(1-\alpha)} \left\{ \left(1 + \frac{\epsilon_c}{\rho(m)}\right)^\alpha - 1 \right\}} \right] \quad (2.31)$$

where  $\zeta_\alpha = 1 - 2(2^\alpha - 1)$ .

**Proposition 2.18** *For all  $0 < \alpha < \tilde{\alpha}_+$ , for  $m \in ]-1, 1[$ , for  $(\epsilon_0, \epsilon_s, \epsilon_c)$  that satisfy (2.27) and (2.28) there exists a  $\beta_5(\alpha, m)$  such that for all  $\beta \geq \beta_5(\alpha, m)$ , if  $|\Lambda|$  is large enough, we have*

$$\mu_\Lambda^+ [\mathcal{S}_\Lambda^1(\rho(m), \epsilon_c) | S_\Lambda(m, \epsilon_0)] = \mu_\Lambda^+ [\mathcal{S}_\Lambda^B(\rho(m), \epsilon_c) | S_\Lambda(m, \epsilon_0)] \times \left[ 1 \pm \frac{e^{-\frac{\beta}{\alpha(1-\alpha)} \frac{\zeta_\alpha}{4} (\epsilon_c |\Lambda|)^\alpha}}{1 - 2e^{-\frac{\beta}{\alpha(1-\alpha)} \frac{\epsilon_0}{\rho(m)} c(\alpha) \{\rho(m)|\Lambda|\}^\alpha}} \right] \quad (2.32)$$

where  $c(\alpha)$  is given in (4.11).

The proof of proposition 2.18 will be done in section 4, the one of proposition 2.17 in section 5.

### 3 Preparatory lemmata

In this section we first give an estimate from below of the energy cost to fragmentize a large triangle. Then we give an estimate for the Laplace transform of the probability distribution of the empirical magnetization for the Gibbs measure conditioned to some specific subsets of configurations.

Since the system is ferromagnetic and the strength of the interaction between two spins decays with their distance, conditionally on  $S_\Lambda(m, \epsilon_0)$ , one can expect that at low temperature the system prefers to form a single interval of consecutive minuses (therefore a triangle) of size of order  $\rho(m)|\Lambda|$  instead of various intervals whose sum of lengths will be of order  $\rho(m)|\Lambda|$ . We first show that it is true at the level of the energy.

So let us start with  $\{I_1, I_2, I_3, \dots, I_k\}$  a family of disjoint intervals in  $\Lambda$ , labelled in such a way that  $I_i$  is on the left of  $I_{i+1}$  for all  $i \in \{1, \dots, k-1\}$  and let  $\underline{\sigma}_\Lambda(I_1, \dots, I_k)$  be a spin configuration where  $\underline{\sigma}_\Lambda(I_1, \dots, I_k)$  is minus in each of the intervals  $I_j$  and plus on  $\Lambda \setminus \cup_{j=1}^k I_j$ , then we have

$$h^{++}(\underline{\sigma}_\Lambda(I_1, \dots, I_k)) = \sum_{i=1}^k h^{++}(\sigma_{I_i} \circ 1_{\Lambda \setminus I_i}) - 2 \sum_{1 \leq i < j \leq k} W(I_i, I_j) \quad (3.1)$$

where  $\sigma_{I_i} \circ 1_{\Lambda \setminus I_i}$  is the configuration which is minus on  $I_j$  and plus on  $\Lambda \setminus I_i$  and

$$W(I_i, I_j) = \sum_{\ell_1 \in I_i} \sum_{\ell_2 \in I_j} \frac{1}{|\ell_1 - \ell_2|^{2-\alpha}}. \quad (3.2)$$

Let  $d(i, j)$  be the inter-distance between  $I_i$  and  $I_j$ , since  $W(I_i, I_j)$  is a decreasing function of  $d(i, j)$  any transformation that decreases some  $d(i, j)$  and keeps constant the others decreases the energy (3.1). In particular if we set all the  $d(i, j)$  equal to zero, that is we merge all the intervals  $\{I_1, I_2, I_3, \dots, I_k\}$  in a single one, say  $I_{1, \dots, k}^*$  then we have

$$h^{++}(\underline{\sigma}_\Lambda(I_1, \dots, I_k)) - h^{++}(\underline{\sigma}_\Lambda(I_{1, \dots, k}^*)) > 0. \quad (3.3)$$

In the following Lemma we exploit the structure of triangles to get an explicit lower bound for this difference in some specific cases.

Given  $\rho \in Q_\Lambda$ , see (2.14) and  $\epsilon_c$  that satisfies (2.27) and (2.28) we can assume that  $|\Lambda|$  is large enough to have  $\rho \geq 8\epsilon_c$ , recalling (2.12), let us define

$$\mathcal{T}_\Lambda^B(\rho) \equiv \{\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho) : \sup_{T \in \underline{T}^E} |T| \geq (\rho - 6\epsilon_c)|\Lambda|\}. \quad (3.4)$$

Note that with this definition, recalling (2.19) and remark 2.12, we have  $\mathcal{S}_\Lambda^B(\rho) = \{\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda : \underline{T}^E(\underline{\sigma}_\Lambda) \in \mathcal{T}_\Lambda^B(\rho)\}$ .

We introduce a discretisation of an interval of size  $2\epsilon_c$  around a generic point  $\rho$ :

$$B(\rho, \epsilon_c) = [\rho - \epsilon_c, \rho + \epsilon_c] \cap Q_\Lambda, \quad (3.5)$$

in the special case where  $\rho = \rho(m)$  defined in (2.26), for simplicity we denote

$$B(m, \epsilon_c) = B(\rho(m), \epsilon_c). \quad (3.6)$$

**Lemma 3.1** *For all  $\rho \in B(m, \epsilon_c)$ , with  $\epsilon_c$  that satisfies (2.27) and (2.28), for all  $\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho) \setminus \mathcal{T}_\Lambda^B(\rho)$ , let  $T_0$  be an arbitrary triangle with  $|T_0| = \rho(m)|\Lambda|$ , recalling the definition of  $\bar{\sigma}(\underline{T}^E)$  in (2.8), we have*

$$h^{++}(\bar{\sigma}(\underline{T}^E)) - h^{++}(\bar{\sigma}(T_0)) \geq \frac{\zeta_\alpha}{2\alpha(1-\alpha)} (\epsilon_c |\Lambda|)^\alpha \quad (3.7)$$

where  $\zeta_\alpha = 1 - 2(2^\alpha - 1)$  which is strictly positive if  $\alpha < \alpha_+ = (\log 3)/(\log 2) - 1$ .

**Proof:**

Given an interval  $\mathcal{J} \subset \mathbb{Z}$ , and  $\underline{T}^E \in \mathcal{T}_\Lambda^E$  let

$$\mathcal{Q}(\mathcal{J}, \underline{T}^E) = |\mathcal{J} \cap \Delta(\underline{T}^E)| \quad (3.8)$$

be the number of points in  $\Delta(\underline{T}^E) = \cup_{T \in \underline{T}^E} \Delta(T)$  that belong to  $\mathcal{J}$ . Recall that  $\underline{T}^E$  is made of mutually external triangles, therefore the previous union is over disjoint triangles. Note that  $\mathcal{Q}(\mathcal{J}, \underline{T}^E) \leq |\mathcal{J}|$  and  $\mathcal{Q}(\mathcal{J}, \underline{T}^E)$  is additive with respect to  $\mathcal{J}$ , that is if  $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$  with  $\mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset$  we have  $\mathcal{Q}(\mathcal{J}, \underline{T}^E) = \mathcal{Q}(\mathcal{J}_1, \underline{T}^E) + \mathcal{Q}(\mathcal{J}_2, \underline{T}^E)$ .

Let  $T_1$  be a triangle in  $\underline{T}^E$  with the largest mass. Since  $\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho) \setminus \mathcal{T}_\Lambda^B(\rho)$ , we have  $|T_1| < (\rho - 6\epsilon_c)|\Lambda|$ .

Let us first assume that  $|T_1| \geq 2\epsilon_c|\Lambda|$ , the other case will be treated later. Let  $I^> = \{x \in \mathbb{Z} : x > \Delta(T_1)\}$  and  $I^< = \{x \in \mathbb{Z} : x < \Delta(T_1)\}$  be the semi-infinite intervals respectively on the right and on the left of  $T_1$ . Since  $\sum_{T \in \underline{T}^E} |T| = \rho|\Lambda|$  we have

$$\mathcal{Q}(I^>, \underline{T}^E) + \mathcal{Q}(I^<, \underline{T}^E) = \sum_{T \in \underline{T}^E} \mathbb{1}_{\{T \neq T_1\}} |T| = \rho|\Lambda| - |T_1| \geq 6\epsilon_c|\Lambda| \quad (3.9)$$

then either  $\mathcal{Q}(I^>, \underline{T}^E)$  or  $\mathcal{Q}(I^<, \underline{T}^E)$  should be larger than  $3\epsilon_c|\Lambda|$ , let us assume that it is  $\mathcal{Q}(I^>, \underline{T}^E)$ . Given a positive number  $\kappa < 3\epsilon_c$ , to be fixed later, let  $I_\kappa^> = \{x \in \mathbb{Z} : x - \kappa|\Lambda| > \Delta(T_1)\}$  be the semi-infinite interval at distance  $\kappa|\Lambda|$  of  $T_1$  and on its right. Let  $I_\kappa^+ = \{x \in \mathbb{Z} : x > \Delta(T_1), x - \kappa|\Lambda| < \Delta(T_1)\}$ , the interval of length  $\kappa|\Lambda|$  on the right of  $T_1$  and adjacent to it. Since  $\mathcal{Q}(I_\kappa^+, \underline{T}^E) \leq \kappa|\Lambda|$ , by additivity with respect to  $\mathcal{J}$  of  $\mathcal{Q}(\mathcal{J}, \underline{T}^E)$ , we have recalling that by hypothesis  $\mathcal{Q}(I^>, \underline{T}^E) \geq 3\epsilon_c|\Lambda|$ ,

$$\mathcal{Q}(I_\kappa^>, \underline{T}^E) \geq (3\epsilon_c - \kappa)|\Lambda|. \quad (3.10)$$

We define a new configuration  $\Theta(\bar{\sigma}(\underline{T}^E))$  as follows:

- 1) All the spins  $\sigma_i$  with  $i \in I_\kappa^+$  that are minus are changed in plus. This will erase all the triangles in  $\underline{T}^E$  that have basis in  $I_\kappa^+$ . If a triangle, say  $\tilde{T}_1$  in  $\underline{T}^E$  intersect both  $I_\kappa^+$  and  $I_\kappa^>$  then it becomes a smaller triangle with base  $\Delta(\tilde{T}) \cap I_\kappa^>$ .
- 2) We merge all the bases of triangles  $\tilde{T}$  such that  $\Delta(\tilde{T}) \subset I_\kappa^>$  into a single interval, say  $I_2^*$ , that we put at a distance exactly  $\kappa|\Lambda|$  from  $T_1$ , the length of this interval is

$$|I_2^*| = \mathcal{Q}(I^>, \underline{T}^E) \geq (3\epsilon_c - \kappa)|\Lambda| \quad (3.11)$$

and all the spin are minus there.

- 3) We merge all the bases of the triangles  $\hat{T}$  such that  $\Delta(\hat{T}) \subset I^<(T_1)$  with  $\Delta(T_1)$  to get an interval, say  $I_1^*$  where all the spins are minus. Using that  $|T_1| \geq 2\epsilon_c|\Lambda|$ , we have

$$|I_1^*| \geq 2\epsilon_c|\Lambda|. \quad (3.12)$$

We get a spin configuration,  $\Theta(\bar{\sigma}(\underline{T}^E))$  which is minus on  $I_1^* \cup I_2^*$  and plus everywhere else. We have  $\text{dist}(I_1^*, I_2^*) = \kappa|\Lambda|$  with (3.11) and (3.12). If we choose  $\kappa = \epsilon_c$  we get

$$\text{dist}(I_1^*, I_2^*) = \epsilon_c|\Lambda| \leq \min\{|I_1^*|, |I_2^*|\}. \quad (3.13)$$

Recalling Definition 6.1, the family of triangles  $\underline{T}(\Theta(\bar{\sigma}(\underline{T}^E)))$  cannot be made of two triangles with basis  $I_1^*$  and  $I_2^*$  as it follows comparing (3.13) and (6.5).

Therefore it should be made of a large triangle with basis the interval  $[x_-(I_1^*), x_+(I_2^*)]$ , say  $T_1^{**}$  with  $|T_1^{**}| = |I_1^*| + \epsilon_c|\Lambda| + |I_2^*|$ . We have  $|I_1^*| + |I_2^*| \geq (\rho - \epsilon_c)|\Lambda|$  because we erase less than  $\epsilon_c|\Lambda|$  minuses in the volume  $I_\kappa^+$  with  $\kappa = \epsilon_c$ . Therefore we get

$$|T_1^{**}| \geq \rho|\Lambda|. \quad (3.14)$$

Inside this triangle  $T_1^{**}$ , there is a triangle, say  $T_2^{**}$ , of size exactly  $\epsilon_c|\Lambda|$ . We have

$$h^{++}(\bar{\sigma}(\underline{T}^E)) - h^{++}(\Theta(\bar{\sigma}(\underline{T}^E))) \geq 0. \quad (3.15)$$

To get (3.7) it remains to estimate from below  $h^{++}(\Theta(\bar{\sigma}(\underline{T}^E))) - h^{++}(\bar{\sigma}(T_0))$ .

It is precisely here that we use the structure of triangles. Using (2.4), (2.6) and (2.8) in [5], noticing that  $T_2^{**}$  is the smallest triangle and taking into account of the missing  $\alpha(1 - \alpha)$ , see after Lemma 6.2, we get

$$h^{++}(\Theta(\bar{\sigma}(\underline{T}^E))) - h^{++}(\bar{\sigma}(T_0)) \geq \frac{\zeta_\alpha}{\alpha(1 - \alpha)}(\epsilon_c|\Lambda|)^\alpha + h^{++}(T_1^{**}) - h^{++}(\bar{\sigma}(T_0)). \quad (3.16)$$

Since for a configuration made of a single triangle  $T$  in  $\mathbb{Z}$  we have

$$\frac{2|T|^\alpha}{\alpha(1 - \alpha)} - \frac{2}{\alpha} \leq h^{++}(\bar{\sigma}(T)) \leq \frac{2|T|^\alpha}{\alpha(1 - \alpha)} - 2 \left(1 - \frac{1}{\alpha}\right), \quad (3.17)$$

we get, for  $|\Lambda|$  sufficiently large and how large depends on  $m, \alpha$ , and  $\nu$ , see (2.27),

$$\begin{aligned} h^{++}(\Theta(\bar{\sigma}(\underline{T}^E))) - h^{++}(\bar{\sigma}(T_0)) &\geq \frac{\zeta_\alpha}{\alpha(1 - \alpha)}(\epsilon_c|\Lambda|)^\alpha + \frac{2}{\alpha(1 - \alpha)} [(\rho(m) - \epsilon_c)^\alpha - (\rho(m))^\alpha] |\Lambda|^\alpha - 2 \\ &\geq \frac{\zeta_\alpha}{2\alpha(1 - \alpha)}(\epsilon_c|\Lambda|)^\alpha \end{aligned} \quad (3.18)$$

where we used that  $\rho > \rho(m) - \epsilon_c$ .

It remains to consider the case  $|T_1| < 2\epsilon_c|\Lambda|$ , then all the triangles of  $\underline{T}^E$  have a mass that is smaller than  $2\epsilon_c|\Lambda|$ , since by definition  $|T_1|$  is a triangle with maximal mass in  $\underline{T}^E$ .

Given an integer  $t \in [x_-(\underline{T}^E), x_+(\underline{T}^E)]$ , let  $\mathcal{Q}(t, \underline{T}^E) = \mathcal{Q}([x_-(\underline{T}^E)], t, \underline{T}^E)$  where  $[x_-(\underline{T}^E)]$  is the integer part of  $x_-(\underline{T}^E)$ .  $\mathcal{Q}(t, \underline{T}^E)$  is the number site  $i \leq t$  where  $\sigma_i = -1$  in  $\bar{\sigma}(\underline{T}^E)$ . Then,  $\mathcal{Q}(t, \underline{T}^E)$  is a non-strictly increasing function that increases linearly with a slope 1 when  $t \in \Delta(T)$  for some  $T \in \underline{T}^E$  and is constant for  $t$  in between two such triangles. We have also  $\mathcal{Q}([x_-(\underline{T}^E)], \underline{T}^E) = 0$  and  $\mathcal{Q}([x_+(\underline{T}^E)], \underline{T}^E) = \rho|\Lambda|$ . In particular the graph of this function intersects the level  $\rho|\Lambda|/2$  in two possible ways: either in a constant part of the graph or in the linear part of the graph. So let  $p = \inf\{p \in \mathbb{Z} : \mathcal{Q}(p, \underline{T}^E) = \rho|\Lambda|/2\}$ . Let  $\mathcal{J}^+$  be an interval centered at  $p$  and of size  $\epsilon_c|\Lambda|$ , then  $\mathcal{Q}(\mathcal{J}^+, \underline{T}^E) \leq \epsilon_c|\Lambda|$ . On the other hand  $\mathcal{Q}([x_-(\underline{T}^E)], p - \frac{\epsilon_c}{2}|\Lambda|, \underline{T}^E) \geq (\frac{\rho}{2} - \epsilon_c)|\Lambda|$  and  $\mathcal{Q}([p + \frac{\epsilon_c}{2}|\Lambda|, x_+(\underline{T}^E)], \underline{T}^E) \geq (\frac{\rho}{2} - \epsilon_c)|\Lambda|$ . At this point we proceed as before: all the spin  $\sigma_i$  with  $i \in \mathcal{J}^+$  that are minus are changed in plus ; we merge all the bases of the triangles that are on the left of  $\mathcal{J}^+$  and all the triangles that are on the right of  $\mathcal{J}^+$ . The rest of the proof is the same as above. ■

Now we collect some estimates for the Laplace transform of the probability distribution of the empirical magnetization conditioned on two kind of subsets of spin configurations. The first kind of subsets are simply  $\mathcal{S}_{T_0}$  with  $T_0$  a triangle with  $|T_0| = \rho|\Lambda|$ , see (2.9). For the other ones, recalling (2.13), given  $\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho)$ , see (2.12), let

$$\mathcal{S}_\Lambda^{VS}(\underline{T}^E, \rho, \epsilon_s) = \left\{ \underline{\sigma}_\Lambda \in \mathcal{S}_{\underline{T}^E} \cap \mathcal{S}_\Lambda^1(\rho) : \forall \tilde{T} \in \{\underline{T}(\sigma_\Lambda) \setminus \underline{T}^E\}, |\tilde{T}| \leq \epsilon_s|\Lambda| \right\} \quad (3.19)$$

be the set of spin configurations that give rise to the family  $\underline{T}^E$ , but all the associated triangles that are not in  $\underline{T}^E$  are small. This is relevant for the triangles that are internal to  $\underline{T}^E$ .

**Lemma 3.2** *There exists a  $\beta_6(\alpha, m)$  such that for all  $\beta \geq \beta_6(\alpha, m)$ , for all  $t$  such that*

$$|t| \leq \frac{\zeta_\alpha}{4\alpha(1-\alpha)(\epsilon_s|\Lambda|)^{1-\alpha}} \quad (3.20)$$

where  $\zeta_\alpha = 1 - 2(2^\alpha - 1)$  and  $\epsilon_s$  is defined in (2.27), we have

$$\left| \log \mu_\Lambda^+ \left[ e^{\beta t \sum_{i \in \Lambda} \sigma_i} | \mathcal{S}_\Lambda^{VS}(\underline{T}^E, \rho, \epsilon_s) \right] - \beta t |\Lambda| \mu_\Lambda^+ [m_\Lambda(\underline{\sigma}_\Lambda) | \mathcal{S}_\Lambda^{VS}(\underline{T}^E, \rho, \epsilon_s)] \right| \leq \frac{\beta^2}{2} t^2 |\Lambda| e^{-2\beta J}. \quad (3.21)$$

On the other hand for all  $\rho \in ]0, 1]$ , for all  $t$  such that

$$|t| \leq \frac{\zeta_\alpha 3^{1-\alpha}}{4\alpha(1-\alpha)(\rho|\Lambda|)^{1-\alpha}} \quad (3.22)$$

for all  $T_0$  with  $|T_0| = \rho|\Lambda|$ , we have

$$\left| \log \mu_\Lambda^+ \left[ e^{\beta t \sum_{i \in \Lambda} \sigma_i} | \mathcal{S}_{T_0} \right] - \beta t |\Lambda| \mu_\Lambda^+ [m_\Lambda(\underline{\sigma}_\Lambda) | \mathcal{S}_{T_0}] \right| \leq \frac{\beta^2}{2} t^2 |\Lambda| e^{-2\beta J}. \quad (3.23)$$

**Proof:**

It follows from the Taylor formula that for any set of configurations  $\tilde{\mathcal{S}}$  we have

$$\log \mu_\Lambda^+ \left[ e^{\beta t \sum_{i \in \Lambda} \sigma_i} | \tilde{\mathcal{S}} \right] - \beta t |\Lambda| \mu_\Lambda^+ [m_\Lambda(\underline{\sigma}_\Lambda) | \tilde{\mathcal{S}}] = \beta^2 |\Lambda| \int_0^t ds \int_0^s dr \frac{1}{|\Lambda|} \sum_{(i,j) \in \Lambda \times \Lambda} \mu_\Lambda^+(r) [\sigma_i, \sigma_j | \tilde{\mathcal{S}}] \quad (3.24)$$

where for all  $i, j \in \Lambda$

$$\mu_\Lambda^+(r)[\sigma_i, \sigma_j | \tilde{\mathcal{S}}] = \mu_\Lambda^+(r)[\sigma_i \sigma_j | \tilde{\mathcal{S}}] - \mu_\Lambda^+(r)[\sigma_i | \tilde{\mathcal{S}}] \mu_\Lambda^+(r)[\sigma_j | \tilde{\mathcal{S}}] \quad (3.25)$$

and for any cylindrical function  $f$  and  $r \in \mathbb{R}$

$$\mu_\Lambda^+(r)[f | \tilde{\mathcal{S}}] = \frac{\sum_{\underline{\sigma}_\Lambda \in \tilde{\mathcal{S}}} f(\underline{\sigma}_\Lambda) e^{-\beta h^{++}(\underline{\sigma}_\Lambda)} e^{\beta r \sum_{i \in \Lambda} \sigma_i}}{\sum_{\underline{\sigma}_\Lambda \in \tilde{\mathcal{S}}} e^{-\beta h^{++}(\underline{\sigma}_\Lambda)}}. \quad (3.26)$$

We need an estimate uniform in  $|r| \leq t$  of the two point truncated correlation function (3.25) for the conditioned measure (3.26) with a magnetic field  $r$ . Note that in all the considered cases the magnetic field is going to zero when  $|\Lambda| \uparrow \infty$ .

This will be done using the cluster expansion. As shown in chapter 6, a sufficient condition to be able to use the cluster expansion of [7] in presence of a magnetic field  $t$  is simply

$$\frac{\zeta_\alpha}{2\alpha(1-\alpha)} |\tilde{T}|^\alpha - |t| |\tilde{T}| \geq \frac{\zeta_\alpha}{4\alpha(1-\alpha)} |\tilde{T}|^\alpha \quad (3.27)$$

*i.e.*

$$|t| \leq \frac{\zeta_\alpha}{4\alpha(1-\alpha)} \inf_{\tilde{T} \in \tilde{\mathcal{S}}} \frac{1}{|\tilde{T}|^{1-\alpha}} \quad (3.28)$$

In the case  $\tilde{\mathcal{S}} = \mathcal{S}_\Lambda^{VS}(\underline{\mathcal{T}}^E, \rho, \epsilon_s)$  this gives (3.20) while in the case  $\tilde{\mathcal{S}} = \mathcal{S}_{T_0}$  with  $|T_0| = \rho|L|$ , this give (3.22) where we have used (6.6) that implies that all such triangles  $\tilde{T}$  that are internal to some triangle  $T^*(\tilde{T}) \in \underline{\mathcal{T}}^E$  satisfies

$$|\tilde{T}| \leq \frac{1}{3} |T^*(\tilde{T})| \leq \frac{1}{3} |\underline{\mathcal{T}}^E|. \quad (3.29)$$

■

## 4 Proof of the Proposition 2.18

Let us first note that since  $\mathcal{S}_\Lambda^1(\rho(m), \epsilon_c) \supset \mathcal{S}_\Lambda^B(\rho(m), \epsilon_c)$  we have

$$\mu_\Lambda^+[\mathcal{S}_\Lambda^1(\rho(m), \epsilon_c) | S_\Lambda(m, \epsilon_0)] \geq \mu_\Lambda^+[\mathcal{S}_\Lambda^B(\rho(m), \epsilon_c) | S_\Lambda(m, \epsilon_0)]. \quad (4.1)$$

Therefore we need just to prove an upper for  $\mu_\Lambda^+[\mathcal{S}_\Lambda^1(\rho(m), \epsilon_c) | S_\Lambda(m, \epsilon_0)]$ . We start with

$$\begin{aligned} \mu_\Lambda^+[\mathcal{S}_\Lambda^1(\rho(m), \epsilon_c) | S_\Lambda(m, \epsilon_0)] &= \mu_\Lambda^+[\mathcal{S}_\Lambda^B(\rho(m), \epsilon_c) | S_\Lambda(m, \epsilon_0)] \\ &\times \left[ 1 + \frac{\mu_\Lambda^+[\mathcal{S}_\Lambda^1(\rho(m), \epsilon_c) \setminus \mathcal{S}_\Lambda^B(\rho(m), \epsilon_c), S_\Lambda(m, \epsilon_0)]}{\mu_\Lambda^+[\mathcal{S}_\Lambda^B(\rho(m), \epsilon_c), S_\Lambda(m, \epsilon_0)]} \right]. \end{aligned} \quad (4.2)$$

where  $\mu_\Lambda^+[A, B] = \mu_\Lambda^+[A \cap B]$ . Recalling (2.18), let us define for  $\rho \in ]0, 1[$  and  $x \in \Lambda$ ,

$$\mathcal{T}_\Lambda^{E \setminus B}(\rho, x) = \{\underline{\mathcal{T}}^E \in \mathcal{T}_\Lambda^E(\rho) \setminus \mathcal{T}_\Lambda^B(\rho); x_-(\underline{\mathcal{T}}^E) = x\} \quad (4.3)$$

where, see (6.1),  $x_-(\underline{\mathcal{T}}^E) = \min_{T \in \underline{\mathcal{T}}^E} x_-(T)$ .

We have

$$\begin{aligned} \mu_\Lambda^+[\mathcal{S}_\Lambda^1(\rho(m), \epsilon_c) \setminus \mathcal{S}_\Lambda^B(\rho(m), \epsilon_c), S_\Lambda(m, \epsilon_0)] &\leq \\ \frac{1}{Z_\Lambda^{+++}} \sum_{x \in \Lambda} \sum_{\rho \in B(m, \epsilon_c)} \sum_{\underline{\mathcal{T}}^E \in \mathcal{T}_\Lambda^{E \setminus B}(\rho, x)} \sum_{\underline{\sigma}_\Lambda \in \mathcal{S}_{\underline{\mathcal{T}}^E \cap S_\Lambda(m, \epsilon_0)}} e^{-\beta h^{++}(\underline{\sigma}_\Lambda)} \end{aligned} \quad (4.4)$$

where  $B(m, \epsilon_c)$  is defined in (3.6) and  $\mathcal{S}_{\Gamma^E}$  is defined in (2.9).

On the other hand, for  $T_0$  a fixed triangle with basis in  $\Lambda$  with mass  $|T_0| = \rho(m)|\Lambda|$ , recalling (2.13), we denote

$$\mathcal{S}_{T_0}(\rho(m)) = \{\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda^1(\rho(m)); \underline{T}^E(\underline{\sigma}_\Lambda) = T_0\}. \quad (4.5)$$

Then we have

$$\mu_\Lambda^+[\mathcal{S}_\Lambda^B(\rho(m), \epsilon_c), \mathcal{S}_\Lambda(m, \epsilon_0)] \geq \frac{1}{Z_\Lambda^{++}} \sum_{\underline{\sigma}_\Lambda \in \mathcal{S}_{T_0}(\rho(m)) \cap \mathcal{S}_\Lambda(m, \epsilon_0)} e^{-\beta h^{++}(\underline{\sigma}_\Lambda)}. \quad (4.6)$$

For  $\underline{T}^E \in \mathcal{T}_\Lambda^{E \setminus B}(\rho, x)$  with  $\rho \in B(m, \epsilon_c)$ , recalling the definition of  $\bar{\sigma}(\underline{T}^E)$  in (2.8), for simplicity let us denote

$$\tilde{Z}_\Lambda^{++}(\underline{T}^E, m, \epsilon_0) = \sum_{\sigma_\Lambda \in \mathcal{S}_{\underline{T}^E} \cap \mathcal{S}(m, \epsilon_0)} e^{-\beta[h^{++}(\underline{\sigma}_\Lambda) - h^{++}(\bar{\sigma}(\underline{T}^E))]} \quad (4.7)$$

even if  $Z_\Lambda^{++}(\mathcal{S}_{\underline{T}^E}, \mathcal{S}(m, \epsilon_0))$  should be less ambiguous.  $\tilde{Z}_\Lambda^{++}(T_0, m, \epsilon_0)$  is defined analogously when  $\underline{T}^E = T_0$ . To avoid confusion, note that in this constrained partition function not only the triangle  $T_0$  is present but it is the largest external triangle and all the other external triangles are small.

Calling  $\mathcal{R}_1(T_0)$  the ratio of the right hand side of (4.4) over the right hand side of (4.6), we have

$$\mathcal{R}_1(T_0) \leq \sum_{x \in \Lambda} \sum_{\rho \in B(m, \epsilon_c)} \sum_{\underline{T}^E \in \mathcal{T}_\Lambda^{E \setminus B}(\rho, x)} e^{-\beta[h^{++}(\bar{\sigma}(\underline{T}^E)) - h^{++}(\bar{\sigma}(T_0))]} \frac{\tilde{Z}_\Lambda^{++}(\underline{T}^E, m, \epsilon_0)}{\tilde{Z}_\Lambda^{++}(T_0, m, \epsilon_0)}. \quad (4.8)$$

It remains to consider the last ratio in (4.8). Let us denote for  $\underline{T}^E \in \mathcal{T}_\Lambda^E$  an arbitrary family of external triangles,

$$\tilde{Z}_\Lambda^{++}(\underline{T}^E) = \sum_{\sigma_\Lambda \in \mathcal{S}_{\underline{T}^E}} e^{-\beta[h^{++}(\underline{\sigma}_\Lambda) - h^{++}(\bar{\sigma}(\underline{T}^E))]} \quad (4.9)$$

then we have  $\tilde{Z}_\Lambda^{++}(\underline{T}^E, m, \epsilon_0) \leq \tilde{Z}_\Lambda^{++}(\underline{T}^E)$ . We estimate from below  $\tilde{Z}_\Lambda^{++}(T_0, m, \epsilon_0)$ , when  $T_0$  satisfies  $|T_0| = \rho(m)|\Lambda|$ . We claim that there exists a  $\beta_7(\alpha)$  such that for all  $\beta \geq \beta_7(\alpha)$ , we have

$$\tilde{Z}_\Lambda^{++}(T_0, m, \epsilon_0) \geq \tilde{Z}_\Lambda^{++}(T_0) \left(1 - 2e^{-\frac{\beta}{\alpha(1-\alpha)} \frac{\epsilon_0}{\rho(m)} c(\alpha) \{\rho(m)|\Lambda|\}^\alpha}\right) \quad (4.10)$$

where

$$c(\alpha) = \frac{\zeta_\alpha 3^{1-\alpha} m_\beta}{16} \quad (4.11)$$

from which we get

$$\mathcal{R}_2 \equiv \frac{\tilde{Z}_\Lambda^{++}(\underline{T}^E, m, \epsilon_0)}{\tilde{Z}_\Lambda^{++}(T_0, m, \epsilon_0)} \leq \frac{\tilde{Z}_\Lambda^{++}(\underline{T}^E)}{\tilde{Z}_\Lambda^{++}(T_0)} \frac{1}{1 - 2e^{-\frac{\beta}{\alpha(1-\alpha)} \frac{\epsilon_0}{\rho(m)} c(\alpha) \{\rho(m)|\Lambda|\}^\alpha}} \equiv \mathcal{R}_3 \frac{1}{1 - 2e^{-\frac{\beta}{\alpha(1-\alpha)} \frac{\epsilon_0}{\rho(m)} c(\alpha) \{\rho(m)|\Lambda|\}^\alpha}} \quad (4.12)$$

Let us first prove the claim (4.10) and then estimate  $\mathcal{R}_3$ . The claim will be a consequence of an estimate for

$$\frac{\tilde{Z}_\Lambda^{++}(T_0, m, \epsilon_0)}{\tilde{Z}_\Lambda^{++}(T_0)} = \mu_\Lambda^+[\mathcal{S}_\Lambda(m, \epsilon_0) | \mathcal{S}_{T_0}] = 1 - \mu^+[\mathcal{M}_\Lambda^>(m, \epsilon_0) | \mathcal{S}_{T_0}] - \mu^+[\mathcal{M}_\Lambda^<(m, \epsilon_0) | \mathcal{S}_{T_0}] \quad (4.13)$$

where

$$\begin{aligned} \mathcal{M}_\Lambda^>(m, \epsilon_0) &:= \{m_\Lambda(\underline{\sigma}_\Lambda) > m + \epsilon_0 m_\beta\}; \\ \mathcal{M}_\Lambda^<(m, \epsilon_0) &:= \{m_\Lambda(\underline{\sigma}_\Lambda) < m - \epsilon_0 m_\beta\}. \end{aligned} \quad (4.14)$$

Therefore it is enough to get an upper bound for the last two terms in (4.13). Let us consider first  $\mu^+ [\mathcal{M}_\Lambda^\geq(m, \epsilon_0) | S_{T_0}]$ . By Markov inequality, we have for  $t_1 > 0$

$$\mu^+ [\mathcal{M}_\Lambda^\geq(m, \epsilon_0) | S_{T_0}] \leq e^{-\beta t_1 |\Lambda| (m + \epsilon_0 m_\beta)} \mu_\Lambda^+ \left[ e^{\beta t_1 \sum_{x \in \Lambda} \sigma_i} | S_{T_0} \right]. \quad (4.15)$$

Using Lemma 3.2 and Lemma 2.13 to estimate  $\mu_\Lambda^{++} [m_\Lambda(\underline{\sigma}_\Lambda) | S_{T_0}]$  and assuming

$$\frac{\epsilon_0}{2} m_\beta \geq \frac{10\xi^{++}(\beta)}{\alpha(1-\alpha)} \frac{1}{|\Lambda|^{1-\alpha}} + \frac{\beta^2}{2} t_1 e^{-2\beta J}. \quad (4.16)$$

we get, for  $t$  satisfying (3.22) with  $\rho = \rho(m)$ ,

$$\mu^+ [\mathcal{M}_\Lambda^\geq(m, \epsilon_0) | S_{T_0}] \leq e^{-\beta t_1 |\Lambda| \frac{\epsilon_0}{2} m_\beta}. \quad (4.17)$$

Let us now consider the second term in the right hand side of (4.13), by Markov inequality for any  $t_2 \geq 0$ , we have

$$\mu^+ [\mathcal{M}_\Lambda^\leq(m, \epsilon_0) | S_{T_0}] \leq e^{+\beta t_2 |\Lambda| (m - \epsilon_0 m_\beta)} \mu_\Lambda^+ \left[ e^{-\beta t_2 \sum_{x \in \Lambda} \sigma_i} | S_{T_0} \right] \quad (4.18)$$

the estimates are similar to the previous ones and under the same condition (4.16) for  $t_2$  instead of  $t_1$  we get

$$\mu^+ [S_\Lambda^\geq(m, \epsilon_0) | S_{T_0}] \leq e^{-\beta t_2 |\Lambda| \frac{\epsilon_0}{2} m_\beta} \quad (4.19)$$

This end the proof of the claim (4.10).

Recalling (4.12), it remains to estimate  $\mathcal{R}_3 = \tilde{Z}_\Lambda^{++}(\underline{T}^E) / \tilde{Z}_\Lambda^{++}(T_0)$  for  $\underline{T}^E \in \mathcal{T}_\Lambda^{E \setminus B}(\rho, x)$ . This will be done using a cluster expansion. By Lemma 6.7, used for  $t = 0$ , we consider first the leading terms of  $\log \mathcal{R}_3$  which is

$$\mathcal{E}(\underline{T}^E, T_0) = \sum_{x \in \Lambda} \left( \xi^{\bar{\sigma}(\underline{T}^E)}(x) - \xi^{\bar{\sigma}(T_0)}(x) \right) \quad (4.20)$$

where

$$\xi^{\bar{\sigma}(\underline{T}^E)}(x) = \begin{cases} e^{-\beta \left( 2J + \sum_{y \in \underline{T}^E \setminus \{x\}} J(x-y) \right)}, & \text{if } x \in \Delta(\underline{T}^E) \setminus \text{sf}(\underline{T}^E); \\ e^{-\beta \left( 2J + \sum_{y \in \Lambda \setminus \Delta(\underline{T}^E)} J(x-y) \right)}, & \text{if } x \notin \Delta(\underline{T}^E) \cup \text{sf}(\underline{T}^E); \end{cases} \quad (4.21)$$

and a similar formula holds for  $\xi^{\bar{\sigma}(T_0)}(x)$ . Note first that  $0 \leq \xi^{\bar{\sigma}(\underline{T}^E)}(x) \leq 1$  and  $0 \leq \xi^{\bar{\sigma}(T_0)}(x) \leq 1$ .

Given  $0 < \eta < 1$  to be chosen later, let us denote

$$\partial_\eta T_0 = \{x \in \Lambda : \text{dist}(x, T_0) \leq |\Lambda|^\eta\} \quad (4.22)$$

and

$$\partial_\eta \underline{T}^E = \{x \in \Lambda : \text{dist}(x, \underline{T}^E) \leq |\Lambda|^\eta\}. \quad (4.23)$$

It is easy to check that

$$|\partial_\eta T_0| = 4|\Lambda|^\eta \text{ and } |\partial_\eta \underline{T}^E| \leq 4|\Lambda|^\eta \sharp[\underline{T}^E] \quad (4.24)$$

where  $\sharp[\underline{T}^E]$  is the number of triangles in the family  $\underline{T}^E$ . Note that since  $\underline{T}^E \in \cup_{x \in \Lambda} \mathcal{T}_\Lambda^{E \setminus B}(\rho, x)$  and all triangles in  $\underline{T}^E$  are larger than  $\epsilon_s |\Lambda|$ , where  $\epsilon_s$  satisfies (2.27), we have

$$\sharp[\underline{T}^E] \leq \frac{\rho |\Lambda|}{\epsilon_s |\Lambda|} \leq |\Lambda|^\gamma. \quad (4.25)$$

Therefore

$$\sum_{x \in \partial_\eta T_0 \cup \partial_\eta \underline{T}^E} \left| \xi^{\bar{\sigma}(\underline{T}^E)}(x) - \xi^{\bar{\sigma}(T_0)}(x) \right| \leq 8|\Lambda|^{\gamma+\eta} e^{-2J\beta}. \quad (4.26)$$

To estimate the remaining terms in (4.20), recalling (1.5), we write them as

$$\mathcal{E}^1(\underline{T}^E, T_0) = \xi^{++}(\beta) \sum_{x \in \Lambda \setminus (\partial_\eta T_0 \cup \partial_\eta \underline{T}^E)} \left[ \frac{\xi^{\bar{\sigma}(\underline{T}^E)}(x)}{\xi^{++}(\beta)} - \frac{\xi^{\bar{\sigma}(T_0)}(x)}{\xi^{++}(\beta)} \right]. \quad (4.27)$$

Using (4.21) and (1.5), the two ratios in the bracket in (4.27) are larger than 1. However, given  $x \in \Lambda \setminus (\partial_\eta T_0 \cup \partial_\eta \underline{T}^E)$ , by comparison with an integral we have

$$\sum_{y \in \Lambda} \frac{\mathbb{I}_{\{\text{dist}(x,y) > |\Lambda|^\eta\}}}{|x-y|^{2-\alpha}} \leq \frac{2}{|\Lambda|^{\eta(1-\alpha)}} \frac{1}{1-\alpha} \equiv 2g(|\Lambda|) \quad (4.28)$$

Using that for  $0 \leq z \leq 2g(|\Lambda|)$ , if  $|\Lambda|$  is large enough to have  $2g(|\Lambda|) \leq 1$  then

$$1+z \leq e^z \leq 1+z \left(1 + \frac{z}{2} e^z\right) \leq 1+z(1+3g(|\Lambda|)) \quad (4.29)$$

we get, if  $|\Lambda|$  is large enough and how large depends on  $\beta$  and  $(\alpha, \eta)$ ,

$$\sum_{x \in \Lambda \setminus (\partial_\eta T_0 \cup \partial_\eta \underline{T}^E)} \frac{\xi^{\bar{\sigma}(\underline{T}^E)}(x)}{\xi^{++}(\beta)} \leq |\Lambda| + 2\beta \sum_{x \in \Delta(\underline{T}^E)} \sum_{y \in \Lambda \setminus \Delta(\underline{T}^E)} J(x-y)(1+3g(|\Lambda|)) \quad (4.30)$$

and

$$\sum_{x \in \Lambda \setminus (\partial_\eta T_0 \cup \partial_\eta \underline{T}^E)} \frac{\xi^{\bar{\sigma}(T_0)}(x)}{\xi^{++}(\beta)} \geq |\Lambda| - \left(\frac{6|\Lambda|^\eta}{1-\alpha}\right) + 2\beta \sum_{x \in \Delta(T_0)} \sum_{y \in \Lambda \setminus \Delta(T_0)} J(x-y) \quad (4.31)$$

Therefore, collecting (4.26), (4.30) and (4.31), we get

$$\mathcal{E}(\underline{T}^E, T_0) \leq 2\beta \xi^{++}(\beta) [h^{++}(\bar{\sigma}(\underline{T}^E)) - h^{++}(\bar{\sigma}(T_0))] + \xi^{++}(\beta) \left(\frac{6|\Lambda|^\eta}{1-\alpha}\right) + \xi^{++}(\beta) \frac{6}{\alpha(1-\alpha)^2} |\Lambda|^{\alpha+\gamma-(1-\alpha)\eta}$$

where the last term correspond to  $g(|\Lambda|)$  in (4.30) and comes from the following rough estimate:

$$h^{++}(\bar{\sigma}(\underline{T}^E)) \leq \sum_{\tilde{T} \in \underline{T}^E} h^{++}(\tilde{T}) \leq \frac{2(\rho|\Lambda|)^\alpha}{\alpha(1-\alpha)} \sum_{\tilde{T} \in \underline{T}^E} 1 \leq \frac{2(\rho|\Lambda|)^\alpha \rho}{\alpha(1-\alpha) \epsilon_s} \leq \frac{2\rho^{1+\alpha} |\Lambda|^{\alpha+\gamma}}{\alpha(1-\alpha)} \quad (4.32)$$

and (4.28).

Since we want to use the lower bound (3.7) and  $(\epsilon_c |\Lambda|)^\alpha = |\Lambda|^{(1-\nu)\alpha}$ , we assume

$$\eta \leq (1-\nu)\alpha \text{ and } \alpha + \gamma - \eta(1-\alpha) \leq (1-\nu)\alpha \quad (4.33)$$

that is

$$\frac{\gamma + \nu\alpha}{1-\alpha} \leq \eta \leq (1-\nu)\alpha. \quad (4.34)$$



Adding the error terms of the cluster expansion that are of the form  $(1 \pm e^{-\frac{\beta}{32}(\frac{\zeta_\alpha}{\alpha(1-\alpha)}-3\delta)})$  to  $\log \mathcal{R}_3$ , inserting the result in (4.8), using Lemma 3.1 and Lemma 2.10 we get that there exists a  $\beta_5 = \beta_5(\alpha)$  such that for all  $\beta \geq \beta_5(\alpha)$ , if  $|\Lambda|$  is large enough, we have

$$\mathcal{R}_1(T_0) \leq \frac{e^{-\beta \frac{\zeta_\alpha}{4\alpha(1-\alpha)}(\epsilon_c|\Lambda|)^\alpha}}{1 - 2e^{-\frac{\beta}{\alpha(1-\alpha)}\frac{\epsilon_0}{\rho(m)}c(\alpha)\{\rho(m)|\Lambda\}^\alpha}}. \quad (4.35)$$

Therefore, collecting (4.1), (4.2), (4.8) and (4.35) we get

$$\mu_\Lambda^+[\mathcal{S}_\Lambda^1(\rho(m), \epsilon_c)|S_\Lambda(m, \epsilon_0)] = \mu_\Lambda^+[\mathcal{S}_\Lambda^B(\rho(m), \epsilon_c)|S_\Lambda(m, \epsilon_0)] \left[ 1 \pm \frac{e^{-\beta \frac{\zeta_\alpha}{4\alpha(1-\alpha)}(\epsilon_c|\Lambda|)^\alpha}}{1 - 2e^{-\frac{\beta}{\alpha(1-\alpha)}\frac{\epsilon_0}{\rho(m)}c(\alpha)\{\rho(m)|\Lambda\}^\alpha}} \right] \quad (4.36)$$

this prove (2.32). ■

## 5 Proof of the Proposition 2.17

Let us first give a lower bound for probability of the event we are conditioning on:

**Lemma 5.1** *There exists a  $\beta_8 = \beta_8(\alpha)$  such that for all  $\beta \geq \beta_8$ , if  $|\Lambda|$  is large enough, we have*

$$\mu_\Lambda^+[\mathcal{S}_\Lambda(m, \epsilon_0)] \geq e^{-\frac{2\beta}{\alpha(1-\alpha)}(\rho(m)|\Lambda|)^\alpha[1-\lambda_-(\beta)]}(1 - 2\eta_1(\beta, \Lambda)) \quad (5.1)$$

where

$$\lambda_-(\beta) = 2\xi^{++}(\beta)(1 - e^{-\frac{\beta}{32}(\zeta_\alpha-3\delta)}) \quad (5.2)$$

with  $\xi^{++}(\beta)$  defined in (1.5),  $\zeta_\alpha$  and  $\delta$  defined in Lemma 6.7, and

$$\eta_1(\beta, \Lambda) = e^{-\frac{\beta}{8}\epsilon_0 m \beta \zeta_\alpha |\Lambda|^{\alpha+\gamma(1-\alpha)}}. \quad (5.3)$$

### Proof

Given a triangle  $T$  with  $|T| = \rho(m)|\Lambda|$  and  $\Delta(T) \subset \Lambda$ , recalling (2.9), (2.13), and using (3.19) in the particular case  $\underline{T}^E = T$ , let

$$\mathcal{S}_\Lambda^{VS}(T, \rho(m), \epsilon_s) = \{\underline{\alpha}_\Lambda \in \mathcal{S}_T \cap \mathcal{S}_\Lambda^1(\rho(m)) : \forall \tilde{T} \neq T, |\tilde{T}| < \epsilon_s |\Lambda|\}. \quad (5.4)$$

We have

$$\mu_\Lambda^+[\mathcal{S}_\Lambda(m, \epsilon_0)] \geq [1 - \mu_\Lambda^{++}[(\mathcal{S}_\Lambda(m, \epsilon_0))^c | \mathcal{S}_\Lambda^{VS}(T, \rho(m), \epsilon_s)]] \mu_\Lambda^+[\mathcal{S}_\Lambda^{VS}(T, \rho(m), \epsilon_s)]. \quad (5.5)$$

Let us start with a lower bound for  $\mu_\Lambda^+[\mathcal{S}_\Lambda^{VS}(T, \rho(m), \epsilon_s)]$ . Using (2.8) with  $\underline{T}^E = T$  and  $|T| = \rho(m)|\Lambda|$ , we have

$$\mu_\Lambda^+[\mathcal{S}_\Lambda^{VS}(T, \rho(m), \epsilon_s)] = \frac{1}{Z_\Lambda^{++}(\beta)} e^{-\beta h^{++}(\bar{\sigma}_T)} \sum_{\underline{\alpha}_\Lambda \in \mathcal{S}_\Lambda^{VS}(T, \rho(m), \epsilon_s)} e^{-\beta[h^{++}(\underline{\alpha}_\Lambda) - h^{++}(\bar{\sigma}_T)]} \quad (5.6)$$

using the cluster expansion to estimate

$$-\log Z_\Lambda^{++}(\beta) + \log \left[ \sum_{\underline{\alpha}_\Lambda \in \mathcal{S}_\Lambda^{VS}(T, \rho(m), \epsilon_s)} e^{-\beta[h^{++}(\underline{\alpha}_\Lambda) - h^{++}(\bar{\sigma}_T)]} \right] \quad (5.7)$$

we get

$$\begin{aligned}\mu_{\Lambda}^{+}[\mathcal{S}_{\Lambda}^{VS}(\rho(m), \epsilon_s)] &\geq e^{-\beta h^{++}(\bar{\sigma}_T)} e^{\xi^{++}(\beta) 2\beta h^{++}(\bar{\sigma}_T)(1-e^{-\frac{\beta}{32}(\zeta_{\alpha}-3\delta)})} \\ &\geq e^{-\beta h^{++}(\bar{\sigma}_T)[1-\lambda_{-}(\beta)]}\end{aligned}\quad (5.8)$$

where  $\lambda_{-}(\beta)$  is defined in (5.2). Using (3.17) we get

$$\mu_{\Lambda}^{+}[\mathcal{S}_{\Lambda}^{VS}(\rho(m), \epsilon_s)] \geq \exp\left\{-\frac{2\beta}{\alpha(1-\alpha)}(\rho(m)|\Lambda|)^{\alpha}[1-\lambda_{-}(\beta)]\right\}.\quad (5.9)$$

Recalling (5.5), let us now give an upper bound for  $\mu_{\Lambda}^{++}[(\mathcal{S}_{\Lambda}(m, \epsilon_0))^c | \mathcal{S}_{\Lambda}^{VS}(T, \rho(m), \epsilon_s)]$ . Recalling (4.14), we have

$$(\mathcal{S}_{\Lambda}(m, \epsilon_0))^c = \mathcal{M}_{\Lambda}^{\geq}(m, \epsilon_0) \cup \mathcal{M}_{\Lambda}^{\leq}(m, \epsilon_0).\quad (5.10)$$

We use again the Markov inequality to get on the one hand for  $t_3 > 0$

$$\mu_{\Lambda}^{++}[\mathcal{M}_{\Lambda}^{\geq}(m, \epsilon_0) | \mathcal{S}_{\Lambda}^{VS}(T, \rho(m), \epsilon_s)] \leq e^{-\beta t_3(m+\epsilon_0 m_{\beta})|\Lambda|} \mu_{\Lambda}^{++}[e^{+\beta t_3 m_{\Lambda}(\underline{z}_{\Lambda})|\Lambda|} | \mathcal{S}_{\Lambda}^{VS}(T, \rho(m), \epsilon_s)]\quad (5.11)$$

and on the other hand, for  $t_4 > 0$

$$\mu_{\Lambda}^{++}[\mathcal{M}_{\Lambda}^{\leq}(m, \epsilon_0) | \mathcal{S}_{\Lambda}^{VS}(T, \rho(m), \epsilon_s)] \leq e^{+\beta t_4(m-\epsilon_0 m_{\beta})|\Lambda|} \mu_{\Lambda}^{++}[e^{-\beta t_4 m_{\Lambda}(\underline{z}_{\Lambda})|\Lambda|} | \mathcal{S}_{\Lambda}^{VS}(T, \rho(m), \epsilon_s)].\quad (5.12)$$

Using Lemma 3.2 and Lemma 2.13 after some easy computations, if  $|\Lambda|$  is large enough and how large depends on  $\beta, \alpha$ , we have

$$\mu_{\Lambda}^{++}[\mathcal{S}_{\Lambda}^{\geq}(m, \epsilon_0) | \mathcal{S}_{\Lambda}^{VS}(T, \rho(m), \epsilon_s)] \leq e^{-\frac{\beta}{8}\epsilon_0 m_{\beta} \zeta_{\alpha} |\Lambda|^{\alpha+\gamma(1-\alpha)}} \equiv \eta_1(\beta, \Lambda)\quad (5.13)$$

and by similar arguments, we get

$$\mu_{\Lambda}^{++}[\mathcal{S}_{\Lambda}^{\leq}(m, \epsilon_0) | \mathcal{S}_{\Lambda}^{VS}(T, \rho(m), \epsilon_s)] \leq \eta_1(\beta, \Lambda).\quad (5.14)$$

Recalling (5.5), (5.9), (5.13) and (5.14) we get (5.1). ■

To prove (2.31), recalling (2.16), let us define the partition

$$(\mathcal{S}_{\Lambda}^1(\rho(m), \epsilon_c))^c = \mathcal{S}_{\Lambda}^{\leq}(\rho(m), \epsilon_c) \cup \mathcal{S}_{\Lambda}^{\geq}(\rho(m), \epsilon_c)\quad (5.15)$$

where

$$\mathcal{S}_{\Lambda}^{\leq}(\rho(m), \epsilon_c) = \bigcup_{\rho < \rho(m) - \epsilon_c} \mathcal{S}_{\Lambda}^1(\rho)\quad (5.16)$$

and

$$\mathcal{S}_{\Lambda}^{\geq}(\rho(m), \epsilon_c) = \bigcup_{\rho > \rho(m) + \epsilon_c} \mathcal{S}_{\Lambda}^1(\rho)\quad (5.17)$$

with the same conventions that are mentioned after (2.16).

We consider first the set (5.17), we have

$$\mu_{\Lambda}^{+} [\mathcal{S}_{\Lambda}^{>}(\rho(m), \epsilon_c), \mathcal{S}_{\Lambda}(m, \epsilon_0)] \leq \sum_{\rho \geq \rho(m) + \epsilon_c} \mu_{\Lambda}^{+} [S_{\Lambda}^1(\rho)] \quad (5.18)$$

where the sum is merely over  $\{\rho \geq \rho(m) + \epsilon_c\} \cap Q_{\Lambda}$ . By similar computation as in the proof of Proposition 2.18 we have

$$\mu_{\Lambda}^{+} [S_{\Lambda}^1(\rho)] \leq \mu_{\Lambda}^{+} [S_{\Lambda}^B(\rho)] [1 + \eta_2(\beta, \Lambda)] \quad (5.19)$$

where  $S_{\Lambda}^B(\rho)$  is defined in (2.19),  $\eta_2(\beta, \Lambda) = \frac{e^{-\beta \frac{\zeta \alpha}{4\alpha(1-\alpha)} (\epsilon_c |\Lambda|)^{\alpha}}}{1 - 2e^{-\frac{\beta}{\alpha(1-\alpha)} \epsilon_0 c(\alpha) |\Lambda|^{\alpha}}}$  and  $c(\alpha)$  is defined in (4.11). We have

$$\mu_{\Lambda}^{+} [S_{\Lambda}^B(\rho)] \leq \sum_{\underline{T}^E \in \mathcal{T}_{\Lambda}^B(\rho)} e^{-\beta h^{++}(\overline{\sigma}(\underline{T}^E))} \frac{\tilde{Z}_{\Lambda}^{++}(\underline{T}^E)}{Z_{\Lambda}^{++}} \quad (5.20)$$

where  $\tilde{Z}_{\Lambda}^{++}(\underline{T}^E)$  is defined in (4.9). Using cluster expansion, we get

$$\mu_{\Lambda}^{+} [S_{\Lambda}^B(\rho)] \leq \sum_{\underline{T}^E \in \mathcal{T}_{\Lambda}^B(\rho)} e^{-\beta h^{++}(\overline{\sigma}(\underline{T}^E)) [1 - \lambda_+(\beta)]} \quad (5.21)$$

where

$$\lambda_+(\beta) = 2\xi^{++}(\beta) e^{4\beta\zeta(2-\alpha)} (1 + e^{-\frac{\beta}{32} (\frac{\zeta \alpha}{\alpha(1-\alpha)} - 3\delta)}). \quad (5.22)$$

and note that recalling (1.5), we have  $\lambda_+(\beta) \downarrow 0$  as  $\beta \uparrow \infty$  since  $J$  defined in (1.2) is large and therefore we can assume that  $J > \zeta(2 - \alpha)$ . Using (3.3) we get that for all  $\rho \geq \rho(m) + \epsilon_c$ , for  $\underline{T}^E \in \mathcal{T}_{\Lambda}^B(\rho)$ ,

$$h^{++}(\overline{\sigma}(\underline{T}^E)) \geq \frac{2}{\alpha(1-\alpha)} ((\rho(m) + \epsilon_c) |\Lambda|)^{\alpha} \quad (5.23)$$

and therefore using Lemma 2.10 and lemma 5.1, after a short computation, if  $|\Lambda|$  is large enough and how large depends on  $(m, \alpha, \beta, \nu)$ , we have

$$\mu_{\Lambda}^{+} [\mathcal{S}_{\Lambda}^{>}(\rho(m), \epsilon_c) | \mathcal{S}_{\Lambda}(m, \epsilon_0)] \leq e^{-\frac{2\beta(\rho(m) |\Lambda|)^{\alpha}}{\alpha(1-\alpha)}} \left\{ \left(1 + \frac{\epsilon_c}{\rho(m)}\right)^{\alpha} - 1 \right\}. \quad (5.24)$$

where the  $-1$  in the brackets comes from the lower bound (5.1) and we have assumed

$$\alpha - \nu > \gamma \quad (5.25)$$

to neglect the terms that come from Lemma 2.10 when performing the sum in (5.21). This gives the second term in (2.31).

We consider now the set (5.16) and we write

$$\begin{aligned} \mu_{\Lambda}^{+} [\mathcal{S}_{\Lambda}^{<}(\rho(m), \epsilon_c), \mathcal{S}_{\Lambda}(m, \epsilon_0)] &\leq \mu_{\Lambda}^{+} [\{m_{\Lambda}(\underline{\sigma}_{\Lambda}) \leq m + \epsilon_0 m_{\beta}\}, \mathcal{S}_{\Lambda}^{<}(\rho(m), \epsilon_c)] \\ &\leq |\Lambda| \sup_{\rho \leq \rho(m) - \epsilon_c} [\mu_{\Lambda}^{+} [\{m_{\Lambda}(\underline{\sigma}_{\Lambda}) \leq m + \epsilon_0 m_{\beta}\}, S_{\Lambda}^1(\rho)]] \\ &\leq |\Lambda| \sup_{\rho \leq \rho(m) - \epsilon_c} [\mu_{\Lambda}^{+} [\{m_{\Lambda}(\underline{\sigma}_{\Lambda}) \leq m + \epsilon_0 m_{\beta}\} | S_{\Lambda}^1(\rho)]] \end{aligned} \quad (5.26)$$

since  $\mu_{\Lambda}^{+}[S_{\Lambda}^1(\rho)] \leq 1$ . Note that  $\rho(m) - \epsilon_c > 0$ , by (2.27) if  $|\Lambda|$  is large enough and how large depends only on  $\beta$  and  $m$ .

Using exponential Markov inequality, for  $t > 0$  we get

$$\mu_{\Lambda}^{+}[\{m_{\Lambda}(\underline{\sigma}_{\Lambda}) \leq m + \epsilon_0 m_{\beta}\} | \mathcal{S}_{\Lambda}^1(\rho)] \leq e^{t\beta(m + \epsilon_0 m_{\beta})|\Lambda|} \mu_{\Lambda}^{+}[e^{-\beta t \sum_{i \in \Lambda} \sigma_i} | \mathcal{S}_{\Lambda}^1(\rho)]. \quad (5.27)$$

To estimate the last term in (5.27), we use the following Lemma.

**Lemma 5.2** *There exists a  $\beta_{10} = \beta_{10}(\alpha, \eta)$  such that for all  $0 < t \leq t^*(\epsilon_s)/2 = \frac{\zeta_{\alpha}}{4\alpha(1-\alpha)(\epsilon_s|\Lambda|)^{1-\alpha}}$ , for all  $\beta \geq \beta_{10}$ , for all  $\rho < \rho(m) - \epsilon_c$  we have*

$$\mu_{\Lambda}^{+} \left[ e^{-\beta t \sum_{i \in \Lambda} \sigma_i} | \mathcal{S}_{\Lambda}^1(\rho) \right] \leq \sup_{\underline{T}^E \in \mathcal{T}_{\Lambda}^E(\rho)} \mu_{\Lambda}^{+} \left[ e^{-\beta t \sum_{i \in \Lambda} \sigma_i} | \mathcal{S}_{\Lambda}^{VS}(\underline{T}^E, \rho, \epsilon_s) \right] (1 + \eta_3(\beta, \Lambda)) \quad (5.28)$$

where  $\mathcal{T}_{\Lambda}^E(\rho)$  is defined in (2.12),  $\mathcal{S}_{\Lambda}^{VS}(\underline{T}^E, \rho, \epsilon_s)$  in (3.19) and

$$\eta_3(\beta, \Lambda) = e^{-\frac{\beta \zeta_{\alpha}}{8\alpha(1-\alpha)}(\epsilon_s|\Lambda|^{\alpha})} = e^{-\frac{\beta \zeta_{\alpha}}{8\alpha(1-\alpha)}(|\Lambda|^{\alpha(1-\gamma)})}. \quad (5.29)$$

Let us postpone the proof of this lemma and continue, using Lemma 3.2 and Lemma 2.13, inserting (5.28) in (5.27), using  $\rho \leq \rho(m) - \epsilon_c$ ,  $m(\rho) = (1 - 2\rho)m_{\beta}$ , and  $m = m_{\beta}(1 - 2\rho(m))$ , we get

$$\mu_{\Lambda}^{+}[\{m_{\Lambda}(\underline{\sigma}_{\Lambda}) \leq m + \epsilon_0 m_{\beta}\} | \mathcal{S}_{\Lambda}^1(\rho)] \leq e^{-t\beta(2\epsilon_c - \epsilon_0)m_{\beta}|\Lambda|} (1 + \eta_3(\beta, \Lambda)) \quad (5.30)$$

Taking  $t = t^*(\epsilon_s)/2$ , we get that the exponential in the right hand side of (5.30) is

$$e^{-\frac{\zeta_{\alpha}}{4\alpha(1-\alpha)(\epsilon_s)^{1-\alpha}}\beta(2\epsilon_c - \epsilon_0)m_{\beta}|\Lambda|^{\alpha}} \leq e^{-\frac{\zeta_{\alpha}}{4\alpha(1-\alpha)}\beta m_{\beta}|\Lambda|^{\alpha + \gamma(1-\alpha) - \nu}} \quad (5.31)$$

if  $|\Lambda|$  is large enough and how large depends only on  $(a, \nu)$ , see (2.27).

**Remark:** *We assume here  $\epsilon_c > \epsilon_0$  that is  $a > \nu$ .*

Assuming

$$\alpha + \gamma(1 - \alpha) - \nu > \alpha, \quad (5.32)$$

using (5.1) and recalling (5.26), if  $|\Lambda|$  is large enough, and how large depends on  $(\alpha, \gamma, \nu)$ , we get

$$\mu_{\Lambda}^{+} [\mathcal{S}_{\Lambda}^{\leq}(\rho(m), \epsilon_c) | \mathcal{S}_{\Lambda}(m, \epsilon_0)] \leq e^{-\frac{\zeta_{\alpha}}{8\alpha(1-\alpha)}\beta[(2\epsilon_c - \epsilon_0)m_{\beta}]|\Lambda|^{\alpha + \gamma(1-\alpha)}} \quad (5.33)$$

where we have used a part of (5.31) to bound by 1 the terms coming from the factor  $(1 + \eta_3(\beta, \Lambda))$  and the lower bound (5.1). This gives the first term in (2.31). ■

## Proof of Lemma 5.2

We start with

$$\mu_{\Lambda}^{+}[e^{-\beta t \sum_{i \in \Lambda} \sigma_i} | \mathcal{S}_{\Lambda}^1(\rho)] = \frac{\sum_{\underline{T}^E \in \mathcal{T}_{\Lambda}^E(\rho)} e^{-\beta h^{++}(\overline{\sigma}(\underline{T}^E))} \sum_{\underline{\sigma}_{\Lambda} \in \mathcal{S}_{\underline{T}^E}} e^{-\beta[h^{++}(\underline{\sigma}_{\Lambda}) - h^{++}(\overline{\sigma}(\underline{T}^E))]} e^{-\beta t \sum_{i \in \Lambda} \sigma_i}}{\sum_{\underline{T}^E \in \mathcal{T}_{\Lambda}^E(\rho)} e^{-\beta h^{++}(\overline{\sigma}(\underline{T}^E))} \sum_{\underline{\sigma}_{\Lambda} \in \mathcal{S}_{\underline{T}^E}} e^{-\beta[h^{++}(\underline{\sigma}_{\Lambda}) - h^{++}(\overline{\sigma}(\underline{T}^E))]}]} \quad (5.34)$$

where we used (2.12), (2.8), and (2.9).

Now for each  $\underline{T}^E \in \mathcal{T}_\Lambda^E(r)$ , see (2.12), we have  $\sum_{i \in \Lambda} \bar{\sigma}_i(\underline{T}^E) = |\Lambda|(1-2\rho)$ , because  $\underline{T}^E$  is made of mutually external triangles. Therefore introducing

$$\tilde{Z}_\Lambda^{++}(\underline{T}^E, -t) = \sum_{\underline{\sigma}_\Lambda \in \mathcal{S}_{\underline{T}^E}} e^{-\beta[h^{++}(\underline{\sigma}_\Lambda) - h^{++}(\bar{\sigma}(\underline{T}^E))]} e^{-\beta t \sum_{i \in \Lambda} [\sigma_i - \bar{\sigma}_i(\underline{T}^E)]} \quad (5.35)$$

we get

$$\begin{aligned} \mu_\Lambda^{++}[e^{-\beta t \sum_{i \in \Lambda} \sigma_i} | \mathcal{S}_\Lambda^1(\rho)] &= e^{-\beta t |\Lambda|(1-2\rho)} \times \frac{\sum_{\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho)} e^{-\beta h^{++}(\bar{\sigma}(\underline{T}^E))} \tilde{Z}_\Lambda^{++}(\underline{T}^E, -t)}{\sum_{\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho)} e^{-\beta h^{++}(\bar{\sigma}(\underline{T}^E))} \tilde{Z}_\Lambda^{++}(\underline{T}^E, 0)} \\ &\leq e^{-\beta t |\Lambda|(1-2\rho)} \times \sup_{\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho)} \frac{\tilde{Z}_\Lambda^{++}(\underline{T}^E, -t)}{\tilde{Z}_\Lambda^{++}(\underline{T}^E, 0)} \end{aligned} \quad (5.36)$$

Therefore it remains to estimate the last ratio in (5.36) for  $\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho)$ . In a way similar to (5.4), let us define

$$\mathcal{S}_\Lambda^{VS}(\underline{T}^E, \rho, \epsilon_s) = \left\{ \underline{\sigma}_\Lambda \in \mathcal{S}_{\underline{T}^E} \cap \mathcal{S}_\Lambda^1(\rho) : \forall \tilde{T} \in \{\underline{T}(\sigma_\Lambda) \setminus \underline{T}^E\}, |\tilde{T}| \leq \epsilon_s |\Lambda| \right\} \quad (5.37)$$

and

$$\tilde{Z}_\Lambda^{+,VS}(\underline{T}^E, -t) = \sum_{\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda^{VS}(\underline{T}^E, \rho, \epsilon_s)} e^{-\beta[h^{++}(\underline{\sigma}_\Lambda) - h^{++}(\bar{\sigma}(\underline{T}^E))]} e^{-\beta t \sum_{i \in \Lambda} [\sigma_i - \bar{\sigma}_i(\underline{T}^E)]} \quad (5.38)$$

We claim that, if  $|\Lambda|$  is large enough and how large depends on  $\alpha, \beta, \nu$  see (2.27), we have

$$\tilde{Z}_\Lambda^{++}(\underline{T}^E, -t) \leq \tilde{Z}_\Lambda^{+,VS}(\underline{T}^E, -t)(1 + \eta_3(\beta, \Lambda, \epsilon_s)) \quad (5.39)$$

where  $\eta_3(\beta, \Lambda, \epsilon_s) = e^{-\frac{\beta c_{10}(\alpha)}{C}(\epsilon_s |\Lambda|)^\alpha}$  with the choice  $\epsilon_s = |\Lambda|^{-\gamma}$  we have  $\eta_3(\beta, \Lambda, \epsilon_s = |\Lambda|^{-\gamma}) = \eta_4(\beta, \Lambda) = e^{-\frac{\beta c_{10}(\alpha)}{C} |\Lambda|^{\alpha(1-\gamma)}}$ . Let us assume that (5.39) is true and continue. Then the last ratio in (5.36) can be bounded as follows

$$\frac{\tilde{Z}_\Lambda^{++}(\underline{T}^E, -t)}{\tilde{Z}_\Lambda^{++}(\underline{T}^E, 0)} \leq (1 + \eta_3(\beta, \Lambda, \epsilon_s)) \times \frac{\tilde{Z}_\Lambda^{+,VS}(\underline{T}^E, -t)}{\tilde{Z}_\Lambda^{+,VS}(\underline{T}^E, 0)} \quad (5.40)$$

Recalling (5.36), we get immediately (5.28). ■

It remains to prove the claim (5.39). The proof is based on an estimate of an energy cost when the magnetic field is negative and then a Peierls type argument. Let us start with the energy estimate which is (5.48). Recalling definition 2.8, given  $0 < \rho \leq 1$  and  $\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho)$  let us define

$$\mathcal{S}_{\underline{T}^E}(\rho) = \{ \underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda^1(\rho) : \underline{T}^E(\underline{\sigma}_\Lambda) = \underline{T}^E \}. \quad (5.41)$$

As in section 6, for  $\underline{\sigma}_\Lambda \in \mathcal{S}_{\underline{T}^E}(\rho)$  there is a bijection between

$$\underline{\sigma}_\Lambda \leftrightarrow (\underline{T}^E(\underline{\sigma}_\Lambda), \underline{\Gamma}(\underline{\sigma}_\Lambda)) \quad (5.42)$$

where  $\underline{\Gamma}(\underline{\sigma}_\Lambda)$  is the family of contours obtained by implementing the algorithm  $\mathcal{R}$  on  $\underline{T}(\underline{\sigma}_\Lambda) \setminus \underline{T}^E(\underline{\sigma}_\Lambda)$ . Moreover,

$$\underline{\Gamma}(\underline{\sigma}_\Lambda) = (\tilde{\underline{\Gamma}}(\underline{\sigma}_\Lambda), \Gamma^*(\underline{\sigma}_\Lambda)) \quad (5.43)$$

where  $\tilde{\underline{\Gamma}}(\underline{\sigma}_\Lambda)$  is a family of all contours made of small triangles *i.e.*  $\forall \Gamma \in \tilde{\underline{\Gamma}}(\underline{\sigma}_\Lambda)$ , for all  $T \in \Gamma$ ,  $|T| \leq \epsilon_s |\Lambda|$ . While, if they exist all the large triangles of  $\underline{T}(\underline{\sigma}_\Lambda) \setminus \underline{T}^E(\underline{\sigma}_\Lambda)$ , *i.e.* the  $T^*$  such that  $|T^*| > \epsilon_s |\Lambda|$ , are internal with respect to  $\underline{T}^E$  and belong to the unique contour  $\Gamma^*(\underline{\sigma}_\Lambda)$ .

The uniqueness of  $\Gamma^*(\underline{\sigma}_\Lambda)$  is a consequence of (6.18). In fact, picking up two of them, say  $\Gamma_1^*, \Gamma_2^*$ , using (6.18) and the fact that  $0 < \gamma < 2/3$  we should have

$$\text{dist}(\Gamma_1^*, \Gamma_2^*) > C \min\{|\Gamma_1^*|^3, |\Gamma_2^*|^3\} \geq C \epsilon_s^3 |\Lambda|^3 > |\Lambda| \quad (5.44)$$

which is not possible for two contours having their supports within  $\Lambda$ .

The fluctuations with respect to the ground state  $\bar{\sigma}(\Gamma^E)$  are therefore described by the family of contours  $(\underline{\Gamma}(\underline{\sigma}_\Lambda), \Gamma^*(\underline{\sigma}_\Lambda))$ . Using the bijection between spin configurations and contours we write

$$h^{++}(\underline{\sigma}_\Lambda) - h^{++}(\bar{\sigma}(\Gamma^E)) + t \sum_{i \in \Lambda} \sigma_i = H(\underline{\Gamma}, \Gamma^*) + t \sum_{i \in \Lambda} \sigma_i(\underline{\Gamma}, \Gamma^*) \quad (5.45)$$

Consider now the basis of the triangles belonging to  $\Gamma^*$ , they define a set of intervals whose interior boundary is made of spins with the same sign. Calling  $I^+$  (resp.  $I^-$ ) the union of intervals with interior boundary  $+$  (resp  $-$ ) we have that  $\Delta(\Gamma^*) = I^+ \cup I^-$ . If we define the transformation  $\tau^*$  that depends on  $\Gamma^*$  by : If  $i \in \Delta(\Gamma^*)$

$$\tau_i^*(\underline{\sigma}_\Lambda) = \begin{cases} -\sigma_i, & \text{if } i \in I^+; \\ \sigma_i, & \text{otherwise.} \end{cases} \quad (5.46)$$

while if  $i \in \Lambda \setminus \Delta(\Gamma^*)$  then  $\tau_i^*(\underline{\sigma}_\Lambda) = \sigma_i$ , then we get

$$h^{++}(\tau^*(\underline{\sigma}_\Lambda)) - h^{++}(\bar{\sigma}(\Gamma^E)) + t \sum_{i \in \Lambda} \tau_i^*(\underline{\sigma}_\Lambda) = H(\underline{\Gamma}) + t \sum_{i \in \Lambda} \tau_i^*(\underline{\Gamma}) \quad (5.47)$$

so that the excess of energy when the magnetic field is negative, due to the presence of  $\Gamma^*$  is

$$H(\underline{\Gamma}, \Gamma^*) - H(\underline{\Gamma}) + t \sum_{i \in I^+} [\sigma_i(\underline{\Gamma}, \Gamma^*) - \tau_i^*(\underline{\Gamma})] \geq \frac{c_{10}(\alpha)}{C} \|\Gamma^*\|_\alpha + 2t \sum_{i \in I^+} \sigma_i(\underline{\Gamma}, \Gamma^*) \geq \frac{c_{10}(\alpha)}{C} \|\Gamma^*\|_\alpha \quad (5.48)$$

where we have used (6.25),  $c_{10}(\alpha) = \frac{2\pi^2}{3\alpha(1-\alpha)}$  and the fact that  $\sum_{i \in I^+} \sigma_i(\underline{\Gamma}, \Gamma^*)$  is positive.

To check this last fact, we remark that a necessary condition for a spin  $\sigma_i(\underline{\Gamma}, \Gamma^*)$  at  $i \in I^+$  to be negative is that it belongs to a contour of  $\underline{\Gamma}$ . Since the number of contours of mass  $m$ , see (6.10), that are in  $I^+$  is less or equal to  $|I^+|/(Cm^3)$  because the inter-distance between two such contours is larger than  $Cm^3$ , see (6.18). On the other hand, it is easy to see that the number of negative spins in a contour is smaller or equal to the mass of this contour with equality for contours made of mutually external triangles. Therefore if we denote by  $M^-(I^+)(\underline{\sigma}) = \sum_{i \in I^+} \frac{1-\sigma_i}{2}$  and  $M^+(I^+)(\underline{\sigma}) = \sum_{i \in I^+} \frac{1+\sigma_i}{2}$  the number of negative, respectively positive spins in  $I^+$ , we have

$$M^-(I^+)(\underline{\sigma}(\underline{\Gamma}, \Gamma^*)) \leq \frac{|I^+|}{C} \sum_{m=1}^{\infty} \frac{m}{m^3} = \frac{\pi^2 |I^+|}{6C}. \quad (5.49)$$

Since  $M^-(I^+)(\underline{\sigma}(\underline{\Gamma}, \Gamma^*)) + M^+(I^+)(\underline{\sigma}(\underline{\Gamma}, \Gamma^*)) = |I^+|$ , from (5.49) we get

$$\sum_{i \in I^+} \sigma_i(\underline{\Gamma}, \Gamma^*) = |I^+| - 2M^-(I^+)(\underline{\sigma}(\underline{\Gamma}, \Gamma^*)) \geq |I^+| \left(1 - \frac{\pi^2}{3C}\right) > 0 \quad (5.50)$$

if  $C > \pi^2/3$ , that we can assume. This ends the proof of (5.48).

The above mentioned Peierls type argument runs as follows: Let us write

$$\tilde{Z}_\Lambda^{++}(\underline{T}^E, -t) = \tilde{Z}_\Lambda^{++,G}(\underline{T}^E, -t) + \tilde{Z}_\Lambda^{++,VS}(\underline{T}^E, -t) \quad (5.51)$$

where,

$$\tilde{Z}_\Lambda^{++,G}(\underline{T}^E, -t) = \sum_{\underline{\sigma}_\Lambda \in \mathcal{S}_{\underline{T}^E}^G} e^{-\beta[h^{++}(\underline{\sigma}_\Lambda) - h^{++}(\bar{\sigma}(\underline{T}^E))] e^{-\beta t \sum_{i \in \Lambda} [\sigma_i - \bar{\sigma}_i(\underline{T}^E)]}} \quad (5.52)$$

and

$$\mathcal{S}_{\underline{T}^E}^G = \mathcal{S}_{\underline{T}^E}(\rho) \setminus \mathcal{S}_\Lambda^{VS}(\underline{T}^E, \rho, \epsilon_s), \quad (5.53)$$

see (5.37) and (5.41). We have

$$\tilde{Z}_\Lambda^{++,G}(\underline{T}^E, -t) \leq \sum_{\Gamma^* \sim \underline{T}^E} \sum_{\tilde{\Gamma} \sim \Gamma^* \cup \underline{T}^E} e^{-\beta H(\Gamma^*, \tilde{\Gamma}) + t \sum_i \sigma_i(\Gamma^*, \tilde{\Gamma})} \quad (5.54)$$

where the first sum is over the  $\Gamma^* \sim \underline{T}^E$  that is the set of  $\Gamma^*$  such that there exists a configuration  $\underline{\sigma}_\Lambda$  such that, recalling (5.43)

$$\Gamma^*(\underline{\sigma}_\Lambda) = \Gamma^* \text{ and } \underline{T}^E(\underline{\sigma}_\Lambda) = \underline{T}^E \quad (5.55)$$

and analogous definition for the second sum. Using (5.48)

$$\tilde{Z}_\Lambda^{++,G}(\underline{T}^E, -t) \leq \sum_{\Gamma^* \sim \underline{T}^E} e^{-\beta \frac{\epsilon_{10}(\alpha)}{c} \|\Gamma^*\|_\alpha} \sum_{\tilde{\Gamma} \sim \Gamma^* \cup \underline{T}^E} e^{-\beta H(\tilde{\Gamma}) + t \sum_i \sigma_i(\tilde{\Gamma})} \quad (5.56)$$

and therefore using (6.26)

$$\begin{aligned} \tilde{Z}_\Lambda^{++,G}(\underline{T}^E, -t) &\leq \sum_{\Gamma^* \sim \underline{T}^E} e^{-\beta \frac{\epsilon_{10}(\alpha)}{c} \|\Gamma^*\|_\alpha} Z^{++,VS}(\underline{T}^E, -t) \\ &\leq 2|\Lambda| e^{-\beta \frac{\epsilon_{10}(\alpha)}{c} (\epsilon_s |\Lambda|)^\alpha} Z^{++,VS}(\underline{T}^E, -t). \end{aligned} \quad (5.57)$$

which is (5.39). ■

## 6 Appendix 1: Triangles, contours and Polymers

In this section we regroup all the definitions and estimates that comes from [5] for the Peierls argument and from [7] for the cluster expansion.

### 6.1 Triangles configurations

In this section we start recalling the content of [5]. For all  $i^* \in \Lambda^*$ , we consider an interval  $[i^* - \frac{1}{100}, i^* + \frac{1}{100}] \subset \mathbb{R}$  and choose one point in each interval, say  $r_{i^*} \in \mathbb{R}$  in such a way that for any four distinct points  $r_j, j = 1, \dots, 4$   $|r_1 - r_2| \neq |r_3 - r_4|$ .

Given a spin configuration  $\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda$ , consider the set of its spin flip points  $\mathcal{L}^*(\sigma_\Lambda)$  *i.e.* the set of  $i^* \in \Lambda^*$  such that  $\sigma_{i^* - \frac{1}{2}} = -\sigma_{i^* + \frac{1}{2}}$  and the corresponding points  $(r_{i^*}, i^* \in \mathcal{L}^*(\sigma_\Lambda)) \subset \mathbb{R}$ .

We next embed  $\mathbb{R}$  in  $\mathbb{R}^2$  where the line containing the  $r_{i^*}$  represents the state at  $t = 0$ , and the orthogonal axis represents the evolving time of a process of growing “ $\vee$ -lines”: each point  $r_{i^*}$  branches into two twin lines growing at velocity 1 in the positive half plane, in the directions respectively of angles  $\pi/4$  and  $3/4\pi$ , until one of the two meets another line coming from a different  $r_{j^*}$ . At the instant when two branches of different  $\vee$ -lines meet, they are frozen and stop their growth, at the same instant their twin lines disappear,

while all the other  $\vee$ -line associated to the other points are undisturbed and keep growing. The collision of two lines is represented graphically in the  $(r, t)$  plane by a triangle whose basis is the interval between the two points  $r_{i^*}, r_{j^*}$ , roots of the two lines that collided, and the third vertex is the point representing the collision in the plane  $(r, t)$ .

This construction is a way to construct a pairing of spin flips with a criterion of minimal distance. Our choice of  $r_{i^*}$  makes the definition of triangles non-ambiguous.

For any finite  $\Lambda$  the process stops at a finite time  $t \leq |\Lambda| + 1$  giving rise to a configuration of triangles. The triangles will be denoted by  $T$  and a family of triangles will be denoted by  $\underline{T}$ .

**Definition 6.1** *Given a spin configuration  $\underline{\sigma}_\Lambda \in \mathcal{S}_\Lambda$ , we denote by  $\underline{T}(\underline{\sigma}_\Lambda)$  the configuration of triangles obtained following the above mentioned procedure.*

$x_-, x_+$  will denote respectively the left and right root of the associated  $\vee$ -lines. We will also write:

$$\Delta(T) = [x_-(T), x_+(T)] \cap \mathbb{Z} \quad \text{the basis of the triangle } T; \quad (6.1)$$

$$|T| = \#\{\Delta(T)\} \equiv |\Delta(T)| \quad \text{the mass of the triangle } T; \quad (6.2)$$

$$\text{sf}(T) = \{\inf(\Delta(T)) - 1, \inf(\Delta(T)), \sup(\Delta(T)), \sup(\Delta(T)) + 1\} \quad (6.3)$$

where  $\mathbb{Z}$  is equipped with its natural order;

$$\text{dist}(T, T') = \text{dist}(\text{sf}(T), \text{sf}(T')). \quad (6.4)$$

From our construction it follows that for all triangles  $T_i \neq T_j$ ,

$$\text{dist}(T_i, T_j) \geq \min(|T_i|, |T_j|). \quad (6.5)$$

In particular if a triangle  $\tilde{T}$  is interior to a triangle  $T$ , *i.e.* is such that  $\Delta(\tilde{T}) \subset \Delta(T)$ , then

$$|\tilde{T}| \leq \frac{1}{3}|T|. \quad (6.6)$$

We denote  $\mathcal{T}_\Lambda$  the set of configurations of triangles  $\underline{T} = (T_1, \dots, T_n)$  that satisfy (6.5) and such that  $\Delta(T_i) \subset \Lambda$  for all  $i \in \{1, \dots, n\}$ . Since here the spins at the boundary are specified,  $\bar{\sigma}_{-L-1} = \bar{\sigma}_{L+1} = 1$ , where we recall that  $\Lambda = [-L, +L] \cap \mathbb{Z}$ , the above construction defines a one to one map from  $\mathcal{S}_\Lambda$  to  $\mathcal{T}_\Lambda$ . In particular, if  $\mathbb{1}$  denotes the spin configuration in  $\Lambda$  constantly equals to  $+1$ ,  $\mathbb{1}$  is mapped to the empty configuration of triangles.

We say that two collections of triangles  $\underline{S}' \in \mathcal{T}_\Lambda$  and  $\underline{S} \in \mathcal{T}_\Lambda$  are compatible and we denote it by  $\underline{S}' \simeq \underline{S}$  iff  $\underline{S}' \cup \underline{S} \in \mathcal{T}_\Lambda$  (*i.e.* there exists a configuration in  $\mathcal{S}_\Lambda$  such that its corresponding collection of triangles is the collection made of all triangles in  $\underline{S}'$  and  $\underline{S}$ .)

The basic estimates for a collection of triangles  $\underline{T} \in \mathcal{T}_\Lambda$  is given in Lemma 2.1 and appendix A.1 of [5]: calling

$$H^{++}(\underline{T}) = h^{++}(\sigma_\Lambda(\underline{T})) \quad (6.7)$$

note that from (1.1), the configuration with no triangles has an energy 0 and is a ground state.

**Lemma 6.2** *For all  $0 < \alpha \leq -1 + (\log 3 / \log 2)$ , for  $J$  large enough, for all  $\underline{T} \in \mathcal{T}_\Lambda$*

$$H^{++}(\underline{T}) \geq \frac{2\zeta_\alpha}{\alpha(1-\alpha)} \sum_{T \in \underline{T}} |T|^\alpha \quad (6.8)$$



where  $\zeta_\alpha = 1 - 2(2^\alpha - 1) > 0$ .

The proof is given in Lemma 2.1 and appendix A.1 of [5] by exploiting the property (6.5). Note that in [5] formula (3.4), there is a misprint where  $\leq$  should be replaced by  $\geq$ . Note also the factor  $2/\alpha(1 - \alpha)$  that was present in appendix A in [5], see the proof of Lemma A.1 there, is missing in formulae (3.4) and (2.9) in [5]. Unfortunately this missing factor  $\alpha(1 - \alpha)$  propagate also in [7].

Let us now give the

**Proof of Lemma 2.10** Let us make a partition of the interval  $\Lambda$  in segments of size  $\epsilon_s|\Lambda|/2$ , there are less than  $2|\Lambda|/(\epsilon_s|\Lambda|)$  such segments. Given a family of triangles  $\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho)$ , see (2.12), let  $n(\underline{T}^E)$  be the maximum number of segments contained in  $\Delta(\underline{T}^E) = \cup_{\tilde{T} \in \underline{T}^E} \Delta(\tilde{T})$ . We have  $\rho|\Lambda|/(\epsilon_s|\Lambda|) \leq n(\underline{T}^E) \leq 2\rho|\Lambda|/(\epsilon_s|\Lambda|)$ . Given an integer  $n$  and a family of  $n$  such segments, the number of families of external triangles that can contain this particular family of segments is less than  $(\epsilon_s|\Lambda|)^n$ . Then taking  $\epsilon_s = |\Lambda|^{-\gamma}$  we get

$$\#\{\mathcal{T}_\Lambda^E(\rho)\} \leq \sum_{n=\rho|\Lambda|/(\epsilon_s|\Lambda|)}^{2\rho|\Lambda|/(\epsilon_s|\Lambda|)} \binom{|\Lambda|^\gamma}{n} (\epsilon_s|\Lambda|)^n \leq |\Lambda|^\gamma (|\Lambda|^{1-\gamma})^{2\rho|\Lambda|^\gamma} 2^{|\Lambda|^\gamma} \leq e^{(2-\gamma)|\Lambda|^\gamma \log |\Lambda|} \quad (6.9)$$

if  $|\Lambda|$  is large enough and how large depends only on  $\gamma$ . ■

A contour will be a family of triangles  $\Gamma \equiv \{T : T \in \Gamma\}$  that satisfy the properties listed in the Definition 6.3.

Let us define

$$|\Gamma| \equiv \sum_{T \in \Gamma} |T| \quad \text{the mass of the contour} \quad (6.10)$$

and for  $\alpha > 0$  we define

$$\|\Gamma\|_\alpha \equiv \sum_{T \in \Gamma} |T|^\alpha. \quad (6.11)$$

Recalling (6.1) to (6.4), let us denote

$$\Delta(\Gamma) \equiv \bigcup_{T \in \Gamma} \Delta(T) \quad (6.12)$$

$$x_-(\Gamma) \equiv \min_{T \in \Gamma} x_-(T) \quad (6.13)$$

$$\text{sf}(\Gamma) \equiv \bigcup_{T \in \Gamma} \text{sf}(T) \quad (6.14)$$

$$\text{dist}(\Gamma, \Gamma') \equiv \inf_{\substack{T \in \Gamma \\ T' \in \Gamma'}} \text{dist}(T, T') \quad (6.15)$$

$$\underline{T}(\Gamma) = \{T : T \in \Gamma\} \quad (6.16)$$

**Definition 6.3** Given a configuration of triangles  $\underline{T}$  in  $\mathcal{T}_\Lambda$ , a configuration of contours  $\Gamma = \Gamma(\underline{T})$  is the result of the implementation of an algorithm  $\mathcal{R}$  on the family of triangles  $\underline{T}$ , denoted by  $\underline{\Gamma}(\underline{T}) = \mathcal{R}(\underline{T})$ . It is a partition of  $\underline{T}$  whose atoms, called contours are determined by the following properties P.0, P.1, P.2 :

**P.0** Let  $\mathcal{R}(\underline{T}) \equiv (\Gamma_1, \dots, \Gamma_n)$ ,  $\Gamma_i = \{T_{j,i}, 1 \leq j \leq k_i\}$ , then  $\underline{T} = \{T_{j,i}, 1 \leq i \leq n, 1 \leq j \leq k_i\}$ .

**P.1** Contours are well separated from each other. Any pair  $\Gamma \neq \Gamma'$  in  $\mathcal{R}(\underline{T})$  verifies one of the following two alternatives. (i):  $\Delta(\Gamma) \cap \Delta(\Gamma') = \emptyset$ , or (ii):  $\Delta(\Gamma) \cap \Delta(\Gamma') \neq \emptyset$ , then either  $T(\Gamma) \subset \Delta(\Gamma')$  or  $T(\Gamma') \subset \Delta(\Gamma)$ ; moreover, supposing for instance that the former case is verified, (in which case we call  $\Gamma$  an inner contour,

$\Gamma'$  is external w.r.t  $\Gamma$ ), then for any triangle  $T'_i \in \Gamma'$ , either  $T(\Gamma) \subset T'_i$  or  $T(\Gamma) \cap T'_i = \emptyset$ . Namely either  $\Delta(\Gamma) \cap \Delta(\Gamma') = \emptyset$  or

$$\sum_{T' \in \Gamma'} \mathbb{I}_{\{T(\Gamma) \subset \Delta(T')\}} + \sum_{T \in \Gamma} \mathbb{I}_{\{T(\Gamma') \subset \Delta(T)\}} = 1. \quad (6.17)$$

In both cases

$$\text{dist}(\Gamma, \Gamma') > C \min \{|\Gamma|^3, |\Gamma'|^3\} \quad (6.18)$$

where  $C$  is a constant chosen such that, as in [5], we have

$$\sum_M \frac{4M}{[CM^3]} \leq \frac{1}{2} \quad (6.19)$$

and  $\text{dist}(\Gamma, \Gamma')$  is defined in (6.15).

**P.2 Independence.** Let  $\{\underline{T}^{(1)}, \dots, \underline{T}^{(k)}\}$ , be  $k > 1$  configurations of triangles;  $\mathcal{R}(\underline{T}^{(i)}) = \{\Gamma_j^{(i)}, j = 1, \dots, n_i\}$  the contours of the configuration  $\underline{T}^{(i)}$ . Then, if any distinct pair  $\Gamma_j^{(i)}$  and  $\Gamma_{j'}^{(i')}$  satisfies P.1

$$\mathcal{R}(\underline{T}^{(1)}, \underline{T}^{(2)}, \dots, \underline{T}^{(k)}) = \{\Gamma_j^{(i)}, j = 1, \dots, n_i; i = 1, \dots, k\}. \quad (6.20)$$

**Definition 6.4** (compatibility between contours) We say that two contours  $\Gamma, \Gamma'$  are compatible if (6.18) is verified. We denote

$$\begin{aligned} \Gamma \sim \Gamma' &\iff \Gamma, \Gamma' \text{ are compatible} \\ \Gamma \not\sim \Gamma' &\iff \Gamma, \Gamma' \text{ are incompatible.} \end{aligned} \quad (6.21)$$

Therefore we have a bijection between spin configurations in  $\mathcal{S}_\Lambda$  and triangles in  $\mathcal{T}_\Lambda$  and another one between  $\mathcal{T}_\Lambda$  and its image by  $\mathcal{R}$ , in particular there is a one-to-one correspondence between spin configurations and contour configurations. We denote by  $\mathcal{G}_\Lambda$  the set of all possible configurations of compatible contours associated to  $\mathcal{S}_\Lambda$ .

Moreover denoting by  $\underline{T}(\underline{\Gamma})$  the configuration of triangles that are in  $\underline{\Gamma}$ , we define

$$H^{++}(\underline{\Gamma}) \equiv H^{++}(\underline{T}(\underline{\Gamma})) \quad (6.22)$$

For  $\underline{\Gamma} \in \mathcal{G}_\Lambda$  let  $\underline{\sigma}_\Lambda(\underline{\Gamma})$  be the corresponding spin configuration.

**Definition 6.5** Given a collection of contours  $\underline{\Gamma}$ , a contour  $\Gamma \in \underline{\Gamma}$  is external with respect to  $\underline{\Gamma}$  if each triangle of  $\Delta(\Gamma)$  is not contained in some triangle that belongs to the others contours of  $\underline{\Gamma}$ .

$$\forall T \in \Gamma, \quad \Delta(T) \not\subset \bigcup_{\Gamma' \in \underline{\Gamma}: \Gamma' \neq \Gamma} \Delta(\Gamma') \quad (6.23)$$

A contour which is not external is called internal.

The following Proposition regroups all the basic estimates that we need. It is proved in [5].

**Proposition 6.6** If  $0 \leq \alpha < \alpha_+ = -1 + (\log 3)/(\log 2)$ , then for any contour  $\Gamma$ ,

$$H^{++}(\Gamma) \geq \frac{\zeta_\alpha}{\alpha(1-\alpha)} \|\Gamma\|_\alpha \quad (6.24)$$

with  $\zeta_\alpha$  as in Lemma 6.2. and  $\|\Gamma\|_\alpha$  is defined by (6.11).

For any family of compatible contours  $(\Gamma_0, \underline{\Gamma})$ ,

$$H^{++}(\Gamma_0 \cup \underline{\Gamma}) - H^{++}(\underline{\Gamma}) \geq \delta \|\Gamma_0\|_\alpha \quad (6.25)$$

where if  $0 < \alpha < \alpha_+$ ,  $\delta \equiv c_{10}(\alpha)/C$  with  $C$  as in (6.18) and  $c_{10}(\alpha) = 2\pi^2(3\alpha(1-\alpha))^{-1}$

There exists a  $b_0 \equiv b_0(\alpha)$  such that for all  $b \geq b_0$ , for all integers  $m \geq 1$ ,

$$\sum_{\substack{\Gamma: x_-(\Gamma)=0 \\ |\Gamma|=m}} e^{-b\|\Gamma\|_\alpha} \leq 2e^{-bm^\alpha}. \quad (6.26)$$

### Cluster expansion

Let us recall the result on the cluster expansion given in [7]. Section 4 of [7] gives the derivation of the partition function as that of a gas of polymers with an hard core condition and Proposition 5.4 there proves that the cluster expansion for the logarithm of the partition function is convergent is

$$\sum_{R \ni 0} \xi^{++}(R) \ll 1 \quad (6.27)$$

where  $\xi^{++}(R)$  is the activity of the polymer  $R$  and the sum is over all the polymers containing the origin.

In [7] it is proved that (6.27) follows basically from (6.8), or modification of it as (6.24) by taking  $\beta$  large enough. The consequence of the convergence of the cluster expansion is that we have

$$\log Z_\Lambda^{++} = \sum_{x \in \Lambda} \xi^{++}(R_x^1) [1 + \mathcal{B}(x, ++)] \quad (6.28)$$

where:

1)  $R_x^1$  is the triangle of size 1 located in  $x$ , i.e. is the simplest contour and also the simplest polymer made of a singleton and

$$\xi^{++}(R_x^1) = e^{-2\beta(J+\zeta(2-\alpha))}. \quad (6.29)$$

where  $\zeta(2-\alpha)$  is the Riemann zeta function and  $J$  is defined in (1.2).

2)  $\mathcal{B}(x, ++)$  is an absolutely convergent series

$$|\mathcal{B}(x, ++)| \leq e^{-\frac{\beta}{32}(\frac{\zeta_\alpha}{\alpha(1-\alpha)} - 3\delta)} \quad (6.30)$$

where  $\zeta_\alpha$  is the same as in lemma 6.2 and  $\delta$  is the same as in (6.25).

Here we are considering instead of (6.28) the logarithm of a constrained partition function as

$$\tilde{Z}_\Lambda^{++}(\underline{\mathcal{T}}^E, t) = \sum_{\underline{\sigma}_\Lambda \in \mathcal{S}_{\underline{\mathcal{T}}^E}} e^{-\beta[h^{++}(\underline{\sigma}_\Lambda) - h^{++}(\bar{\sigma}(\underline{\mathcal{T}}^E))] + \beta t \sum_{i \in \Lambda} (\sigma_i - \bar{\sigma}_i(\underline{\mathcal{T}}^E))} \quad (6.31)$$

the partition function restricted to the set of configurations  $\mathcal{S}_{\underline{\mathcal{T}}^E}$ , see (2.9), where an external field  $t$  is present. Here  $\underline{\mathcal{T}}^E \in \mathcal{T}_\Lambda^E(\rho)$ , see (2.12) for some  $0 < \rho < 1$ .

In the case  $t = 0$ , the ground state is  $\bar{\sigma}(\underline{\mathcal{T}}^E)$ , see (2.8) and (2.11) and the energy fluctuations  $\underline{\mathcal{T}}$  are in

$$\mathcal{T}_\Lambda^f(\underline{\mathcal{T}}^E) = \left\{ \tilde{\underline{\mathcal{T}}} \in \mathcal{T}_\Lambda : \tilde{\underline{\mathcal{T}}} \sim \underline{\mathcal{T}}^E, \underline{\mathcal{T}}^E[\underline{\sigma}_\Lambda(\tilde{\underline{\mathcal{T}}} \cup \underline{\mathcal{T}}^E)] = \underline{\mathcal{T}}^E \right\} \quad (6.32)$$

that is the set of family the triangles  $\tilde{\underline{\mathcal{T}}}$  that are compatible with  $\underline{\mathcal{T}}^E$ , their presences does not modify the family of external triangles  $\underline{\mathcal{T}}^E$  in particular the triangles of  $\tilde{\underline{\mathcal{T}}}$  that are external to  $\underline{\mathcal{T}}^E$  are all small.

Let us define, for  $\underline{T} \in \mathcal{T}_\Lambda^f(\underline{T}^E)$

$$H_{\underline{T}^E}^{++}(\underline{T}) = h^{++}(\underline{\sigma}(\underline{T} \cup \underline{T}^E)) - h^{++}(\bar{\sigma}(\underline{T}^E)) \quad (6.33)$$

that will be simply denoted  $H^{++}(\underline{T})$  if no confusion could arise. One can check that if  $\underline{\sigma}_\Lambda \in S_{\underline{T}^E}$  and  $\underline{T} = \underline{T}(\underline{\sigma}_\Lambda)$  is the associated family of triangles in  $\mathcal{T}_\Lambda^f(\underline{T}^E)$  then

$$h^{++}(\underline{\sigma}_\Lambda) - h^{++}(\bar{\sigma}(\underline{T}^E)) - t \sum_{i \in \Lambda} (\sigma_i - \bar{\sigma}_i(\underline{T}^E)) \geq H^{++}(\underline{T}) - |t| \sum_{T \in \underline{T}} |T|. \quad (6.34)$$

Therefore, in presence of a magnetic field  $t$  as in (6.31), an analogous of the condition (6.8) is: For all  $\underline{T} \in \mathcal{T}_\Lambda^f(\underline{T}^E)$

$$H^{++}(\underline{T}) - |t| \sum_{T \in \underline{T}} |T| \geq \frac{\zeta_\alpha}{\alpha(1-\alpha)} \sum_{T \in \underline{T}} |T|^\alpha, \quad (6.35)$$

This condition is satisfied if for all  $\underline{T} \in \mathcal{T}_\Lambda^f(\underline{T}^E)$

$$\frac{2\zeta_\alpha}{\alpha(1-\alpha)} |T|^\alpha - |t| |T| \geq \frac{\zeta_\alpha}{\alpha(1-\alpha)} |T|^\alpha, \quad (6.36)$$

i.e. if

$$|t| \leq \frac{\zeta_\alpha}{\alpha(1-\alpha)} \left(\frac{1}{3} |\underline{T}^E|\right)^{\alpha-1} \leq \inf_{T \in \mathcal{T}_\Lambda^f(\underline{T}^E)} \frac{\zeta_\alpha}{\alpha(1-\alpha)} |T|^{\alpha-1}. \quad (6.37)$$

where we have used (3.29).

We regroup in the following Lemma the contribution of the dominant term of the constrained free energy obtained by cluster expansion. The proof is the same as the one of (6.28) given in [7]. On the other hand since all polymers that can occur in the expansion of  $\log \tilde{Z}_\Lambda^{++}(\underline{T}^E, t)$  occur also in the expansion of (6.28), the error terms satisfy the same bound.

**Lemma 6.7** *There exists a  $\beta_1 = \beta_1(\alpha)$  such that if  $\beta \geq \beta_1(\alpha)$ , for all  $\rho \in [0, 1] \cap Q_\Lambda$ , for  $\epsilon_s = |\Lambda|^{-\gamma}$  with  $0 < \gamma < 2/3$  and for  $\Lambda$  so large to have  $\rho > \epsilon_s$  then, for  $t \leq \zeta_\alpha 3^{1-\alpha} / (4\alpha(1-\alpha)(\rho|\Lambda|)^{1-\alpha}) \equiv t_{\Lambda,1}^*(\rho)$  where  $\zeta_\alpha = 1 - 2(2^\alpha - 1) > 0$  we have, for all  $\underline{T}^E \in \mathcal{T}_\Lambda^E(\rho)$ ,*

$$\log \tilde{Z}_\Lambda^{++}(\underline{T}^E, t) = \sum_{x \in \Lambda} \mathbb{I}_{\{\text{dist}(x, \text{sf}(\underline{T}^E)) \geq 1\}} \xi^{\bar{\sigma}(\underline{T}^E)}(x) e^{-2\beta t \bar{\sigma}_x(\underline{T}^E)} (1 + \mathcal{B}(x, \underline{T}^E, t)) \quad (6.38)$$

where  $\xi^{\bar{\sigma}(\underline{T}^E)} = e^{-\beta[h^{++}(T^{\{x\}}\bar{\sigma}(\underline{T}^E)) - h^{++}(\bar{\sigma}(\underline{T}^E))]}$  and  $T^{\{x\}}\bar{\sigma}(\underline{T}^E)$  is the configuration equals to  $\bar{\sigma}(\underline{T}^E)$  everywhere but at  $x$  where the spin  $\bar{\sigma}_x(\underline{T}^E)$  is reversed and  $\mathcal{B}(x, \underline{T}^E, t)$  can be written as an explicit absolutely convergent series and satisfies

$$|\mathcal{B}(x, \underline{T}^E, t)| \leq e^{-\frac{\beta}{64} \left(\frac{\zeta_\alpha}{\alpha(1-\alpha)}\right)^{-3\delta}} \quad (6.39)$$

where as in Proposition 6.6,

$$\delta = 2\pi^2(3\alpha(1-\alpha))^{-1} C^{-1} \quad (6.40)$$

and  $C$  is the constant that appears in the definition of contours (6.18).

**Remark:** For future reference we notice that: if  $x \in \Lambda$

$$\xi^{\bar{\sigma}(\underline{T}^E)}(x) = \xi^{++}(\beta) \times \begin{cases} e^{2\beta \sum_{y \in \mathbf{z} \setminus \Delta(\underline{T}^E)} J(x-y)}, & \text{if } x \in \Delta(\underline{T}^E) \setminus \text{sf}(\underline{T}^E); \\ e^{2\beta \sum_{y \in \Delta(\underline{T}^E) \setminus \{x\}} J(x-y)}, & \text{if } x \notin \Delta(\underline{T}^E) \cup \text{sf}(\underline{T}^E). \end{cases} \quad (6.41)$$

Let us now prove Lemmata 2.13 and 3.2

For the proof of (2.24), given  $\underline{t} = (t_i, i \in \Lambda) \in \mathbb{R}^\Lambda$ , let us define for  $\underline{\mathcal{T}}^E \in \mathcal{T}_\Lambda^E(\rho)$ ,

$$\tilde{Z}_\Lambda^{++}(\underline{\mathcal{T}}^E, \underline{t}) = \sum_{\underline{\sigma}_\Lambda \in \mathcal{S}_{\underline{\mathcal{T}}^E}} e^{-\beta[h^{++}(\underline{\sigma}_\Lambda) - h^{++}(\bar{\sigma}(\underline{\mathcal{T}}^E))] e^{\beta \sum_{i \in \Lambda} t_i (\sigma_i - \bar{\sigma}_i(\underline{\mathcal{T}}^E))}. \quad (6.42)$$

By elementary computations, one can check that

$$\mu_\Lambda^{++}[\sigma_i | \mathcal{S}_{\underline{\mathcal{T}}^E}] = \bar{\sigma}_i(\underline{\mathcal{T}}^E) + \frac{1}{\beta} \frac{\partial}{\partial t_i} \left[ \ln \tilde{Z}_\Lambda^{++}(\underline{\mathcal{T}}^E, \underline{t}) \right] \Big|_{\underline{t}=0}. \quad (6.43)$$

Then it is immediate to check that

$$\frac{1}{|\Lambda|} \sum_{i \in \Lambda} \bar{\sigma}_i(\underline{\mathcal{T}}^E) = (1 - 2\rho) \quad (6.44)$$

since  $\bar{\sigma}(\underline{\mathcal{T}}^E)$  is made of  $\rho|\Lambda|$  sites in  $\Lambda$  with  $-1$  and what remains is  $+1$ . Now recalling (6.38), the dominant terms that comes from the second term in (6.43) gives a contribution to the mean

$$\frac{1}{|\Lambda|} \sum_{i \in \Lambda} (-2\bar{\sigma}_x(\underline{\mathcal{T}}^E)) \mathbb{1}_{\{\text{dist}(x, \text{sf}(\underline{\mathcal{T}}^E)) \geq 1\}} \xi^{\bar{\sigma}(\underline{\mathcal{T}}^E)}(x). \quad (6.45)$$

Recalling (1.5) and writing  $\xi^{\bar{\sigma}(\underline{\mathcal{T}}^E)}(x) = [\xi^{\bar{\sigma}(\underline{\mathcal{T}}^E)}(x) - \xi^{++}(\beta)] + \xi^{++}(\beta)$  in (6.45), to the term with  $\xi^{++}(\beta)$  corresponds

$$\frac{1}{|\Lambda|} \sum_{i \in \Lambda} (-2\bar{\sigma}_x(\underline{\mathcal{T}}^E)) \xi^{++}(\beta) = -2\xi^{++}(\beta)(1 - 2\rho). \quad (6.46)$$

Recalling (1.4), we get (2.24) since, by similar arguments as in the end of section 4

$$\frac{2}{|\Lambda|} \sum_{i \in \Lambda} \left| \xi^{\bar{\sigma}(\underline{\mathcal{T}}^E)}(x) - \xi^{++}(\beta) \right| \leq \frac{10\xi^{++}(\beta)}{\alpha(1 - \alpha)} |\Lambda|^{\alpha-1}. \quad (6.47)$$

from which we get (2.24). Using (2.22) we get (2.23).

Let us now complete the proof the Lemma 3.2. It is enough to have an estimate uniform in the magnetic field  $r$  which satisfies (3.20) or (3.22) for the two points truncated correlation function (3.25). For notational simplicity, we consider only the case of (3.23), the case of (3.21) can be proved *mutatis mutandis*. Here we have to start with a  $T_0$  with  $|T_0| = \rho|\Lambda|$ . Considering  $\tilde{Z}_\Lambda^{++}(T_0, \underline{t})$  as in (6.42) we take first  $\underline{t} = r + \underline{t}(i, j)$  where  $t_k(i, j) = 0$ , if  $k \neq i, j$  while  $t_i(i, j) = t_i$ ,  $t_j(i, j) = t_j$  and then we get easily

$$\mu_\Lambda^\dagger(r)[\sigma_i, \sigma_j | \mathcal{S}_{T_0}] = \frac{1}{\beta^2} \frac{\partial}{\partial t_i \partial t_j} \log \tilde{Z}_\Lambda^{++}(T_0, r + \underline{t}(i, j)) \Big|_{t(i, j)=0} \quad (6.48)$$

Note first that if the presence of  $T_0$  impose that  $\sigma_i$  or  $\sigma_j$  are fixed, that is when  $i$  or  $j$  are in  $\text{sf}(T_0)$ , see (6.3), then the corresponding truncated correlation is zero. On the other hand, assuming that  $|i - j| \geq 2$ , depending if  $|i - j|$  is smaller or larger than  $C$ , the dominant terms in (6.48) coming from the cluster expansion of  $\log \tilde{Z}_\Lambda^{++}(T_0, r + \underline{t}(i, j))$  are different. In the first case it comes from a single polymer made of a single contour with two unit triangles located at site  $i$  and  $j$ , say  $T_i, T_j$  that should be compatible with the presence of  $T_0$ . In the second case it comes from a single polymer made of two contours each one made of an unit triangle located at site  $i$  and  $j$ , that should be compatible with the presence of  $T_0$ .

It follows from [7], appendix 2, and proposition 5.4 there, that the corresponding activity is

$$\xi^{\bar{\sigma}(T_0)}(i)\xi^{\bar{\sigma}(T_0)}(j)e^{-\beta(r+t_i)2\bar{\sigma}_i(T_0)}e^{-\beta(r+t_j)2\bar{\sigma}_j(T_0)}\left[e^{-\beta[H_{T_0}^{++}(T_i,T_j)-H_{T_0}^{++}(T_i)-H_{T_0}^{++}(T_j)]}-1\right] \quad (6.49)$$

where we have used (6.33). It can be checked by simple algebra that

$$H_{T_0}^{++}(T_i,T_j)-H_{T_0}^{++}(T_i)-H_{T_0}^{++}(T_j)=-\frac{2\bar{\sigma}_i(T_0)\bar{\sigma}_j(T_0)}{|i-j|^{2-\alpha}}. \quad (6.50)$$

Therefore, taking into account the error terms that come from the cluster expansion, we get

$$\mu_{\Lambda}^+(r)[\sigma_i,\sigma_j|\mathcal{S}_{T_0}] = \xi^{\bar{\sigma}(T_0)}(i)\xi^{\bar{\sigma}(T_0)}(j)e^{-\beta r 2\bar{\sigma}_i(T_0)}e^{-\beta r 2\bar{\sigma}_j(T_0)}4\bar{\sigma}_i(T_0)\bar{\sigma}_j(T_0)\left[e^{\frac{2\bar{\sigma}_i(T_0)\bar{\sigma}_j(T_0)}{|i-j|^{2-\alpha}}}-1\right]\left(1\pm e^{-\frac{\beta}{64}\left(\frac{\zeta_{\alpha}}{\alpha(1-\alpha)}-3\delta\right)}\right) \quad (6.51)$$

inserting it in (3.24), it implies (3.23). ■

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