HAL
open science

Recovery of non compactly supported coefficients of an elliptic equation on an infinite waveguide

Yavar Kian

To cite this version:

Yavar Kian. Recovery of non compactly supported coefficients of an elliptic equation on an infinite waveguide. Journal of the Institute of Mathematics of Jussieu, 2020, 19 (5), pp.1573 - 1600. 10.1017/S1474748018000488 . hal-01583151

HAL Id: hal-01583151

https://hal.science/hal-01583151

Submitted on 6 Sep 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

RECOVERY OF NON COMPACTLY SUPPORTED COEFFICIENTS OF AN ELLIPTIC EQUATION ON AN INFINITE WAVEGUIDE

YAVAR KIAN

Abstract. We consider the unique recovery of a non compactly supported and non periodic perturbation of a Schrödinger operator in an unbounded cylindrical domain, also called waveguide, from boundary measurements. More precisely, we prove recovery of general class of electric potentials from the partial Dirichlet-to-Neumann map, where the Dirichlet data is supported on slightly more than half of the boundary and the Neumann data is taken on the other half of the boundary. We apply this result in different context including recovery of some general class of coefficients from measurements on a bounded subset and recovery of an electric potential, supported on an unbounded cylinder, of a Schrödinger operator in a slab.

Keywords : Elliptic equation, scalar potential, unbounded domain, infinite cylindrical waveguide, slab, partial data, Carleman estimate.

Mathematics subject classification 2010 : 35R30, 35J15.

1. Introduction

Let $\Omega := \omega \times \mathbb{R}$, where ω is a bounded open set of $\mathbb{R}^2$, with C^2-boundary. Throughout this paper we denote the point $x \in \Omega$ by $x = (x', x_3)$, where $x_3 \in \mathbb{R}$ and $x' := (x_1, x_2) \in \omega$. Given $q \in L^\infty(\Omega)$ such that 0 is not in the spectrum of $-\Delta + q$ with Dirichlet boundary condition, we consider the following boundary value problem (BVP in short):

$$\begin{cases} (-\Delta + q)v &= 0, \quad \text{in } \Omega, \\ v &= f, \quad \text{on } \Gamma := \partial\Omega. \end{cases} \tag{1.1}$$

Since $\Gamma = \partial\omega \times \mathbb{R}$, the outward unit vector ν normal to Γ reads

$$\nu(x', x_3) = (\nu'(x'), 0), \ x = (x', x_3) \in \Gamma,$$

where ν' is the outer unit normal vector of $\partial\omega$. For simplicity, we refer to ν for both exterior unit vectors normal to $\partial\omega$ and to Γ. For $\theta_0 \in \mathbb{S}^1 := \{y \in \mathbb{R}^2;\ |y| = 1\}$ fixed, we introduce the θ_0-illuminated (resp., θ_0-shadowed) face of $\partial\omega$, as

$$\partial\omega_{\theta_0}^- := \{x \in \partial\omega;\ \theta_0 \cdot \nu(x) \leqslant 0\} \text{ (resp., } \partial\omega_{\theta_0}^+ = \{x \in \partial\omega;\ \theta_0 \cdot \nu(x) \geqslant 0\}). \tag{1.2}$$

Here and in the remaining part of this text, we denote by $x \cdot y := \sum_{j=1}^k x_j y_j$ the Euclidian scalar product of any two vectors $x := (x_1, \ldots, x_k)$ and $y := (y_1, \ldots, y_k)$ of $\mathbb{R}^k$, for $k \in \mathbb{N}^*$, and we put $|x| := (x \cdot x)^{1/2}$.

Set $G := G' \times \mathbb{R}$, where G' is an arbitrary open set of $\partial\omega$ containing the compact set $\partial\omega_{\theta_0}^-$ in $\partial\omega$ and consider $K = K' \times \mathbb{R}$ with K' an arbitrary open set containing the compact set $\partial\omega_{\theta_0}^+$ in $\partial\omega$. In the present paper we seek determination of q from the knowledge of the partial Dirichlet-to-Neumann (DN in short) map

$$\Lambda_q : f \mapsto \partial_\nu v_{|G}, \tag{1.3}$$

where $\partial_\nu v(x) := \nabla v(x) \cdot \nu(x)$ is the normal derivative of the solution v to (1.1), computed at $x \in \Gamma$ and $\text{supp}(f) \subset K$.

1.1. **Physical motivations.** Let us recall that the problem under consideration in this paper is related to the so called electrical impedance tomography (EIT in short) and its several applications in medical imaging and others. Note that the specific geometry of infinite cylinder or closed waveguide can be considered for problems of transmission to long distance or transmission through particular structures, where the ratio length-to-diameter is really high, such as nanostructures. In this context, the problem addressed in this paper can correspond to the unique recovery of an impurity perturbing the guided propagation (see [11, 25]). Let us also observe that in Corollary 1.4, we show how one can apply our result to the problem stated in a slab, which is frequently used for modeling propagation in shallow-ocean acoustics (e.g. [1]), for coefficients supported in an infinite cylinder.

1.2. **Known results.** Since the pioneer work of [7], the celebrated Calderón or the EIT problem has been growing in interest. In [43], Sylvester and Uhlmann provide one of the first and most important results related to this problem. They actually proved, in dimension $n \geqslant 3$, the unique recovery of a smooth conductivity from the full DN map. Since then, several authors extended this result in several way. The determination of an unknown coefficient from partial knowledge of the DN map was first addressed in [6] and extended by Kenig, Sjöstrand and Uhlmann in [27] to the recovery of a potential from restriction of data to the back and the front face illuminated by a point lying outside the convex hull of the domain. In dimension two, similar results with full and partial data have been stated in [5, 21, 22]. We mention also, without being exhaustive, the work of [8, 9, 15, 39, 40] dealing with the stability issue associated to this problem and some results inspired by this approach for other PDEs stated in [13, 20, 29, 30, 31].

Let us remark that all the above mentioned results have been proved in a bounded domain. It appears that only a small number of mathematical papers deal with inverse boundary value problems in an unbounded domain. Combining results of unique continuation with complex geometric optics (CGO in short) solutions and a Carleman estimate borrowed from [6], Li and Uhlmann proved in [37] the unique recovery of compactly supported electric potentials of the stationary Schrödinger operator in a slab from partial boundary measurements. In [34], the authors extended this result to magnetic Schrödinger operators and [10] treated the stability issue for this inverse problem. We mention also [35, 36] dealing with more general Schrödinger equations, the work of [44] for bi-harmonic operators and the recovery of an embedded object in a slab treated by [23, 41]. More recently, [16, 17] proved the stable recovery of coefficients periodic along the axis of an infinite cylindrical domain. Finally, we mention [3, 4, 18, 26, 28, 32, 33] dealing with determination of non-compactly supported coefficients appearing in different PDEs from boundary measurements.

1.3. **Statement of the main result and applications.** Prior to stating the main result of this article we first recall some results stated in [6, 16, 17] related to the well-posedness of the BVP (1.1) in the space $H_\Delta(\Omega) := \{u \in L^2(\Omega);\ \Delta u \in L^2(\Omega)\}$ with the norm

$$\|u\|^2_{H_\Delta(\Omega)} := \|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\,.$$

Since Ω is unbounded, for $X = \omega$ or $X = \partial\omega$ and any $s > 0$, we define the space $H^s(X \times \mathbb{R})$ by

$$H^s(X \times \mathbb{R}) := L^2(X; H^s(\mathbb{R})) \cap L^2(\mathbb{R}; H^s(X)).$$

We define also $H^{-s}(\Gamma)$ to be the dual space of $H^s(\Gamma)$. Combining [6, Lemma 1.1] with [16, Lemma 2.2], we deduce that the map

$$\mathcal{T}_0 u := u_{|\Gamma}\ (\text{resp., } \mathcal{T}_1 u := \partial_\nu u_{|\Gamma}),\ u \in C_0^\infty(\mathbb{R}^3),$$

extend into a continuous function $\mathcal{T}_0 : H_\Delta(\Omega) \to H^{-\frac{1}{2}}(\Gamma)$ (resp., $\mathcal{T}_1 : H_\Delta(\Omega) \to H^{-\frac{3}{2}}(\Gamma)$). We set the space

$$\mathcal{H}(\Gamma) := \mathcal{T}_0 H_\Delta(\Omega) = \{\mathcal{T}_0 u;\ u \in H_\Delta(\Omega)\},$$

and notice from [16, Lemma 2.2] that $\mathcal{T}_0$ is bijective from $B := \{u \in L^2(\Omega);\ \Delta u = 0\}$ onto $\mathscr{H}(\Gamma)$. Thus, with reference to [6, 38], we consider

$$\|f\|_{\mathcal{H}(\Gamma)} := \left\|\mathcal{T}_0^{-1} f\right\|_{H_\Delta(\Omega)} = \left\|\mathcal{T}_0^{-1} f\right\|_{L^2(\Omega)}\,. \tag{1.4}$$

We define also $\mathcal{H}_K(\Gamma) := \{f \in \mathcal{H}(\Gamma) : \ \mathrm{supp}(f) \subset K\}$. Then, in view of [16, Proposition 1.1], assuming that 0 is not in the spectrum of $-\Delta + q$ with Dirichlet boundary condition on Ω, for any $f \in \mathscr{H}(\Gamma)$ we deduce that the BVP (1.1) admits a unique solution $v \in L^2(\Omega)$. Moreover, the DN map $\Lambda_q : f \mapsto \mathcal{T}_1 v_{|G}$ is a bounded operator from $\mathscr{H}_K(\Gamma)$ into $H^{-\frac{3}{2}}(G)$.

The main result of this paper can be stated as follows.

Theorem 1.1. *Let $q_1, q_2 \in L^\infty(\Omega)$ be such that $q_1 - q_2 \in L^1(\Omega)$ and 0 is not in the spectrum of $-\Delta + q_j$, $j = 1, 2$, with Dirichlet boundary condition on Ω. Then the condition*

$$\Lambda_{q_1} = \Lambda_{q_2} \tag{1.5}$$

implies $q_1 = q_2$.

From the main result of this paper, stated in Theorem 1.1, we deduce three other results related to other problems stated in an unbounded domain. The first application that we consider corresponds to the Calderón problem stated in the unbounded domain Ω. In order to state this problem, for $a_* \in (0, +\infty)$ and $a_0 \in W^{2,\infty}(\Omega)$ satisfying $a_0 \geqslant a_*$, we introduce the set of functions

$$\mathcal{A} := \{a \in \mathcal{C}^1(\overline{\Omega}) \cap H^2_{loc}(\Omega) : \ a \geqslant a_*, \ \Delta\left(a^{\frac{1}{2}}\right) - \Delta\left(a_0^{\frac{1}{2}}\right) \in L^1(\Omega) \cap L^\infty(\Omega)\}$$

and, for $a \in \mathcal{A}$, the BVP

$$\begin{cases} -\mathrm{div}(a\nabla u) &= 0, \quad \text{in } \Omega, \\ u &= f, \quad \text{on } \Gamma. \end{cases} \tag{1.6}$$

Recall that for any $a \in \mathcal{A}$ and any $f \in H^{\frac{1}{2}}(\Gamma)$, the BVP (1.6) admits a unique solution $u \in H^1(\Omega)$ for each $f \in H^{\frac{1}{2}}(\Gamma)$. Moreover, the full DN map associated with (1.6), defined by $f \mapsto a\mathcal{T}_1 u$ is a bounded operator from $H^{\frac{1}{2}}(\Gamma)$ to $H^{-\frac{1}{2}}(\Gamma)$. Here, we rather consider the partial DN map,

$$\Sigma_a : f \in H^{\frac{1}{2}}(\Gamma) \cap a^{-\frac{1}{2}}(\mathcal{H}_K(\Gamma)) \mapsto a\mathcal{T}_1 u_{|G}, \tag{1.7}$$

where $a^{-\frac{1}{2}}(\mathcal{H}_K(\Gamma)) := \{a^{-\frac{1}{2}} f; \ f \in \mathcal{H}_K(\Gamma)\}$. The first application of Theorem 1.1 claims unique recovery of a conductivity $a \in \mathcal{A}$, from the knowledge of Σ_a. It is stated as follows.

Corollary 1.2. *Let ω be connected and pick $a_j \in \mathcal{A}$, for $j = 1, 2$, obeying*

$$a_1(x) = a_2(x), \ x \in \Gamma \tag{1.8}$$

and

$$\partial_\nu a_1(x) = \partial_\nu a_2(x), \ x \in K \cap G. \tag{1.9}$$

Then the condition $\Sigma_{a_1} = \Sigma_{a_2}$ implies $a_1 = a_2$.

For our second application we consider the recovery of potentials that are known in the neighborhood of the boundary outside a compact set. In the spirit of [2], we can improve Theorem 1.1 in a quite important way in that case. More precisely, we fix $R > 0$ and we consider γ_1 an arbitrary open subsets of $K' \times (-\infty, -R)$, γ_2 an open subsets of $\partial\omega \times (-\infty, -R)$, γ_1' an open subset of $K' \times (R, +\infty)$ and γ_2' an open subsets of $\partial\omega \times (R, +\infty)$. Then, we consider the partial DN map given by

$$\Lambda^*_{q,R} : \{h \in \mathcal{H}(\Gamma) : \ \mathrm{supp}(h) \subset (K' \times [-R, R]) \cup \gamma_1 \cup \gamma_1'\} \ni f \mapsto \mathcal{T}_1 v_{|(\partial\omega \times [-R,R]) \cup \gamma_2 \cup \gamma_2'}.$$

Our second application can be stated as follows

Corollary 1.3. *Let ω be connected, $R > 0$, $\delta \in (0, R)$, $q_1, q_2 \in L^\infty(\Omega)$ be such that $q_1 - q_2 \in L^1(\Omega)$ and 0 is not in the spectrum of $-\Delta + q_j$, $j = 1, 2$, with Dirichlet boundary condition on Ω. We fix $\omega_{1,*}$, $\omega_{2,*}$ two arbitrary $\mathcal{C}^2$ open and connected subset of ω satisfying $\partial\omega \subset (\partial\omega_{1,*} \cap \partial\omega_{2,*})$. We consider also $\Omega_{j,*}$, $j = 1, 2$, two $\mathcal{C}^2$ open and connected subset of Ω such that*

$$\omega_{1,*} \times (-\infty, -R) \subset \Omega_{1,*} \subset \omega_{1,*} \times (-\infty, \delta - R), \quad \omega_{2,*} \times (R, +\infty) \subset \Omega_{2,*} \subset \omega_{2,*} \times (R - \delta, +\infty)$$

and we assume that

$$q_1(x) = q_2(x), \ x \in \Omega_{1,*} \cup \Omega_{2,*}. \tag{1.10}$$

Then the condition $\Lambda^*_{q_1,R} = \Lambda^*_{q_2,R}$ *implies* $q_1 = q_2$.

In our third application we consider the recovery of potentials, supported in an infinite cylinder, appearing in a stationary Schrödinger equation on a slab. More precisely, for $L > 0$, we consider the set $\mathcal{O} := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 \in (0, L)\}$, then assuming that $q \in L^\infty(\mathcal{O})$ and that 0 is not in the spectrum of $-\Delta + q$ with Dirichlet boundary condition on $\mathcal{O}$, we consider the problem

$$\begin{cases} (-\Delta + q)v &= 0, \quad \text{in } \mathcal{O}, \\ v_{|x_1=0} &= 0, \\ v_{|x_1=L} &= f. \end{cases} \tag{1.11}$$

Fixing $r > 0$, $\partial\mathcal{O}_+ := \{(L, x_2, x_3) : \ x_2, x_3 \in \mathbb{R}\}$ and $\partial\mathcal{O}_{-,r} := \{(0, x_2, x_3) : \ x_2 \in (-r, r), \ x_3 \in \mathbb{R}\}$, we associate to this problem the partial DN map

$$\mathcal{N}_{q,r} : H^{\frac{1}{2}}(\partial\mathcal{O}_+) \ni f \mapsto \partial_{x_1} v_{|\partial\mathcal{O}_{-,r}}$$

Then, we prove the following result.

Corollary 1.4. *Let* $q_1, q_2 \in L^\infty(\mathcal{O})$ *be such that* $q_1 - q_2 \in L^1(\mathcal{O})$ *and* 0 *is not in the spectrum of* $-\Delta + q_j$, $j = 1, 2$, *with Dirichlet boundary condition on* $\mathcal{O}$. *Moreover, assume that there exists* $r \in (0, +\infty)$ *such that*

$$q_1(x_1, x_2, x_3) = q_2(x_1, x_2, x_3) = 0, \quad (x_1, x_2, x_3) \in \{(y_1, y_2, y_3) \in \mathcal{O} : \ |y_2| \geqslant r\}. \tag{1.12}$$

Then, for any $R > r$, *the condition*

$$\mathcal{N}_{q_1,R} = \mathcal{N}_{q_2,R} \tag{1.13}$$

implies $q_1 = q_2$.

1.4. **Comments about the main result and the applications.** To our best knowledge this paper is the first paper proving recovery of coefficients that are neither compactly supported nor periodic for elliptic equations in unbounded domains from boundary measurements. Indeed, beside the present paper it seems that only these two cases have been addressed so far (see [16, 17, 34, 37]).

Like several other papers, the main tools in our analysis are suitable solutions of the equation also called complex geometric optics (CGO in short) solutions combined with Carleman estimates. It has been proved by [16, 17, 34, 37] that for compactly supported or periodic coefficients one can apply unique continuation or Floquet decomposition in order to transform the problem on an unbounded domain into a problem on a bounded domain. Then, one can use the CGO solutions for the problem on the bounded domain in order to prove the recovery of the coefficients under consideration. For more general class of coefficients, one can not apply such arguments and the construction of CGO solutions for the problem on an unbounded domain seems unavoidable. In this paper, using a suitable localization in space, that propagates along the infinite direction of the unbounded cylindrical domain, we introduce, for what seems to be the first time, CGO solutions that can be directly applied to the inverse problem on the unbounded domain. This makes a difference with previous related works and it allows also to derive results like Corollary 1.3 where the recovery of non compactly supported coefficients is proved by mean of measurements on a bounded subset of the unbounded boundary. The construction of the CGO solutions in consideration requires also some extension of arguments, like Carleman estimate and construction of the decaying remainder term, to unbounded domain that we prove in Section 2, 3 and 4.

Let us mention that the arguments used for the construction of the CGO solutions work only if the unbounded domain has one infinite direction (or a cylindrical shape). This approach fails if the unbounded domain has more than one infinite direction like the slab. However, following the approach of [34, 37], by mean of unique continuation properties we prove in Corollary 1.4 the recovery of coefficients supported in an unbounded cylinder. Here the cylinder can be arbitrary and this result extend the one of [34, 37] to non compactly supported coefficients. Note also that, combining the density results stated in Lemma 6.1, used for the proof of Corollary 1.3, with Corollary 1.4, one can check that the data used by [34, 37] for the recovery of compactly supported coefficients allow to recover more general class of coefficients supported in infinite cylinder and known on the neighborhood of the boundary outside a compact set.

In the main result of this paper, stated in Theorem 1.1, we show that the partial DN map Λ_q allows to recover coefficients q which are equivalent modulo integrable functions to a fixed bounded function. This last condition is not fulfilled by the class of potential, periodic along the axis of the cylindrical domain, considered by [16, 17]. However, combining Theorem 1.1 with [16, 17], one can conclude that the partial DN map Λ_q allows to recover the class of coefficients q considered in the present paper as well as potentials q which are periodic along the axis of Ω.

Let us remark that in a similar way to [34, 37], with suitable choice of admissible coefficients q, it is possible to formulate (1.11) with q replaced by $q - k^2$ and k^2 taking some suitable value in the absolute continuous spectrum of the operator $-\Delta + q$ with Dirichlet boundary condition. In this context, (1.11) admits a unique solution satisfying the Sommerfeld radiation condition on the infinite directions of the domain. Assuming that q is chosen in such a way that these conditions are fulfilled for (1.1) and (1.11), one can adapt the argument of the present paper to this problem. In this paper we do not consider such extension of our main result which requires a study of the forward problem.

Let us also observe that like in [34, 37], Corollary 1.4 can be formulated with different kinds of measurements on the side $x_1 = 0$ and $x_1 = L$ of $\partial\mathcal{O}$.

1.5. **Outline.** This paper is organized as follows. In Section 2, we start by considering the CGO solutions, without boundary conditions, for the problem in an unbounded cylindrical domain. For the construction of these solutions we combine different arguments such as localization of the CGO solutions along the axis of the waveguide and some arguments of separation of variables. Then, in the spirit of [6], we introduce in Section 3 a Carleman estimate with linear weight stated in an infinite cylindrical domain. Using this Carleman estimate, we build in Section 4 CGO solutions vanishing on some parts of the boundary. In Section 5, we combine all these results in order to prove Theorem 1.1. Finally, Section 6 is devoted to the applications of the main result stated in Corollary 1.2, 1.3 and 1.4.

2. CGO solutions without conditions

In this section we introduce the first class of CGO solutions of our problem without boundary conditions. These CGO solutions correspond to some specific solutions $u \in H^2(\Omega)$ of $-\Delta u + qu = 0$ in Ω for $q \in L^\infty(\Omega)$. More precisely, we start by fixing $\theta \in \mathbb{S}^1 := \{y \in \mathbb{R}^2 : \ |y| = 1\}$, $\xi' \in \theta^\perp := \{y \in \mathbb{R}^2 : \ y \cdot \theta = 0\}$, $\xi := (\xi', \xi_3) \in \mathbb{R}^3$ with $\xi_3 \neq 0$. Then, we consider $\eta \in \mathbb{S}^2 := \{y \in \mathbb{R}^3 : \ |y| = 1\}$ defined by

$$\eta = \frac{(\xi', -\frac{|\xi'|^2}{\xi_3})}{\sqrt{|\xi'|^2 + \frac{|\xi'|^4}{\xi_3^2}}}.$$

In particular, we have

$$\eta \cdot \xi = (\theta, 0) \cdot \xi = (\theta, 0) \cdot \eta = 0. \tag{2.14}$$

Then, we fix $\chi \in \mathcal{C}_0^\infty(\mathbb{R}; [0, 1])$ such that $\chi = 1$ on a neighborhood of 0 in $\mathbb{R}$ and, for $\rho > 1$, we consider solutions $u \in H^2(\Omega)$ of $-\Delta u + qu = 0$ in Ω taking the form

$$u(x', x_3) = e^{-\rho\theta\cdot x'} \left(e^{i\rho\eta\cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right) e^{-i\xi\cdot x} + w_\rho(x) \right), \quad x = (x', x_3) \in \Omega. \tag{2.15}$$

Here the remainder term $w_\rho \in H^2(\Omega)$ satisfies the decay property

$$\rho^{-1} \|w_\rho\|_{H^2(\Omega)} + \rho \|w_\rho\|_{L^2(\Omega)} \leqslant C\rho^{\frac{7}{8}}, \tag{2.16}$$

with C independent of ρ. This construction can be summarized in the following way.

Theorem 2.1. *There exists $\rho_0 > 1$ such that, for all $\rho > \rho_0$, the equation $-\Delta u + qu = 0$ admits a solution $u \in H^2(\Omega)$ of the form* (2.15) *with w_ρ satisfying the decay property* (2.16).

Remark 2.2. *Comparing to CGO solutions on bounded domains, the main difficulty in the construction of CGO solutions in our context comes from the fact that Ω is not bounded and the CGO solutions should lye in $L^2(\Omega)$. This means that the usual principal parts of the CGO solutions considered by* [6, 27, 43]*, taking the form $e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}e^{-i\xi\cdot x}$ in our context, will be inadequate since it will not be lying in $L^2(\Omega)$. This is the main reason why we introduce the new expression involving the cut-off χ that allows to localize such expressions. The main difficulty in our choice consist of using this expression to localize without loosing the decay properties stated in* (2.16)*. This will be done by assuming that the principal part of the CGO solutions given by*

$$e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}\chi\left(\rho^{-\frac{1}{4}}x_3\right)e^{-i\xi\cdot x}$$

propagates in some suitable way along the axis of the waveguide with respect to the large parameter ρ. Actually, this seems to be one of the main novelty in our construction of CGO solutions comparing to any others.

Clearly, u solves $-\Delta u+qu=0$ if and only if w_ρ solves

$$P_{-\rho}w_\rho=-qw_\rho-e^{\rho\theta\cdot x'}(-\Delta+q)e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}\chi\left(\rho^{-\frac{1}{4}}x_3\right)e^{-i\xi\cdot x}, \tag{2.17}$$

with P_s, $s\in\mathbb{R}$, the differential operator defined by

$$P_s:=-\Delta-2s\theta\cdot\nabla'-s^2, \tag{2.18}$$

where $\nabla'=(\partial_{x_1},\partial_{x_2})^T$. In order to define a suitable set of solutions of (2.17), we start by considering the following equation

$$P_{-\rho}y=F,\quad x\in\Omega. \tag{2.19}$$

Taking the Fourier transform with respect to x_3, denoted by $\mathcal{F}_{x_3}$, on both side of this identity we get

$$P_{k,-\rho}y_k=F_k,\quad k\in\mathbb{R}, \tag{2.20}$$

with $F_k(x')=\mathcal{F}_{x_3}F(x',k)$, $y_k(x')=\mathcal{F}_{x_3}y(x',k)$ and

$$P_{k,-\rho}=-\Delta'+2\rho\theta\cdot\nabla'-\rho^2+k^2.$$

Here $\Delta'=\partial_{x_1}^2+\partial_{x_2}^2$ and $\mathcal{F}_{x_3}$ is defined by

$$\mathcal{F}_{x_3}h(x',k):=(2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}}h(x',x_3)e^{-ikx_3}dx_3,\quad h\in L^1(\Omega).$$

We fix also $p_{k,-\rho}(\zeta)=|\zeta|^2+2i\rho\theta\cdot\zeta+k^2$, $\zeta\in\mathbb{R}^2$, $k\in\mathbb{R}$, such that, for $D_{x'}=-i\nabla'$, we have $p_{k,-\rho}(D_{x'})=P_{k,-\rho}$. Applying some results of [12, 19, 24] about solutions of PDEs with constant coefficients we obtain the following.

Lemma 2.3. *For every $\rho>1$ and $k\in\mathbb{R}$ there exists a bounded operator*

$$E_{k,\rho}:\ L^2(\omega)\to L^2(\omega)$$

such that:

$$P_{k,-\rho}E_{k,\rho}F=F,\quad F\in L^2(\omega), \tag{2.21}$$

$$\|E_{k,\rho}\|_{\mathcal{B}(L^2(\omega))}\leqslant C\rho^{-1}, \tag{2.22}$$

$$E_{k,\rho}\in\mathcal{B}(L^2(\omega);H^2(\omega)) \tag{2.23}$$

and

$$\|E_{k,\rho}\|_{\mathcal{B}(L^2(\omega);H^2(\omega))}+\left\|k^2E_{k,\rho}\right\|_{\mathcal{B}(L^2(\omega))}\leqslant C\rho, \tag{2.24}$$

with $C>0$ depending only on ω.

Proof. In light of [12, Thoerem 2.3] (see also [19, Theorem 10.3.7]), there exists a bounded operator $E_{k,\rho} \in \mathcal{B}(L^2(\omega))$, defined from fundamental solutions associated to $P_{k,-\rho}$ (see Section 10.3 of [19]), such that (2.21) is fulfilled. In addition, fixing

$$\tilde{p}_{k,-\rho}(\zeta) := \left(\sum_{\alpha \in \mathbb{N}^2} |\partial_\zeta^\alpha p_{k,-\rho}(\zeta)|^2 \right)^{\frac{1}{2}}, \quad \zeta \in \mathbb{R}^2,$$

for all differential operator $Q(D_{x'})$ with $\frac{Q(\zeta)}{\tilde{p}_{k,-\rho}(\zeta)}$ a bounded function, we have $Q(D_{x'})E_{k,\rho} \in \mathcal{B}(L^2(\omega))$ and there exists a constant C depending only on ω such that

$$\|Q(D_{x'})E_{k,\rho}\|_{\mathcal{B}(L^2((0,T)\times(-R,R)))} \leqslant C \sup_{\zeta \in \mathbb{R}^2} \frac{|Q(\zeta)|}{\tilde{p}_{k,-\rho}(\zeta)}. \tag{2.25}$$

Note that

$$\tilde{p}_{k,-\rho}(\zeta) \geqslant \sqrt{|\Im \partial_{\zeta_1} p_{k,-\rho}(\mu,\eta)|^2 + |\Im \partial_{\zeta_2} p_{k,-\rho}(\mu,\eta)|^2} = 2\rho, \quad \zeta \in \mathbb{R}^2.$$

Therefore, (2.25) implies

$$\|E_{k,\rho}\|_{\mathcal{B}(L^2(\omega))} \leqslant C \sup_{\zeta \in \mathbb{R}^2} \frac{1}{\tilde{p}_{k,-\rho}(\zeta)} \leqslant C\rho^{-1}$$

and (2.22) is fulfilled. In a same way, for all $\zeta \in \mathbb{R}^2$, assuming that $k^2 + |\zeta|^2 \geqslant 2\rho^2$, we have

$$\tilde{p}_{k,-\rho}(\zeta) \geqslant |\Re p_{k,-\rho}(\zeta)| = k^2 + |\zeta|^2 - \rho^2 \geqslant \frac{k^2 + |\zeta|^2}{2}.$$

Thus, we have

$$\sup_{\zeta \in \mathbb{R}^2} \frac{|\zeta|^2 + k^2}{\tilde{p}_{k,-\rho}(\zeta)} \leqslant \sup_{k^2+|\zeta|^2 \geqslant 2\rho^2} \frac{|\zeta|^2 + k^2}{\tilde{p}_{k,-\rho}(\zeta)} + \sup_{k^2+|\zeta|^2 \leqslant 2\rho^2} \frac{|\zeta|^2 + k^2}{\tilde{p}_{k,-\rho}(\zeta)} \leqslant 2 + 2\rho^2 \sup_{\zeta \in \mathbb{R}^2} \frac{1}{\tilde{p}_{k,-\rho}(\zeta)} \leqslant 3\rho.$$

Then, in view of [12, Theorem 2.3], we deduce (2.23) with

$$\|E_{k,-\rho}\|_{\mathcal{B}(L^2(\omega);H^2(\omega))} + \left\|k^2 E_{k,-\rho}\right\|_{\mathcal{B}(L^2(\omega))} \leqslant C \sup_{\zeta \in \mathbb{R}^2} \frac{1 + |\zeta|^2 + k^2}{\tilde{p}_{k,-\rho}(\zeta)} \leqslant C\rho$$

which implies (2.24). □

Applying this lemma, we can now consider solutions of (2.19) given by the following result.

Lemma 2.4. *For every $\rho > 1$ there exists a bounded operator*

$$E_\rho : \ L^2(\Omega) \to L^2(\Omega)$$

such that:

$$P_{-\rho} E_\rho F = F, \quad F \in L^2(\Omega), \tag{2.26}$$

$$\|E_\rho\|_{\mathcal{B}(L^2(\Omega))} \leqslant C\rho^{-1}, \tag{2.27}$$

$$E_\rho \in \mathcal{B}(L^2(\Omega); H^2(\Omega)) \tag{2.28}$$

and

$$\|E_\rho\|_{\mathcal{B}(L^2(\Omega);H^2(\Omega))} \leqslant C\rho, \tag{2.29}$$

with $C > 0$ depending only on Ω.

Proof. According to Lemma 2.3, we can define E_ρ on $L^2(\Omega)$ by

$$E_\rho F := \Omega \ni (x', x_3) \mapsto \mathcal{F}_k^{-1} \left(E_{k,\rho} \mathcal{F}_{x_3} F(\cdot, k) \right)(x', x_3).$$

It is clear that (2.21) implies (2.26). Moreover, we have

$$\|E_\rho F\|_{L^2(\Omega)}^2 = \int_{\mathbb{R}} \|E_{k,\rho} \mathcal{F}_{x_3} F(\cdot, k)\|_{L^2(\omega)}^2 \, dk$$

and from (2.22) we get

$$\|E_\rho F\|^2_{L^2(\Omega)} \leqslant C^2\rho^{-2}\int_{\mathbb{R}} \|\mathcal{F}_{x_3}F(\cdot,k)\|^2_{L^2(\omega)}\,dk = C^2\rho^{-2}\,\|F\|^2_{L^2(\Omega)}\,.$$

From this estimate we deduce (2.27). In view of (2.23)-(2.24), we have $E_\rho \in \mathcal{B}(L^2(\Omega);H^2(\Omega))$ and, for all $F\in L^2(\Omega)$, we have

$$\begin{aligned}\|E_\rho F\|^2_{H^2(\Omega)} &\leqslant C'\int_{\mathbb{R}}\left[\|E_{k,\rho}\mathcal{F}_{x_3}F(\cdot,k)\|^2_{H^2(\omega)} + \left\|k^2E_{k,\rho}\mathcal{F}_{x_3}F(\cdot,k)\right\|^2_{L^2(\omega)}\right]dk\\ &\leqslant C'C^2\rho^2\int_{\mathbb{R}}\|\mathcal{F}_{x_3}F(\cdot,k)\|^2_{L^2(\omega)}\,dk = C'C^2\rho^2\,\|F\|^2_{L^2(\Omega)}\,,\end{aligned}$$

with C' depending only on ω. This proves (2.28)-(2.29). □

Using this last result, we can build geometric optics solutions of the form (2.15).

Proof of Theorem 2.1. We start by recalling that

$$\begin{aligned}&-e^{\rho\theta\cdot x'}(-\Delta+q)e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}\chi\left(\rho^{-\frac14}x_3\right)e^{-i\xi\cdot x}\\ &= -\left((|\xi|^2+q)\chi\left(\rho^{-\frac14}x_3\right) - 2i\eta_3\rho^{\frac34}\chi'\left(\rho^{-\frac14}x_3\right) + 2i\xi_3\rho^{-\frac14}\chi'\left(\rho^{-\frac14}x_3\right) - \rho^{-\frac12}\chi''\left(\rho^{-\frac14}x_3\right)\right)e^{i\rho\eta\cdot x}e^{-i\xi\cdot x}\end{aligned} \tag{2.30}$$

On the other hand, we have

$$\int_{\mathbb{R}}\left|\chi\left(\rho^{-\frac14}x_3\right)\right|^2dx_3 = \rho^{\frac14}\int_{\mathbb{R}}|\chi(t)|^2\,dt$$

and we deduce that

$$\left\|\chi\left(\rho^{-\frac14}x_3\right)\right\|_{L^2(\Omega)} = \|\chi\|_{L^2(\mathbb{R})}\,|\omega|^{\frac12}\rho^{\frac18}.$$

In the same way, one can check that

$$\left\|\chi\left(\rho^{-\frac14}x_3\right)\right\|_{L^2(\Omega)} + \left\|\chi'\left(\rho^{-\frac14}x_3\right)\right\|_{L^2(\Omega)} + \left\|\chi''\left(\rho^{-\frac14}x_3\right)\right\|_{L^2(\Omega)} \leqslant C\rho^{\frac18},$$

with C depending only on ω and χ. Combining this with (2.30), we find

$$\begin{aligned}&\left\|-e^{\rho\theta\cdot x'}(-\Delta+q)e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}\chi\left(\rho^{-\frac14}x_3\right)e^{-i\xi\cdot x}\right\|_{L^2(\Omega)}\\ &= C\left((|\xi|^2+\|q\|_{L^\infty(\Omega)})\rho^{\frac18} + 2|\eta_3|\rho^{\frac78} + 2|\xi_3|\rho^{-\frac18} + \rho^{-\frac38}\right) \leqslant C\rho^{\frac78},\end{aligned} \tag{2.31}$$

with $C>0$ depending on ω, ξ and $\|q\|_{L^\infty(\Omega)}$. According to Lemma 2.4, we can rewrite equation (2.17) as

$$w_\rho = -E_\rho\left(qw_\rho + e^{\rho\theta\cdot x'}(-\Delta+q)e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}\chi\left(\rho^{-\frac14}x_3\right)e^{-i\xi\cdot x}\right),$$

with $E_\rho\in\mathcal{B}(L^2(\Omega))$ given by Lemma 2.4. For this purpose, we will use a standard fixed point argument associated to the map

$$\begin{aligned}&\mathcal{G}: L^2(\Omega)\to L^2(\Omega),\\ &\qquad G\mapsto -E_\rho\left[qG + e^{\rho\theta\cdot x'}e^{-i\rho\eta\cdot x}(-\Delta+q)e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}\chi\left(\rho^{-\frac14}x_3\right)e^{-i\xi\cdot x}\right].\end{aligned}$$

Indeed, in view of (2.27) and (2.31), we have

$$\|\mathcal{G}w\|_{L^2(\Omega)} \leqslant C\rho^{-\frac18} + C\rho^{-1}\,\|w\|_{L^2(\Omega)}\,,\quad w\in L^2(\Omega),$$

$$\|\mathcal{G}w_1-\mathcal{G}w_2\|_{L^2(\Omega)} \leqslant \|E_\rho[q(w_1-w_2)]\|_{L^2(\Omega)} \leqslant C\rho^{-1}\,\|w_1-w_2\|_{L^2(\Omega)}\,,\quad w\in L^2(\Omega),$$

with C depending on ω, ξ and $\|q\|_{L^\infty(\Omega)}$. Therefore, fixing $M_1 > 0$, there exists $\rho_0 > 1$ such that for $\rho \geqslant \rho_0$ the map $\mathcal{G}$ admits a unique fixed point w_ρ in $\{w \in L^2(\Omega) : \|w\|_{L^2(\Omega)} \leqslant M_1\}$. In addition, condition (2.27)-(2.29) imply that $w_\rho \in H^2(\Omega)$ fulfills (2.16). This completes the proof of Theorem 2.1. □

3. Carleman estimate

In this section we derive a Carleman estimate for the Laplace operator in the unbounded cylindrical domain Ω. We consider first a Carleman inequality similar to [6, Lemma 2.1] for unbounded cylindrical domains.

Proposition 3.1. *Let $\theta \in \mathbb{S}^1$. Then, there exists $d > 0$ depending only on ω such that the estimate*

$$\begin{aligned} &\frac{8\rho^2}{d}\|e^{-\rho\theta\cdot x'}u\|^2_{L^2(\Omega)} + 2\rho\|e^{-\rho\theta\cdot x'}(\theta\cdot\nu)^{1/2}\partial_\nu u\|^2_{L^2(\partial\omega_\theta^+\times\mathbb{R})} \\ \leqslant\ & \|e^{-\rho\theta\cdot x'}\Delta u\|^2_{L^2(\Omega)} + 2\rho\|e^{-\rho\theta\cdot x'}|\theta\cdot\nu|^{1/2}\partial_\nu u\|^2_{L^2(\partial\omega_\theta^-\times\mathbb{R})}, \end{aligned} \tag{3.32}$$

holds for every $u \in H^2(\Omega)$ satisfying $u_{|\Gamma} = 0$.

Proof. We start by proving (3.32) for $u \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ satisfying $u_{|\Gamma} = 0$. The operator $e^{-\rho\theta\cdot x'}\Delta e^{\rho\theta\cdot x'}$ decomposes into the sum $P'_+ + P^3_+ + P_-$, with

$$P'_+ := \Delta' + \rho^2 \text{ and } P_- := 2\rho\theta\cdot\nabla',\ P^3_+ := \partial^2_{x_3},$$

where the symbol Δ' (resp., ∇') stands for the Laplace (resp., gradient) operator with respect to $x' \in \omega$. Thus, we get upon setting $v(x) := e^{-\rho\theta\cdot x'}u(x)$ that

$$\begin{aligned} \|e^{-\rho\theta\cdot x'}\Delta u\|^2_{L^2(\Omega)} &= \|e^{-\rho\theta\cdot x'}\Delta e^{\rho\theta\cdot x'}v\|^2_{L^2(\Omega)} \\ &= \|(P'_+ + P^3_+ + P_-)v\|^2_{L^2(\Omega)} \\ &= \|(P'_+ + P^3_+)v\|^2_{L^2(\Omega)} + \|P_-v\|^2_{L^2(\Omega)} + 2\Re\langle P^3_+v, P_-v\rangle_{L^2(\Omega)} + 2\Re\langle P'_+v, P_-v\rangle_{L^2(\Omega)}, \end{aligned}$$

and hence

$$\|P_-v\|^2_{L^2(\Omega)} + 2\Re\langle P'_+v, P_-v\rangle_{L^2(\Omega)} \leqslant \|e^{-\rho\theta\cdot x'}\Delta u\|^2_{L^2(\Omega)} - 2\Re\langle P^3_+v, P_-v\rangle_{L^2(\Omega)}. \tag{3.33}$$

Moreover, we find upon integrating by parts that

$$2\Re\langle P^3_+v, P_-v\rangle_{L^2(\Omega)} = -\rho\int_{\mathbb{R}}\int_\omega \nabla'\cdot(|\partial_{x_3}v(x)|^2\,\theta)dx'dx_1 = -\rho\int_\Gamma |\partial_{x_3}v(x)|^2\,\theta\cdot\nu(x)d\sigma(x) = 0. \tag{3.34}$$

Here we used the fact that the condition $v_{|\Gamma} = 0$ implies $\partial_{x_3}v_{|\Gamma} = 0$. Next, as the function $w := v(\cdot, x_3) \in C^2(\overline{\omega})$ satisfies $w_{|\partial\omega} = 0$ for a.e. $x_3 \in \mathbb{R}$, applying [6, Lemma 2.1], we deduce that there exists $d > 0$ depending on ω such that

$$\frac{8\rho^2}{d}\|w\|^2_{L^2(\omega)} + 2\rho\int_{\partial\omega} e^{-2\rho\theta\cdot x'}\theta\cdot\nu(x')\left|\partial_\nu e^{\rho\theta\cdot x'}w(x')\right|^2 d\sigma(x') \leqslant \|P_-w\|^2_{L^2(\omega)} + 2\Re\langle P'_+w, P_-w\rangle_{L^2(\omega)}.$$

It follows

$$\begin{aligned} &\frac{8\rho^2}{d}\|e^{-\rho\theta\cdot x'}u(\cdot,x_3)\|^2_{L^2(\omega)} + 2\rho\int_{\partial\omega} e^{-2\rho\theta\cdot x'}\theta\cdot\nu(x)\,|\partial_\nu u(\cdot,x_3)|^2\,d\sigma(x') \\ \leqslant\ & \|P_-v(\cdot,x_3)\|^2_{L^2(\omega)} + 2\Re\langle P'_+v(\cdot,x_3), P_-v(\cdot,x_3)\rangle_{L^2(\omega)}. \end{aligned}$$

Thus, integrating both sides of the above inequality with respect to $x_3 \in \mathbb{R}$, we obtain

$$\frac{8\rho^2}{d}\|e^{-\rho\theta\cdot x'}u\|^2_{L^2(\Omega)} + 2\rho\int_\Gamma e^{-2\rho\theta\cdot x'}\theta\cdot\nu(x)\,|\partial_\nu u(x)|^2\,d\sigma(x) \leqslant \|P_-v\|^2_{L^2(\Omega)} + 2\Re\langle P'_+v, P_-v\rangle_{L^2(\Omega)}. \tag{3.35}$$

Putting (3.33)-(3.35) together, we end up getting (3.32). Finally, using the density of the space of restriction to Ω of function $u \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ satisfying $u_{|\Gamma} = 0$ in the space of function $u \in H^2(\Omega)$ satisfying $u_{|\Gamma} = 0$, we deduce that (3.32) holds for $u \in H^2(\Omega)$. □

Using the fact that

$$|\Delta u|^2 \leqslant 2\left(|(-\Delta+q)u|^2 + \|q\|^2_{L^\infty(\Omega)}|u|^2\right),$$

we get

$$\begin{aligned}\left(\frac{4\rho^2}{d} - \|q\|^2_{L^\infty(\Omega)}\right)\|e^{-\rho\theta\cdot x'}u\|^2_{L^2(\Omega)} + \rho\|e^{-\rho\theta\cdot x'}(\theta\cdot\nu)^{\frac12}\partial_\nu u\|^2_{L^2(\check{\Gamma}^+_\theta)}\\ \leqslant \quad \|e^{-\rho\theta\cdot x'}(-\Delta+q)u\|^2_{L^2(\Omega)} + \rho\|e^{-\rho\theta\cdot x'}|\theta\cdot\nu|^{\frac12}\partial_\nu u\|^2_{L^2(\check{\Gamma}^-_\theta)}.\end{aligned}$$

As a consequence we obtain the following estimate.

Corollary 3.2. *For $M>0$, let $q\in L^\infty(\Omega)$ satisfy $\|q\|_{L^\infty(\Omega)}\leqslant M$. Then, under the conditions of Proposition 3.1, we have*

$$\begin{aligned}\frac{2\rho^2}{d}\|e^{-\rho\theta\cdot x'}u\|^2_{L^2(\Omega)} + \rho\|e^{-\rho\theta\cdot x'}(\theta\cdot\nu)^{\frac12}\partial_\nu u\|^2_{L^2(\check{\Gamma}^+_\theta)}\\ \leqslant \quad \|e^{-\rho\theta\cdot x'}(-\Delta+q)u\|^2_{L^2(\Omega)} + \rho\|e^{-\rho\theta\cdot x'}|\theta\cdot\nu|^{\frac12}\partial_\nu u\|^2_{L^2(\check{\Gamma}^-_\theta)},\end{aligned}$$

provided $\rho\geqslant\rho_1 := M(d/2)^{\frac12}+1$.

4. CGO solutions vanishing on parts of the boundary

In this section we fix $q\in L^\infty(\Omega)$. From now on, for all $y\in\mathbb{S}^1$ and all $r>0$, we set

$$\partial\omega_{+,r,y} = \{x\in\Gamma:\ \nu(x)\cdot y > r\},\quad \partial\omega_{-,r,y} = \{x\in\Gamma:\ \nu(x)\cdot y\leqslant r\}.$$

Here and in the remaining of this text we always assume, without mentioning it, that y and r are chosen in such way that $\partial\omega_{\pm,r,\pm y}$ contain a non-empty relatively open subset of $\partial\omega$. Without lost of generality we assume that there exists $0<\varepsilon<1$ such that for all $\theta\in\{y\in\mathbb{S}^1:|y-\theta_0|\leqslant\varepsilon\}$ we have $\partial\omega_{-,\varepsilon,-\theta}\subset K'$. The goal of this section is to use the Carleman estimate (3.32) in order to build solutions $u\in H_\Delta(\Omega)$ to

$$\begin{cases} -\Delta u + qu = 0 & \text{in } \Omega,\\ u=0, & \text{on } \partial\omega_{+,\varepsilon/2,-\theta}\times\mathbb{R},\end{cases}\tag{4.36}$$

of the form

$$u(x',x_3) = e^{\rho\theta\cdot x'}e^{-i\rho\eta\cdot x}\left(\chi\left(\rho^{-\frac14}x_3\right) + z_\rho(x)\right),\quad x=(x',x_3)\in\Omega.\tag{4.37}$$

Here $\theta\in\{y\in\mathbb{S}^1:|y-\theta_0|\leqslant\varepsilon\}$, $z_\rho\in e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}H_\Delta(\Omega)$ fulfills

$$z_\rho(x',x_3) = -\chi\left(\rho^{-\frac14}x_3\right),\quad (x',x_3)\in\partial\omega_{+,\varepsilon/2,-\theta}\times\mathbb{R}$$

and

$$\|z_\rho\|_{L^2(\Omega)}\leqslant C\rho^{-\frac18},\tag{4.38}$$

with C depending on K', Ω and any $M\geqslant\|q\|_{L^\infty(\Omega)}$. Since $(\partial\omega\setminus K')\subset(\partial\omega\setminus\partial\omega_{-,\varepsilon,-\theta}) = \partial\omega_{+,\varepsilon,-\theta}$, it is clear that condition (4.36) implies $\mathrm{supp}(\mathcal{T}_0u)\subset K$ (recall that for $v\in\mathcal{C}^\infty_0(\overline{\Omega})$, $\mathcal{T}_0v = v_{|\Gamma}$).

The main result of this section can be stated as follows.

Theorem 4.1. *Let $q\in L^\infty(\Omega)$, $\theta\in\{y\in\mathbb{S}^1:|y-\theta_0|\leqslant\varepsilon\}$. For all $\rho>\rho_1$, one can find a solution $u\in H_\Delta(\Omega)$ of (4.36) taking the form (4.37) with z_ρ satisfying (4.38). Here ρ_1 denotes the constant introduced at the end of Corollary 3.2.*

In order to prove existence of such solutions of (4.36) we need some preliminary tools and an intermediate result.

4.1. **Weighted spaces.** In this subsection we give the definition of some weighted spaces. We set $s \in \mathbb{R}$, we fix $\theta \in \{y \in \mathbb{S}^1 : |y - \theta_0| \leqslant \varepsilon\}$ and we denote by γ the function defined on Γ by

$$\gamma(x) = |\theta \cdot \nu(x)|, \quad x \in \Gamma.$$

We introduce the spaces $L_s(\Omega)$, and for all non negative measurable function h on Γ the spaces $L_{s,h,\pm}$ defined respectively by

$$L_s(\Omega) = e^{-s\theta\cdot x'} L^2(\Omega), \quad L_{s,h,\pm} = \{f : \ e^{s\theta\cdot x'} h^{\frac{1}{2}}(x) f \in L^2(\omega_{\pm,\theta} \times \mathbb{R})\}$$

with the associated norm

$$\|u\|_s = \left(\int_\Omega e^{2s\theta\cdot x'} |u|^2 \, dx \right)^{\frac{1}{2}}, \quad u \in L_s(\Omega),$$

$$\|u\|_{s,h,\pm} = \left(\int_{\partial\omega_{\pm,\theta}\times\mathbb{R}} e^{2s\theta\cdot x'} h(x) |u|^2 \, d\sigma(x') dx_3 \right)^{\frac{1}{2}}, \quad u \in L_{s,h,\pm}.$$

4.2. **Completion of the proof.** We set the space

$$\mathcal{D}_0 = \{v_{|\Omega} : \ v \in \mathcal{C}_0^2(\mathbb{R}^3), \ v_{|\Gamma} = 0\}$$

and, in view of Proposition 3.1, applying the Carleman estimate (3.32) to any $g \in \mathcal{D}_0$ we obtain

$$\rho \|g\|_\rho + \rho^{\frac{1}{2}} \|\partial_\nu g\|_{\rho,\gamma,-} \leqslant C(\|(-\Delta + q)g\|_\rho + \|\partial_\nu g\|_{\rho,\rho\gamma,+}), \quad \rho \geqslant \rho_1. \tag{4.39}$$

We introduce also the space

$$\mathcal{M} = \{((-\Delta + q)v_{|\Omega}, \partial_\nu v_{|\partial\omega_{+,\theta}\times\mathbb{R}}) : \ v \in \mathcal{D}_0\}$$

and think of $\mathcal{M}$ as a subspace of $L_\rho(\Omega) \times L_{\rho,\rho\gamma,+}$. Combining the Carleman estimate (4.39) with a classical application of the Hahn Banach theorem (see [27, Proposition 7.1] and [16, Lemma 3.2] for more detail) to a suitable linear form defined on $\mathcal{M}$, we obtain the following intermediate result.

Lemma 4.2. *We fix $\partial\omega^*_{-,\theta} = \{x \in \partial\omega : \ \theta \cdot \nu(x) < 0\}$. Given $\rho \geqslant \rho_1$, with ρ_1 the constant of Corollary 3.2, and*

$$v \in L_{-\rho}(\Omega), \quad v_- \in L_{-\rho,\gamma^{-1},-},$$

there exists $y \in L_{-\rho}(\Omega)$ such that:
1) $-\Delta y + qy = v$ in Ω,
2) $y_{|\partial\omega^*_{-,\theta}\times\mathbb{R}} = v_-$,
3) $\|y\|_{-\rho} \leqslant C \left(\rho^{-1} \|v\|_{-\rho} + \rho^{-\frac{1}{2}} \|v_-\|_{-\rho,\gamma^{-1},-} \right)$ with C depending on Ω, $M \geqslant \|q\|_{L^\infty(\Omega)}$.

Armed with this lemma we are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. We need to consider z_ρ satisfying

$$\begin{cases} z_\rho \in L^2(\Omega) \\ (-\Delta + q)(e^{\rho\theta\cdot x'} e^{-i\rho\eta\cdot x} z_\rho) = -(-\Delta + q) e^{\rho\theta\cdot x'} e^{-i\rho\eta\cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right) \quad \text{in } \Omega \\ z_\rho = -\chi\left(\rho^{-\frac{1}{4}} x_3\right) \quad \text{on } \partial\omega_{+,\varepsilon/2,-\theta} \times \times\mathbb{R}. \end{cases} \tag{4.40}$$

Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ be such that $\text{supp}(\psi) \cap \partial\omega \subset \{x \in \partial\omega : \ \theta \cdot \nu(x) < -\varepsilon/3\}$ and $\psi = 1$ on $\{x \in \partial\omega : \ \theta \cdot \nu(x) < -\varepsilon/2\} = \partial\omega_{+,\varepsilon/2,-\theta}$. Choose $v_-(x', x_3) = -e^{\rho\theta\cdot x'} e^{-i\rho\eta\cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right) \psi(x')$, $x \in \partial\omega_{-,\theta} \times \mathbb{R}$. Since $v_-(x) = 0$ for $x \in \{x \in \Gamma : \ \theta \cdot \nu(x) \geqslant -\varepsilon/3\} \times \mathbb{R}$ we have $v_- \in L_{-\rho,\gamma^{-1},-}$. Fix also

$$v(x) = -(-\Delta + q) e^{\rho\theta\cdot x'} e^{-i\rho\eta\cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right), \quad x \in \Omega.$$

From Lemma 4.2, we deduce that there exists $h \in H_\Delta(\Omega)$ such that

$$\begin{cases} (-\Delta + q)h = v & \text{in } \Omega, \\ h(x) = v_-(x), & x \in \partial\omega_{-,\theta} \times \mathbb{R}. \end{cases}$$

Then, for $z_\rho = e^{-\rho\theta\cdot x'} e^{i\rho\eta\cdot x} h$ condition (4.40) will be fulfilled. Repeating some arguments similar to Theorem 2.1, we obtain

$$\left\| e^{-\rho\theta\cdot x'}(-\Delta + q)e^{\rho\theta\cdot x'} e^{-i\rho\eta\cdot x}\chi\left(\rho^{-\frac{1}{4}}x_3\right)\right\|_{L^2(\Omega)} \leqslant C\rho^{\frac{7}{8}},$$

with C depending only on ω and $M \geqslant \|q\|_{L^\infty(\Omega)}$. Combining this with condition 3) of Lemma 4.2 we get

$$\begin{aligned} \|z_\rho\|_{L^2(\Omega)} = \|h\|_{-\rho} &\leqslant C\left(\rho^{-1}\|v\|_{-\rho} + \rho^{-\frac{1}{2}}\|v_-\|_{-\rho,\gamma^{-1},-}\right) \\ &\leqslant C\left(\rho^{-\frac{1}{8}} + \rho^{-\frac{1}{2}}\left\|\psi\gamma^{-\frac{1}{2}}\right\|_{L^2(\partial\omega_{-,\theta})}\left\|\chi\left(\rho^{-\frac{1}{4}}\cdot\right)\right\|_{L^2(\mathbb{R})}\right) \\ &\leqslant C\left(\rho^{-\frac{1}{8}} + \rho^{-\frac{3}{8}}\left\|\psi\gamma^{-\frac{1}{2}}\right\|_{L^2(\partial\omega_{-,\theta})}\|\chi\|_{L^2(\mathbb{R})}\right) \\ &\leqslant C\rho^{-\frac{1}{8}} \end{aligned}$$

with C depending only on Ω and $\|q\|_{L^\infty(\Omega)}$. Therefore, estimate (4.38) holds. Using the fact that $e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}z_\rho = h \in H_\Delta(\Omega)$, we deduce that u defined by (4.37) is lying in $H_\Delta(\Omega)$ and is a solution of (4.36). This completes the proof of Theorem 4.1. □

5. Uniqueness result

This section is devoted to the proof of Theorem 1.1. From now on we set $q = q_2 - q_1$ on Ω and we assume that $q = 0$ on $\mathbb{R}^3 \setminus \Omega$. Without lost of generality we assume that for all $\theta \in \{y \in \mathbb{S}^1 : |y - \theta_0| \leqslant \varepsilon\}$ we have $\partial\omega_{-,\varepsilon,\theta} \subset G'$ with $\varepsilon > 0$ introduced in the beginning of the previous section. Let $\rho > \max(\rho_0, \rho_1)$ and set $\theta \in \{y \in \mathbb{S}^1 : |y - \theta_0| \leqslant \varepsilon\}$, $\xi := (\xi', \xi_3) \in \mathbb{R}^3$ satisfying $\xi_3 \neq 0$ and $\xi' \in \theta^\perp$. According to Theorem 2.1, we can consider $u_1 \in H^2(\Omega)$ solving $-\Delta u_1 + q_1 u_1 = 0$ on Ω taking the form (2.15) with w_ρ satisfying (2.16). In addition, in view of Theorem 4.1, we can fix $u_2 \in H_\Delta(\Omega)$ a solution of (4.36), with $q = q_2$, of the form (4.37) with $e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}z_\rho \in H_\Delta(\Omega)$ satisfying (4.38). Fix $w_1 \in H_\Delta(\Omega)$ solving

$$\begin{cases} -\Delta w_1 + q_1 w_1 = 0 & \text{in } \Omega, \\ \mathcal{T}_0 w_1 = \mathcal{T}_0 u_2. \end{cases} \tag{5.41}$$

Then, $u = w_1 - u_2$ solves

$$\begin{cases} -\Delta u + q_1 u = (q_2 - q_1)u_2 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.42}$$

and since $(q_2 - q_1)u_2 \in L^2(\Omega)$, in view of [14, Lemma 2.2], we deduce that $u \in H^2(\Omega)$. Using the fact that $u_1 \in H^2(\Omega)$, we can apply the Green formula to get

$$\begin{aligned} \int_\Omega (q_2 - q_1)u_2 u_1 dx &= \int_\Omega u_1(-\Delta u + q_1 u)dx - \int_\Omega u(-\Delta u_1 + q_1 u_1)dx \\ &= -\int_\Gamma \partial_\nu u u_1 d\sigma(x) + \int_\Gamma \partial_\nu u_1 u d\sigma(x). \end{aligned}$$

On the other hand, we have $u_{|\Gamma} = 0$ and, combining (1.5) with the fact that $\text{supp}\mathcal{T}_0 u_2 \subset K$, we deduce that $\partial_\nu u_{|G} = 0$. It follows that

$$\int_\Omega q u_2 u_1 dx = -\int_{\Gamma\setminus G} \partial_\nu u u_1 d\sigma(x). \tag{5.43}$$

In view of (2.16), by interpolation, we have

$$\|w_\rho\|_{L^2(\Gamma)} \leqslant C \|w_\rho\|_{H^{\frac{9}{16}}(\Omega)} \leqslant C \left(\|w_\rho\|_{L^2(\Omega)}\right)^{\frac{23}{32}} \left(\|w_\rho\|_{H^2(\Omega)}\right)^{\frac{9}{32}} \leqslant C\rho^{\frac{7}{16}}.$$

Thus, applying the Cauchy-Schwarz inequality to the first expression on the right hand side of this formula, we get

$$\begin{aligned}
\left|\int_{\Sigma\setminus G} \partial_\nu u u_1 d\sigma(x)\right| &\leqslant \int_{\mathbb{R}}\int_{\partial\omega_{+,\varepsilon,\theta}} \left|\partial_\nu u e^{-\rho x'\cdot\theta}(e^{-i\xi\cdot x}\chi\left(\rho^{-\frac{1}{2}}x_3\right) + w_\rho)\right| d\sigma(x')dx_3 \\
&\leqslant C\left(\int_{\partial\omega_{+,\varepsilon,\theta}\times\mathbb{R}} \left|e^{-\rho x'\cdot\theta}\partial_\nu u\right|^2 d\sigma(x)\right)^{\frac{1}{2}} \left(\left\|\chi\left(\rho^{-\frac{1}{2}}\cdot\right)\right\|_{L^2(\mathbb{R})} + \|w_\rho\|_{L^2(\Gamma)}\right) \\
&\leqslant C\rho^{\frac{7}{16}}\left(\int_{\partial\omega_{+,\varepsilon,\theta}\times\mathbb{R}} \left|e^{-\rho x'\cdot\theta}\partial_\nu u\right|^2 d\sigma(x)\right)^{\frac{1}{2}}
\end{aligned}$$

for some C independent of ρ. Here we have used both (2.16) and the fact that $(\Gamma\setminus G) \subset \partial\omega_{+,\varepsilon,\theta}\times\mathbb{R}$. Combining this estimate with the Carleman estimate stated in Corollary 3.2, we find

$$\begin{aligned}
&\left|\int_\Omega (q_2-q_1)u_2u_1dx\right|^2 \\
&\leqslant C\rho^{\frac{7}{8}}\int_{\partial\omega_{+,\varepsilon,\theta}\times\mathbb{R}} \left|e^{-\rho x'\cdot\theta}\partial_\nu u\right|^2 d\sigma(x) \\
&\leqslant \varepsilon^{-1}C\rho^{\frac{7}{8}}\int_{\partial\omega_{+,\theta}\times\mathbb{R}} \left|e^{-\rho x'\cdot\theta}\partial_\nu u\right|^2 |\nu\cdot\theta| d\sigma(x) \\
&\leqslant \varepsilon^{-1}C\rho^{-\frac{1}{8}}\left(\int_\Omega \left|e^{-\rho x'\cdot\theta}(-\Delta+q_1)u\right|^2 dx\right) \\
&\leqslant \varepsilon^{-1}C\rho^{-\frac{1}{8}}\left(\int_\Omega \left|e^{-\rho x'\cdot\theta}qu_2\right|^2 dx\right) \\
&\leqslant \varepsilon^{-1}C\rho^{-\frac{1}{8}}\left(\|q\|_{L^\infty(\Omega)}\|\chi\|_{L^\infty(\mathbb{R})}\int_\Omega |q(x)|dx + \|q\|^2_{L^\infty(\Omega)}\|z_\rho\|^2_{L^2(\Omega)}\right).
\end{aligned} \tag{5.44}$$

Here $C>0$ stands for some generic constant independent of ρ. Applying the fact that $q\in L^1(\Omega)$, we deduce that

$$\lim_{\rho\to+\infty}\int_\Omega qu_2u_1dx = 0. \tag{5.45}$$

Moreover, we have

$$\int_\Omega qu_1u_2dx = \int_{\mathbb{R}^3}\chi^2(\rho^{-\frac{1}{4}}x_3)q(x)e^{-i\xi\cdot x}dx + \int_\Omega Y(x)dx + \int_\Omega Z(x)dx$$

with $Y(x) = q(x)e^{-i\rho\eta\cdot x}z_\rho(x)w_\rho(x)$ and

$$Z(x) = q(x)\chi\left(\rho^{-\frac{1}{4}}x_3\right)\left[z_\rho e^{-ix\cdot\xi} + w_\rho e^{-i\rho\eta\cdot x}\right].$$

Applying the decay estimate given by (2.16) and (4.38), we obtain

$$\int_\Omega |Y(x)|dx \leqslant \|w_\rho\|_{L^2(\Omega)}\|z_\rho\|_{L^2(\Omega)} \leqslant C\rho^{-\frac{1}{4}}$$

$$\begin{aligned}
\int_\Omega |Z(x)|dx &\leqslant \|q\|_{L^2(\Omega)}\left\|\chi\left(\rho^{-\frac{1}{4}}\cdot\right)\right\|_{L^\infty(\mathbb{R})}\left(\|w_\rho\|_{L^2(\Omega)} + \|z_\rho\|_{L^2(\Omega)}\right) \\
&\leqslant C\|q\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|q\|^{\frac{1}{2}}_{L^1(\Omega)}\|\chi\|_{L^\infty(\mathbb{R})}\rho^{-\frac{1}{8}}.
\end{aligned}$$

with C independent of ρ. Combining this with (5.45), we deduce that

$$\lim_{\rho\to+\infty}\int_{\mathbb{R}^3}\chi^2(\rho^{-\frac{1}{4}}x_3)q(x)e^{-i\xi\cdot x}dx=0.$$

On the other hand, since $q\in L^1(\mathbb{R}^3)$ and $\chi(0)=1$, by the Lebesgue dominate convergence theorem, we find

$$\lim_{\rho\to+\infty}\int_{\mathbb{R}^3}\chi^2(\rho^{-\frac{1}{4}}x_3)q(x)e^{-i\xi\cdot x}dx=\int_{\mathbb{R}^3}q(x)e^{-i\xi\cdot x}dx.$$

This proves that, for all $\theta\in\{y\in\mathbb{S}^1:|y-\theta_0|\leqslant\varepsilon\}$, all $\xi'\in\mathbb{R}^2$ orthogonal to θ and all $\xi_3\in\mathbb{R}\setminus\{0\}$, we have

$$\mathcal{F}\left[\mathcal{F}_{x_3}q(\cdot,\xi_3)\right](\xi')=(2\pi)^{-1}\int_{\mathbb{R}^2}\mathcal{F}_{x_3}q(x',\xi_3)e^{-i\xi'\cdot x'}dx'=0. \tag{5.46}$$

Since $q\in L^1(\mathbb{R}^3)$, $\xi_3\mapsto\mathcal{F}_{x_3}q(\cdot,\xi_3)$ is continuous from $\mathbb{R}$ to $L^1(\mathbb{R}^2)$ and

$$|\mathcal{F}_{x_3}q(\cdot,\xi_3)|\leqslant(2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}}|q(\cdot,x_3)|dx_3,$$

by the Fubini and the Lebesgue dominate convergence theorem, we deduce that (5.46) holds for all $\xi'\in\mathbb{R}^2$ orthogonal to θ and all $\xi_3\in\mathbb{R}$. Now using the fact that for any $\xi_3\in\mathbb{R}$, $\mathcal{F}_{x_3}q(\cdot,\xi_3)$ is supported on $\overline{\omega}$ which is compact, we deduce, by analyticity of $\mathcal{F}\left[\mathcal{F}_{x_3}q(\cdot,\xi_3)\right]$, that $\mathcal{F}_{x_3}q(\cdot,\xi_3)=0$. This proves that $q=0$ which completes the proof of Theorem 1.1.

6. Applications

In this section we will prove the three applications of Theorem 1.1 stated in Corollary 1.2, 1.3 and 1.4.

6.1. **Application to the Calderón problem.** This subsection is devoted to the proof of Corollary 1.2. Applying the Liouville transform, we deduce that for u the solution to (1.6), $v:=a^{\frac{1}{2}}u$ solves the following BVP

$$\begin{cases}(-\Delta+q_a)v &= 0, & \text{in } \Omega,\\ v &= a^{\frac{1}{2}}f, & \text{on } \Gamma,\end{cases}$$

where we recall that $q_a:=a^{-\frac{1}{2}}\Delta(a^{\frac{1}{2}})$. Moreover, one can check that

$$\Sigma_a f=a^{\frac{1}{2}}\Lambda_{q_a}a^{\frac{1}{2}}f-a^{\frac{1}{2}}\left(\partial_\nu a^{\frac{1}{2}}\right)f,\ f\in H^{\frac{1}{2}}(\Gamma)\cap a_1^{-\frac{1}{2}}(\mathcal{H}_K(\Gamma)),$$

where Σ_a is defined by (1.7). From this and (1.8)-(1.9), it then follows for every $f\in H^{\frac{1}{2}}(\Gamma)\cap a_1^{-\frac{1}{2}}(\mathcal{H}_K(\Gamma))$, that

$$\Sigma_{a_j}f=a_1^{\frac{1}{2}}\Lambda_{q_j}a_1^{\frac{1}{2}}f-a_1^{\frac{1}{2}}\left(\partial_\nu a_1^{\frac{1}{2}}\right)f_{|G},\ j=1,2,$$

where, for simplicity, q_j stands for q_{a_j}. As a consequence, the condition $\Sigma_{a_1}=\Sigma_{a_2}$ implies

$$(\Lambda_{q_1}-\Lambda_{q_2})f=a_1^{-\frac{1}{2}}(\Sigma_{a_1}-\Sigma_{a_2})a_1^{-\frac{1}{2}}f=0,\ f\in a_1^{\frac{1}{2}}H^{\frac{1}{2}}(\Gamma)\cap(\mathcal{H}_K(\Gamma)).$$

In particular, this proves that $\Lambda_{q_1}=\Lambda_{q_2}$. Since $a_j\in\mathcal{A}$, $j=1,2$, it is clear that $q_j\in L^\infty(\Omega)$ and $q_1-q_2\in L^1(\Omega)$. Then, according to Theorem 1.1, we have $q_1=q_2$. Fixing $y:=a_1^{\frac{1}{2}}-a_2^{\frac{1}{2}}\in H^2_{loc}(\Omega)$ we deduce that y satisfies

$$\begin{cases}(-\Delta+q_1)y &= -a_2^{\frac{1}{2}}(q_1-q_2)=0, \quad \text{in } \Omega,\\ y_{|K\cap G}=\partial_\nu y_{|K\cap G}=0.\end{cases}$$

Combining this with results of unique continuation for elliptic equations (e.g. [42, Theorem 1]) we have $y=0$ and we deduce that $a_1=a_2$. This completes the proof of Corollary 1.2.

6.2. **Recovery of coefficients that are known in the neighborhood of the boundary outside a compact set.** This subsection is devoted to the proof of Corollary 1.3. For this purpose we assume that the conditions of Corollary 1.3 are fulfilled. Let us also introduce the following sets of functions

$$S_q := \{u \in L^2(\Omega) : \ -\Delta u + qu = 0, \ \mathrm{supp}(\mathcal{T}_0 u) \subset K\}, \quad Q_q := \{u \in L^2(\Omega) : \ -\Delta u + qu = 0\},$$
$$S_{q,\gamma_1,\gamma_1'} := \{u \in L^2(\Omega) : \ -\Delta u + qu = 0, \ \mathrm{supp}(\mathcal{T}_0 u) \subset (K' \times [-R,R]) \cup \gamma_1 \cup \gamma_1'\},$$
$$Q_{q,\gamma_2,\gamma_2'} := \{u \in L^2(\Omega) : \ -\Delta u + qu = 0, \ \mathrm{supp}(\mathcal{T}_0 u) \subset (\partial\omega \times [-R,R]) \cup \gamma_2 \cup \gamma_2'\}.$$

We consider first the following result of density for these spaces.

Lemma 6.1. *The space $Q_{q_1,\gamma_2,\gamma_2'}$ (resp. $S_{q_2,\gamma_1,\gamma_1'}$) is dense in Q_{q_1} (resp. S_{q_2}) for the topology induced by $L^2(\Omega \setminus (\Omega_{1,*} \cup \Omega_{2,*}))$.*

Proof. Due to the similarity of these two results, we consider only the proof of the density of $Q_{q_1,\gamma_2,\gamma_2'}$ in Q_{q_1}. For this purpose, assume the contrary. Then, there exist $g \in L^2(\Omega \setminus (\Omega_{1,*} \cup \Omega_{2,*}))$ and $v_0 \in Q_{q_1}$ such that

$$\int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} gv dx = 0, \quad v \in Q_{q_1,\gamma_2,\gamma_2'}, \tag{6.47}$$

$$\int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} gv_0 dx = 1. \tag{6.48}$$

From now on, we extend g by 0 to Ω. Let $y \in H^2(\Omega)$ be the solution of

$$\begin{cases} (-\Delta + q_1)y &= g, \quad \text{in } \Omega, \\ y &= 0, \quad \text{on } \Gamma. \end{cases}$$

Then, for any $v \in H^2(\Omega) \cap Q_{q_1,\gamma_2,\gamma_2'}$, we find

$$0 = \int_\Omega gv dx = -\int_{\partial\omega\times[-R,R]} \partial_\nu y v d\sigma(x) - \int_{\gamma_2} \partial_\nu y v d\sigma(x) - \int_{\gamma_2'} \partial_\nu y v d\sigma(x).$$

Allowing $v \in H^2(\Omega) \cap Q_{q_1,\gamma_2,\gamma_2'}$ to be arbitrary, we deduce that

$$\partial_\nu y(x) = 0, \quad x \in (\partial\omega \times [-R,R]) \cup \gamma_2 \cup \gamma_2'. \tag{6.49}$$

Therefore, y satisfies

$$\begin{cases} (-\Delta + q_1)y = 0 \ \text{ in } \Omega_{1,*}, \\ y_{|\gamma_2} = \partial_\nu y_{|\gamma_2} = 0 \end{cases}$$

and the unique continuation property for elliptic equations implies that $y_{|\Omega_{1,*}} = 0$. In the same way, we can prove that $y_{|\Omega_{2,*}} = 0$ and we deduce that

$$y_{|\partial\Omega_{j,*}} = \partial_\nu y_{|\partial\Omega_{j,*}} = 0, \quad j = 1,2.$$

Combining this with (6.49), we obtain

$$y(x) = \partial_\nu y(x) = 0, \quad x \in \partial(\Omega \setminus (\overline{\Omega_{1,*}} \cup \overline{\Omega_{2,*}})).$$

Now let us recall that, repeating the arguments used in [6, Corollary 1.2], one can check that, for any $y \in H^2(\Omega)$ and $z \in H_\Delta(\Omega)$, we have

$$\begin{aligned} &\int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} z\Delta y dx - \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} y\Delta z dx \\ &= \langle \mathcal{T}_0 z, \partial_\nu y\rangle_{H^{-\frac12}(\Gamma\setminus(\Omega_{1,*}\cup\Omega_{2,*})),H^{\frac12}(\Gamma\setminus(\Omega_{1,*}\cup\Omega_{2,*}))} - \langle \mathcal{T}_1 z, y\rangle_{H^{-\frac32}(\Gamma\setminus(\Omega_{1,*}\cup\Omega_{2,*})),H^{\frac32}(\Gamma\setminus(\Omega_{1,*}\cup\Omega_{2,*}))}. \end{aligned}$$

Therefore, applying this integration by parts formula, we get

$$\int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} gv_0 dx = \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} (-\Delta + q_1)yv_0 dx = 0.$$

This contradicts (6.48) and completes the proof of the lemma. □

Armed with this lemma we are now in position to complete the proof of Corollary 1.3.

Proof of the Corollary 1.3. Let $u_1 \in Q_{q_1,\gamma_2,\gamma_2'}$ and $u_2 \in S_{q_2,\gamma_1,\gamma_1'}$. Repeating the linearization process described in Section 5 we deduce that $\Lambda^*_{q_1,R} = \Lambda^*_{q_2,R}$ implies

$$\int_\Omega (q_2 - q_1) u_1 u_2 dx = 0.$$

Then, (1.10) implies

$$0 = \int_\Omega (q_2 - q_1) u_1 u_2 dx = \int_{\Omega \setminus (\Omega_{1,*} \cup \Omega_{2,*})} (q_2 - q_1) u_1 u_2 dx. \tag{6.50}$$

Combining this with the density result of Lemma 6.1 and applying again (1.10), we deduce that

$$\int_\Omega (q_2 - q_1) u_1 u_2 dx = \int_{\Omega \setminus (\Omega_{1,*} \cup \Omega_{2,*})} (q_2 - q_1) u_1 u_2 dx = 0, \quad u_1 \in Q_{q_1},\ u_2 \in S_{q_2}.$$

Finally, choosing u_1, u_2 in a similar way to Section 5, we can deduce that $q_1 = q_2$. This completes the proof of the corollary. □

6.3. **Recovery of non-compactly supported coefficients in a slab.** In this subsection we consider Corollary 1.4. Applying the construction of CGO solutions and the Carleman estimate of the previous sections, we will prove how one can extend the result of [37] to coefficients supported on an unbounded cylinder. For this purpose, we start by fixing $\delta \in (0, R - r)$ and ω an open smooth and connected subset of $(0, L) \times \mathbb{R}$ such that $(0, L) \times (-r - \delta, r + \delta) \subset \omega \subset (0, L) \times (-R, R)$. Then, we fix $\Omega := \omega \times \mathbb{R}$ and we consider the set of functions

$$\mathcal{V}_q(\Omega) := \{u \in H^1(\Omega) :\ -\Delta u + qu = 0 \ \text{ in } \Omega\},$$
$$\mathcal{W}_q(\mathcal{O}) := \{u_{|\Omega} :\ u \in H^1(\mathcal{O}),\ -\Delta u + qu = 0 \ \text{ in } \mathcal{O},\ u_{|x_1=0} = 0\},$$
$$\mathcal{W}_q(\Omega) := \{u \in H^1(\Omega) :\ -\Delta u + qu = 0 \ \text{ in } \Omega,\ u_{|x_1=0} = 0\}.$$

Following [37, Lemma 9], one can check the following result of density.

Lemma 6.2. *Let $q \in L^\infty(\Omega)$ be such that 0 is not in the spectrum of $-\Delta + q$ with Dirichlet boundary condition on $\mathcal{O}$. Then the set $\mathcal{W}_q(\mathcal{O})$ is dense in $\mathcal{W}_q(\Omega)$ with respect to the topology of $L^2(\Omega)$.*

In the same way, combining Lemma 6.2 with the Carleman estimate of Corollary 3.2 and [37, Lemma 10], we obtain the following important estimate.

Lemma 6.3. *Let $\theta := (\theta_1, \theta_2) \in \mathbb{S}^1$ be such that $\theta_1 > 0$ and assume that* (1.12)-(1.13) *are fulfilled. Then we have*

$$\begin{aligned} \left| \int_{\mathcal{O}} (q_1 - q_2) v_1 v_2 dx \right| &= \left| \int_\Omega (q_1 - q_2) v_1 v_2 dx \right| \\ &\leqslant C \rho^{-\frac{1}{2}} (\theta_1)^{-\frac{1}{2}} \left(\int_\Omega \left| e^{-\rho x' \cdot \theta} (q_1 - q_2) v_2 \right| dx \right) \left(\int_{\Gamma \cap \{x_1 = L\}} \left| e^{\rho x' \cdot \theta} v_1 \right| d\sigma(x) \right) \end{aligned} \tag{6.51}$$

for all $v_1 \in \mathcal{V}_{q_1}(\Omega)$ and for all $v_2 \in \mathcal{W}_{q_2}(\Omega)$.

Armed with these two results, we will complete the proof of Corollary 1.4 by choosing suitably the solutions v_j, $j = 1, 2$, of the equation $-\Delta v_j + q_j v_j = 0$ in Ω.

Proof of Corollary 1.4. From now on we assume that the condition (1.13) is fulfilled. Let us first start by considering the set $\tilde{\omega} := \{x := (x_1, x_2, x_3) :\ (-x_1, x_2, x_3) \in \omega\} \cup \omega$ and let us extend q_2 by symmetry to $\tilde{\omega} \times \mathbb{R}$ by assuming that

$$q_2(-x_1, x_2, x_3) = q_2(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in \tilde{\omega} \times \mathbb{R}.$$

Applying the results of Section 2, we can consider $u_2 \in H^2(\tilde{\omega} \times \mathbb{R})$ solving $-\Delta u_2 + q_2 u_2 = 0$ in $\tilde{\omega} \times \mathbb{R}$ and taking the form

$$u_2(x) := e^{\rho\theta\cdot x'} \left(e^{-i\rho\eta\cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right) e^{-ix\cdot\xi} + w_{2,\rho}(x) \right), \quad x := (x', x_3) \in \tilde{\omega} \times \mathbb{R}, \tag{6.52}$$

with $\theta := (\theta_1, \theta_2) \in \mathbb{S}^1$ such that $\theta_1 > 0$, $\eta, \xi \in \mathbb{R}^3$ chosen in a similar way to the beginning of Section 2, and $w_{2,\rho} \in H^2(\tilde{\omega} \times \mathbb{R})$ satisfying

$$\rho^{-1} \left\| w_{2,\rho} \right\|_{H^2(\tilde{\omega}\times\mathbb{R})} + \rho \left\| w_{2,\rho} \right\|_{L^2(\tilde{\omega}\times\mathbb{R})} \leqslant C\rho^{\frac{7}{8}}. \tag{6.53}$$

Then, we fix $v_2 \in H^2(\Omega)$ defined by

$$v_2(x_1, x_2, x_3) := u_2(x_1, x_2, x_3) - u_2(-x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in \Omega. \tag{6.54}$$

It is clear that $v_2 \in \mathcal{W}_{q_2}(\Omega)$. In the same way, we fix $v_1 \in \mathcal{V}_{q_1}(\Omega)$

$$v_1(x) := e^{-\rho\theta\cdot x'} \left(e^{i\rho\eta\cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right) + w_{1,\rho}(x) \right), \quad x := (x', x_3) \in \Omega, \tag{6.55}$$

with $w_{1,\rho} \in H^2(\Omega)$ satisfying

$$\rho^{-1} \left\| w_{1,\rho} \right\|_{H^2(\Omega)} + \rho \left\| w_{1,\rho} \right\|_{L^2(\Omega)} \leqslant C\rho^{\frac{7}{8}}. \tag{6.56}$$

Applying (6.51)-(6.56) and the fact that $q_1 - q_2 \in L^\infty(\Omega) \cap L^1(\Omega) \subset L^2(\Omega)$, in a similar way to Section 5 we deduce that

$$\lim_{\rho\to+\infty} \int_\Omega (q_1 - q_2) v_1 v_2 dx = 0.$$

On the other hand, we have

$$\int_\Omega (q_1 - q_2) v_1 v_2 dx = \int_\Omega (q_1 - q_2) \chi^2 \left(\rho^{-\frac{1}{4}} x_3\right) e^{-ix\cdot\xi} dx + \int_\Omega X_\rho dx, \tag{6.57}$$

where

$$\begin{aligned} X_\rho :=& (q_1 - q_2) e^{-\rho\theta\cdot x'} u_2 w_{1,\rho} \\ &- (q_1 - q_2) e^{-2\rho\theta_1 x_1} \left(e^{i\rho\eta\cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right) + w_{1,\rho}(x) \right) \left(e^{-i\rho\eta\cdot s(x)} \chi\left(\rho^{-\frac{1}{4}} x_3\right) e^{-ix\cdot\xi} + w_{2,\rho}(s(x)) \right), \end{aligned}$$

with $s(x_1, x_2, x_3) = (-x_1, x_2, x_3)$. Combining this with the decay estimates (6.53), (6.56) and using the fact that $\theta_1 x_1 > 0$, we deduce that

$$\lim_{\rho\to+\infty} \int_\Omega X_\rho dx = 0.$$

Then, (6.57) and the fact that $q_1 - q_2 \in L^1(\Omega)$ imply that

$$\int_\Omega (q_1 - q_2) e^{-ix\cdot\xi} dx = 0$$

and following the arguments used at the end of the proof of Theorem 1.1 we deduce that $q_1 = q_2$. □

References

[1] D. Ahluwalia and J. Keller, *Exact and asymptotic representations of the sound field in a stratified ocean. Wave propagation and underwater acoustics*, Lecture Notes in Phys., Springer, Berlin, **70** (1977), 14-85.

[2] H. Ammari and G. Uhlmann, *Reconstuction from partial Cauchy data for the Schrödinger equation*, Indiana University Math J., **53** (2004), 169-184.

[3] M. Bellassoued, Y. Kian, E. Soccorsi, *An inverse stability result for non compactly supported potentials by one arbitrary lateral Neumann observation*, J. Diff. Equat., **260** (2016), 7535-7562.

[4] M. Bellassoued, Y. Kian, E. Soccorsi, *An inverse problem for the magnetic Schrödinger equation in infinite cylindrical domains*, preprint, arXiv:1605.06599.

[5] A. Bukhgeim, *Recovering the potential from Cauchy data in two dimensions*, J. Inverse Ill-Posed Probl., 16 (2008), 19-34.

[6] A. L. Bukhgeim and G. Uhlmann, *Recovering a potential from partial Cauchy data*, Commun. Partial Diff. Eqns., **27** (2002), no 3-4, 653-668.

[7] A. P. Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics, Rio de Janeiro, Sociedade Brasileira de Matematica, (1980), 65-73.

[8] P. Caro, D. Dos Santos Ferreira, A. Ruiz, *Stability estimates for the Radon transform with restricted data and applications*, Advances in Math., **267** (2014), 523-564.

[9] P. Caro, D. Dos Santos Ferreira, A. Ruiz, *Stability estimates for the Calderón problem with partial data*, J. Diff. Equat., **260** (2016), 2457-2489.

[10] P. Caro and K. Marinov, *Stability of inverse problems in an infinite slab with partial data*, Commun. Partial Diff. Eqns., **41** (2016), 683-704.

[11] P.-Y. Chang, H.-H. Lin, *Conductance through a single impurity in the metallic zigzag carbon nanotube*, Appl. Phys. Lett., **95** (2009), 082104.

[12] M. Choulli, *Une introduction aux problèmes inverses elliptiques et paraboliques*, Mathématiques et Applications, Vol. 65, Springer-Verlag, Berlin, 2009.

[13] M. Choulli and Y. Kian, *Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from a partial Dirichlet-to-Neumann map. Application to the determination of a nonlinear term*, preprint, arXiv:1605.08672.

[14] M. Choulli, Y. Kian, E. Soccorsi, *Stable determination of time-dependent scalar potential from boundary measurements in a periodic quantum waveguide*, SIAM J. Math. Anal., **47** (2015), no 6, 4536-4558.

[15] M. Choulli, Y. Kian, E. Soccorsi, *Double logarithmic stability estimate in the identification of a scalar potential by a partial elliptic Dirichlet-to-Neumann map*, Bulletin of the South Ural State University, Ser. Mathematical Modelling, Programming and Computer Software (SUSU MMCS), **8** (2015), no 3, 78-95.

[16] M. Choulli, Y. Kian, E. Soccorsi, *Stability result for elliptic inverse periodic coefficient problem by partial Dirichlet-to-Neumann map*, to appear in J. Spec. Theory, arXiv:1601.05355.

[17] M. Choulli, Y. Kian, E. Soccorsi, *On the Calderón problem in periodic cylindrical domain with partial Dirichlet and Neumann data*, to appear in MMAS, arXiv:1601.05358.

[18] M. Choulli and E. Soccorsi, *An inverse anisotropic conductivity problem induced by twisting a homogeneous cylindrical domain*, J. Spec. Theory, **5** (2015), 295-329.

[19] L. Hörmander, *The Analysis of linear partial differential operators*, Vol II, Springer-Verlag, Berlin, Heidelberg, 1983.

[20] G. Hu and Y. Kian, *Determination of singular time-dependent coefficients for wave equations from full and partial data*, preprint, arXiv:1706.07212.

[21] O. Imanuvilov, G. Uhlmann, M. Yamamoto, *The Calderón problem with partial data in two dimensions*, Journal American Math. Society, **23** (2010), 655-691.

[22] O. Imanuvilov, G. Uhlmann, M. Yamamoto, *Partial Cauchy data for general second order elliptic operators in two dimensions*, Publ. Research Institute Math. Sci., **48** (2012), 971-1055.

[23] M. Ikehata, *Inverse conductivity problem in the infinite slab*, Inverse Problems, **17** (2001), 437-454.

[24] V. Isakov, *Completness of products of solutions and some inverse problems for PDE*, J. Diff. Equat., **92** (1991), 305-316.

[25] C. Kane, L. Balents, M. P. A. Fisher, *Coulomb Interactions and Mesoscopic Effects in Carbon Nanotubes*, Phys. Rev. Lett., **79** (1997), 5086-5089.

[26] O. Kavian, Y. Kian, E. Soccorsi, *Uniqueness and stability results for an inverse spectral problem in a periodic waveguide*, Jour. Math. Pures Appl., **104** (2015), no. 6, 1160-1189.

[27] C.E. Kenig, J. Sjöstrand, G. Uhlmann, *The Calderon problem with partial data*, Ann. of Math., **165** (2007), 567-591.

[28] Y. Kian, *Stability of the determination of a coefficient for wave equations in an infinite waveguide*, Inverse Probl. Imaging, **8** (3) (2014), 713-732.

[29] Y. Kian, *Unique determination of a time-dependent potential for wave equations from partial data*, Annales de l'IHP (C) Nonlinear Analysis, **34** (2017), 973-990.

[30] Y. Kian, *Recovery of time-dependent damping coefficients and potentials appearing in wave equations from partial data*, SIAM J. Math. Anal., **48** (6) (2016), 4021-4046.

[31] Y. Kian and L. Oksanen, *Recovery of time-dependent coefficient on Riemanian manifold for hyperbolic equations*, preprint, arXiv:1606.07243.

[32] Y. Kian, Q. S. Phan, E. Soccorsi, *Carleman estimate for infinite cylindrical quantum domains and application to inverse problems*, Inverse Problems, **30**, 5 (2014), 055016.

[33] Y. Kian, Q. S. Phan, E. Soccorsi, *Hölder stable determination of a quantum scalar potential in unbounded cylindrical domains*, Jour. Math. Anal. Appl., **426**, 1 (2015), 194-210.

[34] K. Krupchyk, M. Lassas, G. Uhlmann, *Inverse Problems with Partial Data for a Magnetic Schrödinger Operator in an Infinite Slab or Bounded Domain*, Comm. Math. Phys., **312** (2012), 87-126.

[35] X. Li , *Inverse boundary value problems with partial data in unbounded domains*, Inverse Problems, **28** (2012), 085003.

[36] X. Li , *Inverse problem for Schrödinger equations with Yang-Mills potentials in a slab*, J. Diff. Equat., **253** (2012), 694-726.

[37] X. Li and G. Uhlmann, *Inverse Problems on a Slab*, Inverse Problems and Imaging, **4** (2010), 449-462.

[38] A. Nachman and B. Street, *Reconstruction in the Calderón problem with partial data*, Commun. Partial Diff. Eqns., **35** (2010), 375-390.

[39] L. Potenciano-Machado, *Stability estimates for a Magnetic Schrodinger operator with partial data*, preprint, arXiv:1610.04015.

[40] L. Potenciano-Machado, *Optimal stability estimates for a Magnetic Schrödinger operator with local data*, Inverse Problems, **33** (2017), 095001.

[41] M. Salo and J. N. Wang, *Complex spherical waves and inverse problems in unbounded domains*, Inverse Problems, **22** (2006), 2299-2309.

[42] J. C. Saut and B. Scheurer, *Sur l'unicité du probl'eme de Cauchy et le prolongement unique pour des équations elliptiques à coefficients non localement bornés*, J. Diff. Equat., **43** (1982), 28-43.

[43] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math., **125** (1987), 153-169.

[44] Y. Yang, *Determining the first order perturbation of a bi-harmonic operator on bounded and unbounded domains from partial data*, J. Diff. Equat., **257** (2014), 3607-3639.

Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France.
E-mail address: yavar.kian@univ-amu.fr