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Double spinor Calabi-Yau varieties

Laurent Manivel

September 18, 2017

Abstract

Consider the ten-dimensional spinor variety \mathcal{S} in a projectivized half-spin representation of $Spin_{10}$. This variety is projectively isomorphic to its projective dual \mathcal{S}^\vee in the dual projective space. The intersection $X = \mathcal{S}_1 \cap \mathcal{S}_2$ of two general translates of \mathcal{S} is a smooth Calabi-Yau fivefold, as well as the intersection of their duals $Y = \mathcal{S}_1^\vee \cap \mathcal{S}_2^\vee$. We prove that although X and Y are not birationally equivalent, they are derived equivalent and L-equivalent.

1 Introduction

There has been a lot of recent interest in the relations, for pairs of Calabi-Yau threefolds, between derived equivalence, Hodge equivalence and birationality. The Pfaffian-Grassmannian equivalence provided pairs of non birational Calabi-Yau threefolds which are derived equivalent, but with distinct topologies. Recently, examples were found of non birational Calabi-Yau threefolds which are derived equivalent, and also deformation and Hodge equivalent. They are constructed from the six-dimensional Grassmannian $G(2, V) \subset \mathbf{P}(\wedge^2 V)$, where V denotes a five-dimensional complex vector space. This Grassmannian is well-known to be projectively self dual, more precisely its projective dual is $G(2, V^\vee) \subset \mathbf{P}(\wedge^2 V^\vee)$, where V^\vee denotes the dual space to V . Let Gr_1 and Gr_2 be two translates of $G(2, V)$ in $\mathbf{P}(\wedge^2 V)$, and suppose they intersect transversely. Then Gr_1^\vee and Gr_2^\vee also intersect transversely in $\mathbf{P}(\wedge^2 V^\vee)$. Moreover

$$X = Gr_1 \cap Gr_2 \quad \text{and} \quad Y = Gr_1^\vee \cap Gr_2^\vee$$

are smooth Calabi-Yau threefolds with the required properties. This was established independently in [14] and [1], to which we refer for more details on the general background. We should note however that those remarkable threefolds had already appeared in the literature, see [2, 4, 5].

The purpose of this note is to show that the very same phenomena occur if we replace the Grassmannian $G(2, V) \subset \mathbf{P}(\wedge^2 V)$ by the ten-dimensional spinor variety $\mathcal{S} \subset \mathbf{P}\Delta$, where Δ denotes one of the two half-spin representations of $Spin_{10}$. Recall that \mathcal{S} parametrizes one of the two families of maximal isotropic spaces in a ten-dimensional quadratic vector space. The projective dual of $\mathcal{S} \subset \mathbf{P}\Delta$ is the other such family $\mathcal{S}^\vee \subset \mathbf{P}\Delta^\vee$, which is projectively equivalent to \mathcal{S} . Let \mathcal{S}_1 and \mathcal{S}_2 be two translates of \mathcal{S} in $\mathbf{P}\Delta$, and suppose that they intersect transversely. Then \mathcal{S}_1^\vee and \mathcal{S}_2^\vee also intersect transversely in $\mathbf{P}\Delta^\vee$. Moreover

$$X = \mathcal{S}_1 \cap \mathcal{S}_2 \quad \text{and} \quad Y = \mathcal{S}_1^\vee \cap \mathcal{S}_2^\vee$$

are smooth Calabi-Yau fivefolds of Picard number one which are deformation equivalent, derived or D-equivalent (Proposition 4.2), Hodge equivalent (Corollary 4.3), but not birationally equivalent (Proposition 4.4). Also, the difference of their classes in the Grothendieck ring of varieties is annihilated by a power of the class of the affine line (Proposition 4.5): in the terminology of [9], X and Y are L-equivalent. This confirms their conjecture that for simply connected projective varieties, D-equivalence should imply L-equivalence.

From the point of view of mirror symmetry, X and Y being D-equivalent should have the same mirror, and in this respect their form a double mirror. Of course, from the projective point of view they are also (projective) mirrors one of the other.

The close connection between the Grassmannian $G(2, 5) \subset \mathbf{P}^9$ and the spinor variety $\mathcal{S} \subset \mathbf{P}^{15}$ is classical, and manifests itself at different levels.

1. They are the only two Hartshorne varieties among the rational homogeneous spaces, if we define a Hartshorne variety to be a smooth variety $Z \subset \mathbf{P}^N$ of dimension $n = \frac{2}{3}N$ which is not a complete intersection (recall that Hartshorne's conjecture predicts that $n > \frac{2}{3}N$ is impossible) [17, Corollary 2.16].
2. They are both prime Fano manifolds of index $\iota = \frac{N+1}{2}$, while their topological Euler characteristic is also equal to $N + 1$; this allows their derived categories to admit rectangular Lefschetz decompositions of length ι , based on the similar pairs $\langle \mathcal{O}_Z, U^\vee \rangle$, where U denotes their tautological bundle [7].
3. The Grassmannian $G(2, 5) \subset \mathbf{P}^9$ can be obtained from $\mathcal{S} \subset \mathbf{P}^{15}$ as parametrizing the lines in \mathcal{S} through some given point [11]; conversely, $\mathcal{S} \subset \mathbf{P}^{15}$ can be reconstructed from $G(2, 5) \subset \mathbf{P}^9$ by a simple quadratic birational map defined in terms of the quadratic equations of the Grassmannian [10].

That the story should be more or less the same for our double spinor varieties, as for the double Grassmannians, is therefore not a big surprise. We thought it would nevertheless be useful to check that everything was going through as expected. For that we essentially followed the ideas of [1] and [14], to which this note is of course heavily indebted.

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2 Spinor varieties

2.1 Pure spinors

We start with some basic facts about spin representations. See [12] for more details, and references therein. For convenience we will restrict to even dimensions, so we let $V = V_{2n}$ be a complex vector space of dimension $2n$, endowed with a non degenerate quadratic form. The variety of isotropic n -dimensional subspaces of V , considered as a subvariety of the Grassmannian $G(n, 2n)$, has two connected components $\mathcal{S}_+ = OG_+(n, 2n)$ and $\mathcal{S}_- = OG_-(n, 2n)$, called the spinor varieties, or varieties of pure spinors. Moreover the Plücker line bundle restricted to \mathcal{S}_+ or

\mathcal{S}_- has a square root L , which is still very ample and embeds the spinor varieties into the projectivizations of the two half-spin representations of $Spin_{2n}$, the simply connected double cover of SO_{2n} . We denote the half-spin representations by Δ_+ and Δ_- , in such a way that

$$\mathcal{S}_+ \subset \mathbf{P}\Delta_+ \quad \text{and} \quad \mathcal{S}_- \subset \mathbf{P}\Delta_-.$$

Since the two half-spin representations can be exchanged by an outer automorphism of $Spin_{2n}$, these two embeddings are projectively equivalent. Note that the spinor varieties have dimension $n(n-1)/2$, while the half-spin representations have dimension 2^{n-1} . The half-spin representations are self dual when n is even, and dual one of the other when n is odd.

It follows from the usual Bruhat decomposition that the Chow ring of \mathcal{S}_\pm is free, and the dimension of its k -dimensional component is equal to the number of strict partitions of k with parts smaller than n . In particular the Picard group has rank one, and L is a generator. Denote by U the rank n vector bundle obtained by restricting the tautological bundle of $G(n, 2n)$. Then the tangent bundle to \mathcal{S}_\pm is isomorphic to $\wedge^2 U^\vee$. This implies that \mathcal{S}_\pm is a prime Fano manifold of index $2n-2$.

2.2 The ten dimensional spinor variety

From now on we specialize to $n=5$, and we simply denote $\mathcal{S} \subset \mathbf{P}\Delta$ one of the spinor varieties. This is a ten-dimensional prime Fano manifold of index eight, embedded in codimension five. This case is specific for several reasons, one of the most important being that:

Proposition 2.1. *The action of $Spin_{10}$ on $\mathbf{P}\Delta - \mathcal{S}$ is transitive.*

In particular Δ admits a prehomogeneous action, not of $Spin_{10}$, but of $GL(1) \times Spin_{10}$. This is discussed page 121 of [15].

As any equivariantly embedded rational homogeneous variety, the spinor variety is cut out by quadrics. Moreover $I_{\mathcal{S}}(2) \simeq V$. In fact the spinor variety \mathcal{S} has a beautiful self-dual minimal resolution, which appears in [6, 5.1]:

$$0 \rightarrow \mathcal{O}(-8) \rightarrow V(-6) \rightarrow \Delta^\vee(-5) \rightarrow \Delta(-3) \rightarrow V(-2) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{\mathcal{S}} \rightarrow 0.$$

For future use let us compute the Hilbert polynomial of $\mathcal{S} \subset \mathbf{P}\Delta$. Note that $H^0(\mathcal{O}_{\mathcal{S}}(k))$ is, by the Borel-Weil theorem, the irreducible $Spin_{10}$ -module of highest weight $k\omega_5$ (where ω_5 is the fundamental weight corresponding to the half-spin representation Δ^\vee). Its dimension can thus be computed by a direct application of the Weyl dimension formula, and we get

$$H_{\mathcal{S}}(k) = \frac{1}{2^6 3^3 5^2 7} (k+1)(k+2)(k+3)^2(k+4)^2(k+5)^2(k+6)(k+7).$$

In particular, as is well-known, \mathcal{S} has degree 12. Finally the Poincaré polynomial is also easy to compute:

$$P_{\mathcal{S}}(t) = \frac{(1+t^3)(1-t^8)}{1-t}.$$

2.3 Self duality

Our next statement is a well-known direct consequence of Proposition 2.1:

Corollary 2.2. *The spinor variety $\mathcal{S} \subset \mathbf{P}\Delta$ is self dual.*

To be more precise, the dual variety of the spinor variety $\mathcal{S} \subset \mathbf{P}\Delta$ is the other spinor variety $\mathcal{S}^\vee \subset \mathbf{P}\Delta^\vee$, in the other half-spin representation.

As a consequence, up to the group action there are only two kinds, up to projective equivalence, of hyperplane sections of \mathcal{S} : the smooth and the singular ones. Let us briefly describe their geometries.

Proposition 2.3. *A singular hyperplane section \mathcal{HS}_{sing} of \mathcal{S} is singular along a projective space of dimension four. Moreover \mathcal{HS}_{sing} admits a cell decomposition and its Poincaré polynomial is*

$$P_{\mathcal{HS}_{sing}}(t) = 1 + t + t^2 + 2t^3 + 2t^4 + 2t^5 + 2t^6 + 2t^7 + t^8 + t^9.$$

Proof. Recall that we may consider \mathcal{S} and \mathcal{S}^\vee as the two families of maximal isotropic subspaces of V . Moreover, if E and F are two maximal isotropic spaces, they belong to the same family if and only if their intersection has odd dimension. Given a point of \mathcal{S} , that we identify, with some abuse, to such an isotropic space E , the set of tangent hyperplanes to \mathcal{S} at E defines a subvariety of \mathcal{S}^\vee .

Lemma 2.4. *A point $F \in \mathcal{S}^\vee$ defines a hyperplane in $\mathbf{P}\Delta$ which is tangent to \mathcal{S} at E , if and only if $\dim(E \cap F) = 4$.*

Proof. The stabilizer of E in $Spin(V)$ has only three orbits in \mathcal{S}^\vee , defined by the three possible values for the dimension of the intersection with E . The variety of tangent hyperplanes to \mathcal{S} at E must therefore coincide with the set of $F \in \mathcal{S}^\vee$ such that $\dim(E \cap F) = 4$. \square

Since an isotropic space $E \in \mathcal{S}$ such that $\dim(E \cap F) = 4$ is uniquely determined by $E \cap F$, the hyperplane defined by F is tangent to \mathcal{S} along a subvariety of \mathcal{S} isomorphic to $\mathbf{P}F^\vee$.

For the last assertions, note that a singular hyperplane section of \mathcal{S} is just a Schubert divisor. By general results on the Bruhat decomposition, we know that its complement in \mathcal{S} is precisely the big cell. So \mathcal{HS}_{sing} has a cell decomposition given by all the cells of \mathcal{S} except the big one, and $P_{\mathcal{HS}_{sing}}(t) = P_{\mathcal{S}}(t) - t^{10}$. \square

In the next statement, *locally isomorphic* means isomorphic up to a finite group.

Proposition 2.5. *A smooth hyperplane section \mathcal{HS}_{reg} of \mathcal{S} admits a quasi-homogeneous action of its automorphism group, which is locally isomorphic to $Spin_7 \rtimes \mathbb{G}_a^8$. Moreover \mathcal{HS}_{reg} admits a cell decomposition and its Poincaré polynomial is*

$$P_{\mathcal{HS}_{reg}}(t) = 1 + t + t^2 + 2t^3 + 2t^4 + 2t^5 + 2t^6 + t^7 + t^8 + t^9.$$

Proof. Recall that in order to construct the half-spin representation Δ , we can choose a decomposition $V = E \oplus F$, where E and F are isotropic. Then the quadratic form on V allows to identify F with the dual of E , which we will do in the sequel. In particular the Lie algebra

$$\mathfrak{so}(V) \simeq \wedge^2 V \simeq \wedge^2 E \oplus End(E) \oplus \wedge^2 E^\vee$$

acts naturally on $\Delta = \wedge^0 E \oplus \wedge^2 E \oplus \wedge^4 E$. Moreover [11] the spinor variety \mathcal{S} can be obtained as the image of the rational map

$$\omega \in \wedge^2 E \quad \mapsto \quad [1, \omega, \omega \wedge \omega] \in \mathbf{P}\Delta.$$

Now we choose a general linear form on Δ , as in [15]: we let $h = 1 + f_{1234} \in \wedge^0 E^\vee \oplus \wedge^4 E^\vee \subset \Delta^\vee$, where we have fixed a basis e_1, \dots, e_5 and its dual basis f_1, \dots, f_5 . (We typically use the notation $e_{ij} = e_i \wedge e_j$.) The stabilizer of the corresponding hyperplane section of \mathcal{S} is the stabilizer of the line $[h]$ in $\mathbf{P}\Delta^\vee$, which will coincide with the stabilizer of h in Δ since $Spin(V)$ has no non trivial character. The Lie algebra of this stabilizer is the annihilator of h in $\mathfrak{so}(V)$, which we shall now describe.

Let $X = \alpha + Y + \beta^\vee$ be an element of $\mathfrak{so}(V)$. Then

$$X(h) = tr(Y) + (\alpha(f_{1234}) + \beta^\vee) + Y(f_{1234}).$$

For this to be zero, we need that $tr(Y) = 0$, $\beta^\vee = -\alpha(f_{1234})$ and $Y(f_{1234}) = 0$. We may decompose $Y = f_5 \otimes e + f \otimes e_5 + Z$, where Z does not involve e_5 nor f_5 . Then $Y(f_{1234}) = tr(Z)f_{1234} + f_5 \wedge e(f_{1234})$. So we need $tr(Z) = 0$ and e must be a multiple of e_5 . But since also $tr(Y) = 0$, we finally get $Y = f \otimes e_5 + Z$, where f does not involve f_5 and $tr(Z) = 0$. Of course it is convenient to consider Z as a traceless endomorphism of $G = \langle e_1, e_2, e_3, e_4 \rangle$. So the stabilizer of h is finally $\mathfrak{s} \simeq \wedge^2 E \oplus G^\vee \oplus \mathfrak{sl}(G)$. Since $E = G \oplus \mathbb{C}$, we finally get

$$\mathfrak{s} = \mathfrak{sl}(G) \oplus \wedge^2 G \oplus G \oplus G^\vee.$$

A more careful analysis would allow to check that $\mathfrak{sl}(G) \oplus \wedge^2 G$ is actually a subalgebra of \mathfrak{s} isomorphic to \mathfrak{so}_7 , and acting on $G \oplus G^\vee$ as on the spin representation. But the important point for us will be that $\mathfrak{sl}(G)$ is itself a subalgebra of \mathfrak{s} , of maximal rank. Choose a Cartan subalgebra \mathfrak{h} of $\mathfrak{sl}(G)$; it is the Lie algebra of a three-dimensional torus H acting on our hyperplane section \mathcal{HS}_{reg} . We claim that this torus has only finitely many fixed points on \mathcal{HS}_{reg} . Indeed, as an $\mathfrak{sl}(G)$ -module, the spin representation Δ is isomorphic to the full exterior algebra of G . In particular its weights have multiplicity one, except zero which is the weight of both $\wedge^0 G$ and $\wedge^4 G$. The fixed points of H on \mathcal{HS}_{reg} are given by the intersection of \mathcal{HS}_{reg} with the projectivized weight spaces; each weight of multiplicity one gives at most one fixed point; the zero weight space gives a projective line in $\mathbf{P}\Delta$ which is not contained in the hyperplane $h = 0$, hence also gives at most one fixed point. This proves our claim. Then the Byalinicki-Birula decomposition yields a cell decomposition of \mathcal{HS}_{reg} . The Betti numbers are given by the Lefschetz hyperplane theorem.

Finally, the fact that the automorphism group acts on \mathcal{HS}_{reg} with an open orbit can be verified infinitesimally: it is a simple computation to check that the stabilizer of the point $1 + e_{12} - e_{34} - e_{1234}$ has the correct dimension. \square

The self-duality of \mathcal{S} is preserved at the categorical level, in the sense that $\mathcal{S} \subset \mathbf{P}\Delta$ and $\mathcal{S}^\vee \subset \mathbf{P}\Delta^\vee$ are homologically projectively dual [7, Section 6.2]. As already mentioned in the introduction, the derived category of coherent sheaves on the spinor variety \mathcal{S} has a specially nice rectangular Lefschetz decomposition, defined by eight translates of the exceptional pair $\langle \mathcal{O}_{\mathcal{S}}, U^\vee \rangle$ (where U denotes the rank five tautological bundle).

3 Double spinor varieties

In this section we introduce our main objects of interest, the *double spinor varieties*

$$X = \mathcal{S}_1 \cap \mathcal{S}_2,$$

where $\mathcal{S}_1 = g_1\mathcal{S}$ and $\mathcal{S}_2 = g_2\mathcal{S}$ are translates of \mathcal{S} by $g_1, g_2 \in PGL(\Delta)$. Up to projective equivalence, we can of course suppose that $X = \mathcal{S} \cap g\mathcal{S}$ for $g \in PGL(\Delta)$.

3.1 Local completeness

Let $G = PGL(\Delta)$, with its subgroup $H = Aut(\mathcal{S}) \simeq PSO_{10}$. The family of double spinor varieties is by definition the image of a rational map

$$G/H \times G/H \dashrightarrow Hilb(\mathbf{P}\Delta).$$

Moreover the diagonal left action of G is by projective equivalence, hence factors out when we consider local deformations of a given X . At the global level, the quotient $[(G/H \times G/H)/G]$ should be thought of as the moduli stack of double spinor varieties. One could reproduce the analysis of the similar stack made in [1] for the double Grassmannians, but we will not do that. We will only check the local completeness of our family.

Proposition 3.1. *The family of smooth double spinor varieties is locally complete.*

Proof. We first observe that $H^1(X, T\mathcal{S}_{1|X}) = 0$. This is a direct consequence of the vanishing of $H^q(\mathcal{S}_1, T\mathcal{S}_1(-k))$ for $0 < k < 8$ and $q > 1$, which follows from the Kodaira-Nakano vanishing theorem since this cohomology group is Serre dual to $H^{10-q}(\mathcal{S}_1, \Omega_{\mathcal{S}_1}(k-8))$. Hence the map

$$H^0(X, N_{X/\mathcal{S}_1}) \simeq H^0(X, N_{\mathcal{S}_2/\mathbf{P}|X}) \longrightarrow H^1(X, TX)$$

is surjective. Here we abbreviated $\mathbf{P}\Delta$ by \mathbf{P} . What remains to prove is that the composition

$$H^0(\mathbf{P}, T\mathbf{P}) \xrightarrow{r} H^0(X, T\mathbf{P}|_X) \xrightarrow{s} H^0(X, N_{\mathcal{S}_2/\mathbf{P}|X})$$

is also surjective. The tensor product of the resolutions of $\mathcal{O}_{\mathcal{S}_1}$ and $\mathcal{O}_{\mathcal{S}_2}$ gives a free resolution of \mathcal{O}_X as a $\mathcal{O}_{\mathbf{P}}$ -module, from which the surjectivity of r easily follows. The surjectivity of s follows from the vanishing of $H^1(X, T\mathcal{S}_{2|X})$, which we already verified. \square

3.2 Invariants

Proposition 3.2. *Any smooth double spinor variety $X = \mathcal{S}_1 \cap \mathcal{S}_2$ is a Calabi-Yau fivefold. Moreover $H^q(X, \mathcal{O}_X) = 0$ for $0 < q < 5$, and $H^5(X, \mathbb{Z})$ is torsion free.*

Proof. The smoothness of X is equivalent to the fact that \mathcal{S}_1 and \mathcal{S}_2 meet transversely. Then X has dimension five. Suppose to simplify notations that $\mathcal{S}_1 = \mathcal{S}$. By generic perfection, the $\mathcal{O}_{\mathcal{S}}$ -module \mathcal{O}_X has a free resolution

$$0 \rightarrow \mathcal{O}_{\mathcal{S}}(-8) \rightarrow V_{\mathcal{S}}(-6) \rightarrow \Delta_{\mathcal{S}}^{\vee}(-5) \rightarrow \Delta_{\mathcal{S}}(-3) \rightarrow V_{\mathcal{S}}(-2) \rightarrow \mathcal{O}_{\mathcal{S}} \rightarrow \mathcal{O}_X \rightarrow 0. \quad (1)$$

This immediately implies that $h^0(\mathcal{O}_X) = 1$, so that X is connected, and that the relative dualizing sheaf $\omega_{X/\mathcal{S}} = \mathcal{O}_X(-8)$. Since $\omega_{\mathcal{S}} = \mathcal{O}_{\mathcal{S}}(-8)$, this implies that X has trivial canonical bundle.

Moreover, since \mathcal{S} is Fano of index eight, the line bundles $\mathcal{O}_{\mathcal{S}}(-k)$ are acyclic for $0 < k < 8$. This implies that for $q > 1$, $H^q(X, \mathcal{O}_X) \simeq H^{q+5}(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-8)) = H^{q+5}(\mathcal{S}, \omega_{\mathcal{S}})$, which is zero for $q < 5$.

Finally, applying Sommese's results as in [14, Lemma 3.3], to $A = \mathcal{S}_1$ and $B = \mathcal{S}_2$ in \mathbf{P}^{15} , we get that the relative homotopy groups $\pi_i(\mathcal{S}_1, X) = 0$ for $i \leq 5$. In particular X is simply connected, and by the Bogomolov decomposition theorem, it is Calabi-Yau. Moreover, after passing from homotopy to homology, we deduce from $H_i(\mathcal{S}_1, X, \mathbb{Z}) = 0$ for $i \leq 5$ that $H^5(X, \mathbb{Z})$ is torsion free. \square

Knowing the Hilbert polynomial of \mathcal{S} , we also deduce that the Hilbert polynomial of X is

$$H_X(k) = \frac{2}{5}k(k^2 + 1)(3k^2 + 17).$$

In order to compute the other Hodge numbers of the double spinor varieties, it will be convenient to use the following observation, already made in [5] for the double Grassmannian varieties: when g goes to identity, $X = \mathcal{S} \cap g\mathcal{S}$ degenerates to the zero locus in \mathcal{S} of a global section of its normal bundle $\wedge^4 U^\vee = U(2)$. This bundle is generated by global sections, being homogeneous and irreducible, with

$$H^0(\mathcal{S}, U(2)) = V_{\omega_4 + \omega_5}$$

by the Borel-Weil theorem. So the zero locus of a general section is a smooth Calabi-Yau fivefold which is deformation equivalent to the smooth double spinor varieties; in particular the family of those zero-loci is not locally complete, something that seems to be quite exceptional. The dimension of this family can be checked to coincide with the expected dimension $\dim \mathbf{P}V_{\omega_4 + \omega_5} - \dim SO_{10} = 209 - 45 = 164$. In particular it will define a codimension one subvariety of the local deformation space.

Note that other kinds of degenerations, this time singular, were considered in [2]: typically, joins of two elliptic quintics (which are linear sections of the Grassmannian) in two disjoint \mathbf{P}^4 's in \mathbf{P}^9 . Such degenerations were studied in connection with the Horrocks-Mumford vector bundle, in order to describe the moduli space of $(1, 10)$ -polarized abelian surfaces. It would certainly be interesting to study the similar story in our setting. The analogous singular degenerations are of course the joins of two K3 surfaces of degree 12 (which are linear sections of the spinor variety) in two disjoint \mathbf{P}^7 's in \mathbf{P}^{15} .

Proposition 3.3. *The non zero Hodge numbers of a smooth double spinor variety are $h^{p,p} = 1$ for $0 \leq p \leq 5$, and*

$$h^{0,5} = h^{5,0} = 1, \quad h^{1,4} = h^{4,1} = 165, \quad h^{2,3} = h^{3,2} = 7708.$$

Proof. We may suppose that X is the zero locus in \mathcal{S} of a general section of $U(2) = \wedge^4 U^\vee$. Then we have the conormal exact sequence

$$0 \rightarrow U_X^\vee(-2) \rightarrow \Omega_{\mathcal{S}|X} \rightarrow \wedge^2 U_X \rightarrow \Omega_X \rightarrow 0$$

and a Koszul complex

$$0 \rightarrow \mathcal{O}_{\mathcal{S}}(-8) \rightarrow U(-6) \rightarrow \wedge^2 U(-4) \rightarrow \wedge^2 U^{\vee}(-4) \rightarrow U^{\vee}(-2) \rightarrow \mathcal{O}_{\mathcal{S}} \rightarrow \mathcal{O}_X \rightarrow 0.$$

Using this complex we can compute the cohomology of $\Omega_{\mathcal{S}|X}$ and $U_X^{\vee}(-2)$ from the cohomology of $\Omega_{\mathcal{S}}$ and $U^{\vee}(-2)$ and their twists on \mathcal{S} which is controlled by the Borel-Weil-Bott theorem. We find that $H^q(X, \Omega_{\mathcal{S}|X}) = 0$ for $q = 0, 2, 3$, while $H^1(X, \Omega_{\mathcal{S}|X}) \simeq H^1(\mathcal{S}, \Omega_{\mathcal{S}}) = \mathbb{C}$. Moreover $H^q(X, U_X^{\vee}(-2)) = 0$ for $0 \leq q \leq 4$. This implies that $H^q(X, \Omega_X) = 0$ for $q = 0, 2, 3$, and that the restriction map $H^1(\mathcal{S}, \Omega_{\mathcal{S}}) \rightarrow H^1(X, \Omega_X)$ is an isomorphism. Moreover the dimension of $H^4(X, \Omega_X)$ is the same as that of $H^1(X, TX)$, which readily follows from the discussion of the proof of Proposition 3.1:

$$h^1(X, TX) = \dim G - 2 \dim H = 255 - 2 \times 45 = 165.$$

Let us now consider the bundle of two-forms on X , which is resolved by the skew-symmetric square of the conormal sequence, that is

$$0 \rightarrow S^2 U_X^{\vee}(-4) \rightarrow \wedge^2 U_X \otimes U_X^{\vee}(-2) \rightarrow S_{211} U_X \rightarrow \Omega_X^2 \rightarrow 0.$$

A tedious application of Borel-Weil-Bott allows to check that

$$H^2(S_{211} U_X) = \mathbb{C}, \quad H^3(\wedge^2 U_X \otimes U_X^{\vee}(-2)) = 0, \quad H^4(S^2 U_X^{\vee}(-4)) = 0.$$

By a standard spectral sequence argument, this implies that $H^2(\Omega_X^2)$ is at most one dimensional, hence in fact one dimensional. At this point, the only missing Hodge number is $h^3(\Omega_X^2) = 1 + \chi(\Omega_X^2)$. We compute this Euler characteristic as $\chi(S_{211} U_X) - \chi(\wedge^2 U_X \otimes U_X^{\vee}(-2)) + \chi(S^2 U_X^{\vee}(-4))$, by reducing once again to a computation on the spinor variety. \square

Corollary 3.4. *The Picard group of X is free of rank one, generated by $\mathcal{O}_X(1)$.*

3.3 Uniqueness

In this section we prove that the only translates of \mathcal{S} that contain $X = \mathcal{S} \cap g\mathcal{S}$ are \mathcal{S} itself, and $g\mathcal{S}$. In particular there is a unique way to represent X as an intersection of two translates of the spinor variety. We follow the approach of [1].

Proposition 3.5. *Let N denote the normal bundle to \mathcal{S} in $\mathbf{P}\Delta$. Then the restriction of N to X is slope stable.*

Proof. Recall that we denoted by U the rank five tautological bundle on \mathcal{S} . There is an isomorphism $N \simeq \wedge^4 U^{\vee} = U(2)$, so we just need to prove that U_X^{\vee} is stable. Let \mathcal{E} be a subsheaf of rank e of U_X^{\vee} , with $1 \leq e \leq 4$. Then $c_1(\mathcal{E}) = tL$ for some integer t . Taking double duals we get a non zero section of $\wedge^e U_X^{\vee}(-t)$, so we just need to prove that $H^0(X, \wedge^e U_X^{\vee}(-t)) = 0$ for any $t > 0$. This would follow from the resolution of \mathcal{O}_X as an $\mathcal{O}_{\mathcal{S}}$ -module, if we knew that $H^q(\mathcal{S}, \wedge^e U^{\vee}(-t)) = 0$ for $t > 0$ and $q \leq 9$. This is not exactly true. A careful application of the Borel-Weil-Bott theorem yields the following statement.

Lemma 3.6. *Suppose that $1 \leq e \leq 4$, $0 \leq q \leq 9$ and $t > 0$.*

Then $H^q(\mathcal{S}, \wedge^e U^{\vee}(-t)) = 0$, except for the three following cohomology groups:

1. $H^9(\mathcal{S}, \wedge^2 U^\vee(-8)) = \mathbb{C}$,
2. $H^1(\mathcal{S}, \wedge^3 U^\vee(-2)) = \mathbb{C}$,
3. $H^0(\mathcal{S}, \wedge^4 U^\vee(-1)) = \Delta$.

We can nevertheless conclude the proof of the proposition.

Indeed, $H^q(\mathcal{S}, U^\vee(-t)) = H^q(\mathcal{S}, \wedge^2 U^\vee(-t)) = 0$ for all $t > 0$ and all $q \leq 5$, hence $H^0(X, U_X^\vee(-t)) = H^0(X, \wedge^2 U_X^\vee(-t)) = 0$ for all $t > 0$ since the resolution of \mathcal{O}_X as an $\mathcal{O}_\mathcal{S}$ -module has length five. The same conclusion holds for $H^0(X, \wedge^3 U_X^\vee(-t))$ and any $t > 0$, since the non vanishing of $H^1(\mathcal{S}, \wedge^3 U^\vee(-2))$ could only transfer to $H^0(X, \wedge^3 U_X^\vee)$. Finally, $H^0(X, \wedge^4 U_X^\vee(-t)) = 0$ for $t > 0$ except possibly for $t = 1$. This means that U_X^\vee could possibly contain a rank four subsheaf \mathcal{E} such that $c_1(\mathcal{E}) = L$. Then its slope

$$\mu_L(\mathcal{E}) = c_1(\mathcal{E})L^4/4 = L^5/4 < \mu_L(U_X^\vee) = c_1(U_X^\vee)L^4/5 = 2L^5/5.$$

So we can nevertheless conclude that U_X^\vee is indeed slope stable. \square

The next step is to prove the following statement:

Proposition 3.7. *Suppose $X \subset \mathcal{S}_1$, where \mathcal{S}_1 is a translate of \mathcal{S} . Let N_1 is the normal bundle to \mathcal{S}_1 in $\mathbf{P}\Delta$. Then the restriction of N_1 to X determines \mathcal{S}_1 .*

Proof. Suppose $\mathcal{S}_1 = \mathcal{S}$ to simplify the notations. Since \mathcal{S} is cut out by quadrics, our strategy will be to reconstruct its quadratic equations from N_X , or equivalently, from U_X . The key observation is that there is a natural isomorphism

$$H^0(\mathcal{S}, U(1)) \simeq H^0(\mathcal{S}, \mathcal{O}_\mathcal{S}(1))^\vee.$$

Indeed, U^\vee is the irreducible bundle associated to the representation of highest weight $\omega_1 = \epsilon_1$. The highest weight of its dual U is $-\epsilon_5 = \omega_4 - \omega_5$. Therefore the highest weight of $U(1)$ is ω_4 , and the assertion follows from the Borel-Weil theorem.

Now there are natural morphisms

$$H^0(\mathcal{S}, U^\vee) \otimes H^0(\mathcal{S}, U(1)) \longrightarrow H^0(\mathcal{S}, U^\vee \otimes U(1)) \longrightarrow H^0(\mathcal{S}, \mathcal{O}_\mathcal{S}(1)),$$

the right hand side being induced by the trace map $U^\vee \otimes U \rightarrow \mathcal{O}_\mathcal{S}$. This determines the quadratic equations of \mathcal{S} , as the image of the induced map

$$\begin{aligned} V = V^\vee = H^0(\mathcal{S}, U^\vee) &\longrightarrow H^0(\mathcal{S}, U(1))^\vee \otimes H^0(\mathcal{S}, \mathcal{O}_\mathcal{S}(1)) \simeq \\ &\simeq H^0(\mathcal{S}, \mathcal{O}_\mathcal{S}(1))^{\otimes 2} \longrightarrow \text{Sym}^2 H^0(\mathcal{S}, \mathcal{O}_\mathcal{S}(1)). \end{aligned}$$

What we just need to do is therefore prove that the restriction morphisms to X for those spaces of sections are isomorphisms. More precisely, we need that

$$H^0(\mathcal{S}, U^\vee) \longrightarrow H^0(X, U_X^\vee), \quad H^0(\mathcal{S}, U(1)) \longrightarrow H^0(X, U_X(1))$$

are isomorphisms. This again follows from a straightforward application of the Borel-Weil-Bott theorem. \square

By the same argument as in [1], we deduce that:

Proposition 3.8. *If $X = \mathcal{S}_1 \cap \mathcal{S}_2$ is a transverse intersection, then the only translates of \mathcal{S} that contain X are \mathcal{S}_1 and \mathcal{S}_2 .*

4 Double mirrors

Recall that the spinor variety $\mathcal{S} \subset \mathbf{P}\Delta$ is projectively dual to the other spinor variety $\mathcal{S}^\vee \subset \mathbf{P}\Delta^\vee$. We may therefore associate to $X = \mathcal{S}_1 \cap \mathcal{S}_2 \subset \mathbf{P}\Delta$, the other double spinor variety $Y = \mathcal{S}_1^\vee \cap \mathcal{S}_2^\vee \subset \mathbf{P}\Delta^\vee$. When X is smooth, its presentation as the intersection of two translated spinor varieties is unique, and therefore Y is uniquely defined.

4.1 Derived equivalence

Proposition 4.1. *The double spinor varieties X and Y are simultaneously smooth.*

Proof. Suppose $\mathcal{S}_1 = g_1\mathcal{S}$ and $\mathcal{S}_2 = g_2\mathcal{S}$ and let $x \in X$. Then $x = g_1E_1 = g_2E_2$ for some E_1, E_2 in \mathcal{S} . The intersection of \mathcal{S}_1 and \mathcal{S}_2 fails to be transverse at x if and only if there is a point $y \in \mathbf{P}\Delta^\vee$ such that the corresponding hyperplane H_y in $\mathbf{P}\Delta$ is tangent to both \mathcal{S}_1 and \mathcal{S}_2 at x . By Lemma 2.4, this means that $y = g_1F_1 = g_2F_2$ for some F_1, F_2 in \mathcal{S}^\vee , such that $\dim(E_1 \cap F_1) = \dim(E_2 \cap F_2) = 4$. In particular y belongs to $g_1\mathcal{S}^\vee \cap g_2\mathcal{S}^\vee = \mathcal{S}_1^\vee \cap \mathcal{S}_2^\vee = Y$ and by symmetry, the intersection of \mathcal{S}_1^\vee and \mathcal{S}_2^\vee fails to be transverse at y . This implies the claim. \square

Proposition 4.2. *When they are smooth, the double spinor varieties X and Y are derived equivalent.*

Proof. This is a direct application of the Main Theorem in [3]. As we already mentioned, the fact that $(\mathcal{S}_1, \mathcal{S}_1^\vee)$ and $(\mathcal{S}_2, \mathcal{S}_2^\vee)$ are pairs of homologically projectively dual varieties was established in [7, Section 6.2]. In fact the results of [3] imply the stronger statement that X and Y are derived equivalent as soon as they have dimension five, even if they are singular. \square

As the authors informed us, Proposition 4.2 should also follow from the results of [8]. From the point of view of mirror symmetry, X and Y being D-equivalent should have the same mirror: they form an instance of a double mirror.

Applying Proposition 2.1 of [14], we deduce (recall from Proposition 3.2 that $H^5(X, \mathbb{Z})$ and $H^5(Y, \mathbb{Z})$ are torsion free):

Corollary 4.3. *The polarized Hodge structures on $H^5(X, \mathbb{Z})$ and $H^5(Y, \mathbb{Z})$ are equivalent.*

4.2 Non birationality

Now we sketch a proof of the following result, according to the ideas of [14, Proof of Lemma 4.7].

Proposition 4.4. *Generically, the mirror double spinors X and Y are not birationally equivalent.*

Proof. Since X and Y have free Picard groups of rank one, both generated by the hyperplane divisor, we are reduced to proving that they are not projectively equivalent. Since they are both contained in a unique pair of translates \mathcal{S}_1 and \mathcal{S}_2 of the spinor variety, we need to show that in general, there is no projective

isomorphism $u : \mathbf{P}\Delta \simeq \mathbf{P}\Delta^\vee$ such that $u(\mathcal{S}_1) = \mathcal{S}_1^\vee$ and $u(\mathcal{S}_2) = \mathcal{S}_2^\vee$, or $u(\mathcal{S}_1) = \mathcal{S}_2^\vee$ and $u(\mathcal{S}_2) = \mathcal{S}_1^\vee$.

Let us fix once and for all a linear isomorphism $u_0 : \mathbf{P}\Delta \simeq \mathbf{P}\Delta^\vee$ such that $u_0(\mathcal{S}_1) = \mathcal{S}_1^\vee$. There is a linear automorphism g of $\mathbf{P}\Delta$ such that $\mathcal{S}_1 = g(\mathcal{S}_2)$. It is easy to check that the existence of u is equivalent to the existence of v, w in $\text{Aut}(\mathcal{S}_1)$ such that either

$$u_0 g^t u_0^{-1} = v g^{-1} w \quad \text{or} \quad u_0 g^t u_0^{-1} = v g w.$$

We follow the approach of [14] to prove that for a general g , such elements of $H = \text{Aut}(\mathcal{S}_1)$ do not exist.

First case. In order to exclude the possibility that $u_0 g^t u_0^{-1} = v g^{-1} w$, one might exhibit an $H \times H$ -invariant function on $g = SL(\Delta)$ such that $F(g^{-1}) \neq F(u_0 g^t u_0^{-1})$. To do this, recall that the quadratic equations of the spinor variety $\mathcal{S} \subset \mathbf{P}\Delta$ are parametrized by $V \simeq V^\vee \subset S^2 \Delta^\vee$. The invariant quadratic form $q \in S^2 V^\vee$ is thus mapped to an invariant element $Q \in S^2 \Delta^\vee \otimes S^2 \Delta^\vee$. (In fact this element belongs to the kernel of the product map to $S^4 \Delta^\vee$, since the latter contains no invariant.) Dually there is an invariant element $Q^\vee \in S^2 \Delta \otimes S^2 \Delta$, and the function we use is $F(g) = \langle Q^\vee, gQ \rangle$.

Second case. As observed in [14], it suffices to show that there exists some partition λ such that the space of H -invariants in $S_\lambda \Delta$ is at least two-dimensional. We provide an abstract argument for that. Suppose the contrary. Let $G = SL(\Delta)$. By the Peter-Weyl theorem, the multiplicity of $S_\lambda \Delta$ inside $\mathbb{C}[G/H]$ is the dimension of its subspace of H -invariants. If this dimension is always smaller or equal to one, $\mathbb{C}[G/H]$ is multiplicity free, which means that H is a *spherical subgroup* of G . Then by [16, Theorem 1], H has an open orbit in the complete flag variety $Fl(\Delta)$. But the dimension of H is just too small for that to be true, and we get a contradiction. \square

4.3 L-equivalence

Recall that \mathbf{L} denotes the class of the affine line in the Grothendieck ring of complex varieties.

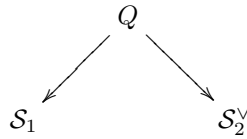
Proposition 4.5. *The double spinor varieties X and Y are such that*

$$([X] - [Y])\mathbf{L}^7 = 0$$

in the Grothendieck ring of varieties.

Note that when X and Y are not birational, $[X] - [Y] \neq 0$ in the Grothendieck ring (see [9, Proposition 2.2]).

Proof. The proof is the same as for Theorem 1.6 in [1]. We consider the incidence correspondence



where Q is the variety of pairs $x \in \mathcal{S}_1$, $y \in \mathcal{S}_2^\vee$ such that x belongs to the hyperplane H_y . The fiber of p_2 over y is $\mathcal{S}_1 \cap H_y$; it is singular if and only if y also belongs to \mathcal{S}_1^\vee , hence to Y . In this case the fiber is isomorphic to \mathcal{HS}_{sing} , otherwise it is isomorphic to \mathcal{HS}_{reg} . This yields two fibrations with constant fibers, which may not be Zariski locally trivial but must be *piecewise trivial*, like in [13, Lemme 3.3]. We deduce that in the Grothendieck ring of varieties,

$$[Q] = [Y][\mathcal{HS}_{sing}] + [\mathcal{S}_2^\vee - Y][\mathcal{HS}_{reg}].$$

The same analysis for the other projection yields the symmetric relation

$$[Q] = [X][\mathcal{HS}_{sing}] + [\mathcal{S}_1 - X][\mathcal{HS}_{reg}].$$

Taking the difference (recall that \mathcal{S}_1 and \mathcal{S}_2^\vee are isomorphic varieties), we get

$$0 = ([X] - [Y])([\mathcal{HS}_{sing}] - [\mathcal{HS}_{reg}]).$$

But \mathcal{HS}_{sing} and \mathcal{HS}_{reg} both have cell decompositions (Propositions 2.3 and 2.5), with the same numbers of cells except that \mathcal{HS}_{reg} has one less in dimension seven. Hence $[\mathcal{HS}_{sing}] - [\mathcal{HS}_{reg}] = \mathbf{L}^7$. \square

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