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A KINETIC SELECTION PRINCIPLE FOR CURL-FREE VECTOR FIELDS OF UNIT NORM

PIERRE BOCHARD[†] AND PAUL PEGON[‡]

ABSTRACT. This article is devoted to the generalization of results obtained in 2002 in [9] by Jabin, Otto and Perthame. In their article they proved that planar vector fields taking value into the unit sphere of the euclidean norm and satisfying a given kinetic equation are locally Lipschitz. Here, we study the same question replacing the unit sphere of the euclidean norm by the unit sphere of *any* norm. Under natural assumptions on the norm, namely smoothness and a qualitative convexity property, that is to be of power type p , we prove that planar vector fields taking value into the unit sphere of such a norm and satisfying a certain kinetic equation are locally $\frac{1}{p-1}$ -Hölder continuous. Furthermore we completely describe the behaviour of such a vector field around singular points as a *vortex* associated to the norm. As our kinetic equation implies for the vector field to be curl-free, this can be seen as a selection principle for curl-free vector fields valued in spheres of general norms which rules out line-like singularities.

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PRELIMINARIES AND NOTATIONS

Our notations in this article will be standard. Beware that we will consider in this article maps instead of equivalence classes for the almost everywhere relation; all maps will be taken Borel measurable. More specifically, let $\Omega, E \subset \mathbb{R}^2$. We will denote by $\mathcal{L}_{\text{loc}}^1(\Omega, E)$ the set of Borel maps m which are locally integrable and such that $m(x) \in B$ for all x in Ω . Of course we do not lose generality with

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this choice. For $x \in \mathbb{R}^2$, $x^\perp$ will denote its rotation by an angle $+\pi/2$. The euclidean norm will be denoted by $|\cdot|$ and a general norm by $\|\cdot\|$.

1. INTRODUCTION

The goal of this paper is the generalization of regularity results on solutions of a kinetic equation obtained by Jabin, Otto and Perthame in their article [9]. In order for the reader to understand where we are starting from, we will begin by briefly recalling their results. Let $\Omega \subset \mathbb{R}^2$ be an open set. Jabin, Otto and Perthame were interested in the study of the following micromagnetic energy on $H^1(\Omega, \mathbb{R}^2)$:

$$(1.1) \quad E_\varepsilon(m) = \varepsilon \int_{\Omega} |Dm|^2 + \frac{1}{\varepsilon} \int_{\Omega} (1 - |m|^2)^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla^{-1} \operatorname{curl} m|^2,$$

where m is extended by zero outside of Ω . They associate to the measurable vector field $m: \Omega \rightarrow \mathbb{R}^2$ the quantity:

$$\begin{aligned} \chi_m: \quad \Omega \times \mathbb{S}^1 &\longrightarrow \{0, 1\} \\ (x, s) &\longmapsto \mathbb{1}_{\{m(x) \cdot s > 0\}}. \end{aligned}$$

The introduction of such a quantity is motivated by the following *averaging formula*: let $u \in \mathbb{S}^1$, then

$$(1.2) \quad u = \frac{1}{2} \int_{\mathbb{S}^1} \mathbb{1}_{\{u \cdot s > 0\}} s \, d\mathcal{H}^1(s)$$

An obvious consequence of (1.2) is that if we suppose that $|m| = 1$ almost everywhere, then for almost every x in Ω ,

$$m(x) = \frac{1}{2} \int_{\mathbb{S}^1} \chi_m(x, s) s \, d\mathcal{H}^1(s).$$

They proved then the following theorem:

Theorem 1.1 (Jabin, Otto, Perthame). *Let m be a zero-state energy, that is a limit of a sequence $(m_\varepsilon)_{\varepsilon>0} \subset H^1(\Omega, \mathbb{R}^2)$ with $E_\varepsilon(m_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$. Then $|m| = 1$ almost everywhere and m satisfies the kinetic equation:*

$$(1.3) \quad \forall s \in \mathbb{S}^1, \quad s^\perp \cdot \nabla \chi_m(\cdot, s) = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Classical results on the regularity of the average of solutions of kinetic equations tell us (see [7] for example) that equation (1.3) implies for m to be in $H^{\frac{1}{2}}(\Omega, \mathbb{R}^2)$. But in the case of this particular kinetic equation, it turns out that combining (1.3) with $|m| = 1$ implies a much stronger regularity. More precisely, the following holds:

Theorem 1.2 (Jabin, Otto, Perthame). *Let m be a measurable vector field on Ω satisfying $|m| = 1$ almost everywhere and equation (1.3). Then m is locally Lipschitz continuous inside Ω except at a locally finite set of singular points. Furthermore, for every singular point p , there is $\alpha \in \{-1, 1\}$ such that in any convex neighborhood of p ,*

$$m(x) = \alpha \frac{x - p}{|x - p|} \quad \text{a.e.}$$

Before we state our own results, let us give a bit more insight into the two previous theorems. Looking at Theorem 1.1, we expect a zero-state energy m to be of modulus one and curl-free because of the energy (1.1). As a consequence, the curl-free information about m has to be encompassed into equation (1.3). It is indeed easy to see that if a modulus one vector field checks equation (1.3) then it is curl-free; we will prove this in our more general setting in Proposition 4.1. Keeping this in

mind, we can now look at Theorem 1.2 in the following way: equation (1.3) is a *selection principle* among curl-free vector fields taking values into $\mathbb{S}^1$ which, roughly speaking, does not allow for line-like singularities.

The question we are addressing here is the following: can we find a kinetic equation which will act as a selection principle for curl-free vector field taking value into the unit sphere of any norm instead of $\mathbb{S}^1$? The purpose of this article is to prove that if we make simple geometric assumptions on our norm, the answer to this question is yes. In the spirit of Theorem 1.2, we will then prove that this kinetic equation implies some regularity on the vector field itself.

Our paper will be divided into three parts. The first one will be devoted to prerequisites on general norms and their modulus of convexity, the second one to the study of a generalized averaging formula in the spirit of (1.2) and the last one to our main result, i.e. the regularity of solutions to the kinetic equation we are going to introduce.

Before stating our results, we need some definitions on norms and on the way to measure their convexity.

Definition 1.1. Let $\|\cdot\|$ be a norm and B its closed unit ball. We call *modulus of convexity* associated to B the quantity $\omega_B: [0, 2] \rightarrow [0, 1]$ defined by:

$$(1.4) \quad \omega_B(\delta) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1 \text{ and } \|x-y\| \geq \delta \right\}.$$

Definition 1.2. Let $p \geq 0$. We say that a norm is of *power type* p if there is a constant $K > 0$ such that:

$$(1.5) \quad \forall \delta \in [0, 2], \quad K\delta^p \leq \omega_B(\delta)$$

As an example, one should keep in mind that the euclidean norm is of power type 2 and that more generally, for $p > 1$, l^p norms are of power type $\max\{p, 2\}$: this is a consequence of Clarkson's inequalities and can be found in [11]. The main reason for introducing these two notions is that we *need* some property on the norm in order to obtain a kinetic equation which acts as a selection principle excluding line-like singularities. It turns out that the *power type* p property is exactly the good property to look at. We refer the reader to the beginning of our third section for a more detailed discussion.

The first important result of our article is an averaging formula in the spirit of (1.2):

Theorem 1.3. Let $B \subset \mathbb{R}^2$ be a symmetric convex body, that is the unit ball of some norm, then:

$$(1.6) \quad \forall x \in \partial B, \quad x = \frac{1}{2} \int_{\partial B^\perp} \mathbb{1}_{\{x \cdot s > 0\}} n_{B^\perp}(s) d\mathcal{H}^1(s),$$

where $n_{B^\perp}(s)$ stands for the unit normal to $\partial B^\perp$ at s .

Remark 1.1. Note that formula (4.1) generalizes easily in any dimension in the case of the euclidean sphere. We refer the reader to [1] for such a formula and its use to prove a rigidity result obtained in the study of an analog of equation (1.3) in dimension greater than 2. It would be nice to know if we can obtain an averaging formula like (1.6) in greater dimension for a general symmetric convex body.

The following corollary of Theorem 1.3, which is crucial to prove our main result, is immediate.

Corollary 1.4. Let $\|\cdot\|$ be a norm, B its closed unit ball and $m: \Omega \rightarrow \mathbb{R}^2$ a Borel vector field such that $\|m\| = 1$. We associate to m the quantity:

$$\begin{aligned} \chi_m: \quad \Omega \times \partial B^\perp &\longrightarrow \{0, 1\} \\ (x, s) &\longmapsto \mathbb{1}_{\{m(x) \cdot s > 0\}}. \end{aligned}$$

Then, for all x in Ω ,

$$(1.7) \quad m(x) = \frac{1}{2} \int_{\partial B^\perp} \chi_m(x, s) n_{B^\perp}(s) d\mathcal{H}^1(s).$$

We need a last definition to state our main result:

Definition 1.3. Let $\|\cdot\|$ be a norm with unit ball B and $\|\cdot\|_\star$ its dual norm defined as usual by the equality:

$$\|x\|_\star = \sup_{\|y\| \leq 1} \langle x, y \rangle.$$

We call *vortex* associated to the norm $\|\cdot\|$ the function $V_B : x \mapsto \nabla_x \|\cdot\|_\star$ at the points where this quantity makes sense. The map $x \mapsto \|\cdot\|$ being convex, V_B is well-defined almost everywhere on $\mathbb{R}^2$ according to Rademacher's theorem.

We can now state the main theorem of our article.

Theorem 1.5. Let $\|\cdot\|$ be a norm of power type p and B its closed unit ball such that ∂B is a C^1 submanifold in $\mathbb{R}^2$. Let $m : \Omega \rightarrow \mathbb{R}^2$ be a Borel vector field such that $\|m\| = 1$ and satisfying:

$$(1.8) \quad \forall s \in \partial B^\perp, \quad n_{B^\perp}(s)^\perp \cdot \nabla \chi_m(\cdot, s) = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Then m is locally $\frac{1}{p-1}$ -Hölder continuous outside a locally finite set of singular point. Furthermore, for every singular point p , there is $\alpha \in \{-1, 1\}$ such that in any convex neighborhood of p ,

$$m(x) = \alpha V_B(x - p) \quad \text{almost everywhere,}$$

where V_B is the vortex associated to $\|\cdot\|$ defined in Definition 1.3.

Remark 1.2. Notice that the two hypotheses on the norm are not redundant: we can find norms which are not of power type p for any p and whose unit sphere is a C^1 submanifold. Take for example the norm associated to the l^1 unit ball whose angles have been smoothed out. On the other hand, an example of a type p norm whose unit ball is *not* a C^1 submanifold is given by $\|x\| = \|x\|_1 + \|x\|_p$ with $p > 2$. The reader interested in the links between convexity and smoothness of the ball of a norm will consult with profit Section 1.e. of [11].

In order to see how Theorem 1.5 could be used, let us note that Theorem 1.2 itself has interesting consequences. For example, together with clever commutator estimates, De Lellis and Ignat proved in [3] the following:

Theorem 1.6 (De Lellis, Ignat). Let Ω be an open set in $\mathbb{R}^2$ and $r \in [1, 3]$. Then every m in

$$W_{curl}^{\frac{1}{r}, r}(\Omega, \mathbb{S}^1) := \left\{ m \in W^{\frac{1}{r}, r}(\Omega, \mathbb{R}^2) : \operatorname{curl} m = 0 \text{ in } \mathcal{D}'(\Omega) \text{ and } |m| = 1 \text{ a.e.} \right\}$$

is locally Lipschitz continuous inside Ω except at a locally finite set of singular points, where $W^{\frac{1}{r}, r}(\Omega, \mathbb{R}^2)$ stands for the usual fractional Sobolev space.

A first natural question would be to ask if we can obtain such a theorem when $\mathbb{S}^1$ is changed into the unit sphere of a norm satisfying the hypotheses of Theorem 1.5. Another natural question, in view of Theorem 1.1, would be to ask if we can obtain an analog of Theorem 1.1 if we change the energy (1.1) into:

$$\tilde{E}_\varepsilon(m) = \varepsilon \int_\Omega |Dm|^2 + \frac{1}{\varepsilon} \int_\Omega (1 - \|m\|^2)^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla^{-1} \operatorname{curl} m|^2,$$

where $\|\cdot\|$ is now a norm satisfying the hypotheses of Theorem 1.5. Note that the proof of both Theorem 1.6 and Theorem 1.1 rely on the notion of *entropy*, that is, in this context, smooth functions $\Phi: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ such that for all open Ω in $\mathbb{R}^2$,

$$\forall m \in C_{curl}^\infty(\Omega, \mathbb{S}^1), \quad \nabla \cdot [\Phi \circ m] = 0.$$

We refer the reader to [4] and [8] for more details and applications of this notion. Therefore a first step in order to obtain analogs of Theorems 1.1 and 1.6 would be to develop an adapted notion of entropy. We will address this question in a paper to come. The reader interested in questions related to the regularity of solutions of the eikonal equation may consult [2].

2. MODULUS OF CONVEXITY AND VORTICES

Our main sources for this section are [10] and [11] on the geometry of norms and [5] on convex analysis. For the reader's comfort, let us collect some general facts on convex functions and convex bodies:

- A function $f: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is said *convex* if $f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$ for all $x, y \in \mathbb{R}^d$, $t \in [0, 1]$. Its domain is defined as $\text{Dom}(f) := \{x : f(x) < \infty\}$.
- The subdifferential of f at x is defined as $\partial f(x) := \{p : \forall y, f(y) \geq f(x) + \langle p, y - x \rangle\}$. It is nonempty if $x \in \text{Dom}(f)$. In that case, f is differentiable at x if and only if its subdifferential is a singleton, in which case $\partial f(x) = \{\nabla f(x)\}$.
- The Legendre-Fenchel transform of f is defined as

$$f^*(p) = \sup_x x \cdot p - f(x).$$

If f is a proper ($\neq +\infty$) convex lsc function, so is f^* , and $f = f^{**}$.

- In that case, ∂f^* is the inverse of ∂f in the sense that

$$(2.1) \quad p \in \partial f(x) \iff x \in \partial f^*(p).$$

- If B is a convex body and $\|\cdot\|$ its associated norm, then $\chi_B^* = \|\cdot\|_*$.
- If $x \in \partial B$, we say that u is a *normal vector to B at x* if

$$\forall y \in B, \quad u \cdot (y - x) \leq 0.$$

Notice that the set of normal vectors, denoted by $\mathcal{N}_B(x)$, is a convex cone which is precisely $\partial \chi_B(x)$.

- If $x \in \partial B$, the following relation between the subdifferentials of $\|\cdot\|$ and χ_B holds:

$$(2.2) \quad \partial_x \|\cdot\| = \partial \chi_B(x) \cap B^*, \quad \text{or equivalently} \quad \partial_x \|\cdot\| = \mathcal{N}_B(x) \cap B^*.$$

- There is a unique unit normal vector at $x \in \partial B$ if and only if the associated norm is differentiable at x . In general this is true for $\mathcal{H}^1$ -almost every $x \in \partial B$, and it is the case for all $x \in \partial B$ for instance if ∂B is a $\mathcal{C}^1$ submanifold.

Before getting to the heart of the matter, let us explain why the norm should satisfy some property for the kinetic equation to act as a selection principle ruling out line-like singularities. For that purpose, let us take another look at the vortex V_B associated to a norm $\|\cdot\|$ of unit ball B as defined in Definition

1.3. It is clear that V_B is curl-free in $\mathcal{D}'(\mathbb{R}^2)$. Indeed, taking $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned} \int_{\mathbb{R}^2} \partial_1 \varphi V_{B,2} - \partial_2 \varphi V_{B,1} \, dx &= \int_{\mathbb{R}^2} \nabla \varphi^\perp \cdot \nabla \|x\|_\star \, dx \\ &= \int_{\mathbb{R}^2} \nabla \cdot [\nabla \varphi^\perp] \|x\|_\star \, dx = 0, \end{aligned}$$

where we used the fact that V_B is the distributional gradient of $\|\cdot\|$ by Rademacher's theorem. Furthermore, V_B takes its values into ∂B . Indeed by (2.1), for almost every x (those where $\|\cdot\|_\star$ is differentiable, i.e. where V_B is defined), one has $p = V_B(x) \iff x \in \partial_p \|\cdot\|$, so that p is a point where the subdifferential $\partial_p \|\cdot\|$ contains a nonzero vector, i.e. a point on the boundary of the ball. Consequently $V_B(x) = p \in \partial B$. We are going to prove that V_B is *always* a solution to the kinetic equation (1.8). As a consequence, if we want this kinetic equation to rule out line-like singularities then V_B should *not* exhibit such singularities, and the regularity of V_B is related to the convexity of B , as we will see in Proposition 2.5. This is the fundamental reason why we need some convexity property on our norm in order to obtain Theorem 1.5.

2.1. Properties of the normal field. We start off by stating some useful properties of the normal field

$$\begin{aligned} n_B : \partial B &\longrightarrow \mathcal{P}(\mathbb{S}^1) \\ x &\longmapsto \mathcal{N}_B(x) \cap \mathbb{S}^1, \end{aligned}$$

where $\mathcal{N}_B(x)$ is the normal cone to B at x .

Lemma 2.1. *Let B be the unit ball of a norm and n_B the application defined above. Then, for $\mathcal{H}^1$ -almost every y in $\mathbb{S}^1$, there is a unique x in ∂B such that $y \in n_B(x)$.*

Proof. Let us consider:

$$\begin{aligned} \tilde{n}_B : \partial B &\longrightarrow \mathcal{P}(\partial B^\star) & \tilde{n}_{B^\star} : \partial B^\star &\longrightarrow \mathcal{P}(\partial B) \\ x &\longmapsto \mathcal{N}_B(x) \cap \partial B^\star, & x &\longmapsto \mathcal{N}_{B^\star}(x) \cap \partial B. \end{aligned}$$

By (2.1) and (2.2), it is easy to see that $p \in \tilde{n}_B(x)$ if and only if $x \in \tilde{n}_{B^\star}(p)$. Since $\partial B^\star$ admits a normal vector for $\mathcal{H}^1$ -almost every p , we get that there is a unique x satisfying $p \in \tilde{n}_B(x)$ for $\mathcal{H}^1$ -almost every p in $\partial B^\star$. We get uniqueness for n_B by using the fact that the projection from $\mathbb{S}^1$ to $\partial B^\star$ is a bi-Lipschitz bijection. $\square$

Remark 2.1. In the rest of the paper, if $y \in \mathbb{S}^1$ is such that there is a unique $x \in \partial B$ satisfying $y \in n_B(x)$, we will abusively denote it by $x = n_B^{-1}(y)$. One can deduce from the previous proof that this is in particular the case if $\partial B^\star$ is a C^1 submanifold of $\mathbb{R}^2$.

Proposition 2.2. *Let $\|\cdot\|$ be a norm of unit ball B , then for almost every x ,*

$$V_B(x) = n_B^{-1} \left(\frac{x}{|x|} \right).$$

Proof. We know that $\|\cdot\|_\star$ is the Legendre-Fenchel transform of χ_B . Consequently $\|\cdot\|_\star$ is differentiable at x if and only if x is in the normal cone of a unique $p \in \partial B$, that is if and only if $n_B^{-1} \left(\frac{x}{|x|} \right)$ is well defined, in which case $n_B^{-1} \left(\frac{x}{|x|} \right) = p = \nabla_x \|\cdot\|_\star \doteq V_B(x)$. By the previous lemma, it is true for almost every x . $\square$

We can now prove as promised that V_B is always solution to the kinetic equation (1.8). Indeed, let $\Omega \subset \mathbb{R}^2$, $\varphi \in C_0^\infty(\Omega)$, and $s \in \partial B$. Using Proposition 2.2, for almost every $x \in \Omega$, $V_B(x) = n_B^{-1}\left(\frac{x}{|x|}\right)$, and

$$\int_{\Omega} \nabla \varphi \cdot n_{B^\perp}(s)^\perp \chi_{V_B}(x, s) dx = \int_{\Omega \cap \{n_B^{-1}\left(\frac{x}{|x|}\right) \cdot s > 0\}} \nabla \varphi \cdot n_{B^\perp}(s)^\perp dx.$$

But for almost every s , $\left\{n_B^{-1}\left(\frac{x}{|x|}\right) \cdot s > 0\right\}$ is an half-space whose boundary is the line L passing through 0 and directed by $n_B(s^\perp)$, whose unit normal is $\pm n_B(s^\perp)^\perp = \mp n_{B^\perp}(s)$. As a consequence, using Stokes formula and the fact that φ is supported in Ω ,

$$\int_{\Omega \cap \{n_B^{-1}\left(\frac{x}{|x|}\right) \cdot s > 0\}} \nabla \varphi \cdot n_{B^\perp}(s)^\perp dx = \pm n_{B^\perp}(s)^\perp \cdot n_{B^\perp}(s) \int_{\Omega \cap L} \varphi dx = 0.$$

Let us define a last quantity that we will need later. For $u, v \in \mathbb{R}^2$ which are not colinear, we define the convex cone $C(u, v)$ generated by u, v as

$$C(u, v) := \left\{ \lambda_1 u + \lambda_2 v : (\lambda_1, \lambda_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \right\}.$$

Lemma 2.3. *Let $\|\cdot\|$ be a strictly convex and differentiable (outside 0) norm of closed unit ball B . Then $n_B : \partial B \rightarrow \mathbb{S}^1$ is monotonic in the sense that*

$$w \in C(u, v) \iff n_B(w) \in C(n_B(u), n_B(v)).$$

Proof. Notice that $C(u, v)$ is (by Hahn-Banach Theorem) the intersection of the half-spaces which are delimited by vector hyperplanes and which contain u and v . First, let us show that, given any such half-space H , n_B is a bijection between $\partial B \cap H$ and a half-circle of $\mathbb{S}^1$. Since n_B is continuous and $\partial B \cap H$ is compact and connected, $n_B(\partial B \cap H)$ is a compact and connected subset of $\mathbb{S}^1$, hence it is a closed arc of $\mathbb{S}^1$. Since $\partial B \cap H$ contains two symmetric points $\pm x \in \partial B$, so does $n_B(\partial B \cap H)$, hence it contains a half-circle $[\theta, \theta + \pi]$. It may not be larger than that, otherwise it would contain a nontrivial arc and its symmetric, hence $\partial B \cap H$ as well since n_B is homeomorphic and antisymmetric. This cannot be true. This proves that n_B sends half-spaces of ∂B to half-circles and it is clear that it realizes a bijection between the set of half-spaces of ∂B and the set of half-circles of $\mathbb{S}^1$.

Now $w \in C(u, v)$ means that w belongs to any half-space (delimited by a vector hyperplane) containing u, v , which means that $n_B(w)$ belongs to any half-circle containing $n_B(u), n_B(v)$ by what we just proved. This exactly means that $n_B(w) \in C(n_B(u), n_B(v))$. $\square$

2.2. Modulus of convexity. In this section we give an alternate definition of the modulus of convexity, which offers some advantages: an intuitive interpretation is given in Proposition 2.4, and a characterization in terms of the regularity of the vortex is given in Proposition 2.5. This will be useful to prove the sharpness of our main theorem. Of course, it is essentially equivalent to Definition 1.1 which is given in the introduction, as showed in Remark 2.3.

Definition 2.1. We define the *modulus of convexity* of B as the greatest nondecreasing function $\rho_B : [0, 2] \rightarrow \mathbb{R}_+$ such that for all $x, y \in \partial B$ and all $u \in \mathcal{N}_B(x)$,

$$(2.3) \quad u \cdot y \leq u \cdot x - \|u\|_* \rho_B(\|y - x\|).$$

We say that B is of *power type* $p \geq 2$ if $\rho_B(\delta) \geq C\delta^p$ for some $C > 0$ and that it is *elliptic* if it is of power type 2.

Remark 2.2. Notice that

$$\begin{aligned} \forall y \in \partial B, \quad u \cdot y &\leq u \cdot x - \|u\|_{\star} \rho_B(\|y - x\|) \\ &\text{if and only if} \\ \forall y \in \mathbb{R}^d, \quad \chi_B(y) &\geq \chi_B(x) + u \cdot (y - x) + \|u\|_{\star} \rho_B(\|y - x\|). \end{aligned}$$

This last inequality is reminiscent of the inequality $f(y) \geq f(x) + p \cdot (y - x)$ for $p \in \partial f(x)$ which holds for arbitrary convex functions. The extra term $\|u\|_{\star} \rho_B(\|y - x\|)$ measures how much B is convex around its boundary.

Remark 2.3. The two definitions of modulus of convexity are equivalent in the sense that:

$$\rho_B(\delta/2) \leq \omega_B(\delta) \leq \rho_B(\delta)/2.$$

Proof. The proof is quite easy. Take $x, y \in \partial B$ and $u \in \mathcal{N}_B((x + y)/2)$ such that $\|u\|_{\star} = 1$. Notice that $u \in \partial_{\frac{x+y}{2}} \|\cdot\|$ (because $\|u\|_{\star} = 1$), hence by convexity:

$$1 = \|x\| \geq \left\| \frac{x + y}{2} \right\| + u \cdot \frac{x - y}{2}.$$

By definition of ρ_B one has:

$$u \cdot y \leq u \cdot \frac{x + y}{2} - \rho_B \left(\left\| y - \frac{x + y}{2} \right\| \right),$$

or equivalently

$$u \cdot \frac{y - x}{2} \leq -\rho_B \left(\frac{\|y - x\|}{2} \right).$$

Putting this in the previous inequality yields:

$$\left\| \frac{x + y}{2} \right\| \leq 1 - u \cdot \frac{x - y}{2} \leq 1 - \rho_B \left(\frac{\|y - x\|}{2} \right).$$

Since ω_B is precisely the largest map for which the inequality $\left\| \frac{x + y}{2} \right\| \leq 1 - \omega_B(\|y - x\|)$ holds for all $x, y \in \partial B$, it follows that:

$$\rho_B(\delta/2) \leq \omega_B(\delta)$$

for all $\delta \in [0, 2]$.

Now let us prove the other inequality. This time we take $x, y \in \partial B$, $u \in \mathcal{N}(x)$ with $\|u\|_{\star} = 1$. By definition we know that:

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \omega_B(\|y - x\|),$$

but since $\|u\|_{\star} = 1$, one has:

$$\left\| \frac{x + y}{2} \right\| \geq u \cdot \frac{x + y}{2} = u \cdot x + u \cdot \frac{y - x}{2} = 1 + u \cdot \frac{y - x}{2}.$$

Putting these two together, one gets:

$$u \cdot y \leq u \cdot x - 2\omega_B(\|y - x\|),$$

and by homogeneity if u has arbitrary norm:

$$u \cdot y \leq u \cdot x - \|u\|_{\star} \cdot 2\omega_B(\|y - x\|).$$

Since ρ_B is the largest function for which such an inequality holds for all x, y and $u \in \mathcal{N}_B(x)$, one has

$$2\omega_B(\delta) \leq \rho_B(\delta)$$

which concludes the proof. $\square$

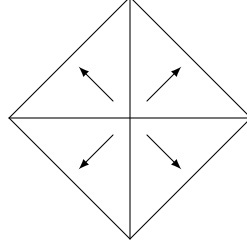


FIGURE 1. Unit ball of $\|\cdot\|_1$ and vortex associated to $\|\cdot\|_\infty$, singular along the axes.

As indicated by its name, the greater ρ_B is, the more convex B is. For example, taking a look at Figure 1, it is easy to see that the modulus of convexity associated to the l^1 norm is constant equal to 0 on a neighborhood of 0. On the other side, the euclidean ball is the most convex of all in the sense that if ω_2 is its modulus of convexity, for all symmetric convex body B ,

$$(2.4) \quad \forall \delta \in [0, 2], \quad \omega_B(\delta) \leq \omega_2(\delta).$$

We refer the reader to [12] for a proof of this inequality.

Remark 2.4. Using the symmetry of $\mathbb{S}^1$, ρ_2 can easily be computed and one finds:

$$\rho_2(\delta) = 1 - \sqrt{1 - \delta^2} \underset{\delta \rightarrow 0}{\sim} \frac{\delta^2}{2}.$$

As a consequence of this equality, of inequality (2.4) and Remark 2.3, there is *no* norm of power type p with $p < 2$.

Throughout the rest of this paper, we will use the modulus of convexity ρ_B as defined in this section. The following proposition gives an intuitive characterization of convex bodies of power type p . It states that they are convex bodies whose boundary is locally the graph of a map which is above that of the map $t \mapsto |t|^p$ around 0.

Proposition 2.4. *Let $x \in \partial B$, u a normal vector of unit euclidean norm. Take $\gamma :] - \varepsilon, +\varepsilon[\rightarrow \partial B$ a local parameterization of ∂B around $x = \gamma(0)$, with direct orientation and unit speed. We denote by $\tau = \dot{\gamma}(0)$ the unit tangent at x and $\nu = \tau^\perp$ the inner normal. In the basis (τ, ν) , we write $\gamma(t) = x + a(t)\tau + b(t)\nu$. Then B is of type p if and only if*

$$(2.5) \quad b(t) \geq C|a(t)|^p.$$

If B is of class C^2 and $\kappa(x)$ denotes the curvature at x , B is elliptic if and only if for some $C > 0$:

$$\forall x \in \partial B, \quad \kappa(x) \geq C.$$

Proof. Writing inequality (2.3) with $y = \gamma(t)$ one gets

$$-b(t) \doteq u \cdot (\gamma(t) - x) \leq -C|\gamma(t) - x|^p \leq -C|a(t)|^p$$

which yields (2.5). Conversely, since $b = o(a)$ the inequality $b(t) \geq C|a(t)|^p$ implies (2.3) for x close to y , which is enough to get it for all x, y .

If ∂B is of class C^2 and so is γ , by definition one has

$$\gamma(t) = x + t\dot{\gamma}(0) + t^2/2 \cdot \ddot{\gamma}(0) + o(t^2) = x + t\tau + \kappa(x)t^2/2 \cdot \nu + o(t^2).$$

Consequently $a(t) = t + o(t^2)$ and $b(t) = \kappa t^2/2 + o(t^2)$, thus B being elliptic, one has:

$$C(t + o(t^2))^2 \leq \kappa t^2/2 + o(t^2).$$

Dividing by t^2 and $t \rightarrow 0$ yields $\kappa \geq 2C$ and ∂B has positive curvature. The converse is straightforward. $\square$

The next proposition establishes a link between the power type p property of a norm and the smoothness of its vortex.

Proposition 2.5. *Let B be a convex body associated to the norm $\|\cdot\|$. Then B is of power type p if and only if $\|\cdot\|_\star$ is differentiable everywhere outside 0 and $V_B \doteq \nabla\|\cdot\|_\star$ is $\frac{1}{p-1}$ -Hölder continuous far from 0, in the sense that for some constant $H > 0$ one has:*

$$\|V_B(v) - V_B(u)\| \leq \frac{H}{\delta^{\frac{1}{p-1}}} \|u - v\|_\star^{\frac{1}{p-1}},$$

for all $\|u\|, \|v\| \geq \delta$ and all $\delta > 0$.

Proof. Take $x, y \in \partial B$ and $u \in \mathcal{N}_B(x), v \in \mathcal{N}_B(y)$ with u, v nonzero. One has

$$u \cdot y \leq u \cdot x - \|u\|_\star \rho_B(\|y - x\|),$$

$$v \cdot x \leq v \cdot y - \|v\|_\star \rho_B(\|y - x\|),$$

and by summing these one gets:

$$2 \min(\|u\|_\star, \|v\|_\star) \rho_B(\|y - x\|) \leq (v - u) \cdot (y - x).$$

Now if B is of power type p then

$$C\|y - x\|^p \leq \rho_B(\|y - x\|) \leq \frac{(v - u) \cdot (y - x)}{\min(\|u\|_\star, \|v\|_\star)} \leq \frac{\|v - u\|_\star \|y - x\|}{\min(\|u\|_\star, \|v\|_\star)},$$

thus

$$\|y - x\| \leq \left(\frac{\|u - v\|_\star}{C \min(\|u\|_\star, \|v\|_\star)} \right)^{\frac{1}{p-1}},$$

which yields

$$\|y - x\| \leq \frac{H}{\delta^{\frac{1}{p-1}}} \|u - v\|_\star^{\frac{1}{p-1}}.$$

provided that $\|u\|_\star, \|v\|_\star \geq \delta$, for some constant H depending only on the norm.

Conversely, take $x \in \partial B$, $u \in \mathcal{N}_B(x)$ with $\|u\|_\star = 1$. We want to show that for all $y \in B$:

$$u \cdot y \leq u \cdot x - C|y - x|^p.$$

Take x_0 a minimizer of $g(y) = \chi_B(y) + (u \cdot (x - y) - C \frac{|y - x|^p}{p})$. We want to show that for small enough C , $g(x_0) \geq 0$. The necessary optimality condition reads

$$u_0 := u + C(x_0 - x)^{p-1} \in \partial\chi_B(x_0).$$

Necessarily, $x_0 \in B$ thus

$$\|u_0 - u\|_* \leq \delta C \|x_0 - x\|^{p-1} \leq \delta C 2^{p-1}$$

where $\delta := \sup_{\|v\|=1} \|v\|_*$. If C is small enough, say $C\delta 2^{p-1} \leq 1/2$ then both $\|u\|_*, \|u_0\|_* \geq 1/2$. On the other hand one has by hypothesis:

$$\|x_0 - x\|^{p-1} \leq 2H^{p-1} \|u_0 - u\|_* \leq 2\delta CH^{p-1} \|x_0 - x\|^{p-1}.$$

Now if one takes C even smaller, so that $2\delta CH^{p-1} < 1$, one must have $x = x_0$ and $u = u_0$, which implies that $\min_{\mathbb{R}^2} g = g(x_0) = g(x) = 0$ and the following holds:

$$\forall y \in B, \quad u \cdot y \leq u \cdot x - C|y - x|^p/p \quad \text{with } C = \min(2^{-p}, 1/(2\delta H^{p-1})).$$

If u has arbitrary norm, this becomes:

$$\forall y \in B, \quad u \cdot y \leq u \cdot x - C' \|u\|_* \|y - x\|^p$$

where $C' = C/p$. □

As a corollary, one gets the following regularity theorem.

Theorem 2.6. *Let $\|\cdot\|$ be a norm with closed unit ball B . It is of power type p if and only if $n_B^{-1} : \mathbb{S}^1 \rightarrow \partial B$ is well-defined and $\frac{1}{p-1}$ -Hölder continuous. In particular it is elliptic iff n_B^{-1} is Lipschitz continuous.*

3. AVERAGING FORMULA

This section is devoted to the proof of Theorem 1.3. A first interesting remark is that if we want an averaging formula in the spirit of formula (1.6) to be true, we *need* the convex to be symmetric. More precisely, we have the following proposition:

Proposition 3.1. *Let $B \subset \mathbb{R}^2$ be a convex body with 0 in its interior. Assume there is a Borel vector measure $\mu \in \mathcal{M}^2(\mathbb{R}^2 \setminus \{0\})$ such that the following representation formula holds:*

$$(3.1) \quad \forall x \in \partial B, \quad x = \int_{\mathbb{R}^2} \mathbb{1}_{\{x \cdot s > 0\}} \mu(ds).$$

Then B is symmetric and one may replace μ in the formula by the measure $\bar{\mu}$ defined by

$$\bar{\mu}(A) := \frac{\mu(A) - \mu(-A)}{2}$$

for all Borel set A , which is an antisymmetric measure in the sense that $\bar{\mu}(-A) = -\bar{\mu}(A)$, and which does not give mass to vector lines.

Proof. Let us set

$$(3.2) \quad x^+ := \{s \in \mathbb{R}^2 : \langle x, s \rangle > 0\}, \quad x^- := \{s \in \mathbb{R}^2 : \langle x, s \rangle < 0\}, \quad x^\perp := \{s \in \mathbb{R}^2 : \langle x, s \rangle = 0\}.$$

One may define the gauge of B as

$$j_B : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & \inf\{t > 0 : x \in tB\}, \end{array}$$

the radial projection onto ∂B and radial symmetry with respect to B respectively as:

$$p_B : \begin{array}{ccc} \mathbb{R}^2 \setminus \{0\} & \longrightarrow & \partial B \\ x & \longmapsto & \frac{x}{j_B(x)}, \end{array} \quad s_B : \begin{array}{ccc} \mathbb{R}^2 \setminus \{0\} & \longrightarrow & \mathbb{R}^2 \\ x & \longmapsto & j_B(x) p_B(-x). \end{array}$$

First, let us assume that μ does not charge lines of the plane, i.e. $\mu(D) = 0$ for all vector line $D \subset \mathbb{R}^2$. Writing $\mathbb{R}^2 = x^+ \sqcup x^- \sqcup x^\perp$ and noticing that $s_B(x)^+ = x^-$, we obtain:

$$\mu(x^+) + \mu(s_B(x)^+) = \mu(x^+) + \mu(x^-) + \mu(x^\perp) = \mu(\mathbb{R}^2) =: v \in \mathbb{R}.$$

If x is in ∂B , according to (3.1) one has $x = \mu(x^+)$ and $s_B(x) = \mu(s_B(x)^+)$ since $s_B(x) \in \partial B$, which yields

$$(3.3) \quad \forall x \in \partial B, \quad x + s_B(x) = v.$$

But $s_B(x)$ is colinear to x , and since $\overset{\circ}{B} \neq \emptyset$, we can find two non colinear vector $x_1, x_2 \in \partial B$. Now writing (3.3) with x_1 and x_2 implies $v = 0$; consequently, for all $x \in \partial B$, $x = -s_B(x) = -p_B(-x)$, which exactly means that B is symmetric.

Now we may get rid of the hypothesis that μ does not charge lines. If μ gives a positive mass to some vector lines, it may only charge countably many of them since μ is finite and these lines only intersect at 0 which is μ -negligible. Let us denote by L the reunion of the perpendiculars to these lines. The previous reasoning shows that $x + s_B(x) = v$ for all $x \in \partial B \setminus L$. But $\partial B \cap L$ is countable hence $\partial B \setminus L$ is dense in ∂B and by continuity of s_B the identity

$$x + s_B(x) = 0 = v$$

holds for all $x \in \partial B$, which implies that B is symmetric and μ does not charge vector lines. Thus if $x \in \partial B$ then $-x \in \partial B$, which implies:

$$-x = \int_{\mathbb{R}^2} \mathbb{1}_{\{-x \cdot s > 0\}} \mu(ds) = \mu(-(x^+))$$

and

$$x = \frac{x - (-x)}{2} = \frac{\mu(x^+) - \mu(-(x^+))}{2} := \bar{\mu}(x^+).$$

Obviously, $\bar{\mu}$ is antisymmetric and does not charge lines since μ does not either. $\square$

Now let us pass on to the proof of the representation formula as stated in Theorem 1.3. It turns out that checking such a formula is quite easy once one has found the right candidate for μ , but the following theorem is more precise as it states that μ is essentially unique.

Theorem 3.2. *Let $B \subset \mathbb{R}^2$ be a symmetric convex body, that is the unit ball of some norm, then there is a unique antisymmetric measure μ supported on $\partial B^\perp$ such that:*

$$(3.4) \quad \forall x \in \partial B, \quad x = \int_{\mathbb{R}^2} \mathbb{1}_{\{x \cdot s > 0\}} \mu(ds),$$

which is

$$\mu = 1/2 \cdot n_{B^\perp} \mathcal{H}_{\partial B^\perp}^1$$

where $n_{B^\perp}(s)$ stands for the unit normal to $\partial B^\perp$ at s .

Proof. Let B be a symmetric convex body and suppose that there is $\mu \in \mathcal{M}^2(\mathbb{R}^2)$ supported in $\partial B^\perp$ and satisfying the representation formula:

$$(3.5) \quad \forall x \in \partial B, \quad x = \int_{\partial B^\perp} \mathbb{1}_{\{x \cdot s > 0\}} \mu(ds).$$

We want to prove that $\mu = \frac{n_{B^\perp}}{2} \mathcal{H}_{\partial B^\perp}^1$. Recalling the notations of (3.2), notice that this rewrites more concisely as

$$(3.6) \quad \forall x \in \partial B, \quad x = \mu(x^+).$$

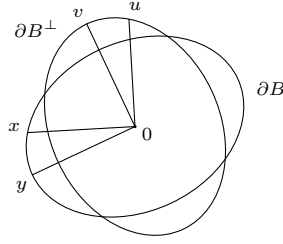


FIGURE 2.

If u, v are two points of $\partial B^\perp$, we denote by $]u, v]$ the oriented arc of $\partial B^\perp$ delimited by u and v (v included and u not included). Let us set $x = u^\perp, y = v^\perp$, which both belong to ∂B since it is symmetric.

Notice that if the oriented angle between u and v is such that $0 < \angle(u, v) < \pi$ then $]u, v] = x^+ \setminus y^+$. Finally we set $E :=]u, v]$ and $F := x^+ \setminus E$, so that

$$x^+ = E \sqcup F, \quad y^+ = F \sqcup (-E),$$

thus

$$\mu(x^+) = \mu(E) + \mu(F), \quad \mu(y^+) = \mu(F) + \mu(-E).$$

Since μ is antisymmetric, $\mu(-E) = -\mu(E)$ hence subtracting the last equality to the previous one yields $\mu(x^+) - \mu(y^+) = 2\mu(E)$. Using (3.6) one gets

$$x - y = 2\mu([u, v]), \quad i.e. \quad \mu([u, v]) = -\frac{1}{2}(v^\perp - u^\perp) = \frac{1}{2}R^{-1}(v - u).$$

In short, setting $F = \frac{1}{2}R^{-1}$, for all u, v such that $0 < \angle(u, v) < \pi$, we have:

$$(3.7) \quad \mu([u, v]) = F(v) - F(u).$$

This relation¹ allows us to assert that μ is the following measure on $\partial B^\perp$:

$$(3.8) \quad \mu(ds) = D_s F \tau(s) ds = F \tau(s) ds$$

where $\tau(s)$ is the unit vector tangent to $\partial B^\perp$ at s (with direct orientation). By definition, $n_{B^\perp}(s) = R^{-1}\tau(s)$ hence (3.8) rewrites $\mu = \frac{n_{B^\perp}}{2} \mathcal{H}_{\partial B^\perp}^1$, which is what we shall prove now. Setting $\nu = n_{B^\perp} \mathcal{H}_{\partial B^\perp}^1$, it is easy to check that $\nu([u, v]) = F(v) - F(u) = \mu([u, v])$ for all u, v such that $0 < \angle(u, v) < \pi$. Indeed, take a curve $\gamma : [0, l] \rightarrow L^\perp$ which describes the oriented arc $[u, v]$ and is

¹Notice that (3.7) would match the definition of the Stieltjes measure associated to a given function F if μ was a measure on the real line. In this case, μ is given by the distributional derivative of F and we are merely transposing this fact to our case, where μ is a measure on a closed curve.

parameterized by arc-length. Then one has

$$\begin{aligned}\nu([u, v]) &= \int_0^l n_{B^\perp}(\gamma(t)) \frac{dt}{2} = \int_0^l R^{-1}(\dot{\gamma}(t)) \frac{dt}{2} \\ &= \frac{1}{2} R^{-1}(\gamma(l) - \gamma(0)) \\ &= F(v) - F(u).\end{aligned}$$

Since the Borel sets $[u, v]$ form a π -system generating the Borel σ -algebra of $\partial B^\perp$, $\mu = \nu$. Moreover, since ν satisfies relation (3.7), it satisfies the representation formula (3.6) by taking $v \rightarrow -u$. This is what we wanted to prove. $\square$

4. KINETIC FORMULATION

4.1. Kinetic formulation and curl-free vector fields of norm one. This subsection is devoted to justifying the heuristic explanation of our introduction, that is that our kinetic equation acts as a selection principle for norm one curl-free vector fields. First, we have the following easy proposition:

Proposition 4.1. *Let Ω be an open set in $\mathbb{R}^2$, B the closed unit ball associated to the norm $\|\cdot\|$ and m a Borel vector field such that $\|m\| = 1$. If m satisfies the kinetic equation (1.8), then m is curl-free in the distributional sense.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$. Then,

$$\begin{aligned}\int_\Omega \partial_1 \varphi m_2 - \partial_2 \varphi m_1 \, dx &= \int_\Omega \nabla \varphi^\perp \cdot m \, dx = \int_\Omega \nabla \varphi^\perp \cdot \left(\frac{1}{2} \int_{\partial B^\perp} \chi_m(x, s) n_{B^\perp}(s) \, d\mathcal{H}^1(s) \right) dx \\ &= \frac{1}{2} \int_{\partial B^\perp} \int_\Omega \nabla \varphi \cdot n_{B^\perp}(s)^\perp \chi_m(x, s) \, dx \, d\mathcal{H}^1(s) = 0,\end{aligned}$$

where we use the Corollary 1.4 in the first line and the kinetic equation (1.8) in the second, which concludes the proof. $\square$

The next proposition gives kind of a reciprocal if the field m is smooth.

Proposition 4.2. *Let Ω be an open convex set in $\mathbb{R}^2$, $x \mapsto \|x\|$ a norm in $C^2(\mathbb{R}^2 \setminus \{0\})$, and $m \in C^1(\Omega, \mathbb{R}^2)$ a curl-free vector field such that $\|m\| = 1$. Then m satisfies the kinetic equation (1.8).*

Proof. Let m be a vector field satisfying our hypothesis. Using the characteristics method, it is easy to see that such a field is constant along lines. Indeed, writing $F(x) := \|x\|$, differentiating the relation $F(m(x)) = 1$ in Ω and using the fact that m is curl-free, one gets:

$$(4.1) \quad D_x m \cdot \nabla F(m(x)) = D_x^T m \cdot \nabla F(m(x)) = 0.$$

Now let $t \mapsto \gamma(t)$ be a smooth curve taking values in Ω and $p(t) := m(\gamma(t))$. Differentiating p , one obtains $\dot{p}(t) = D_{\gamma(t)} m \cdot \dot{\gamma}(t)$. Therefore, choosing γ to be a solution of the differential equation $\dot{\gamma}(t) = \nabla F(m(\gamma(t)))$ and using (4.1) the couple (γ, p) is now solution of the system:

$$\begin{cases} \dot{p}(t) = 0, \\ \dot{\gamma}(t) = \nabla F(p(t)). \end{cases}$$

Thus p is constant equal to $m(\gamma(0)) = m(x)$, γ satisfies $\dot{\gamma} = \nabla F(m(x))$ and m is constant along γ , that is m is constant on $L_x \cap \Omega$ where L_x is the line passing through x and directed by $\nabla F(m(x))$. Set $s \in \partial B^\perp$. If $m(x_0) \cdot s = 0$, then $m(x_0) = \pm s^\perp$, say $m(x_0) = s^\perp$. Since m is constant along the line $\gamma : t \mapsto x_0 + t \nabla F(s^\perp)$ and equal to $s^\perp$, then $\chi_m(\gamma(t), s) = 0$ and by differentiation at $t = 0$ one gets

$\nabla F(s^\perp) \cdot \nabla \chi_m(x_0, s) = 0$ (we write $\xi \cdot \nabla f$ to designate the partial derivative of f in the direction ξ provided it exists). But $\nabla F(s^\perp)$ is the normal vector ξ to B at $s^\perp$ such that $\|\xi\|_* = 1$, in particular it is colinear to $n_B(s^\perp)$, thus $n_B(s^\perp) \cdot \nabla \chi_m(x_0, s) = 0$. Now if $m(x_0) \cdot s > 0$, then $m(x) \cdot s > 0$ along γ , otherwise it must vanish somewhere along that line and $m \equiv \pm s^\perp$ on that line, which is a contradiction. Consequently, $\chi_m(\cdot, s)$ is constant along γ , thus we get again by differentiation along γ at $t = 0$:

$$n_B(s^\perp) \cdot \nabla \chi_m(x_0, s) = 0.$$

The same reasoning works for $m(x_0) \cdot s < 0$. Noticing that $n_{B^\perp}(s)^\perp = n_B(s^\perp)$, this implies that

$$(1.8) \quad \forall s \in \partial B^\perp, \forall x \in \Omega, \quad n_{B^\perp}(s)^\perp \cdot \nabla \chi_m(x, s) = 0.$$

□

4.2. Direction conservation and trace theorem. We are now entering the heart of the proof of Theorem 1.5, which is quite similar in spirit to the proof of Theorem 1.2 by Jabin Otto and Perthame. In order to use equation (1.8) to gain regularity, an important point will be to be able to define the trace of m along a line.

Remark 4.1. Note that starting from here, we will *always* place ourselves under the assumptions of Theorem 1.5, that is the unit sphere ∂B associated to a norm will be a C^1 submanifold and the norm itself will be of power type p .

Definition 4.1. Let $m \in \mathcal{L}_{\text{loc}}^1(\Omega, \mathbb{R}^2)$. We say that x_0 is a *Lebesgue point* of m if:

$$\lim_{r \rightarrow 0} \oint_{B_r(x_0)} |m(x) - m(x_0)| \, dx = 0.$$

We denote by $\text{Leb}(m) \subset \Omega$ the set of Lebesgue point of m . It is well known (see [6] for example) that $|\Omega \setminus \text{Leb}(m)| = 0$.

Let us state a useful lemma which relates Lebesgue points of m to those of $\chi_m(\cdot, s)$.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^2$ be an open set and $m \in \mathcal{L}_{\text{loc}}^1(\Omega, \partial B)$.*

- (i) *If x_0 is a Lebesgue point of $\chi_m(\cdot, s)$ for $\mathcal{H}^1$ -almost every $s \in \partial B^\perp$ then it is a Lebesgue point of m .*
- (ii) *If x_0 is a Lebesgue point of m then x_0 is a Lebesgue point of $\chi_m(\cdot, s)$ for all $s \in \partial B$ such that $m(x_0) \cdot s \neq 0$.*

In particular x_0 is a Lebesgue point of m if and only if it is a Lebesgue point of $\chi_m(\cdot, s)$ for almost every s .

Proof. The proof of (i) is a simple consequence of the averaging formula (1.7). Indeed, if x_0 is a Lebesgue point of $\chi_m(\cdot, s)$ for $\mathcal{H}^1$ -a.e. $s \in \partial B^\perp$:

$$\begin{aligned} \oint_{B_r(x_0)} |m(x) - m(x_0)| \, dx &= \frac{1}{2} \oint_{B_r(x_0)} \left| \int_{\partial B^\perp} (\chi_m(x, s) - \chi_m(x_0, s)) n_{B^\perp}(s) \, d\mathcal{H}^1(s) \right| \, dx \\ &\leq \frac{1}{2} \oint_{B_r(x_0)} \int_{\partial B^\perp} |\chi_m(x, s) - \chi_m(x_0, s)| \, d\mathcal{H}^1(s) \, dx \\ &= \frac{1}{2} \int_{\partial B^\perp} \oint_{B_r(x_0)} |\chi_m(x, s) - \chi_m(x_0, s)| \, dx \, d\mathcal{H}^1(s) \xrightarrow{r \rightarrow 0} 0, \end{aligned}$$

by the dominated convergence theorem.

Now let us prove (ii). Take a Lebesgue point x_0 of m and $s \in \partial B^\perp$ such that $m(x_0) \cdot s \neq 0$, say for example $m(x_0) \cdot s > 0$. First, let us remark that

$$\begin{aligned} m(x_0) \cdot s \frac{|B_r(x_0) \cap \{m \cdot s \leq 0\}|}{|B_r(x_0)|} &\leq \frac{1}{|B_r(x_0)|} \int_{B_r(x_0) \cap \{m \cdot s \leq 0\}} (m(x_0) \cdot s - m(x) \cdot s) dx \\ &\leq \int_{B_r(x_0)} |m(x_0) - m(x)| dx. \end{aligned}$$

Dividing by $m(x_0) \cdot s$ and taking the limit when r goes to 0, one gets

$$\frac{|B_r(x_0) \cap \{m \cdot s \leq 0\}|}{|B_r(x_0)|} \xrightarrow{r \rightarrow 0} 0,$$

but

$$\begin{aligned} \int_{B_r(x_0)} |\chi_m(x, s) - \chi_m(x_0, s)| dx &= \int_{B_r(x_0)} |\chi_m(x, s) - 1| dx \\ &= \frac{|B_r(x_0) \cap \{m \cdot s \leq 0\}|}{|B_r(x_0)|} \xrightarrow{r \rightarrow 0} 0, \end{aligned}$$

hence x_0 is a Lebesgue point of $\chi_m(\cdot, s)$. The case $m(x_0) \cdot s < 0$ is done in a similar way. $\square$

We can now prove the following proposition, which plays a crucial role in the proof of Theorem 1.5.

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^2$ be a convex open set and $m \in \mathcal{L}_{loc}^1(\Omega, \partial B)$ satisfying the kinetic equation (1.8). Assume that $y, z \in \text{Leb}(m)$ and that $s \in \partial B^\perp$ with $z - y \in \text{Vect}(n_{B^\perp}(s)^\perp)$. Then*

$$(4.2) \quad m(y) \cdot s > 0 \quad (\text{resp. } < 0) \implies m(z) \cdot s \geq 0 \quad (\text{resp. } \leq 0).$$

Proof. Let us suppose that $m(y) \cdot s \neq 0$. If $m(z) \cdot s = 0$, then the proof is over; assume otherwise that $m(z) \cdot s \neq 0$. Fix $x \in \Omega$ and define for ε small enough:

$$\chi^\varepsilon(x) := \int_{B_\varepsilon(x)} \chi_m(y, s) dy,$$

which may be written as

$$\chi^\varepsilon(x) = \rho^\varepsilon * \chi_m(\cdot, s) \quad \text{where} \quad \rho^\varepsilon := \frac{\mathbb{1}_{B_\varepsilon(0)}}{|B_\varepsilon(0)|}.$$

χ^ε is continuous and for every Lebesgue point x_0 of $\chi_m(\cdot, s)$, one has $\chi^\varepsilon(x_0) \xrightarrow{\varepsilon \rightarrow 0} \chi_m(x_0, s)$. Now let $(\delta_n)_{n \in \mathbb{N}^*}$ be a sequence of smooth mollifiers supported in $B_{1/n}$ and set

$$\chi_n^\varepsilon := \delta_n * \chi^\varepsilon.$$

We are going to show that $\chi_n^\varepsilon(y) = \chi_n^\varepsilon(z)$ and for that we introduce

$$\begin{aligned} g: \quad [0, 1] &\longrightarrow \mathbb{R} \\ t &\longmapsto \chi_n^\varepsilon(y + t(z - y)). \end{aligned}$$

The function g is smooth and satisfies

$$\begin{aligned} g'(t) &= (z - y) \cdot \nabla \chi_n^\varepsilon(y + t(z - y)) \\ &= (z - y) \cdot \left(\nabla [\delta_n * \rho^\varepsilon] * \chi_m(\cdot, s) \right)(y + t(z - y)) \\ &= \int_{\mathbb{R}^2} (z - y) \cdot \nabla [\delta_n * \rho^\varepsilon](y + t(z - y) - v) \chi_m(v, s) dv. \end{aligned}$$

But $(z - y) \in \text{Vect}(n_{B^\perp}(s)^\perp)$ and $\delta_n * \rho_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$; therefore because of (1.8), $g'(t) = 0$ and

$$\chi_n^\varepsilon(y) = g(0) = g(1) = \chi_n^\varepsilon(z).$$

Taking the limit as n goes to $+\infty$, we obtain

$$\chi^\varepsilon(y) = \chi^\varepsilon(z).$$

But y, z are Lebesgue points of m and $m(y) \cdot s \neq 0, m(z) \cdot s \neq 0$ hence by (ii) of Lemma 4.3 y, z are Lebesgue points of $\chi_m(\cdot, s)$, thus taking the limit as ε goes to 0, we have

$$\chi_m(y, s) = \lim_{\varepsilon \rightarrow 0} \chi^\varepsilon(y) = \lim_{\varepsilon \rightarrow 0} \chi^\varepsilon(z) = \chi_m(z, s),$$

which concludes the proof. $\square$

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^2$ be a convex open set and $m \in \mathcal{L}_{loc}^1(\Omega, \partial B)$ satisfying the kinetic equation (1.8). Let L be a segment of the form $L = \{tv : t \in [-1, 1]\}$ for some $v \in \mathbb{S}^1$. For $r > 0$ set*

$$P_r := \{x_1 v^\perp + x_2 v : (x_1, x_2) \in [-r, r] \times [-1, 1]\}$$

and assume that P_r is included in Ω for small r . Then there exists a measurable function $\tilde{m} : [-1, 1] \rightarrow \mathbb{R}^2$ satisfying

- (i) $\lim_{r \rightarrow 0} \int_{P_r} |m(x) - \tilde{m}(x_2)| dx = 0$,
- (ii) *for almost every x_2 , $\tilde{m}(x_2) \in \partial B$,*
- (iii) *if $x \in \text{Leb}(m) \cap L$ then $x_2 \in \text{Leb}(\tilde{m})$ and $m(x) = \tilde{m}(x_2)$,*
- (iv) *if $x_2 \in \text{Leb}(\tilde{m})$, then there exists a sequence $x_n \in \text{Leb}(m)$ such that*

$$x_n \rightarrow x \quad \text{and} \quad m(x_n) \rightarrow \tilde{m}(x_2).$$

The vector field $\tilde{m}$ is called the trace of m on the line L .

Remark 4.2. We only assume the form of $L = \{tv : t \in [-1, 1]\}$ for commodity but the result obviously holds for any general segment in Ω . This will be used several times in the rest of the article.

Equation (1.8) means that χ_m only depends on $n_{B^\perp}(s) \cdot x$ and s . The next lemma asserts that χ_m may be written as a Borel function of these quantities.

Lemma 4.6. *If m satisfies (1.8), for any convex set V (for instance a convex neighborhood of L) there exists a Borel function $\tilde{\chi}_m : \mathbb{R} \times \partial B^\perp \rightarrow \{0, 1\}$ such that for a.e. x in V ,*

$$(4.3) \quad \chi_m(x, s) = \tilde{\chi}_m(n_{B^\perp}(s) \cdot x, s).$$

Proof. We know that m and χ_m are Borel maps. For s fixed, we denote by $x = u_1 n_{B^\perp}(s) + u_2 n_{B^\perp}(s)^\perp$ the decomposition of x on $n_{B^\perp}(s), n_{B^\perp}(s)^\perp$. Taking arbitrary smooth functions ϕ, ψ of the variables u_1, u_2 with (suitable) compact support and testing the kinetic equation against $\rho(x) = \phi(u_1)\psi(u_2)$, one gets

$$\int \phi(u_1)\psi'(u_2)\chi_m(x, s) du_2 du_1 = 0,$$

which implies that for a.e. u_1 , the map $u_2 \mapsto \chi_m(u_1 n_{B^\perp}(s) + u_2 n_{B^\perp}(s)^\perp, s)$ is a.e. equal to a constant, namely to the mean value of $\chi_m(\cdot, s)$ over the slice $V_{s, x \cdot n_{B^\perp}(s)}$ where $V_{s, a} = \{a n_{B^\perp}(s) + t n_{B^\perp}(s)^\perp : t \in \mathbb{R}\} \cap V$. Therefore, setting

$$\tilde{\chi}_m(a, s) = \int_{V_{s, a}} \chi_m(x, s) dx,$$

one has

$$\chi_m(x, s) = \tilde{\chi}_m(n_{B^\perp}(s) \cdot x, s)$$

for all s and almost every x in V . Let us justify briefly that $\tilde{\chi}_m$ is Borel. Setting $F(s, a, t) = an_{B^\perp}(s) + tn_{B^\perp}(s)^\perp$, which is continuous, and $U_{s,a} = \{t : F(s, a, t) \in V\}$, one has

$$\tilde{\chi}_m(a, s) = \frac{1}{|U_{s,a}|} \int_{U_{s,a}} \chi_m(an_{B^\perp}(s) + tn_{B^\perp}(s)^\perp, s) dt = \frac{\int_{\mathbb{R}} \chi_m(F(s, a, t), s) \mathbb{1}_V(F(s, a, t)) dt}{\int_{\mathbb{R}} \mathbb{1}_V(F(s, a, t)) dt},$$

and it is now clear that it is Borel as an integral of a Borel map with parameter. $\square$

We can now prove Theorem 4.5.

Proof of Theorem 4.5. We proceed in several steps.

Step 1: Trace for χ_m .

Provided that $v \cdot n_{B^\perp}(s) \neq 0$, the function $[-1, 1] \ni x_2 \mapsto \tilde{\chi}_m(x_2 v \cdot n_{B^\perp}(s), s)$ is the trace of $\chi_m(\cdot, s)$ on L in the sense that:

$$(4.4) \quad \lim_{r \rightarrow 0} \oint_{P_r} |\chi_m(x, s) - \tilde{\chi}_m(x_2 v \cdot n_{B^\perp}(s), s)| dx = 0.$$

Indeed, using Lemma 4.6:

$$\begin{aligned} 2 \oint_{P_r} |\chi_m(x, s) - \tilde{\chi}_m(n_{B^\perp}(s) \cdot vx_2, s)| dx \\ &= \int_{-1}^1 \oint_{-r}^r |\chi_m(x_1 v^\perp + x_2 v, s) - \tilde{\chi}_m(x_2 v \cdot n_{B^\perp}(s), s)| dx_1 dx_2 \\ &= \int_{-1}^1 \oint_{-r}^r |\tilde{\chi}_m((x_1 v^\perp + x_2 v) \cdot n_{B^\perp}(s), s) - \tilde{\chi}_m(x_2 v \cdot n_{B^\perp}(s), s)| dx_1 dx_2 \\ &\leq \frac{1}{|v \cdot n_{B^\perp}(s)|} \int_{-1}^1 \oint_{-r}^r |\tilde{\chi}_m(x_1 v^\perp \cdot n_{B^\perp}(s) + y_2, s) - \tilde{\chi}_m(y_2, s)| dx_1 dy_2 \\ &\leq \frac{1}{|v \cdot n_{B^\perp}(s)|} \sup_{|y_1| \leq r} \int_{-1}^1 |\tilde{\chi}_m(y_1 + y_2, s) - \tilde{\chi}_m(y_2, s)| dy_2. \end{aligned}$$

This last quantity goes to 0 as r goes to 0 because of the continuity of the translation in L^1 . Note that for a fixed $v \in \mathbb{S}^1$, there are only two vectors $s \in \partial B^\perp$ for which $v \cdot n_{B^\perp}(s) = 0$ because B is a convex body which is strictly convex (the norm being of type p). As a consequence equality (4.4) is true for $\mathcal{H}^1$ -a.e. s in $\partial B^\perp$.

Step 2: Trace for m .

For $x_2 \in [-1, 1]$, we define the trace $\tilde{m}$ of m on L by the following equality :

$$\tilde{m}(x_2) = \frac{1}{2} \int_{\partial B^\perp} \tilde{\chi}_m(x_2 v \cdot n_{B^\perp}(s), s) n_{B^\perp}(s) d\mathcal{H}^1(s).$$

Then, thanks to the averaging formula (1.7),

$$\begin{aligned} 2 \oint_{P_r} |m(x) - \tilde{m}(x_2)| dx &\leq \oint_{P_r} \int_{\partial B^\perp} |(\chi_m(x, s) - \tilde{\chi}_m(x_2 v \cdot n_{B^\perp}(s), s)) n_{B^\perp}(s)| d\mathcal{H}^1(s) dx \\ &\leq \int_{\partial B^\perp} \left(\oint_{P_r} |\chi_m(x, s) - \tilde{\chi}_m(x_2 v \cdot n_{B^\perp}(s), s)| dx \right) d\mathcal{H}^1(s). \end{aligned}$$

Using (4.4) and the dominated convergence theorem, (i) is proved. Moreover,

$$\begin{aligned} \int_{-1}^1 |1 - \|\tilde{m}(x_2)\|| \, dx_2 &\leq 2 \int_{P_r} |\|m(x)\| - \|\tilde{m}(x_2)\|| \, dx \\ &\leq C \int_{P_r} |m(x) - \tilde{m}(x_2)| \, dx, \end{aligned}$$

the last equality resulting from the equivalence of norms on $\mathbb{R}^2$ and the reverse triangle inequality. This last quantity going to 0 as r goes to 0, (ii) is proved.

Step 3: Proof of (iii).

Take x a Lebesgue point of m . We know by Lemma 4.3 that x is a Lebesgue point of $\chi_m(\cdot, s)$ for almost all s . We define the cube

$$Q_r^s(x) := x + \{an_{B^\perp(s)} + bn_{B^\perp(s)}^\perp : (a, b) \in [-r, r]^2\},$$

and write $y = y_1^s n_{B^\perp(s)} + y_2^s n_{B^\perp(s)}^\perp$ for arbitrary y . Notice that:

$$\int_{Q_r^s(x)} |\chi_m(y, s) - \chi_m(x, s)| \, dy = \int_{Q_r^s(x)} |\tilde{\chi}_m(y_1^s, s) - \tilde{\chi}_m(x_1^s, s)| \, dy = \int_{x_1^s - r}^{x_1^s + r} |\tilde{\chi}_m(u, s) - \tilde{\chi}_m(x_1^s, s)| \, du.$$

The quantity on the left tends to 0 as r goes to 0 hence the quantity on the right as well, which means that $x_1^s = x \cdot n_{B^\perp(s)} \in \text{Leb}(\tilde{\chi}_m)$ for almost all s . Moreover, by definition of the trace, one has for all $y \in L$:

$$\tilde{m}(y_2) = \frac{1}{2} \int_{\partial B^\perp} \tilde{\chi}_m(y_2 v \cdot n_{B^\perp(s)}, s) n_{B^\perp(s)} \, d\mathcal{H}^1(s),$$

thus

$$\int_{x_2 - r}^{x_2 + r} |\tilde{m}(y_2) - \tilde{m}(x_2)| \, dy_2 \leq \int_{\partial B^\perp} \underbrace{\int_{x_2 - r}^{x_2 + r} |\tilde{\chi}_m(y_2 v \cdot n_{B^\perp(s)}, s) - \tilde{\chi}_m(x_2 v \cdot n_{B^\perp(s)}, s)| \, dy_2}_{J_s(r)} \, ds.$$

Provided that $v \cdot n_{B^\perp(s)} \neq 0$, which is true for a.e. s :

$$J_s(r) = \int_{x_1^s - rv \cdot n_{B^\perp(s)}}^{x_1^s + rv \cdot n_{B^\perp(s)}} |\tilde{\chi}_m(u, s) - \tilde{\chi}_m(x_1^s, s)| \, du \xrightarrow{r \rightarrow 0} 0,$$

hence by the dominated convergence theorem, $J_s(r)$ being bounded by 2, one gets:

$$\int_{x_2 - r}^{x_2 + r} |\tilde{m}(y_2) - \tilde{m}(x_2)| \, dy_2 \rightarrow 0,$$

and x_2 is a Lebesgue point of $\tilde{m}$. Moreover:

$$\begin{aligned}
\tilde{m}(x_2) &= \lim_{r \rightarrow 0} \frac{1}{2} \int_{x_2-r}^{x_2+r} \tilde{m}(t) dt \\
&= \lim_{r \rightarrow 0} \frac{1}{2} \int_{\partial B^\perp} \int_{x_2-r}^{x_2+r} \tilde{\chi}_m(tv \cdot n_{B^\perp}(s), s) dt n_{B^\perp}(s) ds \\
&= \lim_{r \rightarrow 0} \frac{1}{2} \int_{\partial B^\perp} \int_{x_2-r}^{x_2+r} \int_{x_1-r}^{x_1+r} \tilde{\chi}_m((tv + un_{B^\perp}(s)^\perp) \cdot n_{B^\perp}(s), s) du dt n_{B^\perp}(s) ds \\
&= \lim_{r \rightarrow 0} \frac{1}{2} \int_{\partial B^\perp} \int_{x_2-r}^{x_2+r} \int_{x_1-r}^{x_1+r} \chi_m(tv + un_{B^\perp}(s)^\perp, s) du dt n_{B^\perp}(s) ds \\
&= \lim_{r \rightarrow 0} \frac{1}{2} \int_{\partial B^\perp} \int_{L_r^s(x)} \chi_m(y, s) dy n_{B^\perp}(s) ds
\end{aligned}$$

where we have defined the parallelogram (which is non-flat for a.e. s):

$$L_r^s(x) = x + \{tv + un_{B^\perp}(s) : (t, v) \in [-r, r]\}.$$

Since x is a Lebesgue point of m , it is a Lebesgue point of $\chi_m(\cdot, s)$ for almost all s , hence

$$\int_{L_r^s(x)} \chi_m(y, s) dy \rightarrow \chi_m(x, s).$$

Passing to the limit as r goes to 0 in the integral:

$$\tilde{m}(x_2) = \frac{1}{2} \int_{\partial B^\perp} \chi_m(x, s) n_{B^\perp}(s) ds = m(x),$$

using again the representation formula.

Step 4: Proof of (iv).

Fix $\varepsilon > 0$, and take $\delta > 0$ to be suitably chosen later in terms of ε . For a, r positive and smaller than ε , we look at the quantity $E(a, r) = \int_{x_2-a}^{x_2+a} \int_{-r}^r |m(y) - \tilde{m}(x_2)| dy$. This quantity is bounded from above as follows:

$$\begin{aligned}
E(a, r) &= \int_{x_2-a}^{x_2+a} \int_{-r}^r |m(y) - \tilde{m}(x_2)| dy \leq \int_{x_2-a}^{x_2+a} \int_{-r}^r |m(y) - \tilde{m}(y_2)| dy \\
&\quad + \int_{x_2-a}^{x_2+a} \int_{-r}^r |\tilde{m}(y_2) - \tilde{m}(x_2)| dy \\
&\leq \int_{P_{r,a}(x)} |m(y) - \tilde{m}(y_2)| dy + \int_{x_2-a}^{x_2+a} |\tilde{m}(y_2) - \tilde{m}(x_2)| dy_2,
\end{aligned}$$

where

$$P_{r,a} := x + \{u_1 v^\perp + u_2 v : (u_1, u_2) \in [-r, r] \times [-a, a]\}.$$

Since x_2 is a Lebesgue point of $\tilde{m}$, one may find a small enough such that

$$\int_{x_2-a}^{x_2+a} |\tilde{m}(y_2) - \tilde{m}(x_2)| dy_2 \leq \delta.$$

Then for this fixed a , by (i) the quantity $\int_{P_{r,a}} |m(y) - \tilde{m}(y_2)| dy$ goes to 0 as r goes to 0, thus one may find r small such that it is less than δ , yielding $E(a, r) \leq 2\delta$. Now $E(a, r)$ may be bounded from below

as follows:

$$E(a, r) = \int_{P_{r,a}} |m(y) - \tilde{m}(x_2)| \geq \varepsilon \frac{|y \in \text{Leb}(m) \cap P_{r,a} : |m(y) - \tilde{m}(x_2)| > \varepsilon|}{|P_{r,a}|},$$

which implies that

$$\frac{|y \in \text{Leb}(m) \cap P_{r,a} : |m(y) - \tilde{m}(x_2)| > \varepsilon|}{|P_{r,a}|} \leq \frac{2\delta}{\varepsilon}.$$

Choosing $\delta = \varepsilon/4$ this quantity is strictly less than 1 hence the set

$$\{y \in \text{Leb}(m) \cap P_{r,a} : |m(y) - \tilde{m}(x_2)| \leq \varepsilon\}$$

is not empty. Any point x^ε in this set is a Lebesgue point such that $|x^\varepsilon - x| \leq \sqrt{2}\varepsilon$ (because $x^\varepsilon \in P_{\varepsilon, \varepsilon}(x)$), and satisfying $|m(x^\varepsilon) - \tilde{m}(x_2)| \leq \varepsilon$. Taking $\varepsilon = 1/n$ gives the desired conclusion. $\square$

With this definition of the trace, we obtain as a corollary an extension of Proposition 4.4.

Corollary 4.7. *Assume that $\Omega \subset \mathbb{R}^2$ is a convex open set and $m \in \mathcal{L}_{\text{loc}}^1(\Omega, \partial B)$ satisfies the kinetic equation (1.8). If L is a given line segment in Ω and $z \in L$ is such that z_2 is a Lebesgue point of the trace $\tilde{m}$ on L , then for all $y \in \text{Leb}(m)$ such that $z - y \in \text{Vect}(n_{B^\perp}(s)^\perp)$ one has:*

$$(4.5) \quad m(y) \cdot s > 0 \quad (\text{resp. } < 0) \implies \tilde{m}(z_2) \cdot s \geq 0 \quad (\text{resp. } \leq 0), .$$

Proof. The key point lies in (iv) of Theorem 4.5. With the notations of this theorem, if z is a point of L such that z_2 is a Lebesgue point of $\tilde{m}$, one may find a sequence $z^n \in \text{Leb}(m)$ such that $z^n \rightarrow z$ and $m(z^n) \rightarrow \tilde{m}(z_2)$. We know that $(z - y) \cdot n_{B^\perp}(s) = 0$. The map $n_{B^\perp} : \partial B^\perp \rightarrow \mathbb{S}^1$ being a homeomorphism, one may find a sequence $s^n \in \partial B^\perp$ such that $(z^n - y) \cdot n_{B^\perp}(s^n) = 0$ and $s^n \rightarrow s$ (set $s^n = \pm n_{B^\perp}^{-1} \left(\frac{(z^n - y)^\perp}{|z^n - y|} \right)$). Applying Proposition 4.4, one gets

$$m(z^n) \cdot s^n \geq 0,$$

and passing to the limit $n \rightarrow \infty$ yields $\tilde{m}(z_2) \cdot s \geq 0$. $\square$

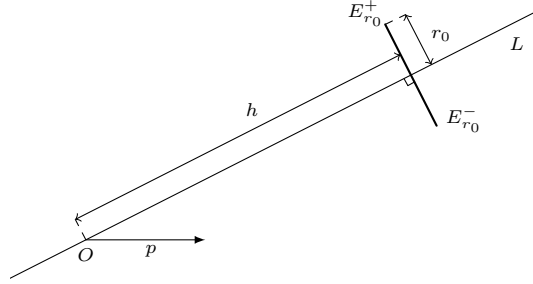
4.3. The regularity theorem. Now that we have defined a trace of our vector field along lines, our goal is to prove a form of invariance along lines just like what would happen in the smooth case thanks to the characteristic method.

Proposition 4.8. *Let $m \in \mathcal{L}^1(\Omega, \partial B)$ satisfying the kinetic equation (1.8). Suppose that x_0 is a Lebesgue point of m and denote by L be the line passing through x_0 and directed by $v := n_B(p) \in \mathbb{S}^1$ where $m(x_0) = p$. Then for almost every x_2 , $\tilde{m}(x_2) \in \{\pm p\}$ where $\tilde{m}$ is the trace of m on L in the sense of Theorem 4.5.*

Proof. By translation of the domain and rotation of the target space, we may assume without loss of generality that $x_0 = 0$ and $p = (\lambda, 0)$ with $\lambda > 0$. Let $h > 0$, $r_0 > 0$ be fixed and set

$$E_{r_0}^\pm := \{x \in \Omega \cap \text{Leb}(m) : x = hv \pm tv^\perp \quad 0 < t \leq r_0\}.$$

For $x^\pm \in E_{r_0}^\pm$, we will note $s^\pm = (s_1^\pm, s_2^\pm) := n_{B^\perp}^{-1} \left(\frac{(x^\pm)^\perp}{|x^\pm|} \right) \in \partial B^\perp$ and $\alpha^\pm = (\alpha_1^\pm, \alpha_2^\pm) := n_{B^\perp}^{-1} \left(\frac{hv^\perp \mp r_0 v}{|hv^\perp \mp r_0 v|} \right)$. Remark that $s^\pm$ is chosen in a way such that $x^\pm \in \text{Vect}(n_{B^\perp}(s^\pm)^\perp)$. We claim

FIGURE 3. The set $E_{r_0}^\pm$

that for r_0 small enough, $s_2^\pm \geq \frac{\lambda}{2} > 0$ and there is $C > 0$ such that:

$$(0 > s_1^+ \geq C\alpha_1^+ \text{ and } 0 < s_1^- \leq C\alpha_1^-)$$

or

$$(0 < s_1^+ \leq C\alpha_1^+ \text{ and } 0 > s_1^- \geq C\alpha_1^-),$$

Let us prove this claim. First note that $n_{B^\perp}(s^\pm)$ is in the cone $C(v^\perp, n_{B^\perp}(\alpha^\pm))$ as defined above Lemma 2.3. Indeed,

$$n_{B^\perp}(s^\pm) = \frac{(x^\pm)^\perp}{|(x^\pm)^\perp|} = \frac{hv^\perp \mp tv}{|hv^\perp \mp tv|},$$

and

$$\begin{aligned} \frac{1}{|hv^\perp \mp tv|} (hv^\perp \mp tv) &= \frac{1}{|hv^\perp \mp tv|} \left(\frac{t}{r_0} (hv^\perp \mp r_0 v) + h \left(1 - \frac{t}{r_0} \right) v^\perp \right), \\ &= \frac{1}{|hv^\perp \mp tv|} \left(\frac{t|hv^\perp \mp r_0 v|}{r_0} n_{B^\perp}(\alpha^\pm) + h \left(1 - \frac{t}{r_0} \right) v^\perp \right). \end{aligned}$$

Using Lemma 2.3, this implies $s^\pm \in C(n_{B^\perp}^{-1}(v^\perp), \alpha^\pm)$. But $n_{B^\perp}^{-1}(v^\perp) = n_B^{-1}(v)^\perp = p^\perp$ so that:

$$s^\pm \in C(p^\perp, \alpha^\pm),$$

meaning that there are t_1, t_2 in $\mathbb{R}^{+*}$ such that

$$\begin{pmatrix} s_1^\pm \\ s_2^\pm \end{pmatrix} = t_1 \begin{pmatrix} 0 \\ \lambda \end{pmatrix} + t_2 \alpha^\pm.$$

Because of the continuity of $n_{B^\perp}^{-1}$ proved in Theorem 2.6, $\alpha^\pm \xrightarrow{r_0 \rightarrow 0} p^\perp = (0, \lambda)$, which implies that for r_0 small enough $s_2^\pm \geq \frac{\lambda}{2} > 0$. To conclude the proof of our claim, we just have to remark that there are only two possibilities for the position of $\alpha^\pm$ with respect to $p^\perp$ since $p^\perp \in C(\alpha^+, \alpha^-)$ by Lemma 2.3.

In the first configuration, because $s^\pm, \alpha^\pm \in \partial B^\perp$, $s^+ \in C(p^\perp, \alpha^+)$ implies $0 > s_1^+ \geq C\alpha_1^+$ and $s^- \in C(p^\perp, \alpha^-)$ implies $0 < s_1^- \leq C\alpha_1^-$. In the second configuration, using the same trick one gets $s^+ \in C(p^\perp, \alpha^+)$ implies $0 < s_1^+ \leq C\alpha_1^+$ and $s^- \in C(p^\perp, \alpha^-)$ implies $0 > s_1^- \geq C\alpha_1^-$. This concludes the proof of our claim.

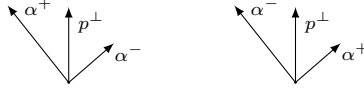


FIGURE 4. Two different configurations

Without loss of generality, we will place ourselves in the second case that is we will chose r_0 small enough such that $s_2^\pm \geq \frac{\lambda}{2} > 0$, $0 < s_1^+ \leq C\alpha_1^+$ and $0 > s_1^- \geq C\alpha_1^-$. Then, using Proposition 4.4, for $x^+ \in E_{r_0}^+$,

$$\begin{aligned} m(0) \cdot s^+ = \lambda s_1^+ > 0 &\implies m(x^+) \cdot s^+ = m_1(x^+)s_1^+ + m_2(x^+)s_2^+ \geq 0 \\ &\implies m_2(x^+) \geq -\frac{m_1(x^+)s_1^+}{s_2^+} \geq -\tilde{C}\alpha_1^+. \end{aligned}$$

Identically, we get for $x^- \in E_{r_0}^-$:

$$\begin{aligned} m(0) \cdot s^- = \lambda s_1^- < 0 &\implies m(x^-) \cdot s^- = m_1(x^-)s_1^- + m_2(x^-)s_2^- \leq 0 \\ &\implies m_2(x^-) \leq -\frac{m_1(x^-)s_1^-}{s_2^-} \leq -\tilde{C}\alpha_1^-. \end{aligned}$$

Notice that Theorem 4.5 implies that

$$\int_{-1}^1 \int_0^r |m(x) - \tilde{m}(x_2)| dx_1 dx_2 = 2 \int_{P_r} |m(x) - \tilde{m}(x_2)| dx \xrightarrow{r \rightarrow 0} 0$$

and

$$\int_{-1}^1 \int_{-r}^0 |m(x) - \tilde{m}(x_2)| dx_1 dx_2 = 2 \int_{P_r} |m(x) - \tilde{m}(x_2)| dx \xrightarrow{r \rightarrow 0} 0.$$

Consequently there is a sequence $r_n \rightarrow 0$ such that:

$$\int_0^{r_n} |m(x) - \tilde{m}(x_2)| dx_1 \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad \int_{-r_n}^0 |m(x) - \tilde{m}(x_2)| dx_1 \xrightarrow{n \rightarrow +\infty} 0$$

for almost every x_2 . Moreover we know that for almost every x_2 , $x_1 v^\perp + x_2 v$ is a Lebesgue point of m for almost every x_1 , so that $m_2(x_1 v^\perp + x_2 v) \geq -\tilde{C}\alpha_1^+$ if $x_1 \geq 0$ and $m_2(x_1 v^\perp + x_2 v) \leq -\tilde{C}\alpha_1^-$ if $x_1 \leq 0$. Thus it follows that for almost every x_2 :

$$\tilde{m}_2(x_2) = \lim_{n \rightarrow \infty} \int_0^{r_n} m_2(x_1 v^\perp + x_2 v) dx_1 \geq -\tilde{C}\alpha_1^+ \xrightarrow{r \rightarrow 0} 0,$$

and

$$\tilde{m}_2(x_2) = \lim_{n \rightarrow \infty} \int_{-r_n}^0 m_2(x_1 v^\perp + x_2 v) dx_1 \leq -\tilde{C}\alpha_1^- \xrightarrow{r \rightarrow 0} 0.$$

This shows that $\tilde{m}_2(x_2) = 0$ for a.e. x_2 , i.e. $\tilde{m}(x_2) \in \{\pm p\}$ which is what we wanted to prove. $\square$

We can now finish the proof of Theorem 1.5.

Proof of Theorem 1.5. Consider any open convex $\omega \subset \Omega$ with $d = d(\omega, \partial\Omega) > 0$. It is enough to prove Theorem 1.5 in any such ω . Let x, y in $\omega \cap \text{Leb}(m)$, L_x be the line passing through x and directed by $n_B(m(x))$ and L_y be the line passing through y and directed by $n_B(m(y))$. If L_x and L_y are parallel and distinct, we claim that $m(x) = m(y)$. Indeed, if it is not the case, $m(x) = -m(y)$. We choose $s \in \partial B^\perp$ such that $y - x \in \text{Vect}(n_{B^\perp}(s)^\perp)$. Note that $m(y) \cdot s \neq 0$, otherwise s and $m(y)^\perp$ are colinear and $y - x \in \text{Vect}(n_B(m(y)))$ and $L_x = L_y$. Now using Proposition 4.4 $\text{sign}(m(y) \cdot s) = \text{sign}(m(x) \cdot s)$ hence $m(x) = m(y)$.

If L_x and L_y intersect each other, let O be their intersection point. Up to a change of variable in the target space, we can suppose that O is the origin, $L_x = \text{Vect}(\frac{x}{|x|})$ and $L_y = \text{Vect}(\frac{y}{|y|})$. Now, up to a change of sign for m , we can suppose that $m(x) = n_B^{-1}(\frac{x}{|x|})$ and $m(y) = \pm n_B^{-1}(\frac{y}{|y|})$. We are going to prove that the trace of m along L_x and L_y in the sense of Theorem 4.5 is completely determined by $m(x)$ and $m(y)$. More precisley,

$$\begin{aligned} \text{for a.e. } t \in \mathbb{R} \text{ such that } t \frac{x}{|x|} \in L_x \cap \omega, \quad \tilde{m}_{L_x}(t) &= n_B^{-1}\left(\frac{x}{|x|}\right), \\ \text{for a.e. } t \in \mathbb{R} \text{ such that } t \frac{y}{|y|} \in L_y \cap \omega, \quad \tilde{m}_{L_y}(t) &= n_B^{-1}\left(\frac{y}{|y|}\right), \end{aligned}$$

where $\tilde{m}_{L_x}$ and $\tilde{m}_{L_y}$ stand for the trace of m along L_x (resp L_y) in the sense of Theorem 4.5. Let $z = t \frac{y}{|y|} \in L_y \cap \omega$ chosen such that t is a Lebesgue point of $\tilde{m}_{L_y}$ and choose $s \in \partial B^\perp$ such that $z - x \in \text{Vect}(n_{B^\perp}(s)^\perp)$. Because of Proposition 4.8, $\tilde{m}(t) = \pm m(y) = \pm n_B^{-1}\left(\frac{y}{|y|}\right)$. But

$$m(x) \cdot s = n_B^{-1}\left(\frac{x}{|x|}\right) \cdot n_{B^\perp}^{-1}\left(\frac{(z-x)^\perp}{|z-x|}\right) = n_B^{-1}\left(\frac{x}{|x|}\right) \cdot n_B^{-1}\left(\frac{(z-x)}{|z-x|}\right)^\perp.$$

Because z and x are not colinear, $\frac{(z-x)}{|z-x|} \notin \text{Vect}(x)$ and n_B^{-1} being bijective, $m(x) \cdot s \neq 0$. Because of Proposition 4.7, this implies $\text{sign}(\tilde{m}_{L_y}(t) \cdot s) = \text{sign}(m(x) \cdot s)$. But $\frac{y}{|y|} = \frac{z}{|z|} \in C\left(\frac{x}{|x|}, \frac{z-x}{|z-x|}\right)$ and by Lemma 2.3, this implies $n_B^{-1}\left(\frac{y}{|y|}\right) \in C\left(n_B^{-1}\left(\frac{x}{|x|}\right), n_B^{-1}\left(\frac{z-x}{|z-x|}\right)\right)$, that is there is α, β in $\mathbb{R}^+$ such that:

$$n_B^{-1}\left(\frac{y}{|y|}\right) = \alpha n_B^{-1}\left(\frac{x}{|x|}\right) + \beta n_B^{-1}\left(\frac{z-x}{|z-x|}\right).$$

Now,

$$\begin{aligned} n_B^{-1}\left(\frac{y}{|y|}\right) \cdot s &= n_B^{-1}\left(\frac{y}{|y|}\right) \cdot n_B^{-1}\left(\frac{z-x}{|z-x|}\right)^\perp \\ &= \alpha n_B^{-1}\left(\frac{x}{|x|}\right) \cdot n_B^{-1}\left(\frac{z-x}{|z-x|}\right)^\perp = \alpha m(x) \cdot s, \end{aligned}$$

so that $\text{sign}(\tilde{m}_{L_y}(t) \cdot s) = \text{sign}(n_B^{-1}\left(\frac{y}{|y|}\right) \cdot s)$. This forces $\tilde{m}_{L_y}(t) = n_B^{-1}\left(\frac{y}{|y|}\right)$ and we proved that $\tilde{m}_{L_y}$ is determined along L_y . We show in the same way that $\tilde{m}_{L_x} = n_B^{-1}\left(\frac{x}{|x|}\right)$ along L_x . Note that if we started with $m(x) = -n_B^{-1}\left(\frac{x}{|x|}\right)$ instead of $n_B^{-1}\left(\frac{x}{|x|}\right)$, the trace of m along L_x and L_y would be determined in the same way, but with the opposite sign. We will use this in the end of our proof.

We are now going to distinguish two cases:

Case 1: $\text{dist}(O, \omega) \leq d$. We claim that m is a vortex in ω . First note that up to a change of sign for m ,

we can suppose that $m(x) = n_B^{-1}\left(\frac{x}{|x|}\right)$ and $m(y) = n_B^{-1}\left(\frac{y}{|y|}\right)$. Let us take $z \in \omega \cap \text{Leb}(m) \cap (L_x \cup L_y)^c$ and let L_z be the line directed by $m(z)$ and passing through z . If we prove that L_z intersects L_x and L_y in O , we are done. Suppose now that this is not the case. Using the same argument than in the beginning, L_z cannot be parallel neither to L_x nor to L_y . Let $P_1 = L_z \cap L_x$ and $P_2 = L_z \cap L_y$. By convexity, there is a small portion of L_z in Ω . But using the remark in our first part to the lines L_x and L_z , one gets:

$$\text{for a.e. } t \in \mathbb{R} \text{ such that } t \frac{x - P_1}{|x - P_1|} \in L_x \cap \omega, \quad \tilde{m}_{L_x}(t) = -n_B^{-1}\left(\frac{x - P_1}{|x - P_1|}\right),$$

$$\text{for a.e. } t \in \mathbb{R} \text{ such that } t \frac{z - P_1}{|z - P_1|} \in L_y \cap \omega, \quad \tilde{m}_{L_z}(t) = -n_B^{-1}\left(\frac{z - P_1}{|z - P_1|}\right),$$

and using it to the lines L_y and L_z , one gets:

$$\text{for a.e. } t \in \mathbb{R} \text{ such that } t \frac{y - P_2}{|y - P_2|} \in L_x \cap \omega, \quad \tilde{m}_{L_x}(t) = -n_B^{-1}\left(\frac{y - P_2}{|y - P_2|}\right),$$

$$\text{for a.e. } t \in \mathbb{R} \text{ such that } t \frac{z - P_2}{|z - P_2|} \in L_y \cap \omega, \quad \tilde{m}_{L_z}(t) = -n_B^{-1}\left(\frac{z - P_2}{|z - P_2|}\right).$$

But $n_B^{-1}(-x) = -n_B^{-1}(x)$ and the two traces obtained for $\tilde{m}_{L_z}$ are opposed on the segment $[P_1, P_2]$, which leads to a contradiction (see Figure 5).

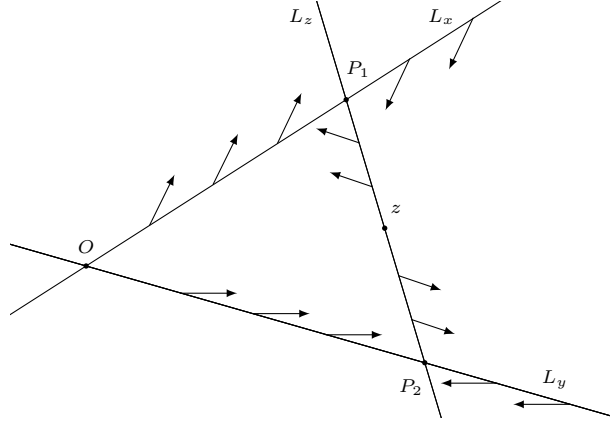


FIGURE 5. An impossible configuration

Case 2: $\text{dist}(O, \omega) > d$. Using the first part, we know that up to a change of sign: $m(x) = n_B^{-1}\left(\frac{x}{|x|}\right) = V_B(x)$ and $m(y) = n_B^{-1}\left(\frac{y}{|y|}\right) = V_B(y)$, so that

$$|m(x) - m(y)| = |V_B(x) - V_B(y)| \leq \frac{K}{d^{\frac{1}{p-1}}} |x - y|^{\frac{1}{p-1}},$$

by Proposition 2.5. This concludes our proof. $\square$

Remark 4.3. Notice that this result is sharp, in the sense that we cannot hope to get a better regularity for vector fields satisfying the kinetic equation and valued in the sphere of a power type p norm. Indeed, as shown in Section 2 the vortex V_B associated to the ball B of this norm is always solution,

and Proposition 2.5 shows that it is $\frac{1}{p-1}$ -Hölder continuous far from 0 if and only if the norm is of power type p .

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