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Gravity as an SU(1,1) gauge theory in four dimensions

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We start with the Hamiltonian formulation of the first order action of pure gravity with a full $\mathfrak{sl}(2,\mathbb{C})$ internal gauge symmetry. We make a partial gauge-fixing which reduces $\mathfrak{sl}(2,\mathbb{C})$ to its sub-algebra $\mathfrak{su}(1,1)$. This case corresponds to a splitting of the space-time $\mathcal{M} = \Sigma \times \mathbb{R}$ where Σ inherits an arbitrary Lorentzian metric of signature $(-,+,+)$. Then, we find a parametrization of the phase space in terms of an $\mathfrak{su}(1,1)$ commutative connection and its associated conjugate electric field. Following the techniques of Loop Quantum Gravity, we start the quantization of the theory and we consider the kinematical Hilbert space on a given fixed graph Γ whose edges are colored with unitary representations of $\mathfrak{su}(1,1)$. We compute the spectrum of area operators acting of the kinematical Hilbert space: we show that space-like areas have discrete spectra, in agreement with usual $\mathfrak{su}(2)$ Loop Quantum Gravity, whereas time-like areas have continuous spectra. We conclude on the possibility to make use of this formulation of gravity to construct a holographic description of black holes in the framework of Loop Quantum Gravity.

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I. INTRODUCTION

Loop quantum gravity was founded on the observation by Ashtekar [1] that working only with the self-dual part (or equivalently the anti-self-dual part) of the Hilbert-Palatini action leads to a simplified parametrization of the phase space of pure gravity. Indeed, the canonical variables are very similar to those of Yang-Mills gauge theory, there is no second class constraints and the first class constraints associated to the local symmetries are polynomial functionals of the the canonical variables. The drawback of the original Ashtekar's approach is that the phase space becomes complex and then one requires the imposition of reality conditions in order to recover the phase space of real general relativity. Of course, if one imposes the reality conditions at the classical level, prior to quantization, one looses all the beauty of the Ashtekar formulation, and recovers the standard Palatini formulation of general relativity, which we do not know how to quantize. Unfortunatelly, so far no one knows how to go the other way around, and implement the reality conditions after quantization of the Ashtekar theory. This difficulty motivated the work of Barbero [2] and, later on, Immirzi [3], who introduced a family of canonical transformations, parametrized by the so-called Barbero-Immirzi parameter γ, and leading to a canonical theory in terms of a real $\mathfrak{su}(2)$ connection kown as the Ashtekar-Barbero connection. The action that leads to this canonical formulation was finally found by Holst [4].

In fact, the Holst action is a first order formulation of gravity with a full $\mathfrak{sl}(2,\mathbb{C})$ internal symmetry and an explicit dependency on the parameter γ which appears as a coupling constant for a topological term. One uses a partial gauge fixing in this action in order to derive a canonical

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theory in terms of the Ashtekar-Barbero. This choice of gauge is referred to as the time gauge, and, by doing so, the Lorentz gauge algebra in the internal space is reduced to its rotational $\mathfrak{su}(2)$ subalgebra. Finally, Loop Quantum Gravity is a canonical quantization of this gauge fixed first order formulation of gravity which lead to a beautiful construction of the space of quantum geometry states at the kinematical level. At this stage, one can naturally ask the question whether the construction of Loop Quantum Gravity deeply relies on the time gauge or not. A related question would be whether the physical predictions of Loop Quantum Gravity are changed or not when one makes another partial gauge fixing or no gauge fixing at all in the Holst action prior to quantization. Indeed, the discreteness of the quantum geometry at the Planck scale predicted in Loop Quantum Gravity can be interpreted as a direct consequence of the compactness (via Harmonic analysis) of the residual symmetry group $SU(2)$ in the time gauge. These important problems have been studied quite a lot the last twenty years but it is fair to say that no definitive conclusion have closed the debates so far.

Most of the approaches to address this issue are based on attempts to quantize the Holst action without any partial gauge fixing, and then keeping the full Lorentz internal invariance of the theory. Now if one performs the canonical analysis of the $\mathfrak{sl}(2,\mathbb{C})$ Holst action, second class constraints appear simply because the connection has more components than the tetrad field. The appearance of second class constraints makes the classical analysis and then the quantization of the theory much more involved. In the analysis of constrained systems, there are two ways of dealing with second class constraints: one can either solve them explicitly, or implement them in the symplectic structure by working with the Dirac bracket. These two methods are totally equivalent. Using the Dirac bracket, Alexandrov and collaborators [5–8] were able to construct a two-parameters family of Lorentz-covariant connections (which are diagonal under the action of the area operator, and transform properly under the action of spatial diffeomorphisms). Generically, these connections are non-commutative and therefore the theory becomes very difficult to quantize. The alternative route to deal with covariant connections was initiated by Barros e Sa in [9] who solved explicitly the second class constraints. In this approach, the phase space is parametrized by two pairs of canonical variables: the generalization (A, E) of the usual Ashtekar-Barbero connection and its conjugate densitized triad E; and a new pair of canonically conjugated fields (χ, ζ), where χ and ζ both take values in $\mathbb{R}^3$. Then, Barros e Sa expressed the remaining boost, rotation, diffeomorphism and scalar constraints in terms of these variables. The elegance of this approach is that it enables one to have a simple symplectic structure with commutative variables, and a tractable expression for the boost, rotation and diffeomorphism generators. Although the scalar constraint becomes more complicated, this structure is enough to study the kinematical structure of loop quantum gravity with a fully Lorentz invariance. This has precisely been done in [10, 11] where one constructed the unique spatial connection which is not only commutative but also transforms covariantly under the action of boosts and rotations. In fact, this connection coincides with the commutative Lorentz connection studied earlier in [8] and the one found in [12]. Furthermore, it has been shown to be gauge related to the Ashtekar-Barbero connection via a pure boost parametrized by the vector χ viewed as a velocity. Hence, the construction proposed in [10, 11] works only when $\chi^2 < 1$. Thus, the pairs of canonical variables formed with the $\mathfrak{sl}(2,\mathbb{C})$ connection and its conjugate electric field parametrize only a part of the fully covariant phase space of the Holst action.

This paper enables us to explore the sector $\chi^2 > 1$ while studying a partial gauge fixing of the Holst action that reduces $\mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{su}(1,1)$. Hence, we start with the Lorentz covariant parametrization of the Holst action found by Barros e Sa [9]. We find a partial gauge fixing which breaks the $\mathfrak{sl}(2,\mathbb{C})$ internal symmetry into $\mathfrak{su}(1,1)$ and this is possible if and only if $\chi^2 > 1$. Such a partial gauge fixing corresponds to a canonical splitting of the space-time $\mathcal{M} = \Sigma \times \mathbb{R}$ where Σ is no more space-like (as it is the case in the usual Ashtekar-Barbero parametrization) but inherits a Lorentzian metric of signature is $(-,+,+)$. As a consequence, only three out of the initial six

first class constraints remain after the partial gauge fixing, and they generate as expected the local $\mathfrak{su}(1,1)$ gauge transformations. The other three constraints form with the three gauge fixing conditions a set of second class constraints that we solve explicitly. Then, we construct an $\mathfrak{su}(1,1)$ connection which appears to be commutative in the sense of the Poisson bracket. This remarkable construction allows us to investigate the loop quantization of the theory and to build the kinematical Hilbert space on a given graph Γ whose edges are associated to $SU(1,1)$ holonomies. It is well-known that [13] the non-compactness of the gauge group prevents us from defining the projective limit of spin-networks and then the sum over all graphs of kinematical Hilbert space is ill-defined. Nonetheless, if one restricts the study to one given graph Γ, it is possible to define the action of the area operator and one easily finds that a space-like area has a discrete spectrum whereas the spectrum of a time-like area is continuous. In other words, if one considers a spin-network defined on a graph Γ dual to a discretization $\Delta = \Gamma^*$ of a $(2+1)$-dimensional manifold, edges e of Γ are colored with representations in the discrete series (resp. in the continuous series) if the dual face $f = e^*$ of Δ is space-like (resp. time-like). The spectrum of space-like areas is in total agreement with the one obtained in the usual Ashtekar-Barbero formalism for space-like surfaces.

The paper is organized as follows. After the introduction, we start in Section II with a brief summary of the canonical analysis à la Barros e sa of the fully Lorentz invariant Holst action. In Section III, we present the partial gauge fixing that breaks $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{su}(1,1)$ before constructing the $\mathfrak{su}(1,1)$ connection and its associated electric field. In Section IV we explore the kinematical quantization of the theory on a given graph and we compute the spectra of area operators which act unitarily in the kinematical Hilbert space. We conclude in Section V with a brief summary of the most important results and a discussion on the consequences of this new parametrization for the description of black holes in Loop Quantum Gravity.

II. FIRST ORDER LORENTZ-COVARIANT GRAVITY

In this section, we summarize the main results of the Hamiltonian analysis of the fully Lorentz invariant Holst action. We start recalling the main steps of the constraints analysis and present the solutions of the second class constraints proposed by Barros e Sa [9]. Then, we describe the parametrization of the Lorentz covariant phase space that will serve to build the $\mathfrak{su}(1,1)$ connection in the next Section. Finally, we discuss the structure of the first class constraints focussing mainly on the generators of the internal Lorentz symmetry.

A. Action and constraints analysis

The Holst action [4] is a generalization of the Hilbert-Palatini first order action with a Barbero-Immirzi parameter γ. In terms of the co-tetrad $e^I_\alpha(x)$ and the Lorentz connection one-form $\omega^{IJ}_\alpha(x)$, the corresponding Lagrangian density is

$$\mathcal{L}[e,\omega] = \frac{1}{2}\epsilon_{IJKL}\, e^I \wedge e^J \wedge F^{KL} + \frac{1}{\gamma} e^I \wedge e^J \wedge F_{IJ}\,,$$

where $F[\omega] = d\omega + \omega\wedge\omega$ is the curvature two-form of the connection ω, ϵ_{IJKL} the fully antisymmetric symbol which defines an invariant non-degenerate bilinear form on $\mathfrak{sl}(2,\mathbb{C})$, and internal indices are lowered and raised with the flat metric η_{IJ} and its inverse η^{IJ}. It is well known that the Holst action is equivalent to the Hilbert-Palatini action. Indeed, if the co-tetrad is not degenerated (i.e. if its determinant is not vanishing), one can uniquely solve ω in terms of e (from the torsionless equation for ω) and find that ω^{IJ}_μ are nothing but the components of the Levi-Civita connection.

Plugging back this solution into the action eliminates the Barbero-Immirzi parameter γ by virtue of the Bianchi identities and leads to the second order Einstein-Hilbert action.

Now, we recall basic results on the canonical analysis of the Holst Lagrangian. For this purpose, it is convenient to introduce the notation

$$ {}^{\gamma}\xi_{IJ} = \xi_{IJ} - \frac{1}{2\gamma}\epsilon_{IJKL}\,\xi^{KL}\,, \tag{2.1} $$

for any element $\xi \in \mathfrak{sl}(2,\mathbb{C})$. After performing a 3+1 decomposition (based on a splitting $\mathcal{M} = \Sigma\times\mathbb{R}$ of the space-time) in order to distinguish between temporal and spatial coordinates (0 is the time label and small latin letters from the beginning of the alphabet $a, b, c, \cdots$ hold for spacial indices), a straightforward calculation leads to the following canonical expression of the Lagrangian density

$$ \mathcal{L}[e,\omega] = {}^{\gamma}\pi^a_{IJ}\,\dot{\omega}^{IJ}_a - g^{IJ}\mathcal{G}_{IJ} - N\mathcal{H} - N^a\mathcal{H}_a, \tag{2.2} $$

where we have introduced the notations $\dot{\omega} = \partial_0\omega$ for the time derivative of ω, g^{IJ} for $-\omega^{IJ}_0$, N for the lapse function N, and N^a for the shift vector. All these functions are Lagrange multipliers which enforce respectively the Gauss, Hamiltonian, and diffeomorphism constraints

$$ \mathcal{G}_{IJ} = D_a{}^{\gamma}\pi^a_{IJ}, \qquad \mathcal{H} = \pi^a_{IK}\pi^{bK}_J{}^{\gamma}F^{IJ}_{ab}\,, \qquad \mathcal{H}_a = \pi^b_{IJ}{}^{\gamma}F^{IJ}_{ab}\,. \tag{2.3} $$

These constraints are expressed in terms of the spatial connection components ω^{IJ}_a, and the canonical momenta defined by

$$ \pi^a_{IJ} \equiv \epsilon_{IJKL}\,\epsilon^{abc}\,e^K_b\,e^L_c. \tag{2.4} $$

Since $\pi^a_{IJ} = -\pi^a_{JI}$ contains 18 components, and the co-tetrad has only 12 independent components, we need to impose 6 primary constraints often called the simplicity constraints

$$ \mathcal{C}^{ab} = \epsilon^{IJKL}\pi^a_{IJ}\pi^b_{KL} \approx 0, \tag{2.5} $$

in order to parametrize the space of momenta in terms of the π variables instead of the co-tetrad variables. Classically, it is equivalent to work with the 12 components e^I_a or with the 18 components π^a_{IJ} constrained to satisfy the 6 relations $\mathcal{C}^{ab} \approx 0$. Hence, at this stage, the non-physical Hamiltonian phase space is parametrized by the 18 pairs of canonically conjugated variables $(\omega^{IJ}_a, \pi^a_{IJ})$, with the set of 10 constraints (2.3) to which we add the 6 constraints $\mathcal{C}^{ab} \approx 0$.

Studying the stability under time evolution of these "primary" constraints is rather standard and has been performed first for the Hilbert-Palatini action in [14] and for the Holst action in [9]. Here we will not reproduce all the steps of this analysis, but only focus on the structure of the second class constraints and their resolution. Details with our notations can be found in [10]. Notice first that in order to recover the 4 phase space degrees of freedom (per space-time points) of gravity, the theory needs to have secondary constraints, which in addition have to be second class. This is indeed the case. Technically, this comes from the fact that the algebra of constraints fails to close because the scalar constraint $\mathcal{H}$ does not commute weakly with the simplicity constraint $\mathcal{C}^{ab}$. Hence, requiring their stability under time evolution generates the following 6 additional secondary constraints

$$ \mathcal{D}^{ab} = \epsilon_{IJMN}\,\pi^{cMN}\left(\pi^{aIK}D_c\pi^{bJ}{}_K + \pi^{bIK}D_c\pi^{aJ}{}_K\right) \approx 0. $$

The Dirac algorithm closes here with 18×2 phase space variables (parametrized by the components of π and ω), and 22 constraints $\mathcal{H}$, $\mathcal{H}_a$, $\mathcal{G}_{IJ}$, $\mathcal{C}^{ab}$ and $\mathcal{D}^{ab}$. Among these constraints, the first 10 are first class (up to adding second class constraints) as expected, and the remaining 12 are second

class. One can check explicitly that $\mathcal{C}^{ab} \approx 0$ and $\mathcal{D}^{ab} \approx 0$ form a set of second class constraints (their associated Dirac matrix is invertible), and that the first class constraints generate the symmetries of the theory, namely the space-time diffeomorphisms and the Lorentz gauge symmetry. Finally, we are left with the expected 4 phase space degrees of freedom per spatial point:

$$18 \times 2(\text{dynamical variables}) - 10(\text{first class constraints}) \times 2 - 12(\text{second class constraints}).$$

We recover the two gravitational modes.

B. Parametrization of the phase space

Now that we have clarified the Hamiltonian structure of the theory, we are going to show how to solve the second class constraints following [9]. First, one writes the 18 components of π_a^{IJ} as

$$\pi_{0i}^a = 2E_i^a, \qquad \pi_{ij}^a = 2(E_i^a \chi_j - E_j^a \chi_i)\,, \tag{2.6}$$

where $\chi_i = e_i^a e_a^0$ (which encodes the deviation of the normal to the hypersurfaces from the time direction) and E_i^a (which corresponds to the usual densitized triad of loop gravity) are now twelve independent variables. Note that e_i^a is the inverse of e_a^i viewed as a 3×3 matrix. This is trivially a solution of the simplicity constraints (2.5) because somehow we have returned to the co-tetrad parametrization (2.4).

Then, we plug the solution (2.6) into the canonical term of the Lagrangian (2.2) which gives

$$\gamma\pi_{IJ}^a \dot{\omega}_a^{IJ} = E_i^a \dot{A}_a^i + \zeta^i \dot{\chi}_i \qquad \text{where} \quad A_a^i = \gamma\omega_a^{0i} + \gamma\omega_a^{ij}\chi_j \quad \text{and} \qquad \zeta^i = \gamma\omega_a^{ij} E_j^a. \tag{2.7}$$

This result strongly suggests that the 18 components of the connection could be expressed in terms of the 12 independent variables (A_a^i, ζ^i) when one solves the 6 secondary second class constraints. This is indeed the case and it can be seen by inverting the relation (2.7) as follows

$$\gamma\omega_a^{0i} = A_a^i - \gamma\omega_a^{ij}\chi_j, \qquad \gamma\omega_a^{ij} = \frac{1}{2}\left(Q_a^{ij} - E_a^i\zeta^j - E_a^j\zeta^i\right), \tag{2.8}$$

where E_a^i is the inverse of E_i^a, and $Q_a^{ij} = Q_a^{ji}$ has a vanishing action on E_i^a. The explicit form of Q_a^{ij} can be obtained from $\mathcal{D}^{ab} \approx 0$ as shown in [9]. Furthermore, when $\gamma^2 \neq 1$, one can uniquely express ω in terms of $\gamma\omega$ using the inverse of the map (2.1).

As a consequence, the phase space can be parametrized by the twelve pairs of canonical variables (A_a^i, E_i^a) and (χ_i, ζ^i) with the (non-trivial) Poisson brackets given by

$$\left\{A_a^i(x), E_j^b(y)\right\} = \delta_j^i \delta_a^b\, \delta^3(x-y) \quad \text{and} \quad \left\{\chi_i(x), \zeta^j(y)\right\} = \delta_i^j\, \delta^3(x-y). \tag{2.9}$$

Remark that if we work in the time gauge (i.e. $\chi = 0$), the variable A_a^i coincides exactly with the usual Ashtekar-Barbero connection.

C. First class constraints

It remains to express the first class constraints (2.3) in terms of the new phase space variables (2.9). This is an easy task using the defining relations (2.6) and (2.8). This was done by Barros e Sa. The constraints have quite a simple form except the Hamiltonian constraint whose expression

is more involved: it can be found in [9] and we will not consider this constraint in this paper. The vector constraint $\mathcal{H}_a$ takes the form

$$\begin{aligned}\mathcal{H}_a &= E^b\cdot(\partial_a A_b-\partial_b A_a)+\zeta\cdot\partial_a\chi+\frac{\gamma^2}{1+\gamma^2}\Big[(E^b\cdot A_b)(A_a\cdot\chi)-(E^b\cdot A_a)(A_b\cdot\chi)\\ &\quad+(A_a\cdot\chi)(\zeta\cdot\chi)-(A_a\cdot\zeta)+\frac{1}{\gamma}\Big(E^b\cdot(A_b\times A_a)+\zeta\cdot(\chi\times A_a)\Big)\Big]\,,\end{aligned}\tag{2.10}$$

where $\cdot$ denotes the scalar product $\lambda\cdot\mu=\lambda_i\mu^i$ and $\times$ denotes the cross product $(\lambda\times\mu)^i=\epsilon^{ijk}\lambda_j\mu_k$ for any two pairs of vectors λ and μ in $\mathbb{R}^3$. Concerning, the Lorentz constraints $\mathcal{G}_{IJ}$, they can be split into its boost part $\mathcal{B}_i\equiv\mathcal{G}_{0i}$, and its rotational part $\mathcal{R}_i\equiv(1/2)\epsilon_i{}^{jk}\mathcal{G}_{jk}$ whose expressions are

$$\mathcal{B} = \partial_a\left(E^a-\frac{1}{\gamma}\chi\times E^a\right)-(\chi\times E^a)\wedge A_a+\zeta-(\zeta\cdot\chi)\chi\,,\tag{2.11a}$$

$$\mathcal{R} = -\partial_a\left(\chi\times E^a+\frac{1}{\gamma}E^a\right)+A_a\times E^a-\zeta\times\chi\,.\tag{2.11b}$$

One can check that these constraints satisfy indeed the Lorentz algebra

$$\{\mathcal{B}\cdot u,\mathcal{B}\cdot v\}=-\mathcal{R}\cdot u\times v,\quad\{\mathcal{R}\cdot u,\mathcal{R}\cdot v\}=\mathcal{R}\cdot u\times v,\quad\{\mathcal{B}\cdot u,\mathcal{R}\cdot v\}=\mathcal{B}\cdot u\times v\,,\tag{2.12}$$

where u and v are arbitrary vectors.

In the time gauge, one immediately recovers the constraints structure of the formulation of gravity in terms of the Astekar-Barbero connection. In that case, $\chi\approx 0$ drastically simplifies the boost constraints which become equivalent to $\zeta-\partial_a E^a\approx 0$. The conditions $\chi\approx 0$ and $\zeta-\partial_a E^a\approx 0$ form a set of second class constraints that can be solved explicitly for χ and ζ. By doing so, the variables (χ,ζ) are eliminated from the theory, and the vectorial, the rotational and also the Hamiltonian constraints are those of Loop Quantum Gravity.

Now our task is to use the phase space variables (2.9) to make a partial gauge fixing which reduces the original Lorentz algebra to $\mathfrak{su}(1,1)$.

III. GRAVITY AS AN SU(1,1) GAUGE THEORY

In this section, we first show how to make a partial gauge fixing of the full Lorentz invariant Holst action which reduces the internal $\mathfrak{sl}(2,\mathbb{C})$ gauge symmetry to $\mathfrak{su}(1,1)$. At the same time, we keep the invariance under diffeomorphisms on Σ. In that case, we will see that the splitting of the space-time $\mathcal{M}=\Sigma\times\mathbb{R}$ is such that Σ is no more a space-like hypersurface as it is the case in the time gauge but inherits instead a Lorentzian structure. Then, we construct a parametrization of the phase space in terms of an $\mathfrak{su}(1,1)$ connection and its conjugate electric field which transforms in the adjoint representation of $\mathfrak{su}(1,1)$. Furthermore, we show that these variables are Darboux coordinates for the phase space, which paves the way towards a quantization of the theory explored in the following Section.

A. Breaking the internal symmetry: from $\mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{su}(1,1)$

As we have already underlined in the previous section, imposing the time gauge $\chi\approx 0$ in the fully covariant Holst action breaks the boost invariance and only the rotational parts of the constraints remain first class among the original 6 internal symmetries. Hence, we get an $\mathfrak{su}(2)$ invariant theory of gravity. In fact, we proceed in a very similar way to construct an $\mathfrak{su}(1,1)$

invariant theory from the Holst action: we find a partial gauge fixing such that two components of the boosts constraints and one of the rotational constraints remain first class whereas the three others form with the gauge fixing conditions a second class system. Naturally, we consider a gauge fixing condition of the form

$$\mathcal{X} \equiv \chi - \chi_0 \approx 0 \tag{3.1}$$

where χ_0 is a fixed non-dynamical vector. Inspiring ourselves with what happens in the time gauge, we expect (3.1) to form a second class system with three out of the six constraints (2.11). These three second class components of the Lorentz generators are supposed to be

$$\mathcal{R}\cdot u \approx 0\,, \quad \mathcal{R}\cdot v \approx 0\,, \quad \mathcal{B}\cdot n \approx 0\,, \tag{3.2}$$

where u and v are two given normalized orthogonal vectors and $n = v \times u$. The reason is that we are left with two boosts and one rotations which are expected to reproduce (up to the addition of second class constraints) an $\mathfrak{su}(1,1)$ Poisson algebra. To derive the conditions for this to happen, we start rewriting (3.2) as a linear system of equations for ζ:

$$M\zeta = \begin{pmatrix} \zeta\cdot U \\ \zeta\cdot V \\ \zeta\cdot W \end{pmatrix} \approx \begin{pmatrix} \mathcal{R}\cdot u|_{\zeta=0} \\ \mathcal{R}\cdot v|_{\zeta=0} \\ \mathcal{B}\cdot n|_{\zeta=0} \end{pmatrix} \quad \text{with} \quad M \equiv \begin{pmatrix} {}^tU \\ {}^tV \\ {}^tW \end{pmatrix} \text{ and } \begin{cases} U \;\equiv\; \chi\times u \\ V \;\equiv\; \chi\times v \\ W \;\equiv\; -n + (\chi\cdot n)\chi \end{cases} \tag{3.3}$$

The system admits an unique solution for ζ if and only if

$$\det M = U\times V\cdot W = (1-\chi^2)(\chi\cdot n)^2 \;\neq 0\;, \tag{3.4}$$

which implies that $\chi^2 \neq 1$ and $\chi\cdot n \neq 0$. When we assume this is the case, the solution ζ_0 can be easily expressed in terms of the components of χ_0, E and A inverting (3.3) as follows

$$\zeta_0 \;=\; M^{-1}\begin{pmatrix} \mathcal{R}\cdot u|_{\zeta=0} \\ \mathcal{R}\cdot v|_{\zeta=0} \\ \mathcal{B}\cdot n|_{\zeta=0} \end{pmatrix} = \frac{(\mathcal{B}\cdot n + \mathcal{R}\cdot\chi\times n)\,\chi - (1-\chi^2)\mathcal{R}\times n}{(1-\chi^2)n\cdot\chi}|_{\zeta=0}\,, \tag{3.5}$$

where we used the expression

$$M^{-1} \;=\; \frac{1}{U\times V\cdot W}\big(V\times W\,,\, W\times U\,,\, U\times V\big)\,. \tag{3.6}$$

Hence, the three constraints (3.2) are equivalent to the three conditions

$$\mathcal{Z} \equiv \zeta - \zeta_0(\chi_0, E, A) \approx 0\,. \tag{3.7}$$

Now, it becomes clear that the gauge fixing conditions $\mathcal{X} \approx 0$ (3.1) and the three constraints $\mathcal{Z} \approx 0$ form a second class system because their associated 6×6 Dirac matrix Δ

$$\Delta(x,y) \equiv \begin{pmatrix} X(x,y) & Y(x,y) \\ -{}^tY(x,y) & Z(x,y) \end{pmatrix} \text{ with } \begin{cases} X^i_j(x,y) \;\equiv\; \{\chi^i(x),\chi_j(y)\} = 0 \\ Y^i_j(x,y) \;\equiv\; \{\mathcal{X}^i(x),\mathcal{Z}_j(y)\} \;=\; \delta^j_i\delta^3(x-y) \\ Z^i_j(x,y) \;\equiv\; \{\mathcal{Z}^i(x),\mathcal{Z}_j(y)\} \end{cases} \tag{3.8}$$

is invertible whatever Z is. These two constraints allow to eliminate the variables χ and ζ from the phase space provided that one introduces the external non dynamical field χ_0.

We are left with three constraints from (2.11) which are required to satisfy an $\mathfrak{su}(1,1)$ Poisson algebra once one replaces χ by χ_0 and ζ by ζ_0. These constraints are denoted

$$\mathcal{J}_u \equiv \mathcal{B}\cdot u|_{\chi_0,\zeta_0}\,, \quad \mathcal{J}_v \equiv \mathcal{B}\cdot v|_{\chi_0,\zeta_0}\,, \quad \mathcal{J}_n \equiv \mathcal{R}\cdot n|_{\chi_0,\zeta_0}\,. \tag{3.9}$$

From now on, we will omit to mention the index 0 for χ to lighten the notations. However, χ has to be understood as an external non dynamical field, and not as the initial dynamical variable in the fully Lorentz invariant Holst action.

A long but standard calculation shows that the three constraints (3.9) form a closed Poisson algebra only when

$$u \cdot \chi = v \cdot \chi = 0\,. \tag{3.10}$$

This is equivalent to the condition that $\chi = \pm|\chi| n$ where $|\chi| \equiv \sqrt{\chi \cdot \chi}$ is the norm of χ. Without loss of generality, we choose $\chi = |\chi| n$. As a consequence, the partial gauge fixing (3.1) leaves the remaining three constraints (3.9) first class only when (3.10) is satisfied. In that case, the expressions of (3.9) simplify a lot and they can be written as

$$\mathcal{J}_0 \,\equiv\, \mathcal{J}_n = n \cdot \tilde{\mathcal{J}}\,, \quad \mathcal{J}_1 \,\equiv\, C\mathcal{J}_v = Cu \cdot \tilde{\mathcal{J}}, \quad \mathcal{J}_2 \,\equiv\, C\mathcal{J}_u = \,-Cv \cdot \tilde{\mathcal{J}}\,, \tag{3.11}$$

where $C = 1/\sqrt{|\chi^2 - 1|}$ is a normalization function and we introduced the vector field

$$\tilde{\mathcal{J}} \;\equiv\; -\frac{1}{\gamma}\big(\partial_a E^a + \partial_a (E^a \times \chi) \times \chi\big) + \tilde{A}_a \times E^a \tag{3.12}$$

given in terms of the $\mathfrak{su}(2)$-valued one form $\tilde{A}$ defined by

$$\tilde{A}_a = A_a - (A_a \cdot \chi)\chi - \partial_a \chi\,. \tag{3.13}$$

Finally, one shows that the constraints algebra reduces to the simple form

$$\{\mathcal{J}_0, \mathcal{J}_1\} \,=\, \mathcal{J}_2\,, \qquad \{\mathcal{J}_0, \mathcal{J}_2\} \,=\, -\mathcal{J}_1\,, \qquad \{\mathcal{J}_1, \mathcal{J}_2\} \,=\, \sigma \mathcal{J}_0\,, \tag{3.14}$$

where

$$\sigma \equiv \frac{1-\chi^2}{|1-\chi^2|} = \mathrm{sg}(1-\chi^2). \tag{3.15}$$

The function $\mathrm{sg}(x)$ denotes the sign of $x \neq 0$. As a consequence, the remaining three constraints form an $\mathfrak{su}(2)$ Poisson algebra when $\chi^2 < 1$ and an $\mathfrak{su}(1,1)$ Poisson algebra when $\chi^2 > 1$ (the case $\chi^2 = 1$ is excluded from the scope of our method and should be studied in a different way[1]). We can write the constraints algebra in the more compact form

$$\{\mathcal{J}_\alpha, \mathcal{J}_\beta\} \;=\; \epsilon_{\alpha\beta}{}^{\tau}\, \mathcal{J}_\tau \tag{3.16}$$

where $\alpha, \beta, \tau \in (0,1,2)$ and $\epsilon_{\alpha\beta\tau}$ is the totally antisymmetric symbol with $\epsilon_{012} = +1$. Furthermore, the indices are lowered and raised with the flat metric and its inverse $\mathrm{diag}(\sigma, +1, +1)$: it is the flat Euclidean metric $\delta_{\alpha\beta}$ when $\sigma = +1$ and the flat Minkowski metric $\eta_{\alpha\beta} \equiv \mathrm{diag}(-1, +1, +1)$ when $\sigma = -1$. Hence, as announced above, one recognizes respectively the $\mathfrak{su}(2)$ and the $\mathfrak{su}(1,1)$ Lie algebras.

Let us close this analysis with one remark. The gauge fixing condition (3.1) makes the three constraints (3.2) (which are first class in the full Lorentz invariant Holst action) second class. Hence, we have left two boosts and one rotation first class in order to get an $\mathfrak{su}(1,1)$ gauge symmetry at the

[1] This case corresponds to a slicing of the space-time in a light like direction. Our analysis based on a partial gauge fixing can be adapted to that situation. Such a Hamiltonian description could provide us with a new formulation (eventually simpler) of gravity in the light front related to [15].

end of the process. This is what we arrive at when $\chi^2 > 1$ but we obtain an $\mathfrak{su}(2)$ gauge symmetry when $\chi^2 < 1$ even though we kept two boosts among the remaining first class constraints. The reason is that, at the end of the gauge fixing process, the remaining first class constraints are non-trivial linear combinations of the six initial first class constraints and the gauge fixing conditions. Hence, they could form either an $\mathfrak{su}(1,1)$ or an $\mathfrak{su}(2)$ algebra. The two most important ingredients in our construction is that, first, we replace three out of the initial six first class constraints by constraints of the type (3.7) which fix ζ, and second we impose that the remaining constraints (when ζ and χ are replaced from $\mathcal{X} \approx 0$ and $\mathcal{Z} \approx 0$) form a closed Poisson algebra. In that respect, we could have considered the conditions $\mathcal{B}.u \approx \mathcal{B}.v \approx \mathcal{B}.n \approx 0$ instead of (3.2): we would have obtained another set of conditions fixing ζ and then, following the same strategy, we would have shown that the remaining three constraints are generators of a closed algebra provided that (3.10) is satisfied. The remaining symmetry would have been $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$ depending on the sign of σ exactly as in the previous analysis.

B. On the space-time foliation

Let us discuss the reason why the sign σ of $(\chi^2 - 1)$ determines the signature of the symmetry algebra $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$. For that purpose, it is very instructive to study the properties of the metric g_{ab} induced on the hypersurface Σ whose expression is

$$g_{ab} \equiv e^I_a \eta_{IJ} e^J_b = e^i_a \gamma_{ij} e^j_b \quad \text{with} \quad \gamma_{ij} \equiv \delta_{ij} - \chi_i \chi_j \tag{3.17}$$

where we inverted the defining relation $\chi_i = e^a_i e^0_a$ to replace e^0_a by $e^i_a \chi_i$. It is immediate to notice that this formula is compatible with the expression of the inverse metric given in [9, 11]

$$\det(g)\, g^{ab} \;=\; (1-\chi^2)\, E^a_i \gamma^{ij} E^b_j \,, \quad \gamma^{ij} \equiv \delta_{ij} - \frac{\chi_i \chi_j}{1-\chi^2} \,, \tag{3.18}$$

due to the properties

$$E^a_i = \det(e) e^a_i \,, \quad \det(g) = (1-\chi^2)\det(e)^2 \,, \quad \gamma^{ij}\gamma_{jk} = \delta^i_k \,. \tag{3.19}$$

Thus, the identity (3.17) implies immediately that the metric induced on Σ has the same signature as γ_{ij}. This latter metric can be easily diagonalized and its eigenvalues/eigenvectors are easily obtained from

$$\gamma_{ij} u^j = u_i \text{ when } u \cdot \chi = 0 \,, \quad \text{and} \quad \gamma_{ij}\chi^j = (1-\chi^2)\chi_i \,. \tag{3.20}$$

Therefore, the signature of the metric depends on the sign of $(\chi^2 - 1)$: Σ is spacelike when $\chi^2 < 1$ whereas it inherits a Lorentzian metric when $\chi^2 > 1$. This clearly explains the presence of σ in the constraints algebra (3.15) and the nature of the gauge symmetry. When the symmetry algebra is $\mathfrak{su}(2)$, the space-time is foliated as usual into hypersurfaces orthogonal to a timelike vector whereas it is foliated in a space-like direction when the symmetry algebra is $\mathfrak{su}(1,1)$. This latest case is not conventional but it is the one we are interested in.

C. Phase space parametrization

From now on, we will mainly focus on the case $\chi^2 > 1$ which has never been studied so far (we will shortly discuss the case $\chi^2 < 1$ at the end of this Section). As the theory admits $\mathfrak{su}(1,1)$ as a gauge symmetry algebra, it is natural to look for a parametrization of the phase space adapted to

this symmetry. More precisely, we look for conjugate variables which transform in a covariant way under the Poisson action of the $\mathfrak{su}(1,1)$ generators. In a first part, we exhibit an unique $\mathfrak{su}(1,1)$-valued connection which is commutative in the sense of the Poisson bracket. This connection is the $\mathfrak{su}(1,1)$ analogous of the generalized Ashtekar-Barbero connection defined for $\chi \neq 0$ in [10, 11] for instance. In a second part, we show that it is canonically conjugate to an electric field which transforms as a vector under the action of the first class constraints. Hence, the $\mathfrak{su}(1,1)$-connection together with its conjugate electric field provide us with a very useful and natural parametrization of the phase space. We finish with computing the action of the vectorial constraints on these variables which transform as expected under the action of the generators of diffeomorphisms.

1. The connection

Now, we address the problem of finding an $\mathfrak{su}(1,1)$ connection defined by

$$\mathcal{A} = \mathcal{A}^0\, J_0 + \mathcal{A}^1\, J_1 + \mathcal{A}^2\, J_2 \quad \text{with} \quad [J_\alpha, J_\beta] = \epsilon_{\alpha\beta}{}^{\tau} J_\tau \tag{3.21}$$

which satisfies the following requirements. First, is constructed from the components of A (such that it is commutative in the sense of the Poisson bracket) and the non-dynamical vectors (χ, u and v) only. Second it transforms as

$$\delta_\varepsilon \mathcal{A} \;=\; d\varepsilon + [\mathcal{A}, \varepsilon]\,, \tag{3.22}$$

under the action of the gauge transformations where $\varepsilon = \varepsilon^\alpha(x) J_\alpha$ is an arbitrary $\mathfrak{su}(1,1)$-valued function on Σ. For this relation to make sense, we have to precise the definition of δ_ε in terms of the gauge generators. In particular, we have to establish the link between the parameter $\varrho \in \mathbb{R}^3$ entering in the smeared constraint $\tilde{\mathcal{J}}(\varrho)$ and the parameter ε defining the $\mathfrak{su}(1,1)$ infinitesimal gauge transformations of $\mathcal{A}$. From (3.11), it is natural to expect that

$$\delta_\varepsilon \mathcal{A} \;=\; \{\tilde{\mathcal{J}}(\varrho), \mathcal{A}\} \quad \text{with} \quad \varepsilon^0 = \varrho \cdot n,\; \varepsilon^1 = c_1 \varrho \cdot u,\; \varepsilon^2 = c_2 \varrho \cdot v\,, \tag{3.23}$$

where c_1 and c_2 are functions of χ. Now, the problem consists in finding the components of $\mathcal{A}$ and the functions c_1 and c_2 such that $\mathcal{A}$ transforms as an $\mathfrak{su}(1,1)$ connection under the action of the first class constraints.

We are going to propose an ansatz for $\mathcal{A}$. As the expressions of the gauge generators are simpler with $\tilde{A}$ instead of A itself, we also look for an $\mathfrak{su}(1,1)$ connection $\mathcal{A}$ written in terms of $\tilde{A}$. This is possible because, when $\chi^2 \neq 1$, $\tilde{A}$ can be uniquely expressed in terms of A and χ inverting the relation (3.13) as follows:

$$A_a \;=\; \tilde{A}_a + \partial_a \chi + \chi \cdot (\tilde{A}_a + \partial_a \chi) \frac{\chi}{1-\chi^2}\,. \tag{3.24}$$

Inspiring ourselves from the decomposition (3.11) of the first class constraints into $\mathfrak{su}(1,1)$ gauge generators, we propose the following form for the components of $\mathcal{A}$:

$$\mathcal{A}^0 = p_0\,(\tilde{A} \cdot n) + q_0\,, \quad \mathcal{A}^1 = p_1\,(\tilde{A} \cdot u) + q_1\,, \quad \mathcal{A}^2 = p_2\,(\tilde{A} \cdot v) + q_2\,, \tag{3.25}$$

where (p_0, p_1, p_2) are functions of χ whereas (q_0, q_1, q_2) are one-forms constructed from $d\chi$, du and dv only.

Hence, the problem reduces now in finding the functions (c_1, c_2) and (p_0, p_1, p_2) together with the one-forms (q_0, q_1, q_2) which solve the equations (3.23). These equations can be more explicitly

written as

$$p_0\{\tilde{\mathcal{J}}(\varrho),(\tilde{A}\cdot n)\} = d(\varrho\cdot n)+c_1(p_2\,(\tilde{A}\cdot v)+q_2)\varrho\cdot u-c_2(p_1\,(\tilde{A}\cdot u)+q_1)\varrho\cdot v\,, \tag{3.26}$$
$$p_1\{\tilde{\mathcal{J}}(\varrho),(\tilde{A}\cdot u)\} = d(c_1\varrho\cdot u)+(p_2\,(\tilde{A}\cdot v)+q_2)\varrho\cdot n-c_2(p_0\,(\tilde{A}\cdot n)+q_0)\varrho\cdot v\,, \tag{3.27}$$
$$p_2\{\tilde{\mathcal{J}}(\varrho),(\tilde{A}\cdot v)\} = d(c_2\varrho\cdot v)-(p_1\,(\tilde{A}\cdot u)+q_1)\varrho\cdot n+c_1(p_0\,(\tilde{A}\cdot n)+q_0)\varrho\cdot u\,, \tag{3.28}$$

where each Poisson brackets on the l.h.s. are easily deduced from

$$\{\tilde{\mathcal{J}}(\varrho),\tilde{A}\} = -\frac{1}{\gamma}(1-\chi^2)d\varrho+\tilde{A}\times\varrho-\frac{1}{\gamma}\chi\times(d\chi\times\varrho)+(\tilde{A}\cdot\chi\times\varrho)\chi\,. \tag{3.29}$$

A straightforward calculations show that the previous system reduces to the following three sets of equations:

$$\begin{array}{ll} p_0(1-\chi^2)=c_1p_2=c_2p_1=-\gamma\,, & dn+c_1q_2u-c_2q_1v=0\,,\\ p_1=-p_2=-c_2p_0=\gamma c_1/(\chi^2-1)\,, & d(c_1u)+q_2n-c_2q_0v+p_1[(u\cdot d\chi)\chi-(\chi\cdot d\chi)u]/\gamma=0\,,\\ p_1=-p_2=c_1p_0=-\gamma c_2/(\chi^2-1)\,, & d(c_2v)-q_1n+c_1q_0u+p_2[(v\cdot d\chi)\chi-(\chi\cdot d\chi)v]/\gamma=0\,. \end{array}$$

This is clearly an overcomplete set of conditions for the unkowns of the problem. However, an immediate analysis shows that (up to a simple sign ambiguity), the system admits an unique solution given by

$$p_0=\frac{\gamma}{\chi^2-1}\,,\quad p_1=\frac{\gamma}{\sqrt{\chi^2-1}}\,,\quad p_2=-\frac{\gamma}{\sqrt{\chi^2-1}}\,, \tag{3.30}$$
$$q_0=dv\cdot u\,,\quad q_1=-\frac{1}{\sqrt{\chi^2-1}}v\cdot dn\,,\quad q_2=-\frac{1}{\sqrt{\chi^2-1}}u\cdot dn\,, \tag{3.31}$$

with $c_1=-c_2=\sqrt{\chi^2-1}$.

As a conclusion, let us summarize the main results of this part. The theory admits an $\mathfrak{su}(1,1)$ gauge connection $\mathcal{A}=\mathcal{A}^0J_0+\mathcal{A}^1J_1+\mathcal{A}^2J_2$ whose components are

$$\mathcal{A}^0 = \frac{\gamma}{\chi^2-1}\tilde{A}\cdot n+u\cdot dv\,, \tag{3.32}$$
$$\mathcal{A}^1 = \frac{1}{\sqrt{\chi^2-1}}\Big(\gamma\tilde{A}\cdot u-v\cdot dn\Big)\,, \tag{3.33}$$
$$\mathcal{A}^2 = -\frac{1}{\sqrt{\chi^2-1}}\Big(\gamma\tilde{A}\cdot v+u\cdot dn\Big)\,. \tag{3.34}$$

We have just proved that it transforms as follows

$$\delta_\varepsilon\mathcal{A}=\{\tilde{\mathcal{J}}(\varrho),\mathcal{A}\}=d\varepsilon+[\mathcal{A},\varepsilon]\quad\text{with}\quad \varrho=\varepsilon^0n+\frac{\varepsilon^1u-\varepsilon^2v}{\sqrt{\chi^2-1}} \tag{3.35}$$

under the action of the first class constraints. Note that this transformation law is totally consistent with the fact that

$$\tilde{\mathcal{J}}(\varrho)=\mathcal{J}_0(\varepsilon^0)+\mathcal{J}_1(\varepsilon^1)+\mathcal{J}_2(\varepsilon^2)\,, \tag{3.36}$$

where the components of $\tilde{\mathcal{J}}$ are the smeared $\mathfrak{su}(1,1)$ generators introduced in (3.11).

Let us close this analysis with two remarks.
First, one can reproduce exactly the same analysis when $\chi^2<1$. In that case, one obtains an $\mathfrak{su}(2)$ connection whose expression is very similar to the previous one obtained for $\mathfrak{su}(1,1)$:

everything happens as if one makes the replacement $\sqrt{\chi^2-1} \mapsto -\sqrt{1-\chi^2}$ in the components of the connection. The $\mathfrak{su}(2)$-valued connection is certainly related to the generalized Ashtekar-Barbero connection obtained in different ways [8, 11, 12]. In the limit $\chi \to 0$ with n constant, one recovers the usual Ashtekar-Barbero connection in the time-gauge written in the orthonormal basis $(n, -u, v)$:

$$\mathcal{A} = \mathcal{A}^0\, n - \mathcal{A}^1\, u + \mathcal{A}^2\, v = \gamma A\,. \tag{3.37}$$

Second, by construction, the limit $\chi \to 0$ does not exist for the $\mathfrak{su}(1,1)$-valued connection. The analogous of the time gauge is defined by the limit $|\chi| \to \infty$ where the direction n tends to a constant. Let us study this limit, and for simplicity, we assume that the direction n is constant. Starting from the relation

$$\tilde{A}^i_a = {}^\gamma\omega^{ij}_a \chi_j + {}^\gamma\omega^{0i}_a - {}^\gamma\omega^{0i}_a \chi_j \chi^i\,, \tag{3.38}$$

we obtain the following limits for the components of $\mathcal{A}$

$$\mathcal{A}^0_a \to -\gamma\, {}^\gamma\omega^{0i}_a n_i\,, \qquad \mathcal{A}^1_a \to \gamma\, {}^\gamma\omega^{ij}_a n_j u_i\,, \qquad \mathcal{A}^2_a \to -\gamma\, {}^\gamma\omega^{ij}_a n_j v_i\,. \tag{3.39}$$

One recognizes the components of the spin-connection in what we could call the "space-gauge" which would be defined by the choice $e^a_i n^i = 0$ (instead of $e^a_0 = 0$ for the usual time gauge). As a consequence, the limit $|\chi| \to \infty$ with n constant is well-defined and consists in a foliation of the space-time $\mathcal{M} = \Sigma \times \mathbb{R}$ where the slices Σ are orthogonal to the space-like vector $(0, n)$.

2. *The electric field*

We follow the same strategy to construct an electric field $\mathcal{E}$ which transforms as an $\mathfrak{su}(1,1)$ under the gauge transformations. More precisely, we are looking for $\mathcal{E} = \mathcal{E}^0 J_0 + \mathcal{E}^1 J_1 + \mathcal{E}^2 J_2$ which satisfies two conditions. First we require its components to be constructed from E, χ, u and v only and we consider the natural ansatz

$$\mathcal{E}^0 \;=\; r_0(E\cdot n)\,, \quad \mathcal{E}^1 \;=\; r_1(E\cdot u)\,, \quad \mathcal{E}^2 \;=\; r_2(E\cdot v)\,, \tag{3.40}$$

where (r_0, r_1, r_2) are functions of χ only. Second we require $\mathcal{E}$ to transform as a vector

$$\delta_\varepsilon \mathcal{E} \equiv \{\tilde{\mathcal{J}}(\varrho), \mathcal{E}\} = [\mathcal{E}, \varepsilon] \quad \text{with} \quad \varrho = \varepsilon^0 n + \frac{\varepsilon^1 u - \varepsilon^2 v}{\sqrt{\chi^2-1}}\,, \tag{3.41}$$

in adequacy with what has been done in the previous part for the connection. A simple calculation shows that these conditions implies necessarily

$$r_1 = \sqrt{\chi^2-1}\, r_0\,, \quad r_2 = -\sqrt{\chi^2-1}\, r_0\,, \tag{3.42}$$

where, at this point, r_0 is free because equations (3.41) form a linear system for the unknowns (r_0, r_1, r_2).

Let us close this analysis with three remarks.
First, the free parameter r_0 can be fixed requiring in addition that $\mathcal{E}$ is canonically conjugate to $\mathcal{A}$ according to

$$\{\mathcal{A}^1, \mathcal{E}^1\} \;=\; \{\mathcal{A}^2, \mathcal{E}^2\} \;=\; 1 \quad \text{and} \quad \{\mathcal{A}^0, \mathcal{E}^0\} \;=\; -1, \tag{3.43}$$

which easily leads to $r_0 = 1/\gamma$.

Second, it will be useful to express the (inverse of the) induced metric q^{ab} on Σ in terms of the $\mathfrak{su}(1,1)$-covariant electric field. A direct calculation shows that

$$\det(g)\, g^{ab} \;=\; -\gamma^2\, \mathcal{E}^{\alpha a}\, \eta_{\alpha\beta}\, \mathcal{E}^{\beta b}\,. \tag{3.44}$$

Note that this formula makes very clear that the metric g_{ab} is Lorentzian and its signature is $(-1,+1,+1)$ as we have already seen in a previous analysis (3.20).

The final remark concerns the $\mathfrak{su}(1,1)$ gauge generators $\mathcal{J}_\alpha$. It is immediate to see that one can express them in terms of $\mathcal{A}$ and $\mathcal{E}$ only as follows

$$\mathcal{J}_\alpha(x)\, J^\alpha \;=\; \partial_a \mathcal{E}^a(x) \,+\, [\mathcal{A}_a(x)\,,\, \mathcal{E}^a(x)]\,. \tag{3.45}$$

We recover the usual Gauss-like form of the constraints, and this expression makes very clear that $\mathcal{A}$ and $\mathcal{E}$ transforms respectively as a connection and a vector under the action of the gauge generators.

3. Transformations under diffeomophisms

As for the Ashtekar-Barbero connection (or its generalization), we do not expect $\mathcal{A}$ to be a fully space-time connection on $\mathcal{M}$. However, it must transform correctly under diffeomorphisms induced on the hypersurface Σ. To see this is indeed the case, we first need to identify the generators of diffeomorphisms on Σ. A direct calculation shows that they are given by the following linear combination of the $\mathfrak{su}(1,1)$ gauge generators and the vectorial constraints:

$$\tilde{\mathcal{H}}(N^a) \;\equiv\; \mathcal{H}(N^a) - \frac{\gamma}{(1+\gamma^2)\chi^2}\tilde{\mathcal{J}}(N^a\Omega_a) \quad \text{with} \quad \Omega_a \equiv \gamma\chi \times A_a - (A_a\cdot\chi)\chi\,, \tag{3.46}$$

which, after some calculations, reduces to

$$\begin{aligned}\tilde{\mathcal{H}}(N^a) \;&=\; \int d^3x\, N^a \Big(E^b\cdot(\partial_a A_b - \partial_b A_a) - A_a\cdot\partial_b E^b + \zeta_0\cdot\partial_a\chi\Big)\\ &=\; \int d^3x\, N^a \eta_{\alpha\beta}\, \Big(\mathcal{E}^{\alpha b}\cdot(\partial_a \mathcal{A}^\alpha_b - \partial_b \mathcal{A}^\alpha_a) - \mathcal{A}^\alpha_a\cdot\partial_b\mathcal{E}^{\alpha b}\Big)\,.\end{aligned} \tag{3.47}$$

Hence, it is immediate to see from this last expression that the constraints $\tilde{\mathcal{H}}(N^a)$ form the algebra of diffeomorphisms. Furthermore, their actions on $\mathcal{A}$ and $\mathcal{E}$ is exactly the lie derivative along the vector field N^a:

$$\{\tilde{\mathcal{H}}(N^a), \mathcal{A}_b\} = -\mathcal{L}_{N^a}\mathcal{A}_b\,, \quad \{\tilde{\mathcal{H}}(N^a), \mathcal{E}_b\} = -\mathcal{L}_{N^a}\mathcal{E}_b\,. \tag{3.48}$$

Thus, as announced above, $\mathcal{A}$ is an $\mathfrak{su}(1,1)$-valued connection on Σ.

IV. ON THE QUANTIZATION

We have now all the ingredients to start the quantization of gravity formulated in terms of the $\mathfrak{su}(1,1)$ gauge connection. Following the standard construction of Loop Quantum Gravity, we assume that quantum states are polymer states, and then we build the kinematical Hilbert space from holonomies of the connection along edges on Σ.

A. Quantum states on a fixed graph

As usual, to any closed graph $\Gamma \subset \Sigma$ with N nodes and E edges, one associates a kinematical Hilbert space $\mathcal{H}_{kin}(\Gamma)$ which is isomorphic to

$$\mathcal{H}_{kin}(\Gamma) \;\simeq\; \big(\mathrm{Fun}[SU(1,1)^{\otimes E}]/SU(1,1)^{\otimes N};\, d\mu^{\otimes E}\big)\;, \tag{4.1}$$

where $d\mu$ is the Haar measure on $SU(1,1)$. Due to the non-compactness of the gauge group, such a Hilbert space needs a regularization to be well-defined (which consists basically in "dividing" by the infinite volume of the group). The details of the regularization of non-compact spin-networks has been well studied in [13]. However, it is well-known that the "projective sum" $\oplus_\Gamma \mathcal{H}_{kin}(\Gamma)$ on the space of all graphs on Σ is ill-defined and, up to our knowledge, no one knows how to construct a non-compact Ashtekar-Lewandowski measure. Thus, only the kinematical Hilbert space on a fixed graph Γ is mathematically well-defined and we limit the study of quantum states as elements of $\mathcal{H}_{kin}(\Gamma)$ only. Hence, a quantum state is a function $\psi_\Gamma[A] \equiv f(U_1, \cdots, U_E)$ of the holonomies

$$U_e \;\equiv\; P\exp\int_e \mathcal{A} \;\in SU(1,1) \tag{4.2}$$

along the edges e of Γ. The electric field $\mathcal{E}$ is promoted as an operator whose action on ψ_Γ is formally given by

$$\hat{\mathcal{E}}^a_i(x)\,\psi_\Gamma[A] \;=\; i\ell_p^2 \frac{\delta}{\delta \mathcal{A}^i_a(x)}\psi_\Gamma[A]\,, \tag{4.3}$$

where ℓ_p is the Planck length. Note that the flux of $\mathcal{E}$ across a surface is a well-defined operator on $\mathcal{H}_{kin}(\Gamma)$: it acts as a vector field on the space of $SU(1,1)$ functions.

The Peter-Weyl theorem implies that ψ_Γ can be formally decomposed as follows

$$\Psi_\Gamma[A] \;=\; \sum_{s_1,\cdots,s_E} \mathrm{tr}\left(\tilde{f}(s_e)\bigotimes_{e=1}^{E}\pi_{s_e}(U_e)\right) \tag{4.4}$$

where

$$\pi_s : SU(1,1) \to \mathrm{End}(V_s) \quad \text{and} \quad \tilde{f} \in \bigotimes_{e=1}^{N} V^*_{s_e}\,. \tag{4.5}$$

The sum runs over unitary irreducible representations of $SU(1,1)$ labelled generically by s_e. We used the notation V_{s_e} for the modulus of the representation, $V^*_{s_e}$ for its dual, and tr denotes the pairing between $\otimes_e V_{s_e}$ and its dual $\otimes_e V^*_{s_e}$. Due to the gauge invariance of ψ_Γ, the Fourier modes $\tilde{f}$ are in fact $SU(1,1)$ intertwiners and the expression of $\psi[A]$ needs a regularization to be well-defined [13]. Furthermore, unitary irreducible representations of $SU(1,1)$, which are classified into the two discrete series (both labelled with integers) and the continuous series (labelled with real numbers), are infinite dimensional (see [16] for a review on representations theory of $\mathfrak{su}(1,1)$).

B. Area operators

Thus, edges of $SU(1,1)$ spin-networks can be colored with discrete or real numbers. The geometrical interpretation is clear: these two different types of colors label edges which are normal to either time-like or space-like surfaces. To see how to link the representations to the time-like or

space-like natures of the surfaces, we have to compute the spectrum of the area operators in terms of the quadratic Casimir of $\mathfrak{su}(1,1)$. For that purpose, we start with the expression (3.44) of the inverse metric g^{ab} that we contract twice with the normal n_a to a given surface S. This leads to the formula

$$\det(g)\, n^2 \;=\; -\gamma^2\, (n_a \mathcal{E}^{a\alpha})\, \eta_{\alpha\beta}\, (\mathcal{E}^{b\beta} n_b)\,, \tag{4.6}$$

where $n^2 = n_a n_b g^{ab}$. Hence, the determinant of the induced metric h on the surface S is given by

$$\det(h) \;=\; -\gamma^2 (n_a \mathcal{E}^{a\alpha})\, \eta_{\alpha\beta}\, (\mathcal{E}^{b\beta} n_b)\,. \tag{4.7}$$

As a consequence, the action of the area operator $\hat{S}$, punctured by an edge e of the graph Γ colored by a representation s_e, on $\mathcal{H}_{kin}(\Sigma)$ is diagonal and its eigenvalue $S(s)$ is given by the equation

$$S(e)^2 \;=\; -i^2\gamma^2\ell_p^4\, \pi_e(J_1^2 + J_2^2 - J_0^2) \;=\; \gamma^2\ell_p^4\, \pi_e(C) \tag{4.8}$$

where $\pi_e(C)$ is identified with the unique eigenvalue of the Casimir tensor $C \equiv -J_0^2 + J_1^2 + J_2^2$ in the representation s_e. Obviously, the evaluation $\pi_e(C)$ depends on the nature discrete ($s_e = j_e \in \mathbb{N}$) or continuous ($s_e \in \mathbb{R}$) of the representation according to

$$\pi_{j_e}(C) = j_e(j_e+1) \quad \text{and} \quad \pi_{s_e}(C) = -(s_e^2 + \frac{1}{4})\,. \tag{4.9}$$

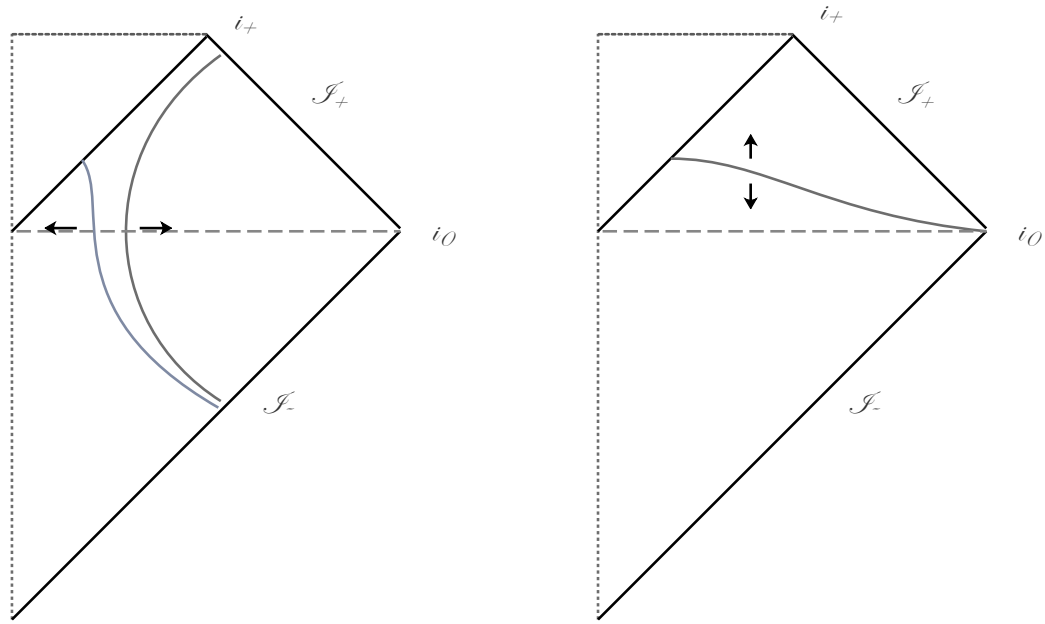
We deduce immediately that $S(e)^2$ is positive when e is colored with a discrete representation whereas $S(e)^2$ is negative when e is colored with a representation in the continuous series. As a consequence, the area operator of any space-like surface has a discrete spectrum and the area operator of any time-like surface has a continuous spectrum. Furthermore, the spectrum of space-like areas is in total agreement of the usual spectrum in Loop Quantum Gravity. Note that a very similar result has been recently derived in the context of twisted geometries [17].

V. DISCUSSION

In this paper, we have formulated gravity as an $SU(1,1)$ gauge theory. We have started with the Hamiltonian formulation of the fully Lorentz invariant Holst action on a space-time manifold of the form $\mathcal{M} = \Sigma \times \mathbb{R}$. Then we have considered a partial gauge fixing which reduces the internal $\mathfrak{sl}(2,\mathbb{C})$ gauge symmetry to $\mathfrak{su}(1,1)$. The 3-dimensional slice Σ inherits a Lorentzian metric of signature $(-,+,+)$. The partial gauge fixing relies on the introduction on an external non-dynamical vector field χ which measures the normal of the hypersurface Σ but it plays in fact no physical role at the end of the process.

Next we found that the phase space of the partially gauge fixed theory is well-parametrized by a pair $(\mathcal{A}, \mathcal{E})$ formed with an $\mathfrak{su}(1,1)$-valued connection on Σ and its canonically conjugate electric field whose components can be identified to vectors in the flat (2+1) Minkowski space-time. The phase space comes with first class constraints: the Gauss constraints which generate $\mathfrak{su}(1,1)$ gauge transformations, the vectorial constraints which have been shown to generate diffeomorphisms on Σ and the usual scalar constraint that we have not studied in this paper.

Finally, we have explored the quantization of the theory studying some aspects of the kinematical Hilbert space $\mathcal{H}_{kin}(\Gamma)$ on a fixed given graph Γ which lies on Σ. Due to the non-compactness of the gauge group $SU(1,1)$, $\mathcal{H}_{kin}(\Gamma)$ needs a regularization to be well-defined and the projective sum over all possible graphs is not under control. This is why we restrict our study to the quantization on a fixed graph only. We compute the spectrum of the area operators acting on $\mathcal{H}_{kin}(\Gamma)$ and found that the spectrum is discrete for space-like surfaces and continuous for time-like surfaces.



(a)Time-like slices (b)Space-like slices

FIG. 1: Different Hamiltonian slicings of a spherical black hole space-time. The picture (b) represents the usual slicing in terms of space-like hypersurfaces which leads to the effective $SU(2)$ Chern-Simons description of the black hole: In that case, the horizon appears as a boundary of Σ. In the picture (a), we have represented two slicings of the black hole space-time where Σ are Lorentzian hypersurfaces: these gauge choices would lead to new descriptions of black holes in Loop Quantum Gravity. In particular, the slicing which does not cross the horizon is interesting in view of a holographic description of black holes in the frame of Loop Quantum Gravity.

Furthermore, the usual quantization of the Holst action in the time-gauge ($\chi = 0$) and the new quantization presented here and based to another totally inequivalent partial gauge fixing ($\chi^2 > 1$) lead to exactly the same spectrum of the area operator (on space-like surfaces) at the kinematical level. This strongly suggests that the time gauge introduces no anomaly in the quantization of gravity, at least at the kinematical level, as it was already underlined in [11] in a different situation.

This formulation of gravity seems very interesting because it offers another point of view on the quantization of gravity in four dimensions. Now, we have a description of the kinematical quantum states of gravity not only on space-like surfaces Σ but also on time-like surfaces (only remains the description of the quantum states on null-surfaces, what we hope to study in the future). Hence, with those space-like and time-like kinematical quantum states, we are not far from having a fully covariant description of quantum gravity. In that respect, it would be very instructive to make a contact between these two canonical quantizations and spin-foam models for covariant quantum gravity. Furthermore, if we understand how to "connect" the time-like and the space-like kinematical quantum states, we could open a new and promising way towards a better understanding of the dynamics in Loop Quantum Gravity.

It is also interesting to notice that the Hamiltonian constraint in the formalism where Σ is space-like becomes a component of the vectorial constraints in the formalism where Σ is time-like. The reverse is also true. As we know very well how to quantize the vectorial constraints on

the kinematical Hilbert space, we think again that understanding the relation between these two Hamiltonian quantizations could lead us to a solution of the Hamiltonian constraint. We hope to study these questions related to the quantum dynamics in the future.

Beside, we deeply think that this new formulation will allow us to understand better the physics of quantum black holes in Loop Quantum Gravity. In the usual treatment [23–30], black holes are considered as isolated horizons and they appear as boundary of a 3 dimensional space-like hypersurface Σ. Their effective dynamics has been shown to be governed by an $SU(2)$ Chern-Simons theory whose quantization leads to the construction and the counting of the quantum microstates for the black holes. With the $\mathfrak{su}(1,1)$ formulation of gravity, it is now possible to start a Hamiltonian quantization of gravity where Σ is time-like. Naturally, one would expect that quantizing black holes with space-like or time-like slices would lead to two equivalent descriptions of the black hole microstates. At first sight, we would say that, starting with a time-like slicing, one would get an $SU(1,1)$ Chern-Simons theory as an effective dynamics for the spherical black hole for instance. Thus, we can ask the question how an $SU(1,1)$ and an $SU(2)$ Chern-Simons theories could provide two equivalent Hilbert spaces when they are quantized. This may be possible when γ becomes complex and equal to $\pm i$ because, in that case, we expect the two gauge group of the Chern-Simons theories to become the same Lorentz group. This would give one more argument in favor of the analytic continuation procedure introduced and studied in [18–22]. However, this idea might be too naive because, on a time-like slicing, the black hole does not appear as a boundary anymore and a particle leaving on the slice Σ now cross the horizon and does not see any border. To finish, this new formulation of Loop Quantum Gravity opens the possibility to define a kind of "holographic" description for black holes in the framework of Loop Quantum Gravity as shown in the picture Fig. 1 above. We hope to study all these very intriguing aspects related to black holes in a future work [31].

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Appendix A: "Time" vs. "Space" gauge in the Holst action

The very well-known "time" gauge refers to the condition e^0_a which breaks $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{su}(2)$ in the Holst action. It corresponds to taking a slicing $\Sigma\times\mathbb{R}$ of the space-time where the hypersurfaces Σ are space-like. In fact, one can easily generalize the time gauge by considering instead the condition $e^\mu_a n_\mu = 0$ where n_μ is a given fixed vector. When n_μ is time-like, the slices Σ are space-like (as for the time gauge where $n_\mu = \delta^0_\mu$) whereas the slices are time-like when n_μ is space-like. We want to study thus latter case in this appendix. To simplify the analysis, we assume (without loss of generality) that $n_\mu = \delta^3_\mu$.

We are going to show that the Hamiltonian analysis of the Holst action such a gauge leads to a phase space which corresponds to the limit $|\chi|\to\infty$ and $n_i\to\delta^3_i$. First, we notice that the only non vanishing components of π^a_{IJ} are $E^a_\alpha \equiv \pi^a_{\alpha 3}$ with $\alpha\in(0,1,2)$. It is immediate to check that the simplicity constraints $\mathcal{C}^{ab}\approx 0$ are satisfied. In this gauge, it is "natural" to choose the third direction to be the "time" parameter because of the slicing. Hence, the "symplectic" term (in the third direction) in the Holst action involves only the component $\omega^{\alpha 3}_a$ of the spin-connection (with

$\alpha \in (0,1,2)$ and $a \in (0,1,2)$ also) according to the formula

$$ {}^{\gamma}\pi^a_{IJ}\,\partial_3\omega^{IJ}_a = E^a_\alpha \partial_3 A^\alpha_a\,, \qquad \text{where} \quad A^\alpha_a \equiv {}^{\gamma}\omega^{\alpha 3}_a\,. \tag{A1} $$

Hence, the connection A is clearly the variable canonically conjugate to E. Finally, one shows that the resolution of the second class constrains $\mathcal{D}^{ab} \approx 0$ leads to the following expression for the gauge generators

$$ \mathcal{J}_0 \;=\; -\frac{1}{\gamma}\partial_a E^{a0} - A^1_a E^{a2} + A^2_a E^{a1}\,, \tag{A2} $$

$$ \mathcal{J}_1 \;=\; -\frac{1}{\gamma}\partial_a E^{a2} + A^0 E^1 - A^1 E^0\,, \tag{A3} $$

$$ \mathcal{J}_2 \;=\; \frac{1}{\gamma}\partial_a E^a 1 - A^0 E^2 + A^2 E^0\,. \tag{A4} $$

They satisfy the constraints algebra

$$ \{\mathcal{J}_0, \mathcal{J}_1\} \;=\; \mathcal{J}_2\,, \qquad \{\mathcal{J}_0, \mathcal{J}_2\} \;=\; -\mathcal{J}_1\,, \qquad \{\mathcal{J}_1, \mathcal{J}_2\} \;=\; -\mathcal{J}_0\,, \tag{A5} $$

which is nothing by the $\mathfrak{su}(1,1)$ algebra. At this point, it is not difficult to see that the associated covariant connection has the following components

$$ \mathcal{A}^0_a = -\gamma\,{}^{\gamma}\omega^{03}_a\,, \qquad \mathcal{A}^1_a = \gamma\,{}^{\gamma}\omega^{23}_a\,, \qquad \mathcal{A}^2_a = -\gamma\,{}^{\gamma}\omega^{13}_a\,. \tag{A6} $$

We recover as announced the same expression of the $\mathfrak{su}(1,1)$-valued connection in the limit $|\chi| \to \infty$ (3.39) a part that we have interchanged the components 0 and 3 of space-time indices.

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