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EFFECTIVE COMPUTATION OF $SO(3)$ AND $O(3)$ LINEAR REPRESENTATIONS SYMMETRY CLASSES

M. OLIVE

ABSTRACT. We propose a general algorithm to compute all the symmetry classes of any $SO(3)$ or $O(3)$ linear representation. This method relies on the introduction of a binary operator between sets of conjugacy classes of closed subgroups, called the *clips*. We compute explicit tables for this operation which allows to solve definitively the problem.

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1. INTRODUCTION

The problem of finding the *symmetry classes* (also called *isotropy classes*) of a given Lie group linear representation is a difficult task in general, even for a compact group, where their number is known to be finite [15, 13].

Some results are known, however, in certain cases. For instance, Ihrig and Golubitsky [9] computed the symmetry classes for all *irreducible* representations of the orthogonal groups $SO(3)$ and $O(3)$. Because of their utility in the effective study of *moduli spaces* of hyperelliptic curves [11], symmetry classes for irreducible representations of $SL(2, \mathbb{C})$ were also computed [1, 12, 3, 4].

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Interests for $\text{SO}(3)$ and $\text{O}(3)$ results in solid mechanics is well-known. Forte and Vianello [5] obtained for the first time the eight symmetry classes for the Elasticity tensor (a *reducible* $\text{SO}(3)$ -representation). Following their approach, other results were obtained in piezoelectricity [20], photoelasticity [6] and flexoelasticity [10].

More generally, the formulation of linear constitutive laws in mechanics naturally leads to general $\text{O}(3)$ or $\text{SO}(3)$ *reducible* linear representations and the *determinacy of their symmetry classes*. Up to the author's knowledge, no *general effective algorithm* existed so far to solve this problem.

In this paper, we propose such an algorithm introducing a binary operation defined over the set of conjugacy classes of a given group G and that we decided to call the *clips operation*. This operation allows to compute the set of *symmetry classes* $\mathfrak{J}(V)$ of a direct sum $V = V_1 \oplus V_2$ of linear representations of a group G , if we know the symmetry classes for each individual representations $\mathfrak{J}(V_1)$ and $\mathfrak{J}(V_2)$.

Finally, we compute clips tables for all closed subgroups of $\text{O}(3)$ and $\text{SO}(3)$. Together with the results of Ihrig and Golubitsky [9], who have computed the symmetry classes for *irreducible* representations, this allows to obtain, in a finite step process, the set of symmetry classes for any *reducible* $\text{O}(3)$ or $\text{SO}(3)$ -representation. This method was first introduced in [16] and led to general results in continuum mechanics [17, 18], where symmetry classes for 5-th, 6-th and 7-th order tensor spaces [14] were obtained for the first time.

In [section 2](#), the theory of *clips* is introduced for a general group G and applied in the context of symmetry classes where it is shown that isotropy classes of a direct sum corresponds to the clips of their respective isotropy classes. In [section 3](#), we recall classical results on the classification of closed subgroups of $\text{SO}(3)$ and $\text{O}(3)$ up to conjugacy. Models for irreducible representations of $\text{O}(3)$ and $\text{SO}(3)$ and their symmetry classes are recalled in [section 4](#). In [section 5](#), clips tables for closed $\text{SO}(3)$ and $\text{O}(3)$ subgroups are exhibited. Finally, in [Appendix A](#) and [Appendix B](#), we provide all the details and proofs to obtain these tables.

2. A GENERAL THEORY OF CLIPS

Given a group G and a subgroup H of G , the conjugacy class of H

$$[H] := \{gHg^{-1}, \quad g \in G\}$$

is a subset of $\mathcal{P}(G)$. We define $\text{Conj}(G)$ to be the set of all conjugacy classes of a given group G :

$$\text{Conj}(G) := \{[H], \quad H \subset G\}.$$

Recall that, on $\text{Conj}(G)$, there is a pre-order relation induced by inclusion. It is defined as follows:

$$[H_1] \preceq [H_2] \quad \text{if } H_1 \text{ is conjugate to a subgroup of } H_2.$$

When restricted to the *closed subgroups* of a topological compact group, this pre-order relation becomes a *partial order* [2, Proposition 1.9] and defines the *poset* (partial ordered set) of conjugacy classes of *closed subgroups* of G .

We now define a binary operation called the *clips operation* on the set $\text{Conj}(G)$.

Definition 2.1. Given two conjugacy classes $[H_1]$ and $[H_2]$ of a group G , we define their *clips* as the following subset of conjugacy classes:

$$[H_1] \odot [H_2] := \{[H_1 \cap gH_2g^{-1}], \quad g \in G\}.$$

This definition immediately extends to two families (finite or infinite) \mathcal{F}_1 and \mathcal{F}_2 of conjugacy classes:

$$\mathcal{F}_1 \odot \mathcal{F}_2 := \bigcup_{[H_i] \in \mathcal{F}_i} [H_1] \odot [H_2].$$

This clips operation defined thus a binary operation on the set $\mathcal{P}(\text{Conj}(G))$ which is *associative* and *commutative*. We have moreover

$$[\mathbf{1}] \odot [H] = \{[\mathbf{1}]\} \quad \text{and} \quad [G] \odot [H] = \{[H]\},$$

for every conjugacy class $[H]$, where $\mathbf{1} := \{e\}$ and e is the identity element of G .

Consider now a linear representation (V, ρ) of the group G . Given $\mathbf{v} \in V$, its *isotropy group* (or *symmetry group*) is defined as

$$G_{\mathbf{v}} := \{g \in G, \quad g \cdot \mathbf{v} = \mathbf{v}\}$$

and its *isotropy class* is the conjugacy class $[G_{\mathbf{v}}]$ of its isotropy group. The *isotropy classes* (or *orbit types*) of the representation V is the family of all isotropy classes of vectors \mathbf{v} in V :

$$\mathfrak{I}(V) := \{[G_{\mathbf{v}}]; \quad \mathbf{v} \in V\}.$$

The central observation is that the isotropy classes of a direct sum of representations is obtained by the clips of their respective isotropy classes.

Lemma 2.2. *Let V_1 and V_2 be two linear representations of G . Then*

$$\mathfrak{I}(V_1 \oplus V_2) = \mathfrak{I}(V_1) \odot \mathfrak{I}(V_2).$$

Proof. Let $[G_{\mathbf{v}}]$ be some isotropy class for $\mathfrak{I}(V_1 \oplus V_2)$ and write $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_i \in V_i$. Note first that $G_{\mathbf{v}_1} \cap G_{\mathbf{v}_2} \subset G_{\mathbf{v}}$. Conversely given $g \in G_{\mathbf{v}}$, we get

$$g \cdot \mathbf{v} = g \cdot \mathbf{v}_1 + g \cdot \mathbf{v}_2 = \mathbf{v}, \quad g \cdot \mathbf{v}_i \in V_i,$$

and thus $g \cdot \mathbf{v}_i = \mathbf{v}_i$. This shows that $G_{\mathbf{v}} = G_{\mathbf{v}_1} \cap G_{\mathbf{v}_2}$ and therefore that

$$\mathfrak{I}(V_1 \oplus V_2) \subset \mathfrak{I}(V_1) \odot \mathfrak{I}(V_2).$$

Conversely, let $[H] = [H_1 \cap gH_2g^{-1}]$ in $\mathfrak{I}(V_1) \odot \mathfrak{I}(V_2)$ where $H_i = G_{\mathbf{v}_i}$ for some vectors $\mathbf{v}_i \in V_i$. Then, if we set

$$\mathbf{v} = \mathbf{v}_1 + g \cdot \mathbf{v}_2,$$

we have $G_{\mathbf{v}} = H_1 \cap gH_2g^{-1}$, as before, which shows that

$$[H_1 \cap gH_2g^{-1}] \in \mathfrak{I}(V_1 \oplus V_2)$$

and achieves the proof. \square

Using this lemma, we deduce a general algorithm to obtain the isotropy classes $\mathfrak{I}(V)$ of a finite dimensional representation of a reductive algebraic group G , provided we know:

- (1) a decomposition $V = \bigoplus_i W_i$ into irreducible representations W_i .
- (2) the isotropy classes $\mathfrak{I}(W_i)$ for the *irreducible representations* W_i ;
- (3) the tables of *clips operations* $[H_1] \odot [H_2]$ between conjugacy classes of closed subgroups $[H_i]$ of G .

In the sequel of this paper, we will apply successfully this program to the linear representations of $\text{SO}(3)$ and $\text{O}(3)$.

3. CLOSED SUBGROUPS OF $\text{O}(3)$

Every closed subgroup of $\text{SO}(3)$ is conjugate to one of the following list [8]

$$\text{SO}(3), \text{O}(2), \text{SO}(2), \mathbb{D}_n (n \geq 2), \mathbb{Z}_n (n \geq 2), \mathbb{T}, \mathbb{O}, \mathbb{I}, \text{ and } \mathbb{1}$$

where:

- $\text{O}(2)$ is the subgroup generated by all the rotations around the z -axis and the order 2 rotation $r : (x, y, z) \mapsto (x, -y, -z)$ around the x -axis;
- $\text{SO}(2)$ is the subgroup of all the rotations around the z -axis;
- \mathbb{Z}_n is the unique cyclic subgroup of order n of $\text{SO}(2)$ ($\mathbb{Z}_1 = \{I\}$);
- \mathbb{D}_n is the *dihedral* group. It is generated by \mathbb{Z}_n and $r : (x, y, z) \mapsto (x, -y, -z)$ ($\mathbb{D}_1 = \{I\}$);
- \mathbb{T} is the *tetrahedral* group, the (orientation-preserving) symmetry group of the tetrahedron \mathcal{T}_0 defined in Figure 8. It has order 12;
- \mathbb{O} is the *octahedral* group, the (orientation-preserving) symmetry group of the cube \mathcal{C}_0 defined in Figure 8. It has order 24;
- \mathbb{I} is the *icosahedral* group, the (orientation-preserving) symmetry group of the dodecahedron \mathcal{D}_0 in Figure 9. It has order 60;
- $\mathbb{1}$ is the trivial subgroup, containing only the unit element.

The *poset* of conjugacy classes of closed subgroups of $\text{SO}(3)$ is completely described by the following inclusion of subgroups [8]:

$$\begin{aligned} \mathbb{Z}_n &\subset \mathbb{D}_n \subset \text{O}(2) && (n \geq 2); \\ \mathbb{Z}_n &\subset \mathbb{Z}_m \text{ and } \mathbb{D}_n \subset \mathbb{D}_m, && (\text{if } n \text{ divides } m); \\ \mathbb{Z}_n &\subset \text{SO}(2) \subset \text{O}(2) && (n \geq 2); \end{aligned}$$

completed by $[\mathbb{Z}_2] \preceq [\mathbb{D}_n]$ ($n \geq 2$) and by the arrows in Figure 1 (note that an arrow between the classes $[H_1]$ and $[H_2]$ means that $[H_1] \preceq [H_2]$), taking account of the exceptional subgroups $\mathbb{O}, \mathbb{T}, \mathbb{I}$.

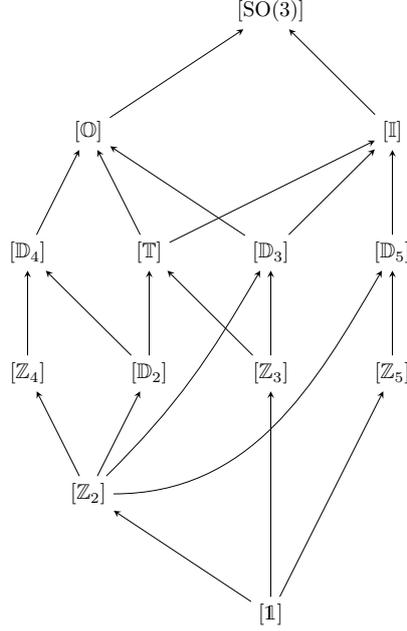
Classification of $\text{O}(3)$ -closed subgroups is more involving [9, 19] and has been described using *three types of subgroups*. Given a closed subgroup Γ of $\text{O}(3)$ this classification runs as follows.

Type I: A subgroup Γ is of type I if it is a subgroup of $\text{SO}(3)$;

Type II: A subgroup Γ is of type II if $-I \in \Gamma$. In that case, Γ is generated by some subgroup K of $\text{SO}(3)$ and $-I$;

Type III: A subgroup Γ is of type III if $-I \notin \Gamma$ and Γ is not a subgroup of $\text{SO}(3)$.

The description of type III subgroups requires more details. We will denote by $\mathbf{Q}(\mathbf{v}; \theta) \in \text{SO}(3)$ the rotation around $\mathbf{v} \in \mathbb{R}^3$ with angle $\theta \in [0, 2\pi[$ and by $\sigma_{\mathbf{v}} \in \text{O}(3)$, the reflection through the plane normal to \mathbf{v} . Finally, we

FIGURE 1. Exceptional conjugacy classes of closed $\text{SO}(3)$ subgroups

fix an arbitrary orthonormal frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, and we introduce the following definitions.

- \mathbb{Z}_2^- is the order 2 reflection group generated by $\sigma_{\mathbf{i}}$ (where $\mathbb{Z}_1^- = \{\mathbf{1}\}$);
- \mathbb{Z}_{2n}^- ($n \geq 2$) is the group of order $2n$, generated by \mathbb{Z}_n and $-\mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right)$ (see (B.1));
- \mathbb{D}_{2n}^h ($n \geq 2$) is the group of order $4n$ generated by \mathbb{D}_n and $-\mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right)$ (see (B.3));
- \mathbb{D}_n^v ($n \geq 2$) is the group of order $2n$ generated by \mathbb{Z}_n and $\sigma_{\mathbf{i}}$ (where $\mathbb{D}_1^v = \{\mathbf{1}\}$);
- $\text{O}(2)^-$ is generated by $\text{SO}(2)$ and $\sigma_{\mathbf{i}}$.

These planar subgroups are completed by the subgroup \mathbb{O}^- which is of order 24 (see subsection B.4 and (B.5) for details).

The poset of conjugacy classes of closed subgroups of $\text{O}(3)$ is given in Figure 2 for \mathbb{O}^- subgroups and in Figure 3 for $\text{O}(3)$ subgroups.

4. SYMMETRY CLASSES FOR IRREDUCIBLE REPRESENTATIONS

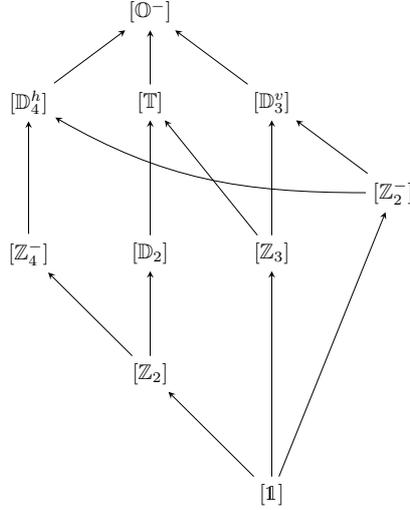
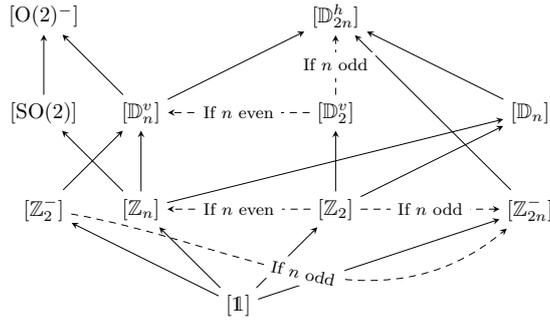
Let $\mathcal{P}_n(\mathbb{R}^3)$ be the space of *homogeneous polynomials* of degree n on \mathbb{R}^3 . We have two natural representations of $\text{O}(3)$ on $\mathcal{P}_n(\mathbb{R}^3)$. The first one, noted ρ_n is given by

$$[\rho_n(p)](\mathbf{x}) := p(g^{-1}\mathbf{x}), \quad g \in G, \quad \mathbf{x} \in \mathbb{R}^3$$

whereas the second one, noted ρ_n^* , is given by

$$[\rho_n^*(p)](\mathbf{x}) := \det(g)p(g^{-1}\mathbf{x}), \quad g \in G, \quad \mathbf{x} \in \mathbb{R}^3.$$

Note that both of them induce the same representation ρ_n of $\text{SO}(3)$.

FIGURE 2. Poset of closed \mathbb{O}^- subgroupsFIGURE 3. Poset of closed $O(3)$ subgroups

Let $\mathcal{H}_n(\mathbb{R}^3) \subset \mathcal{P}_n(\mathbb{R}^3)$ be the subspace of *homogeneous harmonic polynomials* of degree n (polynomials with null Laplacian). It is a classical fact [8] that $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$ and $(\mathcal{H}_n(\mathbb{R}^3), \rho_n^*)$ ($n \geq 0$) are irreducible $O(3)$ -representations, and each irreducible $O(3)$ -representation is isomorphic to one of them. Models for irreducible representations of $SO(3)$ reduce to $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$ ($n \geq 0$).

Remark 4.1. Other classical models for $O(3)$ and $SO(3)$ irreducible representations, used in mechanics [5], are given by spaces of *harmonic tensors* (i.e. totally symmetric traceless tensors).

The isotropy classes for irreducible representations of $SO(3)$ and $O(3)$ were first obtained by Ihrig and Golubitsky [9].

Theorem 4.2 (Ihrig-Golubitsky 1984). *The isotropy classes for the $SO(3)$ -representation $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$ are:*

- (1) $[\mathbf{1}]$ for $n \geq 3$;
- (2) $[Z_k]$ for $2 \leq k \leq n$ if n is odd and $2 \leq k \leq n/2$ if n is even;
- (3) $[D_k]$ for $2 \leq k \leq n$;

- (4) $[\mathbb{T}]$ for $n = 3, 6, 7$ or $n \geq 9$;
- (5) $[\mathbb{O}]$ for $n \neq 1, 2, 3, 5, 7, 11$;
- (6) $[\mathbb{I}]$ for $n = 6, 10, 12, 15, 16, 18$ or $n \geq 20$ and $n \neq 23, 29$;
- (7) $[\mathrm{SO}(2)]$ for n odd;
- (8) $[\mathrm{O}(2)]$ for n even;
- (9) $[\mathrm{SO}(3)]$ for any n .

Remark 4.3. In [9, Theorem 6.6] (for $\mathrm{SO}(3)$ irreducible representations) :

- $[\mathbb{T}]$ is an isotropy class for $n = 6, 7$ and $n \geq 9$;
- $[\mathbb{O}]$ is an isotropy class for $n \neq 1, 2, 5, 7, 11$.

Such lists are different from (4) and (5) in our Theorem 4.2. But according to [9, Proposition 3.7], $[\mathbb{T}]$ is a maximum isotropy class for $n = 3$. We have thus corrected this error in Theorem 4.2.

Theorem 4.4 (Ihrig-Golubistky 1984). *The isotropy classes for the $\mathrm{O}(3)$ -representations $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$ (for n odd) and $(\mathcal{H}_n(\mathbb{R}^3), \rho_n^*)$ (for n even) are:*

- (1) $[\mathbb{1}]$ for $n \geq 3$;
- (2) $[\mathbb{Z}_k]$ for $2 \leq k \leq n/2$;
- (3) $[\mathbb{Z}_{2k}^-]$ for $k \leq \frac{n}{3}$;
- (4) $[\mathbb{D}_k]$ for $2 \leq k \leq n/2$ if n is odd and for $2 \leq k \leq n$ if n is even.
- (5) $[\mathbb{D}_k^v]$ for $2 \leq k \leq n$ if n is odd and $2 \leq k \leq n/2$ if n is even;
- (6) $[\mathbb{D}_{2k}^h]$ for $2 \leq k \leq n$, except \mathbb{D}_4^h for $n = 3$;
- (7) $[\mathbb{T}]$ for $n \neq 1, 2, 3, 5, 7, 8, 11$;
- (8) $[\mathbb{O}]$ for $n \neq 1, 2, 3, 5, 7, 11$;
- (9) $[\mathbb{O}^-]$ for $n \neq 1, 2, 4, 5, 8$;
- (10) $[\mathbb{I}]$ for $n = 6, 10, 12, 15, 16, 18$ or $n \geq 20$ and $n \neq 23, 29$;
- (11) $[\mathrm{O}(2)]$ when n is even ;
- (12) $[\mathrm{O}(2)^-]$ when n is odd.

Remark 4.5. In [9, Theorem 6.8] (for $\mathrm{O}(3)$ irreducible representations), $[\mathbb{T}]$ is an isotropy class for $n \neq 1, 2, 5, 7, 8, 11$ (which is different from the list (7) in our Theorem 4.4 above). But according to [7, 20], $[\mathbb{T}]$ is not an isotropy class in the case $n = 3$, and we corrected this error in the list (7) of Theorem 4.4.

5. CLIPS TABLES

5.1. $\mathrm{SO}(3)$ closed subgroups. The resulting conjugacy classes for the clips operation of closed $\mathrm{SO}(3)$ subgroups are given in Table 1, where the following notations have been used:

$$\begin{aligned}
 d_2 &:= \gcd(n, 2), & k_2 &:= 3 - d_2 & d_3 &:= \gcd(n, 3), \\
 d_5 &:= \gcd(n, 5) & d_2' &:= \gcd(m, 2), \\
 dz &:= 2, \text{ if } d = 1, & dz &:= 1, \text{ otherwise,} \\
 dz' &:= 2, \text{ if } m \text{ and } n \text{ even,} & dz' &:= 1, \text{ otherwise,} \\
 d_4 &:= 4, \text{ if } 4 \mid n, & d_4 &:= 1, \text{ otherwise.}
 \end{aligned}$$

Remark 5.1. The clips operations $[\mathbb{T}] \odot [\mathbb{T}]$ and $[\mathbb{T}] \odot [\mathbb{O}]$ were wrong in [17, 16], since the isotropy class $[\mathbb{D}_2]$ was omitted.

\odot	$[Z_n]$	$[D_n]$	$[T]$	$[O]$	$[I]$	$[SO(2)]$	$[O(2)]$
$[Z_m]$	$[1], [Z_d]$						
$[D_m]$	$[1]$ $[Z_{d_2}]$ $[Z_d]$	$[1], [Z_{d_2}]$ $[Z_{d'_2}], [Z_{dz}], [D_{dz}']$ $[Z_d], [D_d]$					
$[T]$	$[1]$ $[Z_{d_2}]$ $[Z_{d_3}]$	$[1], [Z_2]$ $[Z_{d_3}], [D_{d_2}]$	$[1], [Z_2]$ $[D_2], [Z_3]$ $[T]$				
$[O]$	$[1]$ $[Z_{d_2}]$ $[Z_{d_3}]$ $[Z_{d_4}]$	$[1], [Z_2]$ $[Z_{d_3}], [Z_{d_4}]$ $[D_{d_2}], [D_{d_3}]$ $[D_{d_4}]$	$[1]$ $[Z_2], [D_2]$ $[Z_3]$ $[T]$	$[1], [Z_2]$ $[D_2], [Z_3]$ $[D_3], [Z_4]$ $[D_4], [O]$			
$[I]$	$[1]$ $[Z_{d_2}]$ $[Z_{d_3}]$ $[Z_{d_5}]$	$[1], [Z_2]$ $[Z_{d_3}], [Z_{d_5}]$ $[D_{d_2}]$ $[D_{d_3}], [D_{d_5}]$	$[1]$ $[Z_2]$ $[Z_3]$ $[T]$	$[1], [Z_2]$ $[Z_3], [D_3]$ $[T]$	$[1], [Z_2]$ $[Z_3], [D_3]$ $[Z_5], [D_5]$ $[I]$		
$[SO(2)]$	$[1], [Z_n]$	$[1], [Z_2]$ $[Z_n]$	$[1], [Z_2]$ $[Z_3]$	$[1], [Z_2]$ $[Z_3], [Z_4]$	$[1], [Z_2]$ $[Z_3], [Z_5]$	$[1], [SO(2)]$	
$[O(2)]$	$[1], [Z_{d_2}]$ $[Z_n]$	$[1], [Z_{d_2}]$ $[D_{k_2}], [D_n]$	$[1], [Z_2]$ $[D_2], [Z_3]$	$[1], [Z_2]$ $[D_2], [D_3]$ $[D_4]$	$[1], [Z_2]$ $[D_2], [D_3]$ $[D_5]$	$[1], [Z_2]$ $[SO(2)]$	$[Z_2], [D_2]$ $[O(2)]$

TABLE 1. Clips operations for $SO(3)$

Example 5.2 (Isotropy classes for a family of n vectors). For one vector, we get

$$\mathfrak{J}(\mathcal{H}_1(\mathbb{R}^3)) = \{[SO(2)], [SO(3)]\}.$$

From [Table 1](#), we deduce that the isotropy classes for a family of n vectors ($n \geq 2$) is

$$\mathfrak{J}\left(\bigoplus_{k=1}^n \mathcal{H}_1(\mathbb{R}^3)\right) = \{[1], [SO(2)], [SO(3)]\}.$$

Example 5.3 (Isotropy classes for a family of n quadratic forms). The space of quadratic forms on \mathbb{R}^3 , $S_2(\mathbb{R}^3)$, decomposes into two irreducible components (*deviatoric* and *spherical* tensors for the mechanics):

$$S_2(\mathbb{R}^3) = \mathcal{H}_2(\mathbb{R}^3) \oplus \mathcal{H}_0(\mathbb{R}^3).$$

We get thus

$$\mathfrak{J}(S_2(\mathbb{R}^3)) = \mathfrak{J}(\mathcal{H}_2(\mathbb{R}^3)) = \{[D_2], [O(2)], [SO(3)]\}.$$

The useful part of [Table 1](#), for our purpose, reads:

\odot	$[Z_2]$	$[D_2]$	$[O(2)]$
$[Z_2]$	$\{[1], [Z_2]\}$	$\{[1], [Z_2]\}$	$\{[1], [Z_2]\}$
$[D_2]$		$\{[1], [Z_2], [D_2]\}$	$\{[1], [Z_2], [D_2]\}$
$[O(2)]$			$\{[Z_2], [D_2], [O(2)]\}$

We deduce therefore that the set of isotropy classes for a family of n quadratic forms ($n \geq 2$) is

$$\mathfrak{I} \left(\bigoplus_{k=1}^n S_2(\mathbb{R}^3) \right) = \{[1], [Z_2], [D_2], [O(2)], [SO(3)]\}.$$

5.2. O(3) closed subgroups. Let us first consider an O(3)-representation where $-I$ act as $-\text{Id}$ (meaning that this representation doesn't reduce to some SO(3) representation). In such a case, only the null vector can be fixed by $-\text{Id}$, and so type II subgroups never appear as isotropy subgroups. In that case, we need only to focus on clips operations between type I and type III subgroups, and then between type III subgroups, since clips operations between type I subgroups have already been considered in [Table 1](#). For type III subgroups as detailed in [Appendix B](#) we have:

Lemma 5.4. *Let H_1 be some type III closed subgroup of O(3) and H_2 be some type I closed subgroup of O(3). Then we have*

$$H_1 \cap H_2 = (H_1 \cap SO(3)) \cap H_2,$$

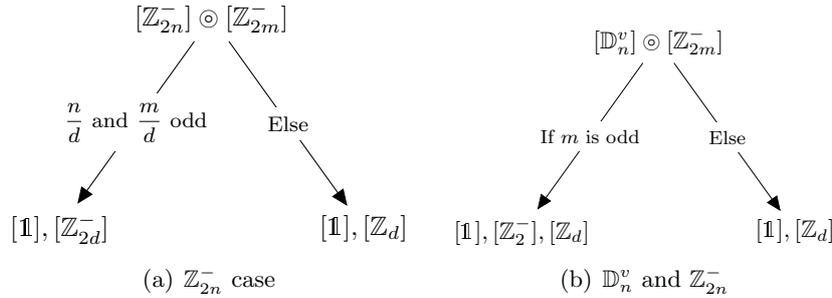
and for every closed subgroup H of SO(3), we get:

$$\begin{aligned} [Z_2^-] \odot [H] &= \{[1]\}, & [Z_{2n}^-] \odot [H] &= [Z_n] \odot [H], \\ [\mathbb{D}_n^v] \odot [H] &= [Z_n] \odot [H], & [\mathbb{D}_{2n}^h] \odot [H] &= [\mathbb{D}_n] \odot [H], \\ [O^-] \odot [H] &= [T] \odot [H], & [O(2)^-] \odot [H] &= [SO(2)] \odot [H]. \end{aligned}$$

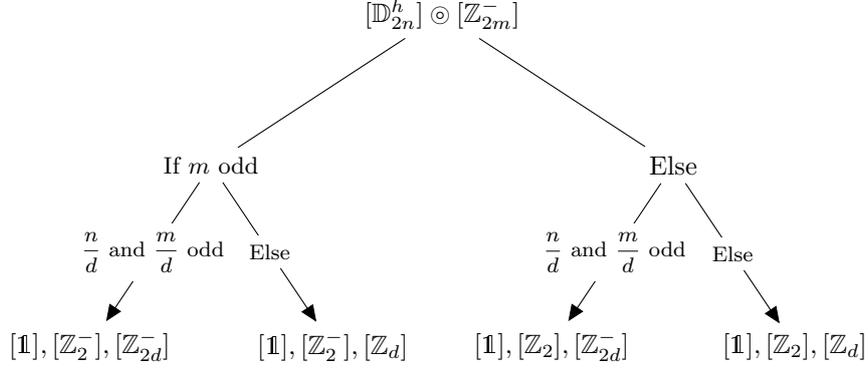
The resulting conjugacy classes for the clips operation for type III subgroups are given in [Table 2](#), where the following notations have been used:

$$\begin{aligned} d &:= \gcd(n, m), & d_2(n) &:= \gcd(n, 2), \\ d_3(n) &:= \gcd(n, 3), & i(n) &:= 3 - \gcd(2, n). \end{aligned}$$

Remark 5.5. One misprint in [\[16, 18\]](#) for clips operation $[O(2)^-] \odot [\mathbb{D}_{2n}^h]$ has been corrected: the conjugacy class $[Z_2]$ appears for m even (and not for m odd).



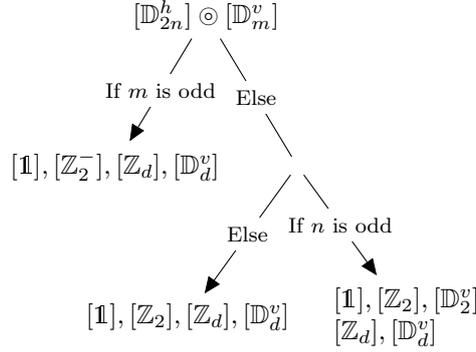
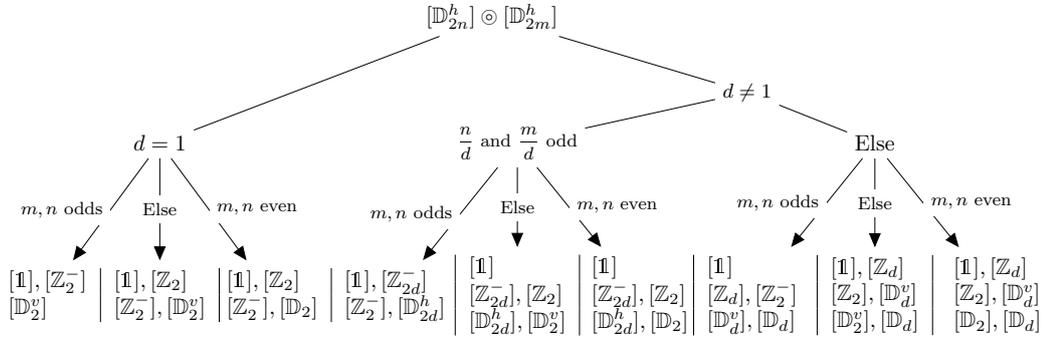
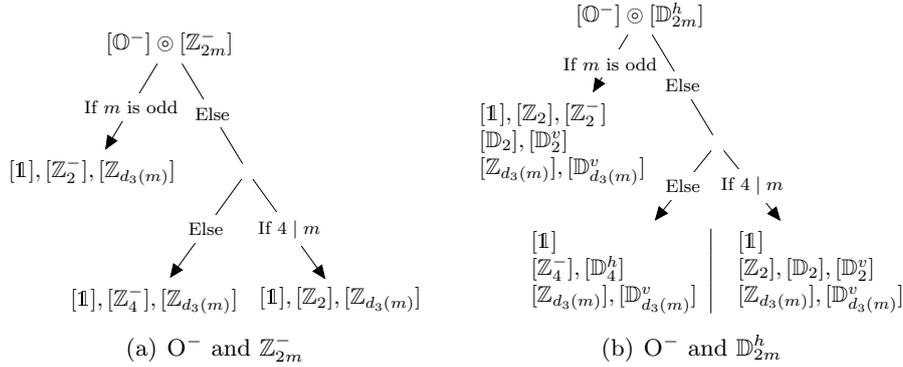
\odot	$[Z_2^-]$	$[Z_{2m}^-]$	$[D_m^v]$	$[D_{2m}^h]$	$[O^-]$	$[O(2)^-]$
$[Z_2^-]$	$[1], [Z_2^-]$					
$[Z_{2n}^-]$	$[1], [Z_{i(n)}^-]$	Figure 4(a)				
$[D_n^v]$	$[1], [Z_2^-]$	Figure 4(b)	$[1], [Z_2^-]$ $[D_d^v], [Z_d]$			
$[D_{2n}^h]$	$[1], [Z_2^-]$	Figure 4	Figure 5	Figure 6		
$[O^-]$	$[1], [Z_2^-]$	Figure 7(a)	$[1], [Z_2^-]$ $[Z_{d_3(m)}]$ $[D_{d_3(m)}^v]$ $[Z_{d_2(m)}]$ $[Z_{d_2(m)}^v]$	Figure 7(b)	$[1], [Z_2^-]$ $[Z_4^-], [Z_3]$	
$[O(2)^-]$	$[1], [Z_2^-]$	$[1], [Z_{i(m)}^-]$ $[Z_m]$	$[1], [Z_2^-], [D_m^v]$	$[1]$ $[Z_{d_2(m)}], [Z_2^-]$ $[D_{i(m)}^v], [D_m^v]$	$[1], [Z_2^-]$ $[D_3^v], [D_2^v]$	$[Z_2^-], [O(2)^-]$

TABLE 2. Clips operations on type III $O(3)$ -subgroupsFIGURE 4. D_{2n}^h and Z_{2n}^- APPENDIX A. PROOFS FOR $SO(3)$

In this section, we provide all the details required to obtain the results in Table 1. We will start by the following definition which was introduced in [8] and happens to be quite useful for this task.

Definition A.1. Let K_1, K_2, \dots, K_s be subgroups of a given group G . We say that G is the *direct union* of the K_i and we write $G = \biguplus_{i=1}^s K_i$ if

$$G = \bigcup_{i=1}^s K_i \quad \text{and} \quad K_i \cap K_j = \{e\}, \quad \forall i \neq j.$$

FIGURE 5. \mathbb{D}_{2n}^h and \mathbb{D}_n^v FIGURE 6. \mathbb{D}_{2n}^h and \mathbb{D}_{2m}^h (a) O^- and Z_{2m}^- (b) O^- and \mathbb{D}_{2m}^h

In the following, we will have to identify repeatedly the conjugacy class of intersections such as

$$(A.1) \quad H_1 \cap (gH_2g^{-1}),$$

where H_1 and H_2 are two closed subgroups of $SO(3)$ and $g \in SO(3)$. A useful observation is that all closed $SO(3)$ subgroups have some *characteristic axes* and that intersection (A.1) depends only on the relative positions of these characteristic axes.

As detailed below, for any subgroup conjugate to \mathbb{Z}_n or \mathbb{D}_n ($n \geq 3$), the axis of an n -th order rotation (in this subgroup) is called its *primary axis*. For subgroups conjugate to \mathbb{D}_n ($n \geq 3$), axes of order two rotations are said to be *secondary axes*. In the special case $n = 2$, the z -axis is the primary axis of \mathbb{Z}_2 , while any of the x , y or z axis can be considered as a primary axis of \mathbb{D}_2 .

A.1. Cyclic subgroup. For any axis a of \mathbb{R}^3 (throughout the origin), we denote by \mathbb{Z}_n^a , the unique cyclic subgroup of order n around the a -axis, which is its primary axis. We have then:

Lemma A.2. *Let $m, n \geq 2$ be two integers and $d = \gcd(n, m)$. Then*

$$[\mathbb{Z}_n] \odot [\mathbb{Z}_m] = \{[\mathbf{1}], [\mathbb{Z}_d]\}.$$

Proof. We have to consider intersections, such as

$$\mathbb{Z}_n \cap (g\mathbb{Z}_m g^{-1}) = \mathbb{Z}_n \cap \mathbb{Z}_m^a,$$

for some axis a , and only two cases occur:

- (1) If $a \neq (Oz)$, then necessarily the intersection reduces to $\mathbf{1}$.
- (2) If $a = (Oz)$, then the order r of a rotation in $\mathbb{Z}_n \cap \mathbb{Z}_m$ divides both n and m and thus divides $d = \gcd(m, n)$. We get therefore: $\mathbb{Z}_n \cap \mathbb{Z}_m \subset \mathbb{Z}_d$. But obviously, $\mathbb{Z}_d \subset \mathbb{Z}_n \cap \mathbb{Z}_m$ and thus $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$. □

A.2. Dihedral subgroup. Let b_1 be the x -axis and b_k ($k = 2, \dots, n$) be the axis recursively defined by

$$b_k := \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right) b_{k-1}.$$

Then, we have

$$(A.2) \quad \mathbb{D}_n = \mathbb{Z}_n \uplus \mathbb{Z}_2^{b_1} \uplus \dots \uplus \mathbb{Z}_2^{b_n},$$

where the z -axis (corresponding to a n -th order rotation) is the primary axis and the b_k -axes (corresponding to order two rotations) are the secondary axes of this dihedral group (see [Figure 7](#)).

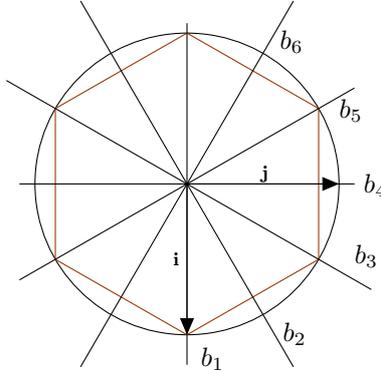


FIGURE 7. Secondary axis of the dihedral group \mathbb{D}_6

Lemma A.3. *Let $m, n \geq 2$ be two integers. Set $d := \gcd(n, m)$ and $d_2(m) := \gcd(m, 2)$. Then, we have*

$$[\mathbb{D}_n] \odot [\mathbb{Z}_m] = \{[\mathbf{1}], [\mathbb{Z}_{d_2(m)}], [\mathbb{Z}_d]\}.$$

Proof. Let $\Gamma = \mathbb{D}_n \cap \mathbb{Z}_m^a$. From decomposition (A.2), we have to consider intersections

$$\mathbb{Z}_n \cap \mathbb{Z}_m^a, \quad \mathbb{Z}_2^{b_j} \cap \mathbb{Z}_m^a.$$

which thus reduce to Lemma A.2. \square

Lemma A.4. *Let $m, n \geq 2$ be two integers. Set*

$$d := \gcd(n, m), \quad d_2(m) := \gcd(m, 2), \quad d_2(n) := \gcd(n, 2),$$

and

$$dz := \begin{cases} 2 & \text{if } d = 1, \\ 1 & \text{otherwise.} \end{cases}, \quad dz' := \begin{cases} 2 & \text{if } m \text{ and } n \text{ even,} \\ 1 & \text{otherwise.} \end{cases}$$

Then, we have

$$[\mathbb{D}_n] \odot [\mathbb{D}_m] = \{[\mathbf{1}], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_{d_2(m)}], [\mathbb{Z}_{dz}], [\mathbb{D}_{dz'}], [\mathbb{Z}_d], [\mathbb{D}_d]\}.$$

Proof. Let $\Gamma = \mathbb{D}_n \cap (g\mathbb{D}_mg^{-1})$.

- (1) If both primary axes and one secondary axis match, $\Gamma = \mathbb{D}_d$ if $d \neq 1$ and $\Gamma \in [\mathbb{Z}_2]$ otherwise;
- (2) if only the primary axes match, $\Gamma = \mathbb{Z}_d$;
- (3) if the primary axis of $g\mathbb{D}_mg^{-1}$ matches with the secondary axis (Ox) of \mathbb{D}_n (or the converse), we obtain $\Gamma \in [\mathbb{D}_2]$ for n and m even and a secondary axis of $g\mathbb{D}_mg^{-1}$ is (Oz), otherwise we obtain $\Gamma \in [\mathbb{Z}_2]$
- (4) in all other cases we have $\Gamma = \mathbf{1}$.

\square

A.3. Tetrahedral subgroup. The (orientation-preserving) symmetry group \mathbb{T} of the tetrahedron $\mathcal{T}_0 := A_1A_3A_7A_5$ (see Figure 8) decomposes as (see [9]):

$$(A.3) \quad \mathbb{T} = \bigoplus_{i=1}^4 \mathbb{Z}_3^{\mathbf{vt}_i} \bigoplus_{j=1}^3 \mathbb{Z}_2^{\mathbf{et}_j}$$

where \mathbf{vt}_i (resp. \mathbf{et}_j) are the *vertices axes* (resp. *edges axes*) of the tetrahedron (see Figure 8):

$$\begin{aligned} \mathbf{vt}_1 &:= (OA_1), & \mathbf{vt}_2 &:= (OA_3), & \mathbf{vt}_3 &:= (OA_5), & \mathbf{vt}_4 &:= (OA_7), \\ \mathbf{et}_1 &:= (Ox), & \mathbf{et}_2 &:= (Oy), & \mathbf{et}_3 &:= (Oz). \end{aligned}$$

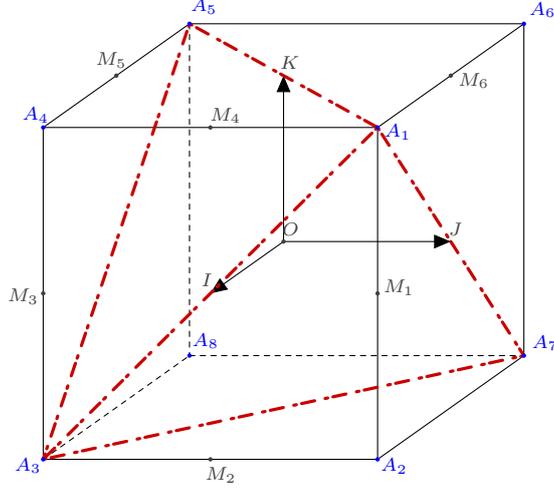
Corollary A.5. *Let $n \geq 2$ be an integer. Set $d_2(n) := \gcd(n, 2)$ and $d_3(n) := \gcd(3, n)$. Then, we have*

$$[\mathbb{Z}_n] \odot [\mathbb{T}] = \{[\mathbf{1}], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_{d_3(n)}]\}.$$

Proof. Consider $\mathbb{T} \cap \mathbb{Z}_n^a$ for some axis a . As a consequence of Lemma A.2, we need only to consider the case where a is an edge axis or a face axis of the tetrahedron, reducing to the clips operations

$$[\mathbb{Z}_2] \odot [\mathbb{Z}_n], \quad [\mathbb{Z}_3] \odot [\mathbb{Z}_n]$$

which directly leads to the Lemma. \square

FIGURE 8. Cube \mathcal{C}_0 and tetrahedron $\mathcal{T}_0 := A_1A_3A_7A_5$

Corollary A.6. *Let $n \geq 2$ be some integer. Set $d_2(n) := \gcd(n, 2)$ and $d_3(n) := \gcd(3, n)$. Then, we have*

$$[\mathbb{D}_n] \odot [\mathbb{T}] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_2(n)}]\}.$$

Proof. Let $\Gamma = \mathbb{T} \cap (g\mathbb{D}_ng^{-1})$. From decomposition (A.3), we need only to consider intersections

$$\mathbb{Z}_3^{\mathbf{vt}_i} \cap (g\mathbb{D}_ng^{-1}) \text{ and } \mathbb{Z}_2^{\mathbf{et}_j} \cap (g\mathbb{D}_ng^{-1})$$

which have already been studied (see Lemma A.3). \square

Lemma A.7. *We have*

$$[\mathbb{T}] \odot [\mathbb{T}] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{T}]\}.$$

Proof. Let $\Gamma = \mathbb{T} \cap (g\mathbb{T}g^{-1})$.

- (1) If no axes match, then $\Gamma = \mathbf{1}$;
- (2) if only one edge axis (resp. one face axis) from both configurations match, then $\Gamma \in [\mathbb{Z}_2]$ (resp. $[\mathbb{Z}_3]$);
- (3) if $g = \mathbf{Q}(\mathbf{k}, \frac{\pi}{2})$, then $\Gamma = \mathbb{D}_2$.

\square

A.4. Octahedral subgroup. The group \mathbb{O} is the (orientation-preserving) symmetry group of the cube \mathcal{C}_0 (see Figure 8) with vertices

$$\{A_i\}_{i=1\dots 8} = (\pm 1, \pm 1, \pm 1).$$

We have the decomposition (see [9]):

$$(A.4) \quad \mathbb{O} = \bigoplus_{i=1}^3 \mathbb{Z}_4^{\mathbf{fc}_i} \bigoplus_{j=1}^4 \mathbb{Z}_3^{\mathbf{vc}_j} \bigoplus_{l=1}^6 \mathbb{Z}_2^{\mathbf{ec}_l}$$

with *vertices*, *edges* and *faces axes* respectively denoted \mathbf{vc}_i , \mathbf{ec}_j and \mathbf{fc}_j . For instance we have

$$\mathbf{vc}_1 := (OA_1), \quad \mathbf{ec}_1 := (OM_1), \quad \mathbf{fc}_1 := (OI).$$

As an application of decomposition (A.4) and Lemma A.2, we obtain the following corollary.

Corollary A.8. *Let $n \geq 2$ be some integer. Set*

$$d_2(n) = \gcd(n, 2), \quad d_3(n) = \gcd(n, 3),$$

and

$$d_4(n) = \begin{cases} 4 & \text{if } 4 \mid n, \\ 1 & \text{otherwise.} \end{cases}$$

Then, we have

$$[\mathbb{Z}_n] \odot [\mathbb{O}] = \{[\mathbf{1}], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_{d_3(n)}], [\mathbb{Z}_{d_4(n)}]\}.$$

Corollary A.9. *Let $n \geq 2$ be some integer. Set*

$$d_2(n) = \gcd(n, 2), \quad d_3(n) = \gcd(n, 3),$$

and

$$d_4(n) = \begin{cases} 4 & \text{if } 4 \mid n, \\ 1 & \text{otherwise.} \end{cases}$$

Then, we have

$$[\mathbb{D}_n] \odot [\mathbb{O}] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3(n)}], [\mathbb{Z}_{d_4(n)}], [\mathbb{D}_{d_2(n)}], [\mathbb{D}_{d_3(n)}], [\mathbb{D}_{d_4(n)}]\}.$$

Proof. Using decomposition (A.4), we have to consider intersections

$$\mathbb{D}_n \cap (g\mathbb{Z}_4^{\mathbf{fc}_i}g^{-1}), \quad \mathbb{D}_n \cap (g\mathbb{Z}_3^{\mathbf{vc}_j}g^{-1}), \quad \mathbb{D}_n \cap (g\mathbb{Z}_2^{\mathbf{ec}_i}g^{-1})$$

which have already been studied in Lemma A.3. \square

Lemma A.10. *We have*

$$[\mathbb{T}] \odot [\mathbb{O}] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{T}]\}.$$

Proof. Let $\Gamma = \mathbb{O} \cap (g\mathbb{T}g^{-1})$. From decompositions (A.3)–(A.4) and Lemma A.2, we only have to consider intersections

$$\mathbb{Z}_4^{\mathbf{fc}_i} \cap (g\mathbb{Z}_2^{\mathbf{et}_j}g^{-1}), \quad \mathbb{Z}_3^{\mathbf{vc}_j} \cap (g\mathbb{Z}_3^{\mathbf{vt}_i}g^{-1}), \quad \mathbb{Z}_2^{\mathbf{ec}_i} \cap (g\mathbb{Z}_2^{\mathbf{et}_j}g^{-1}).$$

Now, we always can find g such that the intersection Γ reduces to some subgroup conjugate to $\mathbf{1}$, \mathbb{Z}_2 or \mathbb{Z}_3 and taking $g = \mathbf{Q}(\mathbf{k}, \frac{\pi}{4})$, we get that Γ is conjugate to \mathbb{D}_2 , which achieves the proof. \square

Lemma A.11. *We have*

$$[\mathbb{O}] \odot [\mathbb{O}] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_4], [\mathbb{D}_4], [\mathbb{O}]\}.$$

Proof. Consider the subgroup $\Gamma = \mathbb{O} \cap (g\mathbb{O}g^{-1}) \subset \mathbb{O}$. From the poset in Figure 1, we deduce that the conjugacy class $[\Gamma]$ belong to the following list

$$\{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_4], [\mathbb{D}_4], [\mathbb{T}], [\mathbb{O}]\}.$$

Now:

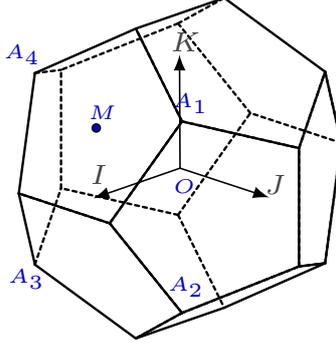
- (1) if g fixes only one edge axis (resp. one vertex axis), then $\Gamma \in [\mathbb{Z}_2]$ (resp. $\Gamma \in [\mathbb{Z}_3]$);

- (2) if $g = \mathbf{Q}\left(\mathbf{i}; \frac{\pi}{6}\right)$, only one face axis is fixed by g and $\Gamma \in [\mathbb{Z}_4]$;
- (3) if $g = \mathbf{Q}\left(\mathbf{i}; \frac{\pi}{4}\right)$, $\Gamma \supset \mathbb{Z}_4^{\mathbf{i}} \uplus \mathbb{Z}_2^{\mathbf{k}}$ and thus $\Gamma \in [\mathbb{D}_4]$;
- (4) if $g = \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{4}\right) \circ \mathbf{Q}\left(\mathbf{i}; \frac{\pi}{4}\right)$, $\Gamma \in [\mathbb{D}_2]$ with characteristic axes $g\mathbf{fc}_3 = \mathbf{ec}_6$, $g\mathbf{ec}_1 = \mathbf{fc}_1$ and $g\mathbf{ec}_2 = \mathbf{ec}_5$;
- (5) if $g = \mathbf{Q}(\mathbf{vc}_1, \pi)$, $\Gamma \in [\mathbb{D}_3]$ with \mathbf{vc}_1 as the primary axis and \mathbf{ec}_5 as the secondary axis of Γ ;
- (6) if $\Gamma \supset \mathbb{T}$, then g fixes the three edge axes of the tetrahedron, and g fix the cube. In that case, $\Gamma = \mathbb{O}$.

□

A.5. Icosahedral subgroup. The group \mathbb{I} is the (orientation-preserving) symmetry group of the dodecahedron \mathcal{D}_0 (Figure 9), where we have

- Twelve vertices: $(\pm\phi, \pm\phi^{-1}, 0), (\pm\phi^{-1}, 0, \pm\phi), (0, \pm\phi, \pm\phi^{-1})$, ϕ being the gold number.
- Eight vertices: $(\pm 1, \pm 1, \pm 1)$ of a cube.

FIGURE 9. Dodecahedron \mathcal{D}_0

We thus have the decomposition

$$(A.5) \quad \mathbb{I} = \biguplus_{i=1}^6 \mathbb{Z}_5^{\mathbf{fd}_i} \biguplus_{j=1}^{10} \mathbb{Z}_3^{\mathbf{vd}_j} \biguplus_{l=1}^{15} \mathbb{Z}_2^{\mathbf{ed}_l}$$

with *vertices*, *edges* and *faces axes* respectively denoted \mathbf{vd}_i , \mathbf{ed}_j and \mathbf{fd}_j . For instance we have

$$\mathbf{vd}_1 := (OA_1), \quad \mathbf{ed}_1 := (OI), \quad \mathbf{fd}_1 := (OM)$$

where M is the center of some face.

From decomposition (A.5) and from Lemma A.2 we obtain the following corollary.

Corollary A.12. *Let $n \geq 2$ be some integer. Set*

$$d_2 := \gcd(n, 2); \quad d_3 := \gcd(n, 3); \quad d_5 := \gcd(n, 5).$$

Then, we have

$$[\mathbb{Z}_n] \odot [\mathbb{I}] = \{[\mathbf{1}], [\mathbb{Z}_{d_2}], [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_5}]\}.$$

Using once again decomposition (A.5) and clips operation $[\mathbb{D}_n] \odot [\mathbb{Z}_m]$ in Lemma A.3 we get the following corollary.

Corollary A.13. *Let $n \geq 2$ be some integer. Set*

$$d_2 := \gcd(n, 2); \quad d_3 := \gcd(n, 3); \quad d_5 := \gcd(n, 5).$$

Then, we have

$$[\mathbb{D}_n] \odot [\mathbb{I}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_5}], [\mathbb{D}_{d_2}], [\mathbb{D}_{d_3}], [\mathbb{D}_{d_5}]\}.$$

Lemma A.14. *We have*

$$[\mathbb{I}] \odot [\mathbb{T}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{T}]\}.$$

Proof. Let $\Gamma = \mathbb{I} \cap (g\mathbb{T}g^{-1})$. From decompositions (A.5)–(A.3) and Lemma A.2, we only have to consider intersections

$$\mathbb{Z}_3^{\mathbf{vd}_j} \cap (g\mathbb{Z}_3^{\mathbf{vt}_i}g^{-1}), \quad \mathbb{Z}_2^{\mathbf{ed}_l} \cap (g\mathbb{Z}_2^{\mathbf{et}_j}g^{-1}).$$

First, note that there always exists g such that Γ reduces to a subgroup conjugate to $\mathbb{1}, \mathbb{Z}_2$ or \mathbb{Z}_3 . Now, if Γ contains a subgroup conjugate to \mathbb{D}_2 , then its three characteristic axes are edge axes of the dodecahedron: say Ox , Oy and Oz . In that case, g fixes these three axes, and also the 8 vertices of the cube \mathcal{C}_0 . The subgroup $g\mathbb{T}g^{-1}$ is thus the (orientation-preserving) symmetry group of a tetrahedron included in the dodecahedron \mathcal{D}_0 . Then, $\Gamma \in [\mathbb{T}]$. \square

The next two Lemmas are more involving.

Lemma A.15. *We have*

$$[\mathbb{O}] \odot [\mathbb{I}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{T}]\}.$$

Proof. Let $\Gamma = \mathbb{I} \cap (g\mathbb{O}g^{-1})$. From the poset in Figure 1, we deduce that the conjugacy class $[\Gamma]$ belongs to the following list

$$\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{T}]\}.$$

First, we can always find $g \in \text{SO}(3)$ such that $\Gamma \in [\mathbb{Z}_3]$, $\Gamma \in [\mathbb{Z}_2]$ or $\Gamma = \mathbb{1}$. Moreover, as in the proof of Lemma A.14, if Γ contains a subgroup conjugate to \mathbb{D}_2 , then $\Gamma \in [\mathbb{T}]$. Finally, we will exhibit some $g \in \text{SO}(3)$ such that $\Gamma \in [\mathbb{D}_3]$. First, recall that

$$A_1(1, 1, 1), \quad A_2(1, 1, -1), \quad A_4(1, -1, 1), \quad A_5(-1, -1, 1)$$

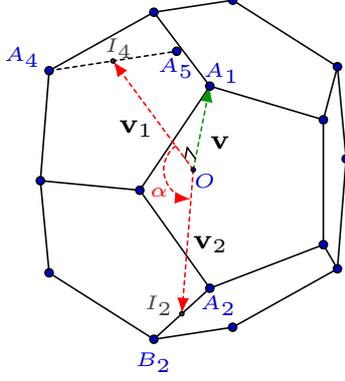
are common vertices of the cube \mathcal{C}_0 and the dodecahedron \mathcal{D}_0 . Let now $B_2(\phi^{-1}, 0, -\phi)$ be a vertex of the dodecahedron and I_2 (resp. I_4) be the middle-point of $[B_2A_2]$ (resp. $[A_4A_5]$ – see Figure 10).

Then, $a_1 = (OI_4)$ and $a_2 = (OI_2)$ are perpendicular axes to $a = (OA_1)$. Choose α such that $\mathbf{Q}(\mathbf{v}, \alpha)\mathbf{v}_1 = \mathbf{v}_2$, with (see Figure 10):

$$\mathbf{v} = \overrightarrow{OA_1}, \quad \mathbf{v}_1 = \overrightarrow{OI_4}, \quad \mathbf{v}_2 = \overrightarrow{OI_2},$$

and set $g = \mathbf{Q}(\mathbf{v}, \alpha)$. From decompositions (A.4) and (A.5), we deduce then, that $\mathbb{I} \cap (g\mathbb{O}g^{-1})$ contains the subgroups

$$\mathbb{Z}_2^{a_2} \cap (g\mathbb{Z}_2^{a_1}g^{-1}) = \mathbb{Z}_2^{a_2}, \quad \mathbb{Z}_3^a \cap (g\mathbb{Z}_3^a g^{-1}) = \mathbb{Z}_3^a.$$

FIGURE 10. Rotation g to obtain $[\mathbb{D}_3]$ in $[\mathbb{O}] \odot [\mathbb{I}]$.

Therefore, Γ contains a subgroup conjugate to \mathbb{D}_3 . Using a maximality argument (see poset in Figure 1), we must have $\Gamma \in [\mathbb{D}_3]$, and this concludes the proof. \square

Lemma A.16. *We have*

$$[\mathbb{I}] \odot [\mathbb{I}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_5], [\mathbb{D}_5], [\mathbb{I}]\}.$$

Proof. Let $\Gamma = \mathbb{I} \cap (g\mathbb{I}g^{-1})$. Considering the subclasses of $[\mathbb{I}]$, we have to check the classes

$$[\mathbb{T}], [\mathbb{D}_3], [\mathbb{D}_5], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{Z}_5], [\mathbb{Z}_2].$$

Note first, that there exist rotations g such that $\Gamma \in [\mathbb{Z}_2]$, $\Gamma \in [\mathbb{Z}_3]$, $\Gamma \in [\mathbb{Z}_5]$ or $\Gamma = \mathbb{1}$.

When Γ contains a subgroup conjugate to \mathbb{T} or \mathbb{D}_2 , using the same argument as in the proof of Lemma A.14, g fixes all the dodecahedron vertices. In that case, $\Gamma = \mathbb{1}$.

We will now exhibit some $g \in \text{SO}(3)$ such that $\Gamma \in [\mathbb{D}_3]$. Consider the dodecahedron \mathcal{D}_0 in Figure 9 and the points $A_3(1, -1, -1)$ and $B_3(\phi, -\phi^{-1}, 0)$. Let I_3 be the middle-point of $[A_3B_3]$ and g be the order two rotation around $a_1 := (OA_1)$ (see Figure 11). Let

$$b_1 := (OI_3), \quad b_2 := \mathbf{Q}\left(a_1, \frac{2\pi}{3}\right)b_1, \quad b_3 := \mathbf{Q}\left(a_1, \frac{2\pi}{3}\right)b_2.$$

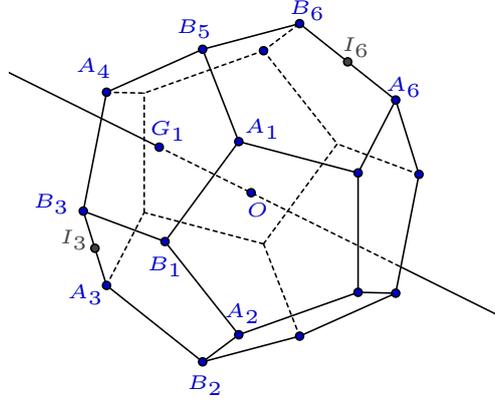
We check directly that a_1, b_i ($i = 1, \dots, 3$) are the only g -invariant characteristic axes of the dodecahedron. We deduce then, from decomposition (A.5) that Γ reduces to

$$\mathbb{Z}_3^{a_1} \left(\bigoplus_{i=1}^3 \mathbb{Z}_2^{b_i} \right) \in [\mathbb{D}_3].$$

In the same way, we can find $g \in \text{SO}(3)$, such that $\Gamma \in [\mathbb{D}_5]$. Let

$$B_1(\phi, \phi^{-1}, 0), \quad B_5(\phi^{-1}, 0, \phi), \quad B_6(-\phi^{-1}, 0, \phi), \quad A_6(-1, 1, 1)$$

be vertices of the dodecahedron \mathcal{D}_0 . Let G_1 be the center of the pentagon $A_1B_1B_3A_4B_5$ (see Figure 11) and I_6 be the middle-point of $[B_6A_6]$. Let g

FIGURE 11. Rotation g to obtain $[\mathbb{D}_3]$ or $[\mathbb{D}_5]$ in $[\mathbb{I}] \odot [\mathbb{I}]$.

be the order two rotation around $f_1 := (OG_1)$ and set

$$c_1 := (OI_6), \quad c_{k+1} := \mathbf{Q}\left(\overrightarrow{OA_1}, \frac{2\pi}{5}\right) c_k, \quad 1 \leq k \leq 4.$$

Then we can check that f_1, c_k ($k = 1, \dots, 5$) are the only g -invariant characteristic axes of the dodecahedron. Using decomposition (A.5), we deduce then, that $\Gamma \in [\mathbb{D}_5]$, which concludes the proof. \square

A.6. Infinite subgroups. The primary axis of both $\text{SO}(2)$ and $\text{O}(2)$ is defined as the z -axis, while any perpendicular axis to (Oz) is a secondary axis for $\text{O}(2)$.

Clips operation between $\text{SO}(2)$ or $\text{O}(2)$ and finite subgroups are obtained using simple arguments on characteristic axes. The same holds for the clips $[\text{SO}(2)] \odot [\text{SO}(2)]$. To compute $[\text{O}(2)] \odot [\text{O}(2)]$, consider the subgroup $\Gamma = \text{O}(2) \cap (g\text{O}(2)g^{-1})$ for some $g \in \text{SO}(3)$.

- (1) If both primary axes are the same, then $\Gamma = \text{O}(2)$;
- (2) if the primary axis of $g\text{O}(2)g^{-1}$ is in the xy -plane, then $\Gamma \in [\mathbb{D}_2]$;
- (3) in all other cases, $\Gamma \in [\mathbb{Z}_2]$, where the primary axis of Γ is perpendicular to the primary axes of $\text{O}(2)$ and $g\text{O}(2)g^{-1}$.

APPENDIX B. PROOFS FOR $\text{O}(3)$

In this Appendix, we provide the details about clips operations between type III closed $\text{O}(3)$ subgroups. The proofs follow the same ideas that has been used for $\text{SO}(3)$ closed subgroups (decomposition into simpler subgroups and discussion about their characteristic axes), but most of them are unfortunately more involving.

We first recall the general structure of type III subgroups Γ of $\text{O}(3)$ (see [9] for details). For each such subgroup Γ , there exists a couple $L \subset H$ of $\text{SO}(3)$ subgroups such that $H = \pi(\Gamma)$, where

$$\pi : g \in \text{O}(3) \mapsto \det(g)g \in \text{SO}(3)$$

and $L = \text{SO}(3) \cap \Gamma$ is an indexed 2 subgroup of H . These characteristic couples are detailed in Table 3. Note that, for a given couple (L, H) , Γ can be recovered as $\Gamma = L \cap gL$, where $-g \in H \setminus L$.

Γ	H	L
\mathbb{Z}_2^-	\mathbb{Z}_2	$\mathbf{1}$
\mathbb{Z}_{2n}^-	\mathbb{Z}_{2n}	\mathbb{Z}_n
\mathbb{D}_n^v	\mathbb{D}_n	\mathbb{Z}_n
\mathbb{D}_{2n}^h	\mathbb{D}_{2n}	\mathbb{D}_n
\mathbb{O}^-	\mathbb{O}	\mathbb{T}
$\text{O}(2)^-$	$\text{O}(2)$	$\text{SO}(2)$

TABLE 3. Characteristic couples for type III subgroups

In the following, we shall use the following convention:

$$\mathbb{Z}_1^\sigma = \mathbb{Z}_1^- = \mathbb{D}_1^v = \mathbf{1}.$$

B.1. \mathbb{Z}_{2n}^- subgroups. Consider the couple $\mathbb{Z}_n \subset \mathbb{Z}_{2n}$ ($n > 1$) in Table 3, where

$$\mathbb{Z}_{2n} = \left\{ I, \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right), \mathbf{Q}\left(\mathbf{k}; \frac{2\pi}{n}\right), \dots \right\}$$

and let $\mathbf{r}_n := \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right) \in \mathbb{Z}_{2n} \setminus \mathbb{Z}_n$. Set

$$(B.1) \quad \mathbb{Z}_{2n}^- := \mathbb{Z}_n \cup (-\mathbf{r}_n \mathbb{Z}_n).$$

The *primary axis* of the subgroup \mathbb{Z}_{2n}^- is defined as the z -axis.

Remark B.1. The subgroup \mathbb{Z}_2^- is generated by $-\mathbf{Q}(\mathbf{k}, \pi)$ which is the reflection through the xy plane. If σ_b is the reflection through the plane with normal axis b , then $\mathbb{Z}_2^{\sigma_b} := \{e, \sigma_b\}$, which is conjugate to \mathbb{Z}_2^- .

We have the following lemma.

Lemma B.2. *Let $m, n \geq 2$ be two integers. Set $d := \text{gcd}(n, m)$ and*

$$\mathbf{r}_n := \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right), \quad \mathbf{r}_m := \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{m}\right).$$

The intersection $(-\mathbf{r}_n \mathbb{Z}_n) \cap (-\mathbf{r}_m \mathbb{Z}_m)$ does not reduce to \emptyset if and only if m/d and n/d are odds. In such a case, we have

$$(-\mathbf{r}_n \mathbb{Z}_n) \cap (-\mathbf{r}_m \mathbb{Z}_m) = -\mathbf{r}_d \mathbb{Z}_d, \quad \mathbf{r}_d = \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{d}\right).$$

Proof. The intersection $(-\mathbf{r}_n \mathbb{Z}_n) \cap (-\mathbf{r}_m \mathbb{Z}_m)$ differs from \emptyset , if and only if, there exist integers i, j such that

$$\frac{2i+1}{n}\pi = \frac{2j+1}{m}\pi, \quad 2i+1 \leq 2n, \quad 2j+1 \leq 2m.$$

Let $n = dn_1$ and $m = dm_1$. The preceding equation can then be recast as $(2i+1)m_1 = (2j+1)n_1$, so that

$$2i+1 = pn_1 \text{ and } 2j+1 = pm_1.$$

Thus, m_1 and n_1 are necessarily odds, in which case (recall that $\mathbb{Z}_1 = \mathbb{1}$)

$$(-\mathbf{r}_n \mathbb{Z}_n) \cap (-\mathbf{r}_m \mathbb{Z}_m) = -\mathbf{r}_d \mathbb{Z}_d, \quad \mathbf{r}_d = \mathbf{Q} \left(\mathbf{k}; \frac{\pi}{d} \right).$$

□

Corollary B.3. *Let $m, n \geq 1$ be two integers. Set $d := \gcd(n, m)$. Then, we have*

$$[\mathbb{Z}_{2n}^-] \odot [\mathbb{Z}_{2m}^-] = \begin{cases} \{[\mathbb{1}], [\mathbb{Z}_{2d}^-]\} & \text{if } n/d \text{ and } m/d \text{ are odd} \\ \{[\mathbb{1}], [\mathbb{Z}_d]\} & \text{otherwise} \end{cases}$$

Proof. Note first that all intersections reduce to $\mathbb{1}$ when the characteristic axes don't match, so we have only to consider the situation where they match. Now, by (B.1), we have only to consider the intersection

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_{2m}^- = (\mathbb{Z}_n \cap \mathbb{Z}_m) \cup ((-\mathbf{r}_n \mathbb{Z}_n) \cap (-\mathbf{r}_m \mathbb{Z}_m)).$$

By Lemma A.2, $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$ and we directly conclude using Lemma B.2. □

B.2. \mathbb{D}_n^v subgroups. Consider the couple $\mathbb{Z}_n \subset \mathbb{D}_n$ in Table 3. Recall that \mathbb{D}_n contains \mathbb{Z}_n and all the second order rotations about the b_j 's axes (see (A.2) and Figure 7). Set

$$(B.2) \quad \mathbb{D}_n^v := \mathbb{Z}_n \bigoplus_{j=1}^n \mathbb{Z}_2^{\sigma_{b_j}}.$$

Given $g \in O(3)$, the primary axis of $g\mathbb{D}_n^v g^{-1}$ is $g(Oz)$, and its secondary axes are gb_j .

Lemma B.4. *Let $n \geq 2$, $m \geq 1$ be two integers. Set $d = \gcd(n, m)$ and*

$$i(m) := 3 - \gcd(2, m) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 2, & \text{if } m \text{ is odd.} \end{cases}$$

Then, we have

$$[\mathbb{D}_n^v] \odot [\mathbb{Z}_{2m}^-] = \{[\mathbb{1}], [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_d]\}.$$

Proof. Let $\Gamma := \mathbb{D}_n^v \cap (g\mathbb{Z}_{2m}^- g^{-1})$ and

$$\mathbb{Z}_{2m}^- = \mathbb{Z}_m \cup (-\mathbf{r}_m \mathbb{Z}_m), \quad \mathbf{r}_m = \mathbf{Q} \left(\mathbf{k}; \frac{\pi}{m} \right).$$

- (1) If both primary axes of \mathbb{D}_n^v and $g\mathbb{Z}_{2m}^- g^{-1}$ (generated by $g\mathbf{k}$) match, then by decomposition (B.2) and Lemma A.2, Γ reduces to $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$.
- (2) If the primary axis of $g\mathbb{Z}_{2m}^- g^{-1}$ matches with a secondary axis of \mathbb{D}_n^v , say (Ox) , then Γ reduces to $\mathbb{Z}_2^{\sigma_{b_0}} \cap (g\mathbb{Z}_{2m}^- g^{-1})$. Such an intersection has already been studied in the clips operation $[\mathbb{Z}_2^-] \odot [\mathbb{Z}_{2m}^-]$ (see Lemma B.3).
- (3) Otherwise, $\Gamma = \mathbb{1}$, which concludes the proof. □

Lemma B.5. *Let $m, n \geq 2$ be two integers and $d = \gcd(n, m)$. Then, we have*

$$[\mathbb{D}_n^v] \odot [\mathbb{D}_m^v] = \{ \mathbf{1}, [\mathbb{Z}_2^-], [\mathbb{D}_d^v], [\mathbb{Z}_d] \}.$$

Proof. Only two cases need to be considered.

- (1) If the primary axes of \mathbb{D}_n^v and $g\mathbb{D}_m^v g^{-1}$ do not match, then we get $\mathbf{1}$.
- (2) If they have the same primary axis, by decomposition (B.2), we have to consider the intersections

$$\mathbb{Z}_n \cap \mathbb{Z}_m, \quad \mathbb{Z}_2^{\sigma_{b_j}} \cap \mathbb{Z}_2^{\sigma_{b'_k}},$$

which reduce to

$$\mathbb{Z}_d \bigcup \mathbb{Z}_2^{\sigma_{c_l}},$$

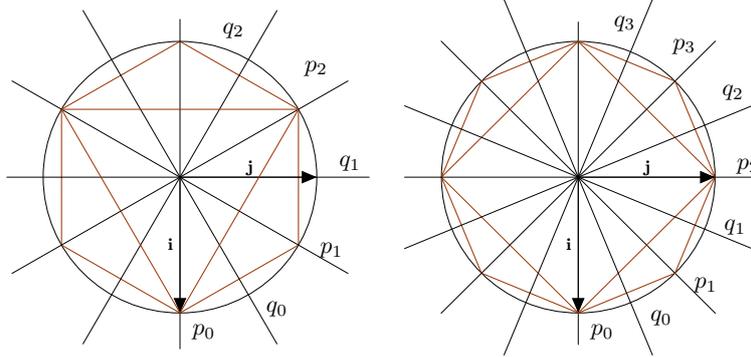
where c_l are the common secondary axis of the two subgroups. Then, we get either \mathbb{Z}_d , \mathbb{D}_d^v or a subgroup conjugate to \mathbb{Z}_2^- (when $d = 1$ and $b_0 = b'_0$), which concludes the proof. \square

B.3. \mathbb{D}_{2n}^h subgroups. Consider the couple $\mathbb{D}_n \subset \mathbb{D}_{2n}$ in Table 3. For $j = 0, \dots, n-1$, let p_j be the axis generated by

$$\mathbf{v}_j := \mathbf{Q} \left(\mathbf{k}; \frac{j\pi}{n} \right) \cdot \mathbf{i},$$

and q_j , be the axis generated by

$$\mathbf{w}_j := \mathbf{Q} \left(\mathbf{k}; \frac{(2j+1)\pi}{2n} \right) \cdot \mathbf{i}.$$



(a) Characteristic axes for $\mathbb{D}_6 \supset \mathbb{D}_3$ (b) Characteristic axes for $\mathbb{D}_8 \supset \mathbb{D}_4$

Set

$$\mathbb{D}_n = \left\{ \mathbf{1}, \mathbf{Q} \left(\mathbf{k}; \frac{2\pi}{n} \right), \mathbf{Q} \left(\mathbf{k}; \frac{4\pi}{n} \right), \dots, \mathbf{Q}(\mathbf{v}_0; \pi), \mathbf{Q}(\mathbf{v}_1; \pi), \dots \right\},$$

and

$$-\mathbf{r}_n \mathbb{D}_n = \left\{ -\mathbf{Q} \left(\mathbf{k}; \frac{\pi}{n} \right), -\mathbf{Q} \left(\mathbf{k}; \frac{3\pi}{n} \right), \dots, -\mathbf{Q}(\mathbf{w}_0; \pi), -\mathbf{Q}(\mathbf{w}_1; \pi), \dots \right\},$$

where $\mathbf{r}_n = \mathbf{Q} \left(\mathbf{k}; \frac{\pi}{n} \right)$. We define

$$(B.3) \quad \mathbb{D}_{2n}^h := \mathbb{D}_n \cup (-\mathbf{r}_n \mathbb{D}_n),$$

which decomposes as

$$(B.4) \quad \mathbb{D}_{2n}^h = \mathbb{Z}_{2n}^- \bigoplus_{j=0}^{n-1} \mathbb{Z}_2^{p_j} \bigoplus_{j=0}^{n-1} \mathbb{Z}_2^{\sigma_{q_j}}.$$

Note that in this decomposition, there are n subgroups conjugate to \mathbb{Z}_2 and n others conjugate to \mathbb{Z}_2^- . The z -axis (resp x -axis) is said to be the primary (resp. secondary) axis of \mathbb{D}_{2n}^h . For each $g \in \mathbf{O}(3)$, the primary (resp. secondary) axis of the subgroup $g\mathbb{D}_{2n}^h g^{-1}$ is generated by $g\mathbf{k}$ (resp. by $g\mathbf{i}$).

Lemma B.6. *Let $m, n \geq 2$ be two integers. Set $d = \gcd(n, m)$, $d_2(m) = \gcd(m, 2)$ and*

$$i(m) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 2, & \text{otherwise.} \end{cases}$$

Then,

- If $\frac{n}{d}$ or $\frac{m}{d}$ is even, we have

$$[\mathbb{D}_{2n}^h] \odot [\mathbb{Z}_{2m}^-] = \left\{ \mathbf{1}, [\mathbb{Z}_{d_2(m)}], [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_d] \right\};$$

- If $\frac{n}{d}$ and $\frac{m}{d}$ are odd, we have

$$[\mathbb{D}_{2n}^h] \odot [\mathbb{Z}_{2m}^-] = \left\{ \mathbf{1}, [\mathbb{Z}_{d_2(m)}], [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_{2d}^-] \right\}.$$

Proof. First of all, if no characteristic axes of \mathbb{D}_{2n}^h and $g\mathbb{Z}_{2m}^- g^{-1}$ match, then their intersection reduces to $\mathbf{1}$.

We have now to consider three cases:

(1) The first case is when \mathbb{D}_{2n}^h and $g\mathbb{Z}_{2m}^- g^{-1}$ have the same primary axis. Then, using decompositions (B.4) and (B.1), we only have to consider the intersection

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_{2m}^-.$$

This has already been studied in the clips operation $[\mathbb{Z}_{2n}^-] \odot [\mathbb{Z}_{2m}^-]$ in Lemma B.3, leading to the conjugacy class $[\mathbb{Z}_d]$ or $[\mathbb{Z}_{2d}^-]$.

(2) The second one is when some secondary axis p_j (say p_0) match with the primary axis of $g\mathbb{Z}_{2m}^- g^{-1}$, then we only have to consider intersection

$$\mathbb{Z}_2^{p_0} \cap (g\mathbb{Z}_{2m}^- g^{-1}) = \mathbb{Z}_2^{p_0} \cap (g\mathbb{Z}_m g^{-1})$$

leading to $\mathbb{Z}_2^{p_0}$ if m is even.

(2) Finally, we have to consider the case when the primary axis of $g\mathbb{Z}_{2m}^- g^{-1}$ is q_0 . In that case the problem reduces to the intersection

$$\mathbb{Z}_2^{\sigma_{q_0}} \cap (g\mathbb{Z}_{2m}^- g^{-1})$$

leading to the conjugacy class $[\mathbb{Z}_2^-]$ for odd m (see Lemma B.3). This concludes the proof. \square

The cases $[\mathbb{D}_{2n}^h] \odot [\mathbb{D}_n^v]$ and $[\mathbb{D}_{2n}^h] \odot [\mathbb{D}_{2m}^h]$ are more involving. We start by formulating the following lemma, without proof (see [Figure 7](#) for an example):

Lemma B.7. *If n is even then there exist $p_k \perp p_l$ and $q_r \perp q_s$, where no axes p_i, q_j are perpendicular. If n is odd, there exist $p_i \perp q_j$ and no axes p_k, p_l , nor q_r, q_s are perpendicular.*

Lemma B.8. *Let $m, n \geq 2$ be two integers. Set $d_2(m) := \gcd(m, 2)$,*

$$i(m, n) := \begin{cases} 2, & \text{if } m \text{ is even and } n \text{ is odd,} \\ 1, & \text{otherwise,} \end{cases}$$

and

$$i(m) := 3 - \gcd(2, m) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 2, & \text{if } m \text{ is odd.} \end{cases}$$

Then, we have

$$[\mathbb{D}_{2n}^h] \odot [\mathbb{D}_m^v] = \left\{ [\mathbb{1}], [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_{d_2(m)}], [\mathbb{D}_{i(m,n)}^v], [\mathbb{Z}_d], [\mathbb{D}_d^v] \right\}.$$

Proof. The only non trivial cases are when $g\mathbb{D}_m^v g^{-1}$ and \mathbb{D}_{2n}^h have no matching characteristic axes. Now we have to distinguish whether the principal axis a of $g\mathbb{D}_m^v g^{-1}$ is (Oz) or not:

- (1) Let first suppose that $a = (Oz)$. In that case, we need to compute the intersection $\mathbb{D}_{2n}^h \cap \mathbb{D}_m^v$. From [\(B.2\)](#) and [\(B.4\)](#), this reduces to study the following three intersections

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_m, \quad \mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b_j}}, \quad \mathbb{Z}_2^{\sigma_{a_k}} \cap \mathbb{Z}_2^{\sigma_{b_j}}.$$

Now:

- (a) The first intersection, $\mathbb{Z}_{2n}^- \cap \mathbb{Z}_m$, reduces to $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$ (from [\(B.1\)](#) and [Lemma A.2](#)).
- (b) The second one, $\mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b_j}}$, reduces to $\mathbb{1}$, since primary axes of \mathbb{Z}_{2n}^- and $\mathbb{Z}_2^{\sigma_{b_j}}$ (conjugate to \mathbb{Z}_2^-) do not match.
- (c) The last one, $\mathbb{Z}_2^{\sigma_{a_k}} \cap \mathbb{Z}_2^{\sigma_{b_j}}$, can reduce to some $\mathbb{Z}_2^{\sigma_{a_k}}$ if $b_0 = q_0$. In that case, $\mathbb{D}_{2n}^h \cap \mathbb{D}_m^v$ contains \mathbb{Z}_d and some $\mathbb{Z}_2^{\sigma_{a_k}}$, which generate \mathbb{D}_d^v (see [\(B.2\)](#)).

This first case thus leads to \mathbb{Z}_d or \mathbb{D}_d^v .

- (2) Consider now the case when $a \neq (Oz)$. Thus the intersections to be considered are

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b'_0}}, \quad \mathbb{Z}_2^{p_j} \cap \mathbb{Z}_m^a, \quad \mathbb{Z}_2^{\sigma_{a_k}} \cap \mathbb{Z}_2^{\sigma_{b'_j}}.$$

where the b'_j axes are the secondary axis of $g\mathbb{D}_m^v g^{-1}$. Now:

- (a) First suppose that $a = p_0$ (for instance) and all the other axis are different. Then, $\mathbb{Z}_2^{p_j} \cap \mathbb{Z}_m^a = \mathbb{Z}_{d_2(m)}^a$.
- (b) Suppose now that $a = p_0$ and $b'_j = q_k$ for some couple (k, j) , so that $\mathbb{Z}_2^{\sigma_{a_k}} \cap \mathbb{Z}_2^{\sigma_{b'_j}} = \mathbb{Z}_2^{\sigma_{a_k}}$. As $q_k \perp p_0$, we deduce from [Lemma B.7](#) that n is odd. All depend on m parity: if m is even then Γ contains $\mathbb{Z}_2^{p_0}$ and $\mathbb{Z}_2^{\sigma_{a_k}}$, and we obtain some subgroup conjugate to \mathbb{D}_2^v . If m is odd, then Γ reduces to $\mathbb{Z}_2^{\sigma_{a_k}}$, which is conjugate to \mathbb{Z}_2^- .

- (c) Finally, suppose that $a \neq p_j$ for all j (and recall that $a \neq (Oz)$), so that the intersections to be considered are

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b'_0}}, \quad \mathbb{Z}_2^{\sigma_{a_k}} \cap \mathbb{Z}_2^{\sigma_{b'_j}}.$$

We thus only obtain some subgroups already considered in the previous cases, which conclude the proof. \square

Lemma B.9. *Let $m, n \geq 2$ be two integers. Set $d = \gcd(n, m)$ and*

$$\Delta = [\mathbb{D}_{2n}^h] \odot [\mathbb{D}_{2m}^h].$$

Then, $[\mathbf{1}] \subset \Delta$ and:

- For every integer d :
 - ◊ If m and n are even, then $\Delta \supset \{[\mathbb{Z}_2], [\mathbb{D}_2]\}$;
 - ◊ If m and n are odds, then $\Delta \supset \{[\mathbb{Z}_2^-]\}$;
 - ◊ Otherwise, $\Delta \supset \{[\mathbb{Z}_2], [\mathbb{D}_2^y]\}$;
- If $d = 1$, then
 - ◊ If m and n are odds, then $\Delta \supset \{[\mathbb{D}_2^y]\}$;
 - ◊ Otherwise m or n is even and $\Delta \supset \{[\mathbb{Z}_2], [\mathbb{Z}_2^-]\}$;
- If $d \neq 1$, then
 - ◊ If $\frac{m}{d}$ and $\frac{n}{d}$ are odds, then $\Delta \supset \{[\mathbb{Z}_{2d}^-], [\mathbb{D}_{2d}^h]\}$;
 - ◊ Otherwise, $\frac{m}{d}$ or $\frac{n}{d}$ is even and $\Delta \supset \{[\mathbb{Z}_d], [\mathbb{D}_d], [\mathbb{D}_d^y]\}$;

Sketch of proof. We consider decomposition (B.3). If no characteristic axes \mathbb{D}_{2n}^h and $g\mathbb{D}_{2m}^h g^{-1}$ match, then their intersection reduces to $\mathbf{1}$. Otherwise, from (B.3) it reduces to

$$\begin{aligned} \mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_{2m}^- g^{-1}), \quad \mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_2^{\sigma_{a'_k}} g^{-1}), \quad \mathbb{Z}_2^{p_j} \cap (g\mathbb{Z}_2^{p'_k} g^{-1}), \\ \mathbb{Z}_2^{\sigma_{a_j}} \cap (g\mathbb{Z}_{2m}^- g^{-1}), \quad \mathbb{Z}_2^{\sigma_{a_j}} \cap (g\mathbb{Z}_2^{\sigma_{a'_k}} g^{-1}), \end{aligned}$$

where all $\mathbb{Z}_2^{\sigma_{a_j}}, \mathbb{Z}_2^{\sigma_{a'_k}}$ are subgroups conjugate to \mathbb{Z}_2^- .

Now all these intersections have already been studied in the clips operation $[\mathbb{Z}_{2r}^-] \odot [\mathbb{Z}_{2s}^-]$. We can thus use Lemma B.3, argue on the characteristic axes and Lemma B.7, to conclude the proof in each case. \square

B.4. \mathbb{O}^- subgroup. Consider the couple $\mathbb{T} \subset \mathbb{O}$ in Table 3 and the following decompositions

$$\mathbb{O} = \biguplus_{i=1}^3 \mathbb{Z}_4^{\mathbf{fc}_i} \biguplus_{j=1}^4 \mathbb{Z}_3^{\mathbf{vc}_j} \biguplus_{l=1}^6 \mathbb{Z}_2^{\mathbf{ec}_l},$$

and

$$\mathbb{T} = \biguplus_{j=1}^4 \mathbb{Z}_3^{\mathbf{vt}_j} \biguplus \mathbb{Z}_2^{\mathbf{et}_1} \biguplus \mathbb{Z}_2^{\mathbf{et}_2} \biguplus \mathbb{Z}_2^{\mathbf{et}_3}, \quad \mathbb{Z}_2^{\mathbf{et}_i} \subset \mathbb{Z}_4^{\mathbf{fc}_i}, \quad i = 1, 2, 3.$$

This leads (see [9] for details) to the decomposition

$$(B.5) \quad \mathbb{O}^- := \biguplus_{i=1}^3 (\mathbb{Z}_4^{\mathbf{fc}_i})^- \biguplus_{j=1}^4 \mathbb{Z}_3^{\mathbf{vc}_j} \biguplus_{l=1}^6 \mathbb{Z}_2^{\sigma_{\mathbf{ec}_l}},$$

where $(\mathbb{Z}_4^{\mathbf{fc}_i})^-$ is the subgroup conjugate to \mathbb{Z}_4^- with \mathbf{fc}_i as primary axis. Note also that $\mathbb{Z}_2^{\sigma_{e_l}}$ are subgroups conjugate to \mathbb{Z}_2^- with \mathbf{ec}_l as primary axis.

Using this decomposition (B.5), and those of type III closed $O(3)$ subgroups previously mentioned directly lead to the following corollaries.

Corollary B.10. *Let $n \geq 2$ be an integer. Set $d_2(n) = \gcd(n, 2)$ and $d_3(n) = \gcd(3, n)$. Then,*

- if n is odd, we have

$$[\mathbb{O}^-] \odot [\mathbb{Z}_{2n}^-] = \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_{d_3(n)}]\};$$

- if $n = 2 + 4k$ for $k \in \mathbb{N}$, we have

$$[\mathbb{O}^-] \odot [\mathbb{Z}_{2n}^-] = \{[\mathbf{1}], [\mathbb{Z}_4^-], [\mathbb{Z}_{d_3(n)}]\};$$

- if n is even and $4 \nmid n$, we have

$$[\mathbb{O}^-] \odot [\mathbb{Z}_{2n}^-] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3(n)}]\}.$$

Moreover in all cases, we have

$$[\mathbb{O}^-] \odot [\mathbb{D}_n^v] = \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_{d_2(n)}^v]\}.$$

Corollary B.11. *Let $n \geq 2$ be an integer and $d_3(n) := \gcd(n, 3)$.*

- If n is even and $n = 2 + 4k$ for $k \in \mathbb{N}$, then we have

$$[\mathbb{O}^-] \odot [\mathbb{D}_{2n}^h] = \{[\mathbf{1}], [\mathbb{Z}_4^-], [\mathbb{D}_4^h], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v]\};$$

- if n is even and $4 \mid n$, then we have

$$[\mathbb{O}^-] \odot [\mathbb{D}_{2n}^h] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{D}_2^v], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v]\}.$$

- if n is odd, then we have

$$[\mathbb{O}^-] \odot [\mathbb{D}_{2n}^h] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{D}_2], [\mathbb{D}_2^v], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v]\}.$$

Corollary B.12. *We have*

$$[\mathbb{O}^-] \odot [\mathbb{O}^-] = \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{Z}_3]\}.$$

B.5. $O(2)^-$ subgroup. Consider the couple $SO(2) \subset O(2)$ in Table 3 and set

$$(B.6) \quad O(2)^- := SO(2) \bigoplus_{v \subset xy\text{-plane}} \mathbb{Z}_2^{\sigma_v}.$$

As $\mathbb{Z}_2^{\sigma_v}$ are subgroups conjugate to \mathbb{Z}_2^- , previous results on clips operation of $[\mathbb{Z}_2^-]$ and Type III subgroups except $[O(2)^-]$ leads to the following lemma.

Lemma B.13. *Let $n \geq 2$ be some integer. Set $d_2(n) := \gcd(2, n)$ and*

$$i(n) := 3 - \gcd(2, n) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Then, we have

$$\begin{aligned} [\mathrm{O}(2)^-] \odot [\mathbb{Z}_{2n}^-] &= \{[\mathbf{1}], [\mathbb{Z}_{i(n)}^-], [\mathbb{Z}_n]\}, \\ [\mathrm{O}(2)^-] \odot [\mathbb{D}_n^v] &= \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{D}_n^v]\}, \\ [\mathrm{O}(2)^-] \odot [\mathbb{D}_{2n}^h] &= \{[\mathbf{1}], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_2^-], [\mathbb{D}_{i(n)}^v], [\mathbb{D}_n^v]\}, \\ [\mathrm{O}(2)^-] \odot [\mathbb{O}^-] &= \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{D}_3^v], [\mathbb{D}_2^v]\}, \\ [\mathrm{O}(2)^-] \odot [\mathrm{O}(2)^-] &= \{[\mathbb{Z}_2^-], [\mathrm{O}(2)^-]\}. \end{aligned}$$

Sketch of proof. We will only focus on the clips operation $[\mathrm{O}(2)^-] \odot [\mathbb{D}_{2n}^h]$ and consider thus intersections

$$\mathrm{O}(2)^- \cap \left(g\mathbb{D}_{2n}^h g^{-1}\right), \quad g \in \mathrm{O}(3).$$

There are only two non trivial cases to work on, whether characteristic axes match or not.

- (1) If primary axes match, then, by (B.4) and (B.6), we have to consider intersections

$$(B.7) \quad \mathbb{Z}^{\sigma_v} \cap \mathbb{Z}_{2n}^-, \quad \mathrm{SO}(2) \cap \mathbb{Z}_2^{p_j}, \quad \mathbb{Z}^{\sigma_b} \cap \mathbb{Z}_2^{\sigma_{q_j}}$$

which reduce to \mathbb{D}_n^v (see decomposition (B.2)).

- (2) Suppose, moreover, that $p_0 = (Oz)$, in which case $\mathrm{SO}(2) \cap \mathbb{Z}_2^{p_j} = \mathbb{Z}_2$.
- (a) For n odd, there exists some secondary axes q_k in the xy plane (see Lemma B.7) and thus $\mathbb{Z}^{\sigma_b} \cap \mathbb{Z}_2^{\sigma_{q_j}}$ reduces to $\mathbb{Z}_2^{\sigma_{q_k}}$. Moreover, $\mathbb{Z}^{\sigma_v} \cap \mathbb{Z}_{2n}^-$ reduces to some $\mathbb{Z}_2^{\sigma_v}$ with v perpendicular to p_0 and q_j and the final result is a subgroup conjugate to \mathbb{D}_2^v .
- (b) For n even, we obtain \mathbb{Z}_2 .
- (3) If now the primary axis a of $g\mathbb{D}_{2n}^h g^{-1}$ is (Ox) , and no other characteristic axes correspond to (Oz) nor (Oy) , then intersections (B.7) reduce to $\mathbb{Z}_2^{\sigma_a} \cap (g\mathbb{Z}_{2n}^- g^{-1})$ which is conjugate to \mathbb{Z}_2^- .

□

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