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# EFFECTIVE COMPUTATION OF $SO(3)$ AND $O(3)$ LINEAR REPRESENTATIONS SYMMETRY CLASSES

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Marc Olive. EFFECTIVE COMPUTATION OF  $SO(3)$  AND  $O(3)$  LINEAR REPRESENTATIONS SYMMETRY CLASSES. 2017. hal-01582262v1

**HAL Id: hal-01582262**

**<https://hal.science/hal-01582262v1>**

Preprint submitted on 5 Sep 2017 (v1), last revised 3 Sep 2019 (v4)

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# EFFECTIVE COMPUTATION OF $SO(3)$ AND $O(3)$ LINEAR REPRESENTATIONS SYMMETRY CLASSES

M. OLIVE

ABSTRACT. We propose a general algorithm to compute all the symmetry classes of any  $SO(3)$  or  $O(3)$  linear representation. This method relies on the introduction of a binary operator between sets of conjugacy classes of closed subgroups, called the *clips*. We compute explicit tables for this operation which allows to solve definitively the problem.

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## 1. INTRODUCTION

The problem of finding the *symmetry classes* (also called *isotropy classes*) of a given Lie group linear representation is a difficult task in general, even for a compact group, where their number is known to be finite [15, 13].

Some results are known, however, in certain cases. For instance, Ihrig and Golubitsky [9] computed the symmetry classes for all *irreducible* representations of the orthogonal groups  $SO(3)$  and  $O(3)$ . Because of their utility in the effective study of *moduli spaces* of hyperelliptic curves [11], symmetry classes for irreducible representations of  $SL(2, \mathbb{C})$  were also computed [1, 12, 3, 4].

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*Date:* August 30, 2017.

2010 *Mathematics Subject Classification.* 20C35, 22C05.

*Key words and phrases.* Isotropy classes; Clips; Linear representations; Symmetry classes; Tensorial representations.

Interests for  $\text{SO}(3)$  and  $\text{O}(3)$  results in solid mechanics is well-known. Forte and Vianello [5] obtained for the first time the eight symmetry classes for the Elasticity tensor (a *reducible*  $\text{SO}(3)$ -representation). Following their approach, other results were obtained in piezoelectricity [20], photoelasticity [6] and flexoelasticity [10].

More generally, the formulation of linear constitutive laws in mechanics naturally leads to general  $\text{O}(3)$  or  $\text{SO}(3)$  *reducible* linear representations and the *determinacy of their symmetry classes*. Up to the author's knowledge, no *general effective algorithm* existed so far to solve this problem.

In this paper, we propose such an algorithm introducing a binary operation defined over the set of conjugacy classes of a given group  $G$  and that we decided to call the *clips operation*. This operation allows to compute the set of *symmetry classes*  $\mathfrak{J}(V)$  of a direct sum  $V = V_1 \oplus V_2$  of linear representations of a group  $G$ , if we know the symmetry classes for each individual representations  $\mathfrak{J}(V_1)$  and  $\mathfrak{J}(V_2)$ .

Finally, we compute clips tables for all closed subgroups of  $\text{O}(3)$  and  $\text{SO}(3)$ . Together with the results of Ihrig and Golubitsky [9], who have computed the symmetry classes for *irreducible* representations, this allows to obtain, in a finite step process, the set of symmetry classes for any *reducible*  $\text{O}(3)$  or  $\text{SO}(3)$ -representation. This method was first introduced in [16] and led to general results in continuum mechanics [17, 18], where symmetry classes for 5-th, 6-th and 7-th order tensor spaces [14] were obtained for the first time.

In [section 2](#), the theory of *clips* is introduced for a general group  $G$  and applied in the context of symmetry classes where it is shown that isotropy classes of a direct sum corresponds to the clips of their respective isotropy classes. In [section 3](#), we recall classical results on the classification of closed subgroups of  $\text{SO}(3)$  and  $\text{O}(3)$  up to conjugacy. Models for irreducible representations of  $\text{O}(3)$  and  $\text{SO}(3)$  and their symmetry classes are recalled in [section 4](#). In [section 5](#), clips tables for closed  $\text{SO}(3)$  and  $\text{O}(3)$  subgroups are exhibited. Finally, in [Appendix A](#) and [Appendix B](#), we provide all the details and proofs to obtain these tables.

## 2. A GENERAL THEORY OF CLIPS

Given a group  $G$  and a subgroup  $H$  of  $G$ , the conjugacy class of  $H$

$$[H] := \{gHg^{-1}, \quad g \in G\}$$

is a subset of  $\mathcal{P}(G)$ . We define  $\text{Conj}(G)$  to be the set of all conjugacy classes of a given group  $G$ :

$$\text{Conj}(G) := \{[H], \quad H \subset G\}.$$

Recall that, on  $\text{Conj}(G)$ , there is a pre-order relation induced by inclusion. It is defined as follows:

$$[H_1] \preceq [H_2] \quad \text{if } H_1 \text{ is conjugate to a subgroup of } H_2.$$

When restricted to the *closed subgroups* of a topological compact group, this pre-order relation becomes a *partial order* [2, Proposition 1.9] and defines the *poset* (partial ordered set) of conjugacy classes of *closed subgroups* of  $G$ .

We now define a binary operation called the *clips operation* on the set  $\text{Conj}(G)$ .

**Definition 2.1.** Given two conjugacy classes  $[H_1]$  and  $[H_2]$  of a group  $G$ , we define their *clips* as the following subset of conjugacy classes:

$$[H_1] \odot [H_2] := \{[H_1 \cap gH_2g^{-1}], \quad g \in G\}.$$

This definition immediately extends to two families (finite or infinite)  $\mathcal{F}_1$  and  $\mathcal{F}_2$  of conjugacy classes:

$$\mathcal{F}_1 \odot \mathcal{F}_2 := \bigcup_{[H_i] \in \mathcal{F}_i} [H_1] \odot [H_2].$$

This clips operation defined thus a binary operation on the set  $\mathcal{P}(\text{Conj}(G))$  which is *associative* and *commutative*. We have moreover

$$[\mathbf{1}] \odot [H] = \{[\mathbf{1}]\} \quad \text{and} \quad [G] \odot [H] = \{[H]\},$$

for every conjugacy class  $[H]$ , where  $\mathbf{1} := \{e\}$  and  $e$  is the identity element of  $G$ .

Consider now a linear representation  $(V, \rho)$  of the group  $G$ . Given  $\mathbf{v} \in V$ , its *isotropy group* (or *symmetry group*) is defined as

$$G_{\mathbf{v}} := \{g \in G, \quad g \cdot \mathbf{v} = \mathbf{v}\}$$

and its *isotropy class* is the conjugacy class  $[G_{\mathbf{v}}]$  of its isotropy group. The *isotropy classes* (or *orbit types*) of the representation  $V$  is the family of all isotropy classes of vectors  $\mathbf{v}$  in  $V$ :

$$\mathfrak{I}(V) := \{[G_{\mathbf{v}}]; \quad \mathbf{v} \in V\}.$$

The central observation is that the isotropy classes of a direct sum of representations is obtained by the clips of their respective isotropy classes.

**Lemma 2.2.** *Let  $V_1$  and  $V_2$  be two linear representations of  $G$ . Then*

$$\mathfrak{I}(V_1 \oplus V_2) = \mathfrak{I}(V_1) \odot \mathfrak{I}(V_2).$$

*Proof.* Let  $[G_{\mathbf{v}}]$  be some isotropy class for  $\mathfrak{I}(V_1 \oplus V_2)$  and write  $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$  where  $\mathbf{v}_i \in V_i$ . Note first that  $G_{\mathbf{v}_1} \cap G_{\mathbf{v}_2} \subset G_{\mathbf{v}}$ . Conversely given  $g \in G_{\mathbf{v}}$ , we get

$$g \cdot \mathbf{v} = g \cdot \mathbf{v}_1 + g \cdot \mathbf{v}_2 = \mathbf{v}, \quad g \cdot \mathbf{v}_i \in V_i,$$

and thus  $g \cdot \mathbf{v}_i = \mathbf{v}_i$ . This shows that  $G_{\mathbf{v}} = G_{\mathbf{v}_1} \cap G_{\mathbf{v}_2}$  and therefore that

$$\mathfrak{I}(V_1 \oplus V_2) \subset \mathfrak{I}(V_1) \odot \mathfrak{I}(V_2).$$

Conversely, let  $[H] = [H_1 \cap gH_2g^{-1}]$  in  $\mathfrak{I}(V_1) \odot \mathfrak{I}(V_2)$  where  $H_i = G_{\mathbf{v}_i}$  for some vectors  $\mathbf{v}_i \in V_i$ . Then, if we set

$$\mathbf{v} = \mathbf{v}_1 + g \cdot \mathbf{v}_2,$$

we have  $G_{\mathbf{v}} = H_1 \cap gH_2g^{-1}$ , as before, which shows that

$$[H_1 \cap gH_2g^{-1}] \in \mathfrak{I}(V_1 \oplus V_2)$$

and achieves the proof.  $\square$

Using this lemma, we deduce a general algorithm to obtain the isotropy classes  $\mathfrak{I}(V)$  of a finite dimensional representation of a reductive algebraic group  $G$ , provided we know:

- (1) a decomposition  $V = \bigoplus_i W_i$  into irreducible representations  $W_i$ .
- (2) the isotropy classes  $\mathfrak{I}(W_i)$  for the *irreducible representations*  $W_i$ ;
- (3) the tables of *clips operations*  $[H_1] \odot [H_2]$  between conjugacy classes of closed subgroups  $[H_i]$  of  $G$ .

In the sequel of this paper, we will apply successfully this program to the linear representations of  $\text{SO}(3)$  and  $\text{O}(3)$ .

### 3. CLOSED SUBGROUPS OF $\text{O}(3)$

Every closed subgroup of  $\text{SO}(3)$  is conjugate to one of the following list [8]

$$\text{SO}(3), \text{O}(2), \text{SO}(2), \mathbb{D}_n (n \geq 2), \mathbb{Z}_n (n \geq 2), \mathbb{T}, \mathbb{O}, \mathbb{I}, \text{ and } \mathbb{1}$$

where:

- $\text{O}(2)$  is the subgroup generated by all the rotations around the  $z$ -axis and the order 2 rotation  $r : (x, y, z) \mapsto (x, -y, -z)$  around the  $x$ -axis;
- $\text{SO}(2)$  is the subgroup of all the rotations around the  $z$ -axis;
- $\mathbb{Z}_n$  is the unique cyclic subgroup of order  $n$  of  $\text{SO}(2)$  ( $\mathbb{Z}_1 = \{I\}$ );
- $\mathbb{D}_n$  is the *dihedral* group. It is generated by  $\mathbb{Z}_n$  and  $r : (x, y, z) \mapsto (x, -y, -z)$  ( $\mathbb{D}_1 = \{I\}$ );
- $\mathbb{T}$  is the *tetrahedral* group, the (orientation-preserving) symmetry group of the tetrahedron  $\mathcal{T}_0$  defined in Figure 8. It has order 12;
- $\mathbb{O}$  is the *octahedral* group, the (orientation-preserving) symmetry group of the cube  $\mathcal{C}_0$  defined in Figure 8. It has order 24;
- $\mathbb{I}$  is the *icosahedral* group, the (orientation-preserving) symmetry group of the dodecahedron  $\mathcal{D}_0$  in Figure 9. It has order 60;
- $\mathbb{1}$  is the trivial subgroup, containing only the unit element.

The *poset* of conjugacy classes of closed subgroups of  $\text{SO}(3)$  is completely described by the following inclusion of subgroups [8]:

$$\begin{aligned} \mathbb{Z}_n &\subset \mathbb{D}_n \subset \text{O}(2) && (n \geq 2); \\ \mathbb{Z}_n &\subset \mathbb{Z}_m \text{ and } \mathbb{D}_n \subset \mathbb{D}_m, && (\text{if } n \text{ divides } m); \\ \mathbb{Z}_n &\subset \text{SO}(2) \subset \text{O}(2) && (n \geq 2); \end{aligned}$$

completed by  $[\mathbb{Z}_2] \preceq [\mathbb{D}_n]$  ( $n \geq 2$ ) and by the arrows in Figure 1 (note that an arrow between the classes  $[H_1]$  and  $[H_2]$  means that  $[H_1] \preceq [H_2]$ ), taking account of the exceptional subgroups  $\mathbb{O}, \mathbb{T}, \mathbb{I}$ .

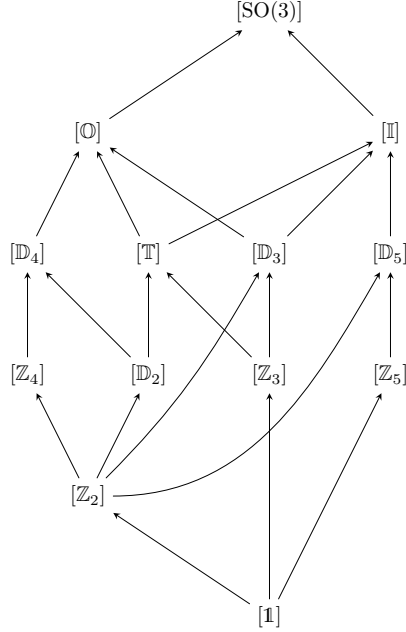
Classification of  $\text{O}(3)$ -closed subgroups is more involving [9, 19] and has been described using *three types of subgroups*. Given a closed subgroup  $\Gamma$  of  $\text{O}(3)$  this classification runs as follows.

**Type I:** A subgroup  $\Gamma$  is of type I if it is a subgroup of  $\text{SO}(3)$ ;

**Type II:** A subgroup  $\Gamma$  is of type II if  $-I \in \Gamma$ . In that case,  $\Gamma$  is generated by some subgroup  $K$  of  $\text{SO}(3)$  and  $-I$ ;

**Type III:** A subgroup  $\Gamma$  is of type III if  $-I \notin \Gamma$  and  $\Gamma$  is not a subgroup of  $\text{SO}(3)$ .

The description of type III subgroups requires more details. We will denote by  $\mathbf{Q}(\mathbf{v}; \theta) \in \text{SO}(3)$  the rotation around  $\mathbf{v} \in \mathbb{R}^3$  with angle  $\theta \in [0, 2\pi[$  and by  $\sigma_{\mathbf{v}} \in \text{O}(3)$ , the reflection through the plane normal to  $\mathbf{v}$ . Finally, we

FIGURE 1. Exceptional conjugacy classes of closed  $\text{SO}(3)$  subgroups

fix an arbitrary orthonormal frame  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ , and we introduce the following definitions.

- $\mathbb{Z}_2^-$  is the order 2 reflection group generated by  $\sigma_{\mathbf{i}}$  (where  $\mathbb{Z}_1^- = \{\mathbf{1}\}$ );
- $\mathbb{Z}_{2n}^-$  ( $n \geq 2$ ) is the group of order  $2n$ , generated by  $\mathbb{Z}_n$  and  $-\mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right)$  (see (B.1));
- $\mathbb{D}_{2n}^h$  ( $n \geq 2$ ) is the group of order  $4n$  generated by  $\mathbb{D}_n$  and  $-\mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right)$  (see (B.3));
- $\mathbb{D}_n^v$  ( $n \geq 2$ ) is the group of order  $2n$  generated by  $\mathbb{Z}_n$  and  $\sigma_{\mathbf{i}}$  (where  $\mathbb{D}_1^v = \{\mathbf{1}\}$ );
- $\text{O}(2)^-$  is generated by  $\text{SO}(2)$  and  $\sigma_{\mathbf{i}}$ .

These planar subgroups are completed by the subgroup  $\mathbb{O}^-$  which is of order 24 (see subsection B.4 and (B.5) for details).

The poset of conjugacy classes of closed subgroups of  $\text{O}(3)$  is given in Figure 2 for  $\mathbb{O}^-$  subgroups and in Figure 3 for  $\text{O}(3)$  subgroups.

#### 4. SYMMETRY CLASSES FOR IRREDUCIBLE REPRESENTATIONS

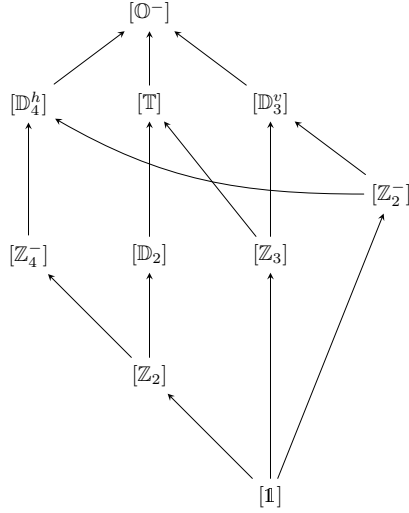
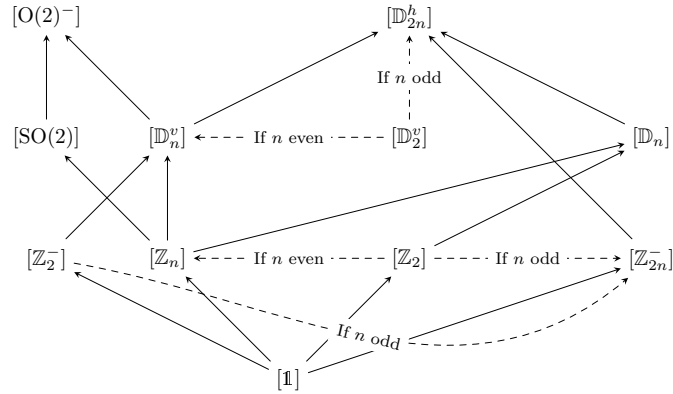
Let  $\mathcal{P}_n(\mathbb{R}^3)$  be the space of *homogeneous polynomials* of degree  $n$  on  $\mathbb{R}^3$ . We have two natural representations of  $\text{O}(3)$  on  $\mathcal{P}_n(\mathbb{R}^3)$ . The first one, noted  $\rho_n$  is given by

$$[\rho_n(p)](\mathbf{x}) := p(g^{-1}\mathbf{x}), \quad g \in G, \quad \mathbf{x} \in \mathbb{R}^3$$

whereas the second one, noted  $\rho_n^*$ , is given by

$$[\rho_n^*(p)](\mathbf{x}) := \det(g)p(g^{-1}\mathbf{x}), \quad g \in G, \quad \mathbf{x} \in \mathbb{R}^3.$$

Note that both of them induce the same representation  $\rho_n$  of  $\text{SO}(3)$ .

FIGURE 2. Poset of closed  $\mathbb{O}^-$  subgroupsFIGURE 3. Poset of closed  $\mathrm{O}(3)$  subgroups

Let  $\mathcal{H}_n(\mathbb{R}^3) \subset \mathcal{P}_n(\mathbb{R}^3)$  be the subspace of *homogeneous harmonic polynomials* of degree  $n$  (polynomials with null Laplacian). It is a classical fact [8] that  $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$  and  $(\mathcal{H}_n(\mathbb{R}^3), \rho_n^*)$  ( $n \geq 0$ ) are irreducible  $\mathrm{O}(3)$ -representations, and each irreducible  $\mathrm{O}(3)$ -representation is isomorphic to one of them. Models for irreducible representations of  $\mathrm{SO}(3)$  reduce to  $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$  ( $n \geq 0$ ).

*Remark 4.1.* Other classical models for  $\mathrm{O}(3)$  and  $\mathrm{SO}(3)$  irreducible representations, used in mechanics [5], are given by spaces of *harmonic tensors* (i.e. totally symmetric traceless tensors).

The isotropy classes for irreducible representations of  $\mathrm{SO}(3)$  and  $\mathrm{O}(3)$  were first obtained by Ihrig and Golubitsky [9].

**Theorem 4.2** (Ihrig-Golubitsky 1984). *The isotropy classes for the  $\mathrm{SO}(3)$ -representation  $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$  are:*

- (1)  $[\mathbb{1}]$  for  $n \geq 3$ ;

- (2)  $[\mathbb{Z}_k]$  for  $2 \leq k \leq n$  if  $n$  is odd and  $2 \leq k \leq n/2$  if  $n$  is even;
- (3)  $[\mathbb{D}_k]$  for  $2 \leq k \leq n$ ;
- (4)  $[\mathbb{T}]$  for  $n = 3, 6, 7$  or  $n \geq 9$ ;
- (5)  $[\mathbb{O}]$  for  $n \neq 1, 2, 3, 5, 7, 11$ ;
- (6)  $[\mathbb{I}]$  for  $n = 6, 10, 12, 15, 16, 18$  or  $n \geq 20$  and  $n \neq 23, 29$ ;
- (7)  $[\text{SO}(2)]$  for  $n$  odd;
- (8)  $[\text{O}(2)]$  for  $n$  even;
- (9)  $[\text{SO}(3)]$  for any  $n$ .

*Remark 4.3.* In [9, Theorem 6.6] (for  $\text{SO}(3)$  irreducible representations) :

- $[\mathbb{T}]$  is an isotropy class for  $n = 6, 7$  and  $n \geq 9$ ;
- $[\mathbb{O}]$  is an isotropy class for  $n \neq 1, 2, 5, 7, 11$ .

Such lists are different from (4) and (5) in our Theorem 4.2. But according to [9, Proposition 3.7],  $[\mathbb{T}]$  is a maximum isotropy class for  $n = 3$ . We have thus corrected this error in Theorem 4.2.

**Theorem 4.4** (Ihrig-Golubistky 1984). *The isotropy classes for the  $\text{O}(3)$ -representations  $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$  (for  $n$  odd) and  $(\mathcal{H}_n(\mathbb{R}^3), \rho_n^*)$  (for  $n$  even) are:*

- (1)  $[\mathbb{1}]$  for  $n \geq 3$ ;
- (2)  $[\mathbb{Z}_k]$  for  $2 \leq k \leq n/2$ ;
- (3)  $[\mathbb{Z}_{2k}^-]$  for  $k \leq \frac{n}{3}$ ;
- (4)  $[\mathbb{D}_k]$  for  $2 \leq k \leq n/2$  if  $n$  is odd and for  $2 \leq k \leq n$  if  $n$  is even.
- (5)  $[\mathbb{D}_k^v]$  for  $2 \leq k \leq n$  if  $n$  is odd and  $2 \leq k \leq n/2$  if  $n$  is even;
- (6)  $[\mathbb{D}_{2k}^h]$  for  $2 \leq k \leq n$ , except  $\mathbb{D}_4^h$  for  $n = 3$ ;
- (7)  $[\mathbb{T}]$  for  $n \neq 1, 2, 3, 5, 7, 8, 11$ ;
- (8)  $[\mathbb{O}]$  for  $n \neq 1, 2, 3, 5, 7, 11$ ;
- (9)  $[\mathbb{O}^-]$  for  $n \neq 1, 2, 4, 5, 8$ ;
- (10)  $[\mathbb{I}]$  for  $n = 6, 10, 12, 15, 16, 18$  or  $n \geq 20$  and  $n \neq 23, 29$ ;
- (11)  $[\text{O}(2)]$  when  $n$  is even ;
- (12)  $[\text{O}(2)^-]$  when  $n$  is odd.

*Remark 4.5.* In [9, Theorem 6.8] (for  $\text{O}(3)$  irreducible representations),  $[\mathbb{T}]$  is an isotropy class for  $n \neq 1, 2, 5, 7, 8, 11$  (which is different from the list (7) in our Theorem 4.4 above). But according to [7, 20],  $[\mathbb{T}]$  is not an isotropy class in the case  $n = 3$ , and we corrected this error in the list (7) of Theorem 4.4.

## 5. CLIPS TABLES

**5.1.  $\text{SO}(3)$  closed subgroups.** The resulting conjugacy classes for the clips operation of closed  $\text{SO}(3)$  subgroups are given in Table 1, where the following notations have been used:

$$\begin{aligned}
 d_2 &:= \gcd(n, 2), & k_2 &:= 3 - d_2 & d_3 &:= \gcd(n, 3), \\
 d_5 &:= \gcd(n, 5) & d_2' &:= \gcd(m, 2), \\
 dz &:= 2, \text{ if } d = 1, & dz &:= 1, \text{ otherwise,} \\
 d_4 &:= 4, \text{ if } 4 \mid n, & d_4 &:= 1, \text{ otherwise.}
 \end{aligned}$$

*Remark 5.1.* The clips operations  $[\mathbb{T}] \odot [\mathbb{T}]$  and  $[\mathbb{T}] \odot [\mathbb{O}]$  were wrong in [17, 16], since the isotropy class  $[\mathbb{D}_2]$  was omitted.



$\odot$	$[Z_n]$	$[\mathbb{D}_n]$	$[\mathbb{T}]$	$[\mathbb{O}]$	$[\mathbb{I}]$	$[\text{SO}(2)]$	$[\text{O}(2)]$
$[Z_m]$	$[\mathbf{1}], [Z_d]$						
$[\mathbb{D}_m]$	$[\mathbf{1}]$ $[Z_{d_2}]$ $[Z_d]$	$[\mathbf{1}], [Z_{d_2}]$ $[Z_{d'_2}], [Z_{dz}]$ $[Z_d], [\mathbb{D}_d]$					
$[\mathbb{T}]$	$[\mathbf{1}]$ $[Z_{d_2}]$ $[Z_{d_3}]$	$[\mathbf{1}], [Z_2]$ $[Z_{d_3}], [\mathbb{D}_{d_2}]$	$[\mathbf{1}], [Z_2]$ $[\mathbb{D}_2], [Z_3]$ $[\mathbb{T}]$				
$[\mathbb{O}]$	$[\mathbf{1}]$ $[Z_{d_2}]$ $[Z_{d_3}]$ $[Z_{d_4}]$	$[\mathbf{1}], [Z_2]$ $[Z_{d_3}], [Z_{d_4}]$ $[\mathbb{D}_{d_2}], [\mathbb{D}_{d_3}]$ $[\mathbb{D}_{d_4}]$	$[\mathbf{1}]$ $[Z_2], [\mathbb{D}_2]$ $[Z_3]$ $[\mathbb{T}]$	$[\mathbf{1}], [Z_2]$ $[\mathbb{D}_2], [Z_3]$ $[\mathbb{D}_3], [Z_4]$ $[\mathbb{D}_4], [\mathbb{O}]$			
$[\mathbb{I}]$	$[\mathbf{1}]$ $[Z_{d_2}]$ $[Z_{d_3}]$ $[Z_{d_5}]$	$[\mathbf{1}], [Z_2]$ $[Z_{d_3}], [Z_{d_5}]$ $[\mathbb{D}_{d_2}]$ $[\mathbb{D}_{d_3}], [\mathbb{D}_{d_5}]$	$[\mathbf{1}]$ $[Z_2]$ $[Z_3]$ $[\mathbb{T}]$	$[\mathbf{1}], [Z_2]$ $[Z_3], [\mathbb{D}_3]$ $[\mathbb{T}]$	$[\mathbf{1}], [Z_2]$ $[Z_3], [\mathbb{D}_3]$ $[Z_5], [\mathbb{D}_5]$ $[\mathbb{I}]$		
$[\text{SO}(2)]$	$[\mathbf{1}], [Z_n]$	$[\mathbf{1}]$ $[Z_2]$ $[Z_n]$	$[\mathbf{1}]$ $[Z_2]$ $[Z_3]$	$[\mathbf{1}], [Z_2]$ $[Z_3], [Z_4]$	$[\mathbf{1}], [Z_2]$ $[Z_3], [Z_5]$	$[\mathbf{1}], [\text{SO}(2)]$	
$[\text{O}(2)]$	$[\mathbf{1}], [Z_{d_2}]$ $[Z_n]$	$[\mathbf{1}], [Z_{d_2}]$ $[\mathbb{D}_{k_2}], [\mathbb{D}_n]$	$[\mathbf{1}], [\mathbb{D}_2]$ $[Z_3]$	$[\mathbf{1}], [\mathbb{D}_2]$ $[\mathbb{D}_3], [\mathbb{D}_4]$	$[\mathbf{1}], [\mathbb{D}_2]$ $[\mathbb{D}_3], [\mathbb{D}_5]$	$[\mathbf{1}], [Z_2]$ $[\text{SO}(2)]$	$[Z_2], [\mathbb{D}_2]$ $[\text{O}(2)]$

TABLE 1. Clips operations for  $\text{SO}(3)$ 

*Example 5.2* (Isotropy classes for a family of  $n$  vectors). For one vector, we get

$$\mathfrak{J}(\mathcal{H}_1(\mathbb{R}^3)) = \{[\text{SO}(2)], [\text{SO}(3)]\}.$$

From [Table 1](#), we deduce that the isotropy classes for a family of  $n$  vectors ( $n \geq 2$ ) is

$$\mathfrak{J}\left(\bigoplus_{k=1}^n \mathcal{H}_1(\mathbb{R}^3)\right) = \{[\mathbf{1}], [\text{SO}(2)], [\text{SO}(3)]\}.$$

*Example 5.3* (Isotropy classes for a family of  $n$  quadratic forms). The space of quadratic forms on  $\mathbb{R}^3$ ,  $S_2(\mathbb{R}^3)$ , decomposes into two irreducible components (*deviatoric* and *spherical* tensors for the mechanics):

$$S_2(\mathbb{R}^3) = \mathcal{H}_2(\mathbb{R}^3) \oplus \mathcal{H}_0(\mathbb{R}^3).$$

We get thus

$$\mathfrak{J}(S_2(\mathbb{R}^3)) = \mathfrak{J}(\mathcal{H}_2(\mathbb{R}^3)) = \{[\mathbb{D}_2], [\text{O}(2)], [\text{SO}(3)]\}.$$

The useful part of [Table 1](#), for our purpose, reads:

$\odot$	$[\mathbb{Z}_2]$	$[\mathbb{D}_2]$	$[\mathbb{O}(2)]$
$[\mathbb{Z}_2]$	$\{[\mathbf{1}], [\mathbb{Z}_2]\}$	$\{[\mathbf{1}], [\mathbb{Z}_2]\}$	$\{[\mathbf{1}], [\mathbb{Z}_2]\}$
$[\mathbb{D}_2]$		$\{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2]\}$	$\{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2]\}$
$[\mathbb{O}(2)]$			$\{[\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{O}(2)]\}$

We deduce therefore that the set of isotropy classes for a family of  $n$  quadratic forms ( $n \geq 2$ ) is

$$\mathfrak{I} \left( \bigoplus_{k=1}^n S_2(\mathbb{R}^3) \right) = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{O}(2)], [\mathbb{SO}(3)]\}.$$

**5.2.  $\mathbb{O}(3)$  closed subgroups.** Let us first consider an  $\mathbb{O}(3)$ -representation where  $-I$  act as  $-\text{Id}$  (meaning that this representation doesn't reduce to some  $\mathbb{SO}(3)$  representation). In such a case, only the null vector can be fixed by  $-\text{Id}$ , and so type II subgroups never appear as isotropy subgroups. In that case, we need only to focus on clips operations between type I and type III subgroups, and then between type III subgroups, since clips operations between type I subgroups have already been considered in [Table 1](#). For type III subgroups as detailed in [Appendix B](#) we have:

**Lemma 5.4.** *Let  $H_1$  be some type III closed subgroup of  $\mathbb{O}(3)$  and  $H_2$  be some type I closed subgroup of  $\mathbb{O}(3)$ . Then we have*

$$H_1 \cap H_2 = (H_1 \cap \mathbb{SO}(3)) \cap H_2,$$

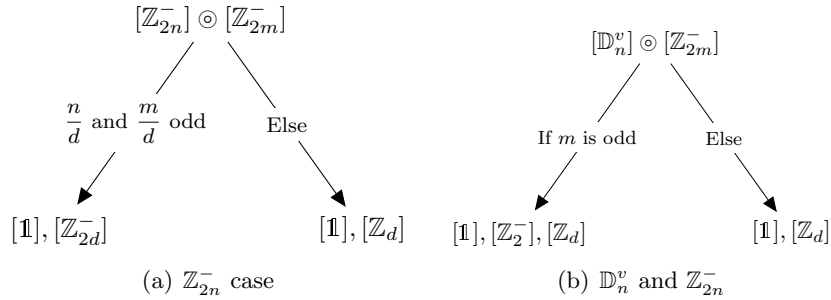
and for every closed subgroup  $H$  of  $\mathbb{SO}(3)$ , we get:

$$\begin{aligned} [\mathbb{Z}_2^-] \odot [H] &= \{[\mathbf{1}]\}, & [\mathbb{Z}_{2n}^-] \odot [H] &= [\mathbb{Z}_n] \odot [H], \\ [\mathbb{D}_n^v] \odot [H] &= [\mathbb{Z}_n] \odot [H], & [\mathbb{D}_{2n}^h] \odot [H] &= [\mathbb{D}_n] \odot [H], \\ [\mathbb{O}^-] \odot [H] &= [\mathbb{T}] \odot [H], & [\mathbb{O}(2)^-] \odot [H] &= [\mathbb{SO}(2)] \odot [H]. \end{aligned}$$

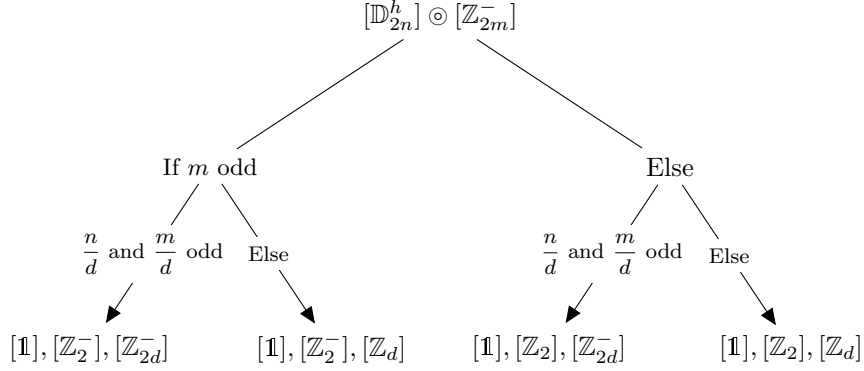
The resulting conjugacy classes for the clips operation for type III subgroups are given in [Table 2](#), where the following notations have been used:

$$\begin{aligned} d &:= \gcd(n, m), & d_2(n) &:= \gcd(n, 2), \\ d_3(n) &:= \gcd(n, 3), & i(n) &:= 3 - \gcd(2, n). \end{aligned}$$

*Remark 5.5.* One misprint in [\[16, 18\]](#) for clips operation  $[\mathbb{O}(2)^-] \odot [\mathbb{D}_{2n}^h]$  has been corrected: the conjugacy class  $[\mathbb{Z}_2]$  appears for  $m$  even (and not for  $m$  odd).



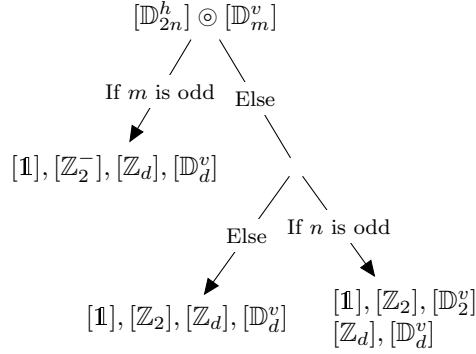
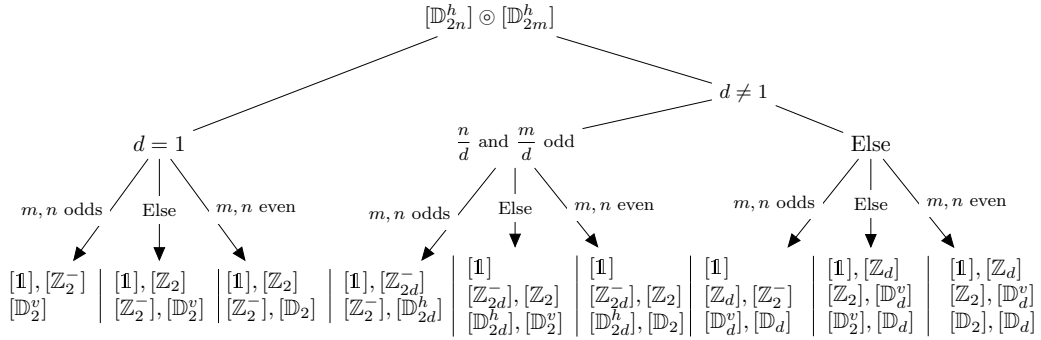
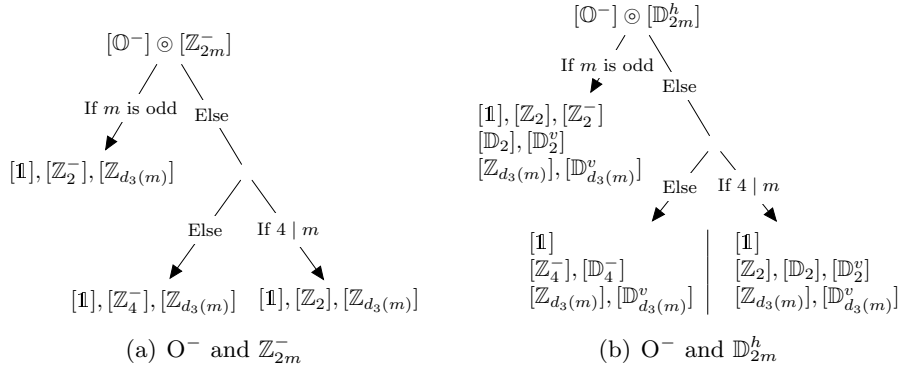
$\odot$	$[Z_2^-]$	$[Z_{2m}^-]$	$[D_m^v]$	$[D_{2m}^h]$	$[O^-]$	$[O(2)^-]$
$[Z_2^-]$	$[1], [Z_2^-]$					
$[Z_{2n}^-]$	$[1], [Z_{i(n)}^-]$	Figure 4(a)				
$[D_n^v]$	$[1], [Z_2^-]$	Figure 4(b)	$[1], [Z_2^-]$ $[D_d^v], [Z_d]$			
$[D_{2n}^h]$	$[1], [Z_2^-]$	Figure 4	Figure 5	Figure 6		
$[O^-]$	$[1], [Z_2^-]$	Figure 7(a)	$[1], [Z_2^-]$ $[Z_{d_3(m)}]$ $[D_{d_3(m)}^v]$ $[Z_{d_2(m)}]$ $[Z_{d_2(m)}^v]$	Figure 7(b)	$[1], [Z_2^-]$ $[Z_4^-], [Z_3]$	
$[O(2)^-]$	$[1], [Z_2^-]$	$[1], [Z_{i(m)}^-]$ $[Z_m]$	$[1], [Z_2^-], [D_m^v]$	$[1]$ $[Z_{d_2(m)}], [Z_2^-]$ $[D_{i(m)}^v], [D_m^v]$	$[1], [Z_2^-]$ $[D_3^v], [D_2^v]$	$[Z_2^-], [O(2)^-]$

TABLE 2. Clips operations on type III  $O(3)$ -subgroupsFIGURE 4.  $D_{2n}^h$  and  $Z_{2n}^-$ APPENDIX A. PROOFS FOR  $SO(3)$ 

In this section, we provide all the details required to obtain the results in Table 1. We will start by the following definition which was introduced in [8] and happens to be quite useful for this task.

**Definition A.1.** Let  $K_1, K_2, \dots, K_s$  be subgroups of a given group  $G$ . We say that  $G$  is the *direct union* of the  $K_i$  and we write  $G = \biguplus_{i=1}^s K_i$  if

$$G = \bigcup_{i=1}^s K_i \quad \text{and} \quad K_i \cap K_j = \{e\}, \quad \forall i \neq j.$$

FIGURE 5.  $\mathbb{D}_{2n}^h$  and  $\mathbb{D}_n^v$ FIGURE 6.  $\mathbb{D}_{2n}^h$  and  $\mathbb{D}_{2m}^h$ (a)  $O^-$  and  $Z_{2m}^-$ (b)  $O^-$  and  $\mathbb{D}_{2m}^h$ 

In the following, we will have to identify repeatedly the conjugacy class of intersections such as

$$(A.1) \quad H_1 \cap (gH_2g^{-1}),$$

where  $H_1$  and  $H_2$  are two closed subgroups of  $SO(3)$  and  $g \in SO(3)$ . A useful observation is that all closed  $SO(3)$  subgroups have some *characteristic axes* and that intersection (A.1) depends only on the relative positions of these characteristic axes.

As detailed below, for any subgroup conjugate to  $\mathbb{Z}_n$  or  $\mathbb{D}_n$  ( $n \geq 3$ ), the axis of an  $n$ -th order rotation (in this subgroup) is called its *primary axis*. For subgroups conjugate to  $\mathbb{D}_n$  ( $n \geq 3$ ), axes of order two rotations are said to be *secondary axes*. In the special case  $n = 2$ , the  $z$ -axis is the primary axis of  $\mathbb{Z}_2$ , while any of the  $x$ ,  $y$  or  $z$  axis can be considered as a primary axis of  $\mathbb{D}_2$ .

**A.1. Cyclic subgroup.** For any axis  $a$  of  $\mathbb{R}^3$  (throughout the origin), we denote by  $\mathbb{Z}_n^a$ , the unique cyclic subgroup of order  $n$  around the  $a$ -axis, which is its primary axis. We have then:

**Lemma A.2.** *Let  $m, n \geq 2$  be two integers and  $d = \gcd(n, m)$ . Then*

$$[\mathbb{Z}_n] \odot [\mathbb{Z}_m] = \{[\mathbf{1}], [\mathbb{Z}_d]\}.$$

*Proof.* We have to consider intersections, such as

$$\mathbb{Z}_n \cap (g\mathbb{Z}_m g^{-1}) = \mathbb{Z}_n \cap \mathbb{Z}_m^a,$$

for some axis  $a$ , and only two cases occur:

- (1) If  $a \neq (Oz)$ , then necessarily the intersection reduces to  $\mathbf{1}$ .
- (2) If  $a = (Oz)$ , then the order  $r$  of a rotation in  $\mathbb{Z}_n \cap \mathbb{Z}_m$  divides both  $n$  and  $m$  and thus divides  $d = \gcd(m, n)$ . We get therefore:  $\mathbb{Z}_n \cap \mathbb{Z}_m \subset \mathbb{Z}_d$ . But obviously,  $\mathbb{Z}_d \subset \mathbb{Z}_n \cap \mathbb{Z}_m$  and thus  $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$ . □

**A.2. Dihedral subgroup.** Let  $b_1$  be the  $x$ -axis and  $b_k$  ( $k = 2, \dots, n$ ) be the axis recursively defined by

$$b_k := \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right) b_{k-1}.$$

Then, we have

$$(A.2) \quad \mathbb{D}_n = \mathbb{Z}_n \uplus \mathbb{Z}_2^{b_1} \uplus \dots \uplus \mathbb{Z}_2^{b_n},$$

where the  $z$ -axis (corresponding to a  $n$ -th order rotation) is the primary axis and the  $b_k$ -axes (corresponding to order two rotations) are the secondary axes of this dihedral group (see [Figure 7](#)).

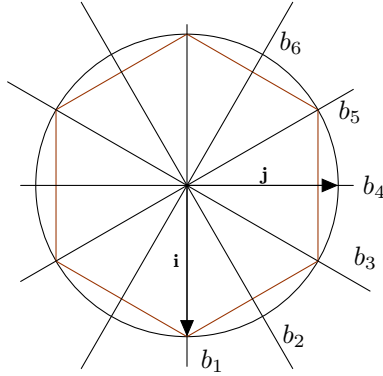


FIGURE 7. Secondary axis of the dihedral group  $\mathbb{D}_6$

**Lemma A.3.** *Let  $m, n \geq 2$  be two integers. Set  $d := \gcd(n, m)$  and  $d_2(m) := \gcd(m, 2)$ . Then, we have*

$$[\mathbb{D}_n] \odot [\mathbb{Z}_m] = \{[\mathbf{1}], [\mathbb{Z}_{d_2(m)}], [\mathbb{Z}_d]\}.$$

*Proof.* Let  $\Gamma = \mathbb{D}_n \cap \mathbb{Z}_m^a$ . From decomposition (A.2), we have to consider intersections

$$\mathbb{Z}_n \cap \mathbb{Z}_m^a, \quad \mathbb{Z}_2^{b_j} \cap \mathbb{Z}_m^a.$$

which thus reduce to Lemma A.2.  $\square$

**Lemma A.4.** *Let  $m, n \geq 2$  be two integers. Set*

$$d := \gcd(n, m), \quad d_2(m) := \gcd(m, 2), \quad d_2(n) := \gcd(n, 2),$$

and

$$dz := \begin{cases} 2 & \text{if } d = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then, we have

$$[\mathbb{D}_n] \odot [\mathbb{D}_m] = \{[\mathbf{1}], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_{d_2(m)}], [\mathbb{Z}_{dz}], [\mathbb{Z}_d], [\mathbb{D}_d]\}.$$

*Proof.* Let  $\Gamma = \mathbb{D}_n \cap (g\mathbb{D}_m g^{-1})$ .

- (1) If both primary axes and one secondary axis match,  $\Gamma = \mathbb{D}_d$  if  $d \neq 1$  and  $\Gamma \in [\mathbb{Z}_2]$  otherwise;
- (2) if only the primary axes match,  $\Gamma = \mathbb{Z}_d$ ;
- (3) finally, if the primary axis of  $\mathbb{D}_n$  matches with some secondary axis of  $g\mathbb{D}_m g^{-1}$  (or the converse),  $\Gamma \in [\mathbb{Z}_2]$  for  $n$  even and  $\Gamma = \mathbf{1}$  otherwise.

$\square$

**A.3. Tetrahedral subgroup.** The (orientation-preserving) symmetry group  $\mathbb{T}$  of the tetrahedron  $\mathcal{T}_0 := A_1 A_3 A_7 A_5$  (see Figure 8) decomposes as (see [9]):

$$(A.3) \quad \mathbb{T} = \bigoplus_{i=1}^4 \mathbb{Z}_3^{\mathbf{vt}_i} \bigoplus_{j=1}^3 \mathbb{Z}_2^{\mathbf{et}_j}$$

where  $\mathbf{vt}_i$  (resp.  $\mathbf{et}_j$ ) are the *vertices axes* (resp. *edges axes*) of the tetrahedron (see Figure 8):

$$\begin{aligned} \mathbf{vt}_1 &:= (OA_1), & \mathbf{vt}_2 &:= (OA_3), & \mathbf{vt}_3 &:= (OA_5), & \mathbf{vt}_4 &:= (OA_7), \\ \mathbf{et}_1 &:= (Ox), & \mathbf{et}_2 &:= (Oy), & \mathbf{et}_3 &:= (Oz). \end{aligned}$$

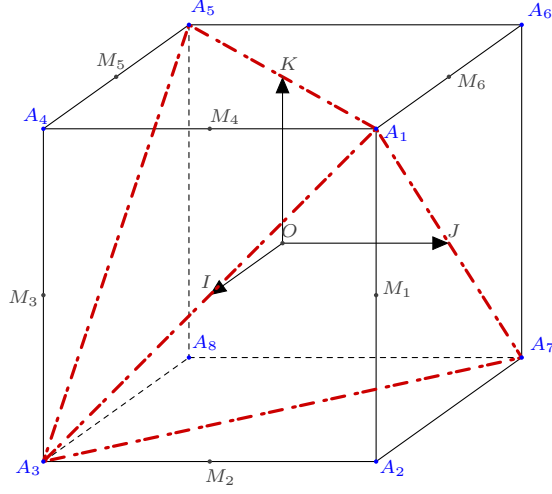
**Corollary A.5.** *Let  $n \geq 2$  be an integer. Set  $d_2(n) := \gcd(n, 2)$  and  $d_3(n) := \gcd(3, n)$ . Then, we have*

$$[\mathbb{Z}_n] \odot [\mathbb{T}] = \{[\mathbf{1}], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_{d_3(n)}]\}.$$

*Proof.* Consider  $\mathbb{T} \cap \mathbb{Z}_n^a$  for some axis  $a$ . As a consequence of Lemma A.2, we need only to consider the case where  $a$  is an edge axis or a face axis of the tetrahedron, reducing to the clips operations

$$[\mathbb{Z}_2] \odot [\mathbb{Z}_n], \quad [\mathbb{Z}_3] \odot [\mathbb{Z}_n]$$

which directly leads to the Lemma.  $\square$

FIGURE 8. Cube  $\mathcal{C}_0$  and tetrahedron  $\mathcal{T}_0 := A_1A_3A_7A_5$ 

**Corollary A.6.** *Let  $n \geq 2$  be some integer. Set  $d_2(n) := \gcd(n, 2)$  and  $d_3(n) := \gcd(3, n)$ . Then, we have*

$$[\mathbb{D}_n] \odot [\mathbb{T}] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_2(n)}]\}.$$

*Proof.* Let  $\Gamma = \mathbb{T} \cap (g\mathbb{D}_ng^{-1})$ . From decomposition (A.3), we need only to consider intersections

$$\mathbb{Z}_3^{\mathbf{vt}_i} \cap (g\mathbb{D}_ng^{-1}) \text{ and } \mathbb{Z}_2^{\mathbf{et}_j} \cap (g\mathbb{D}_ng^{-1})$$

which have already been studied (see Lemma A.3).  $\square$

**Lemma A.7.** *We have*

$$[\mathbb{T}] \odot [\mathbb{T}] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{T}]\}.$$

*Proof.* Let  $\Gamma = \mathbb{T} \cap (g\mathbb{T}g^{-1})$ .

- (1) If no axes match, then  $\Gamma = \mathbf{1}$ ;
- (2) if only one edge axis (resp. one face axis) from both configurations match, then  $\Gamma \in [\mathbb{Z}_2]$  (resp.  $[\mathbb{Z}_3]$ );
- (3) if  $g = \mathbf{Q}(\mathbf{k}, \frac{\pi}{2})$ , then  $\Gamma = \mathbb{D}_2$ .

$\square$

**A.4. Octahedral subgroup.** The group  $\mathbb{O}$  is the (orientation-preserving) symmetry group of the cube  $\mathcal{C}_0$  (see Figure 8) with vertices

$$\{A_i\}_{i=1\dots 8} = (\pm 1, \pm 1, \pm 1).$$

We have the decomposition (see [9]):

$$(A.4) \quad \mathbb{O} = \bigoplus_{i=1}^3 \mathbb{Z}_4^{\mathbf{fc}_i} \bigoplus_{j=1}^4 \mathbb{Z}_3^{\mathbf{vc}_j} \bigoplus_{l=1}^6 \mathbb{Z}_2^{\mathbf{ec}_l}$$

with *vertices*, *edges* and *faces axes* respectively denoted  $\mathbf{vc}_i$ ,  $\mathbf{ec}_j$  and  $\mathbf{fc}_j$ . For instance we have

$$\mathbf{vc}_1 := (OA_1), \quad \mathbf{ec}_1 := (OM_1), \quad \mathbf{fc}_1 := (OI).$$

As an application of decomposition (A.4) and Lemma A.2, we obtain the following corollary.

**Corollary A.8.** *Let  $n \geq 2$  be some integer. Set*

$$d_2(n) = \gcd(n, 2), \quad d_3(n) = \gcd(n, 3),$$

and

$$d_4(n) = \begin{cases} 4 & \text{if } 4 \mid n, \\ 1 & \text{otherwise.} \end{cases}$$

Then, we have

$$[\mathbb{Z}_n] \odot [\mathbb{O}] = \{[\mathbf{1}], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_{d_3(n)}], [\mathbb{Z}_{d_4(n)}]\}.$$

**Corollary A.9.** *Let  $n \geq 2$  be some integer. Set*

$$d_2(n) = \gcd(n, 2), \quad d_3(n) = \gcd(n, 3),$$

and

$$d_4(n) = \begin{cases} 4 & \text{if } 4 \mid n, \\ 1 & \text{otherwise.} \end{cases}$$

Then, we have

$$[\mathbb{D}_n] \odot [\mathbb{O}] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3(n)}], [\mathbb{Z}_{d_4(n)}], [\mathbb{D}_{d_2(n)}], [\mathbb{D}_{d_3(n)}], [\mathbb{D}_{d_4(n)}]\}.$$

*Proof.* Using decomposition (A.4), we have to consider intersections

$$\mathbb{D}_n \cap (g\mathbb{Z}_4^{\mathbf{fc}_i}g^{-1}), \quad \mathbb{D}_n \cap (g\mathbb{Z}_3^{\mathbf{vc}_j}g^{-1}), \quad \mathbb{D}_n \cap (g\mathbb{Z}_2^{\mathbf{ec}_i}g^{-1})$$

which have already been studied in Lemma A.3.  $\square$

**Lemma A.10.** *We have*

$$[\mathbb{T}] \odot [\mathbb{O}] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{T}]\}.$$

*Proof.* Let  $\Gamma = \mathbb{O} \cap (g\mathbb{T}g^{-1})$ . From decompositions (A.3)–(A.4) and Lemma A.2, we only have to consider intersections

$$\mathbb{Z}_4^{\mathbf{fc}_i} \cap (g\mathbb{Z}_2^{\mathbf{et}_j}g^{-1}), \quad \mathbb{Z}_3^{\mathbf{vc}_j} \cap (g\mathbb{Z}_3^{\mathbf{vt}_i}g^{-1}), \quad \mathbb{Z}_2^{\mathbf{ec}_i} \cap (g\mathbb{Z}_2^{\mathbf{et}_j}g^{-1}).$$

Now, we always can find  $g$  such that the intersection  $\Gamma$  reduces to some subgroup conjugate to  $\mathbf{1}$ ,  $\mathbb{Z}_2$  or  $\mathbb{Z}_3$  and taking  $g = \mathbf{Q}(\mathbf{k}, \frac{\pi}{4})$ , we get that  $\Gamma$  is conjugate to  $\mathbb{D}_2$ , which achieves the proof.  $\square$

**Lemma A.11.** *We have*

$$[\mathbb{O}] \odot [\mathbb{O}] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_4], [\mathbb{D}_4], [\mathbb{O}]\}.$$

*Proof.* Consider the subgroup  $\Gamma = \mathbb{O} \cap (g\mathbb{O}g^{-1}) \subset \mathbb{O}$ . From the poset in Figure 1, we deduce that the conjugacy class  $[\Gamma]$  belong to the following list

$$\{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_4], [\mathbb{D}_4], [\mathbb{T}], [\mathbb{O}]\}.$$

Now:

- (1) if  $g$  fixes only one edge axis (resp. one vertex axis), then  $\Gamma \in [\mathbb{Z}_2]$  (resp.  $\Gamma \in [\mathbb{Z}_3]$ );

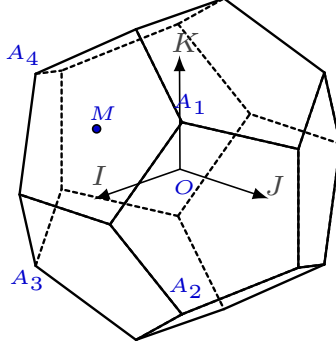


- (2) if  $g = \mathbf{Q}\left(\mathbf{i}; \frac{\pi}{6}\right)$ , only one face axis is fixed by  $g$  and  $\Gamma \in [\mathbb{Z}_4]$ ;
- (3) if  $g = \mathbf{Q}\left(\mathbf{i}; \frac{\pi}{4}\right)$ ,  $\Gamma \supset \mathbb{Z}_4^{\mathbf{i}} \uplus \mathbb{Z}_2^{\mathbf{k}}$  and thus  $\Gamma \in [\mathbb{D}_4]$ ;
- (4) if  $g = \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{4}\right) \circ \mathbf{Q}\left(\mathbf{i}; \frac{\pi}{4}\right)$ ,  $\Gamma \in [\mathbb{D}_2]$  with characteristic axes  $g\mathbf{fc}_3 = \mathbf{ec}_6$ ,  $g\mathbf{ec}_1 = \mathbf{fc}_1$  and  $g\mathbf{ec}_2 = \mathbf{ec}_5$ ;
- (5) if  $g = \mathbf{Q}(\mathbf{vc}_1, \pi)$ ,  $\Gamma \in [\mathbb{D}_3]$  with  $\mathbf{vc}_1$  as the primary axis and  $\mathbf{ec}_5$  as the secondary axis of  $\Gamma$ ;
- (6) if  $\Gamma \supset \mathbb{T}$ , then  $g$  fixes the three edge axes of the tetrahedron, and  $g$  fix the cube. In that case,  $\Gamma = \mathbb{O}$ .

□

**A.5. Icosahedral subgroup.** The group  $\mathbb{I}$  is the (orientation-preserving) symmetry group of the dodecahedron  $\mathcal{D}_0$  (Figure 9), where we have

- Twelve vertices:  $(\pm\phi, \pm\phi^{-1}, 0), (\pm\phi^{-1}, 0, \pm\phi), (0, \pm\phi, \pm\phi^{-1})$ ,  $\phi$  being the gold number.
- Eight vertices:  $(\pm 1, \pm 1, \pm 1)$  of a cube.

FIGURE 9. Dodecahedron  $\mathcal{D}_0$ 

We thus have the decomposition

$$(A.5) \quad \mathbb{I} = \bigoplus_{i=1}^6 \mathbb{Z}_5^{\mathbf{fd}_i} \bigoplus_{j=1}^{10} \mathbb{Z}_3^{\mathbf{vd}_j} \bigoplus_{l=1}^{15} \mathbb{Z}_2^{\mathbf{ed}_l}$$

with *vertices*, *edges* and *faces axes* respectively denoted  $\mathbf{vd}_i$ ,  $\mathbf{ed}_j$  and  $\mathbf{fd}_j$ . For instance we have

$$\mathbf{vd}_1 := (OA_1), \quad \mathbf{ed}_1 := (OI), \quad \mathbf{fd}_1 := (OM)$$

where  $M$  is the center of some face.

From decomposition (A.5) and from Lemma A.2 we obtain the following corollary.

**Corollary A.12.** *Let  $n \geq 2$  be some integer. Set*

$$d_2 := \gcd(n, 2); \quad d_3 := \gcd(n, 3); \quad d_5 := \gcd(n, 5).$$

*Then, we have*

$$[\mathbb{Z}_n] \odot [\mathbb{I}] = \{[\mathbf{1}], [\mathbb{Z}_{d_2}], [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_5}]\}.$$

Using once again decomposition (A.5) and clips operation  $[\mathbb{D}_n] \odot [\mathbb{Z}_m]$  in Lemma A.3 we get the following corollary.

**Corollary A.13.** *Let  $n \geq 2$  be some integer. Set*

$$d_2 := \gcd(n, 2); \quad d_3 := \gcd(n, 3); \quad d_5 := \gcd(n, 5).$$

*Then, we have*

$$[\mathbb{D}_n] \odot [\mathbb{I}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_5}], [\mathbb{D}_{d_2}], [\mathbb{D}_{d_3}], [\mathbb{D}_{d_5}]\}.$$

**Lemma A.14.** *We have*

$$[\mathbb{I}] \odot [\mathbb{T}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{T}]\}.$$

*Proof.* Let  $\Gamma = \mathbb{I} \cap (g\mathbb{T}g^{-1})$ . From decompositions (A.5)–(A.3) and Lemma A.2, we only have to consider intersections

$$\mathbb{Z}_3^{\mathbf{vd}_j} \cap (g\mathbb{Z}_3^{\mathbf{vt}_i}g^{-1}), \quad \mathbb{Z}_2^{\mathbf{ed}_l} \cap (g\mathbb{Z}_2^{\mathbf{et}_j}g^{-1}).$$

First, note that there always exists  $g$  such that  $\Gamma$  reduces to a subgroup conjugate to  $\mathbb{1}, \mathbb{Z}_2$  or  $\mathbb{Z}_3$ . Now, if  $\Gamma$  contains a subgroup conjugate to  $\mathbb{D}_2$ , then its three characteristic axes are edge axes of the dodecahedron: say  $Ox$ ,  $Oy$  and  $Oz$ . In that case,  $g$  fixes these three axes, and also the 8 vertices of the cube  $\mathcal{C}_0$ . The subgroup  $g\mathbb{T}g^{-1}$  is thus the (orientation-preserving) symmetry group of a tetrahedron included in the dodecahedron  $\mathcal{D}_0$ . Then,  $\Gamma \in [\mathbb{T}]$ .  $\square$

The next two Lemmas are more involving.

**Lemma A.15.** *We have*

$$[\mathbb{O}] \odot [\mathbb{I}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{T}]\}.$$

*Proof.* Let  $\Gamma = \mathbb{I} \cap (g\mathbb{O}g^{-1})$ . From the poset in Figure 1, we deduce that the conjugacy class  $[\Gamma]$  belongs to the following list

$$\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{T}]\}.$$

First, we can always find  $g \in \text{SO}(3)$  such that  $\Gamma \in [\mathbb{Z}_3]$ ,  $\Gamma \in [\mathbb{Z}_2]$  or  $\Gamma = \mathbb{1}$ . Moreover, as in the proof of Lemma A.14, if  $\Gamma$  contains a subgroup conjugate to  $\mathbb{D}_2$ , then  $\Gamma \in [\mathbb{T}]$ . Finally, we will exhibit some  $g \in \text{SO}(3)$  such that  $\Gamma \in [\mathbb{D}_3]$ . First, recall that

$$A_1(1, 1, 1), \quad A_2(1, 1, -1), \quad A_4(1, -1, 1), \quad A_5(-1, -1, 1)$$

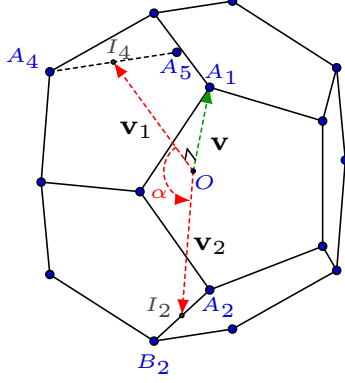
are common vertices of the cube  $\mathcal{C}_0$  and the dodecahedron  $\mathcal{D}_0$ . Let now  $B_2(\phi^{-1}, 0, -\phi)$  be a vertex of the dodecahedron and  $I_2$  (resp.  $I_4$ ) be the middle-point of  $[B_2A_2]$  (resp.  $[A_4A_5]$  – see Figure 10).

Then,  $a_1 = (OI_4)$  and  $a_2 = (OI_2)$  are perpendicular axes to  $a = (OA_1)$ . Choose  $\alpha$  such that  $\mathbf{Q}(\mathbf{v}, \alpha)\mathbf{v}_1 = \mathbf{v}_2$ , with (see Figure 10):

$$\mathbf{v} = \overrightarrow{OA_1}, \quad \mathbf{v}_1 = \overrightarrow{OI_4}, \quad \mathbf{v}_2 = \overrightarrow{OI_2},$$

and set  $g = \mathbf{Q}(\mathbf{v}, \alpha)$ . From decompositions (A.4) and (A.5), we deduce then, that  $\mathbb{I} \cap (g\mathbb{O}g^{-1})$  contains the subgroups

$$\mathbb{Z}_2^{a_2} \cap (g\mathbb{Z}_2^{a_1}g^{-1}) = \mathbb{Z}_2^{a_2}, \quad \mathbb{Z}_3^a \cap (g\mathbb{Z}_3^a g^{-1}) = \mathbb{Z}_3^a.$$

FIGURE 10. Rotation  $g$  to obtain  $[\mathbb{D}_3]$  in  $[\mathbb{O}] \odot [\mathbb{I}]$ .

Therefore,  $\Gamma$  contains a subgroup conjugate to  $\mathbb{D}_3$ . Using a maximality argument (see poset in Figure 1), we must have  $\Gamma \in [\mathbb{D}_3]$ , and this concludes the proof.  $\square$

**Lemma A.16.** *We have*

$$[\mathbb{I}] \odot [\mathbb{I}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_5], [\mathbb{D}_5], [\mathbb{I}]\}.$$

*Proof.* Let  $\Gamma = \mathbb{I} \cap (g\mathbb{I}g^{-1})$ . Considering the subclasses of  $[\mathbb{I}]$ , we have to check the classes

$$[\mathbb{T}], [\mathbb{D}_3], [\mathbb{D}_5], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{Z}_5], [\mathbb{Z}_2].$$

Note first, that there exist rotations  $g$  such that  $\Gamma \in [\mathbb{Z}_2]$ ,  $\Gamma \in [\mathbb{Z}_3]$ ,  $\Gamma \in [\mathbb{Z}_5]$  or  $\Gamma = \mathbb{1}$ .

When  $\Gamma$  contains a subgroup conjugate to  $\mathbb{T}$  or  $\mathbb{D}_2$ , using the same argument as in the proof of Lemma A.14,  $g$  fixes all the dodecahedron vertices. In that case,  $\Gamma = \mathbb{1}$ .

We will now exhibit some  $g \in \text{SO}(3)$  such that  $\Gamma \in [\mathbb{D}_3]$ . Consider the dodecahedron  $\mathcal{D}_0$  in Figure 9 and the points  $A_3(1, -1, -1)$  and  $B_3(\phi, -\phi^{-1}, 0)$ . Let  $I_3$  be the middle-point of  $[A_3B_3]$  and  $g$  be the order two rotation around  $a_1 := (OA_1)$  (see Figure 11). Let

$$b_1 := (OI_3), \quad b_2 := \mathbf{Q}\left(a_1, \frac{2\pi}{3}\right)b_1, \quad b_3 := \mathbf{Q}\left(a_1, \frac{2\pi}{3}\right)b_2.$$

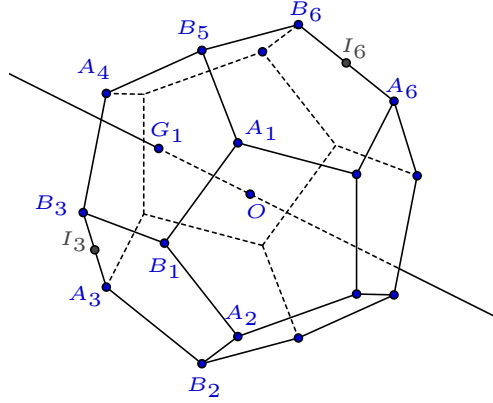
We check directly that  $a_1, b_i$  ( $i = 1, \dots, 3$ ) are the only  $g$ -invariant characteristic axes of the dodecahedron. We deduce then, from decomposition (A.5) that  $\Gamma$  reduces to

$$\mathbb{Z}_3^{a_1} \left[ \bigoplus_{i=1}^3 \mathbb{Z}_2^{b_i} \right] \in [\mathbb{D}_3].$$

In the same way, we can find  $g \in \text{SO}(3)$ , such that  $\Gamma \in [\mathbb{D}_5]$ . Let

$$B_1(\phi, \phi^{-1}, 0), \quad B_5(\phi^{-1}, 0, \phi), \quad B_6(-\phi^{-1}, 0, \phi), \quad A_6(-1, 1, 1)$$

be vertices of the dodecahedron  $\mathcal{D}_0$ . Let  $G_1$  be the center of the pentagon  $A_1B_1B_3A_4B_5$  (see Figure 11) and  $I_6$  be the middle-point of  $[B_6A_6]$ . Let  $g$

FIGURE 11. Rotation  $g$  to obtain  $[\mathbb{D}_3]$  or  $[\mathbb{D}_5]$  in  $[\mathbb{I}] \odot [\mathbb{I}]$ .

be the order two rotation around  $f_1 := (OG_1)$  and set

$$c_1 := (OI_6), \quad c_{k+1} := \mathbf{Q}\left(\overrightarrow{OA_1}, \frac{2\pi}{5}\right) c_k, \quad 1 \leq k \leq 4.$$

Then we can check that  $f_1, c_k$  ( $k = 1, \dots, 5$ ) are the only  $g$ -invariant characteristic axes of the dodecahedron. Using decomposition (A.5), we deduce then, that  $\Gamma \in [\mathbb{D}_5]$ , which concludes the proof.  $\square$

**A.6. Infinite subgroups.** The primary axis of both  $\text{SO}(2)$  and  $\text{O}(2)$  is defined as the  $z$ -axis, while any perpendicular axis to  $(Oz)$  is a secondary axis for  $\text{O}(2)$ .

Clips operation between  $\text{SO}(2)$  or  $\text{O}(2)$  and finite subgroups are obtained using simple arguments on characteristic axes. The same holds for the clips  $[\text{SO}(2)] \odot [\text{SO}(2)]$ . To compute  $[\text{O}(2)] \odot [\text{O}(2)]$ , consider the subgroup  $\Gamma = \text{O}(2) \cap (g\text{O}(2)g^{-1})$  for some  $g \in \text{SO}(3)$ .

- (1) If both primary axes are the same, then  $\Gamma = \text{O}(2)$ ;
- (2) if the primary axis of  $g\text{O}(2)g^{-1}$  is in the  $xy$ -plane, then  $\Gamma \in [\mathbb{D}_2]$ ;
- (3) in all other cases,  $\Gamma \in [\mathbb{Z}_2]$ , where the primary axis of  $\Gamma$  is perpendicular to the primary axes of  $\text{O}(2)$  and  $g\text{O}(2)g^{-1}$ .

## APPENDIX B. PROOFS FOR $\text{O}(3)$

In this Appendix, we provide the details about clips operations between type III closed  $\text{O}(3)$  subgroups. The proofs follow the same ideas that has been used for  $\text{SO}(3)$  closed subgroups (decomposition into simpler subgroups and discussion about their characteristic axes), but most of them are unfortunately more involving.

We first recall the general structure of type III subgroups  $\Gamma$  of  $\text{O}(3)$  (see [9] for details). For each such subgroup  $\Gamma$ , there exists a couple  $L \subset H$  of  $\text{SO}(3)$  subgroups such that  $H = \pi(\Gamma)$ , where

$$\pi : g \in \text{O}(3) \mapsto \det(g)g \in \text{SO}(3)$$

and  $L = \text{SO}(3) \cap \Gamma$  is an indexed 2 subgroup of  $H$ . These characteristic couples are detailed in Table 3. Note that, for a given couple  $(L, H)$ ,  $\Gamma$  can be recovered as  $\Gamma = L \cap gL$ , where  $-g \in H \setminus L$ .

$\Gamma$	$H$	$L$
$\mathbb{Z}_2^-$	$\mathbb{Z}_2$	$\mathbf{1}$
$\mathbb{Z}_{2n}^-$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_n$
$\mathbb{D}_n^v$	$\mathbb{D}_n$	$\mathbb{Z}_n$
$\mathbb{D}_{2n}^h$	$\mathbb{D}_{2n}$	$\mathbb{D}_n$
$\mathbb{O}^-$	$\mathbb{O}$	$\mathbb{T}$
$\text{O}(2)^-$	$\text{O}(2)$	$\text{SO}(2)$

TABLE 3. Characteristic couples for type III subgroups

In the following, we shall use the following convention:

$$\mathbb{Z}_1^\sigma = \mathbb{Z}_1^- = \mathbb{D}_1^v = \mathbf{1}.$$

**B.1.  $\mathbb{Z}_{2n}^-$  subgroups.** Consider the couple  $\mathbb{Z}_n \subset \mathbb{Z}_{2n}$  ( $n > 1$ ) in Table 3, where

$$\mathbb{Z}_{2n} = \left\{ I, \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right), \mathbf{Q}\left(\mathbf{k}; \frac{2\pi}{n}\right), \dots \right\}$$

and let  $\mathbf{r}_n := \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right) \in \mathbb{Z}_{2n} \setminus \mathbb{Z}_n$ . Set

$$(B.1) \quad \mathbb{Z}_{2n}^- := \mathbb{Z}_n \cup (-\mathbf{r}_n \mathbb{Z}_n).$$

The *primary axis* of the subgroup  $\mathbb{Z}_{2n}^-$  is defined as the  $z$ -axis.

*Remark B.1.* When  $n = 2$ , the subgroup  $\mathbb{Z}_2^-$  is generated by  $-\mathbf{Q}(\mathbf{k}, \pi)$  which is the reflection through the  $xy$  plane. If  $\sigma_b$  is the reflection through the plane with normal axis  $b$ , then  $\mathbb{Z}_2^{\sigma_b} := \{e, \sigma_b\}$ , which is conjugate to  $\mathbb{Z}_2^-$ .

We have the following lemma.

**Lemma B.2.** *Let  $m, n \geq 3$  be two integers. Set  $d := \text{gcd}(n, m)$  and*

$$\mathbf{r}_n := \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{n}\right), \quad \mathbf{r}_m := \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{m}\right).$$

*The intersection  $(-\mathbf{r}_n \mathbb{Z}_n) \cap (-\mathbf{r}_m \mathbb{Z}_m)$  does not reduce to  $\emptyset$  if and only if  $m/d$  and  $n/d$  are odds. In such a case, we have*

$$(-\mathbf{r}_n \mathbb{Z}_n) \cap (-\mathbf{r}_m \mathbb{Z}_m) = -\mathbf{r}_d \mathbb{Z}_d, \quad \mathbf{r}_d = \mathbf{Q}\left(\mathbf{k}; \frac{\pi}{d}\right).$$

*Proof.* The intersection  $(-\mathbf{r}_n \mathbb{Z}_n) \cap (-\mathbf{r}_m \mathbb{Z}_m)$  differs from  $\emptyset$ , if and only if, there exist integers  $i, j$  such that

$$\frac{2i+1}{n}\pi = \frac{2j+1}{m}\pi, \quad 2i+1 \leq 2n, \quad 2j+1 \leq 2m.$$

Let  $n = dn_1$  and  $m = dm_1$ . The preceding equation can then be recast as  $(2i+1)m_1 = (2j+1)n_1$ , so that

$$2i+1 = pn_1 \text{ and } 2j+1 = pm_1.$$

Thus,  $m_1$  and  $n_1$  are necessarily odds, in which case (recall that  $\mathbb{Z}_1 = \mathbf{1}$ )

$$(-\mathbf{r}_n \mathbb{Z}_n) \cap (-\mathbf{r}_m \mathbb{Z}_m) = -\mathbf{r}_d \mathbb{Z}_d, \quad \mathbf{r}_d = \mathbf{Q} \left( \mathbf{k}; \frac{\pi}{d} \right).$$

□

**Corollary B.3.** *Let  $m, n \geq 3$  be two integers. Set  $d := \gcd(n, m)$ ,*

$$i1(m, n) := \begin{cases} 2d, & \text{if } n/d \text{ and } m/d \text{ are odd,} \\ 1, & \text{otherwise,} \end{cases}$$

and

$$i2(m, n) := \begin{cases} 1, & \text{if } n/d \text{ and } m/d \text{ are odd,} \\ d, & \text{otherwise.} \end{cases}$$

Then, we have for all integers  $n \geq 1$

$$[\mathbb{Z}_{2n}^-] \odot [\mathbb{Z}_{2m}^-] = \left\{ \mathbf{1}, [\mathbb{Z}_{i1(m,n)}^-], [\mathbb{Z}_{i2(m,n)}^-] \right\},$$

and

$$[\mathbb{Z}_{2n}^-] \odot [\mathbb{Z}_2^-] = \left\{ \mathbf{1}, [\mathbb{Z}_{i1(n,1)}^-] \right\}.$$

*Proof.* Note first that all intersections reduce to  $\mathbf{1}$  when the characteristic axes don't match, so we have only to consider the situation where they match. Now, by (B.1), we have only to consider the intersection

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_{2m}^- = (\mathbb{Z}_n \cap \mathbb{Z}_m) \cup ((-\mathbf{r}_n \mathbb{Z}_n) \cap (-\mathbf{r}_m \mathbb{Z}_m)).$$

By Lemma A.2,  $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$  and we directly conclude using Lemma B.2. □

**B.2.  $\mathbb{D}_n^v$  subgroups.** Consider the couple  $\mathbb{Z}_n \subset \mathbb{D}_n$  in Table 3. Recall that  $\mathbb{D}_n$  contains  $\mathbb{Z}_n$  and all the second order rotations about the  $b_j$ 's axes (see (A.2) and Figure 7). Set

$$(B.2) \quad \mathbb{D}_n^v := \mathbb{Z}_n \bigoplus_{j=0}^{n-1} \mathbb{Z}_2^{\sigma_{b_j}}.$$

Given  $g \in O(3)$ , the primary axis of  $g\mathbb{D}_n^v g^{-1}$  is  $g(Oz)$ , and its secondary axes are  $gb_j$ .

**Lemma B.4.** *Let  $n \geq 2$ ,  $m \geq 1$  be two integers. Set  $d = \gcd(n, m)$  and*

$$i(m) := 3 - \gcd(2, m) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 2, & \text{if } m \text{ is odd.} \end{cases}$$

Then, we have

$$[\mathbb{D}_n^v] \odot [\mathbb{Z}_{2m}^-] = \left\{ \mathbf{1}, [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_d] \right\}.$$

*Proof.* Let  $\Gamma := \mathbb{D}_n^v \cap (g\mathbb{Z}_{2m}^- g^{-1})$  and

$$\mathbb{Z}_{2m}^- = \mathbb{Z}_m \cup (-\mathbf{r}_m \mathbb{Z}_m), \quad \mathbf{r}_m = \mathbf{Q} \left( \mathbf{k}; \frac{\pi}{m} \right).$$

- (1) If both primary axes of  $\mathbb{D}_n^v$  and  $g\mathbb{Z}_{2m}^- g^{-1}$  (generated by  $g\mathbf{k}$ ) match, then by decomposition (B.2) and Lemma A.2,  $\Gamma$  reduces to  $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$ .

- (2) If the primary axis of  $g\mathbb{Z}_{2m}^-g^{-1}$  matches with a secondary axis of  $\mathbb{D}_n^v$ , say  $(Ox)$ , then  $\Gamma$  reduces to  $\mathbb{Z}_2^{\sigma_{b_0}} \cap (g\mathbb{Z}_{2m}^-g^{-1})$ . Such an intersection has already been studied in the clips operation  $[\mathbb{Z}_2^-] \odot [\mathbb{Z}_{2m}^-]$  (see Lemma B.3).
- (3) Otherwise,  $\Gamma = \mathbb{1}$ , which concludes the proof. □

**Lemma B.5.** *Let  $m, n \geq 2$  be two integers and  $d = \gcd(n, m)$ . Then, we have*

$$[\mathbb{D}_n^v] \odot [\mathbb{D}_m^v] = \{ \mathbb{1}, [\mathbb{Z}_2^-], [\mathbb{D}_d^v], [\mathbb{Z}_d] \}.$$

*Proof.* Only two cases need to be considered.

- (1) If the primary axes of  $\mathbb{D}_n^v$  and  $g\mathbb{D}_m^v g^{-1}$  do not match, then we get  $\mathbb{1}$ .
- (2) If they have the same primary axis, by decomposition (B.2), we have to consider the intersections

$$\mathbb{Z}_n \cap \mathbb{Z}_m, \quad \mathbb{Z}_2^{\sigma_{b_j}} \cap \mathbb{Z}_2^{\sigma_{b'_k}},$$

which reduce to

$$\mathbb{Z}_d \uplus \mathbb{Z}_2^{\sigma_{c_l}},$$

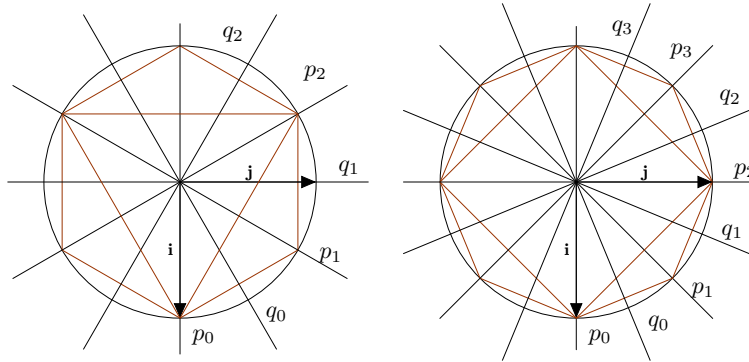
where  $c_l$  are the common secondary axis of the two subgroups. Then, we get either  $\mathbb{Z}_d$ ,  $\mathbb{D}_d^v$  or a subgroup conjugate to  $\mathbb{Z}_2^-$  (when  $d = 1$  and  $b_0 = b'_0$ ), which concludes the proof. □

**B.3.  $\mathbb{D}_{2n}^h$  subgroups.** Consider the couple  $\mathbb{D}_n \subset \mathbb{D}_{2n}$  in Table 3. For  $j = 0, \dots, n-1$ , let  $p_j$  be the axis generated by

$$\mathbf{v}_j := \mathbf{Q} \left( \mathbf{k}; \frac{j\pi}{n} \right) \cdot \mathbf{i},$$

and  $q_j$ , be the axis generated by

$$\mathbf{w}_j := \mathbf{Q} \left( \mathbf{k}; \frac{(2j+1)\pi}{2n} \right) \cdot \mathbf{i}.$$



(a) Characteristic axes for  $\mathbb{D}_6 \supset \mathbb{D}_3$  (b) Characteristic axes for  $\mathbb{D}_8 \supset \mathbb{D}_4$

Set

$$\mathbb{D}_n = \left\{ \mathbf{1}, \mathbf{Q} \left( \mathbf{k}; \frac{2\pi}{n} \right), \mathbf{Q} \left( \mathbf{k}; \frac{4\pi}{n} \right), \dots, \mathbf{Q}(\mathbf{v}_0; \pi), \mathbf{Q}(\mathbf{v}_1; \pi), \dots \right\},$$

and

$$-\mathbf{r}_n \mathbb{D}_n = \left\{ -\mathbf{Q} \left( \mathbf{k}; \frac{\pi}{n} \right), -\mathbf{Q} \left( \mathbf{k}; \frac{3\pi}{n} \right), \dots, -\mathbf{Q}(\mathbf{w}_0; \pi), -\mathbf{Q}(\mathbf{w}_1; \pi), \dots \right\},$$

where  $\mathbf{r}_n = \mathbf{Q} \left( \mathbf{k}; \frac{\pi}{n} \right)$ . We define

$$(B.3) \quad \mathbb{D}_{2n}^h := \mathbb{D}_n \cup (-\mathbf{r}_n \mathbb{D}_n),$$

which decomposes as

$$(B.4) \quad \mathbb{D}_{2n}^h = \mathbb{Z}_{2n}^- \bigoplus_{j=0}^{n-1} \mathbb{Z}_2^{p_j} \bigoplus_{j=0}^{n-1} \mathbb{Z}_2^{\sigma_{q_j}}.$$

Note that in this decomposition, there are  $n$  subgroups conjugate to  $\mathbb{Z}_2$  and  $n$  others conjugate to  $\mathbb{Z}_2^-$ . The  $z$ -axis (resp  $x$ -axis) is said to be the primary (resp. secondary) axis of  $\mathbb{D}_{2n}^h$ . For each  $g \in O(3)$ , the primary (resp. secondary) axis of the subgroup  $g\mathbb{D}_{2n}^h g^{-1}$  is generated by  $g\mathbf{k}$  (resp. by  $g\mathbf{i}$ ).

**Lemma B.6.** *Let  $m, n \geq 2$  be two integers. Set  $d = \gcd(n, m)$ ,  $d_2(m) = \gcd(m, 2)$  and*

$$i(m) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 2, & \text{otherwise.} \end{cases}$$

Then,

- If  $\frac{n}{d}$  or  $\frac{m}{d}$  is even, we have

$$[\mathbb{D}_{2n}^h] \odot [\mathbb{Z}_{2m}^-] = \left\{ \mathbf{1}, [\mathbb{Z}_{d_2(m)}], [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_d] \right\};$$

- If  $\frac{n}{d}$  and  $\frac{m}{d}$  are odd, we have

$$[\mathbb{D}_{2n}^h] \odot [\mathbb{Z}_{2m}^-] = \left\{ \mathbf{1}, [\mathbb{Z}_{d_2(m)}], [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_{2d}^-] \right\}.$$

*Proof.* First of all, if no characteristic axes of  $\mathbb{D}_{2n}^h$  and  $g\mathbb{Z}_{2m}^- g^{-1}$  match, then their intersection reduces to  $\mathbf{1}$ .

We have now to consider three cases:

(1) The first case is when  $\mathbb{D}_{2n}^h$  and  $g\mathbb{Z}_{2m}^- g^{-1}$  have the same primary axis. Then, using decompositions (B.4) and (B.1), we only have to consider the intersection

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_{2m}^-.$$

This has already been studied in the clips operation  $[\mathbb{Z}_{2n}^-] \odot [\mathbb{Z}_{2m}^-]$  in Lemma B.3, leading to the conjugacy class  $[\mathbb{Z}_d]$  or  $[\mathbb{Z}_{2d}^-]$ .

(2) The second one is when some secondary axis  $p_j$  (say  $p_0$ ) match with the primary axis of  $g\mathbb{Z}_{2m}^- g^{-1}$ , then we only have to consider intersection

$$\mathbb{Z}_2^{p_0} \cap (g\mathbb{Z}_{2m}^- g^{-1}) = \mathbb{Z}_2^{p_0} \cap (g\mathbb{Z}_m g^{-1})$$



leading to  $\mathbb{Z}_2^{p_0}$  if  $m$  is even.

(2) Finally, we have to consider the case when the primary axis of  $g\mathbb{Z}_{2m}^-g^{-1}$  is  $q_0$ . In that case the problem reduces to the intersection

$$\mathbb{Z}_2^{\sigma_{q_0}} \cap (g\mathbb{Z}_{2m}^-g^{-1})$$

leading to the conjugacy class  $[\mathbb{Z}_2^-]$  for odd  $m$  (see Lemma B.3). This concludes the proof.  $\square$

The cases  $[\mathbb{D}_{2n}^h] \odot [\mathbb{D}_n^v]$  and  $[\mathbb{D}_{2n}^h] \odot [\mathbb{D}_{2m}^h]$  are more involving. We start by formulating the following lemma, without proof (see Figure 7 for an example):

**Lemma B.7.** *If  $n$  is even then there exist  $p_k \perp p_l$  and  $q_r \perp q_s$ , where no axes  $p_i, q_j$  are perpendicular. If  $n$  is odd, there exist  $p_i \perp q_j$  and no axes  $p_k, p_l$ , nor  $q_r, q_s$  are perpendicular.*

**Lemma B.8.** *Let  $m, n \geq 2$  be two integers. Set  $d_2(m) := \gcd(m, 2)$ ,*

$$i(m, n) := \begin{cases} 2, & \text{if } m \text{ is even and } n \text{ is odd,} \\ 1, & \text{otherwise,} \end{cases}$$

and

$$i(m) := 3 - \gcd(2, m) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 2, & \text{if } m \text{ is odd.} \end{cases}$$

Then, we have

$$[\mathbb{D}_{2n}^h] \odot [\mathbb{D}_m^v] = \left\{ [\mathbb{1}], [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_{d_2(m)}], [\mathbb{D}_{i(m,n)}^v], [\mathbb{Z}_d], [\mathbb{D}_d^v] \right\}.$$

*Proof.* The only non trivial cases are when  $g\mathbb{D}_m^v g^{-1}$  and  $\mathbb{D}_{2n}^h$  have no matching characteristic axes. Now we have to distinguish whether the principal axis  $a$  of  $g\mathbb{D}_m^v g^{-1}$  is  $(Oz)$  or not:

(1) Let first suppose that  $a = (Oz)$ . In that case, we need to compute the intersection  $\mathbb{D}_{2n}^h \cap \mathbb{D}_m^v$ . From (B.2) and (B.4), this reduces to study the following three intersections

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_m, \quad \mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b_j}}, \quad \mathbb{Z}_2^{\sigma_{a_k}} \cap \mathbb{Z}_2^{\sigma_{b_j}}.$$

Now:

- (a) The first intersection,  $\mathbb{Z}_{2n}^- \cap \mathbb{Z}_m$ , reduces to  $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$  (from (B.1) and Lemma A.2).
- (b) The second one,  $\mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b_j}}$ , reduces to  $\mathbb{1}$ , since primary axes of  $\mathbb{Z}_{2n}^-$  and  $\mathbb{Z}_2^{\sigma_{b_j}}$  (conjugate to  $\mathbb{Z}_2^-$ ) do not match.
- (c) The last one,  $\mathbb{Z}_2^{\sigma_{a_k}} \cap \mathbb{Z}_2^{\sigma_{b_j}}$ , can reduce to some  $\mathbb{Z}_2^{\sigma_{a_k}}$  if  $b_0 = q_0$ . In that case,  $\mathbb{D}_{2n}^h \cap \mathbb{D}_m^v$  contains  $\mathbb{Z}_d$  and some  $\mathbb{Z}_2^{\sigma_{a_k}}$ , which generate  $\mathbb{D}_d^v$  (see (B.2)).

This first case thus leads to  $\mathbb{Z}_d$  or  $\mathbb{D}_d^v$ .

(2) Consider now the case when  $a \neq (Oz)$ . Thus the intersections to be considered are

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b'_0}}, \quad \mathbb{Z}_2^{\sigma_{p_j}} \cap \mathbb{Z}_m^a, \quad \mathbb{Z}_2^{\sigma_{a_k}} \cap \mathbb{Z}_2^{\sigma_{b'_j}}.$$

where the  $b'_j$  axes are the secondary axis of  $g\mathbb{D}_m^v g^{-1}$ . Now:

- (a) First suppose that  $a = p_0$  (for instance) and all the other axis are different. Then,  $\mathbb{Z}_2^{p_j} \cap \mathbb{Z}_m^a = \mathbb{Z}_{d_2(m)}^a$ .
- (b) Suppose now that  $a = p_0$  and  $b'j = q_k$  for some couple  $(k, j)$ , so that  $\mathbb{Z}_2^{\sigma_{q_k}} \cap \mathbb{Z}_2^{\sigma_{b'j}} = \mathbb{Z}_2^{\sigma_{q_k}}$ . As  $q_k \perp p_0$ , we deduce from Lemma B.7 that  $n$  is odd. All depend on  $m$  parity: if  $m$  is even then  $\Gamma$  contains  $\mathbb{Z}_2^{p_0}$  and  $\mathbb{Z}_2^{\sigma_{q_k}}$ , and we obtain some subgroup conjugate to  $\mathbb{D}_2^v$ . If  $m$  is odd, then  $\Gamma$  reduces to  $\mathbb{Z}_2^{\sigma_{q_k}}$ , which is conjugate to  $\mathbb{Z}_2^-$ .
- (c) Finally, suppose that  $a \neq p_j$  for all  $j$  (and recall that  $a \neq (Oz)$ ), so that the intersections to be considered are

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b'_0}}, \quad \mathbb{Z}_2^{\sigma_{q_k}} \cap \mathbb{Z}_2^{\sigma_{b'_j}}.$$

We thus only obtain some subgroups already considered in the previous cases, which conclude the proof.  $\square$

**Lemma B.9.** *Let  $m, n \geq 2$  be two integers. Set  $d = \gcd(n, m)$  and*

$$\Delta = [\mathbb{D}_{2n}^h] \odot [\mathbb{D}_{2m}^h].$$

Then,  $[\mathbf{1}] \subset \Delta$  and:

- For every integer  $d$ :
  - ◊ If  $m$  and  $n$  are even, then  $\Delta \supset \{[\mathbb{Z}_2], [\mathbb{D}_2]\}$ ;
  - ◊ If  $m$  and  $n$  are odds, then  $\Delta \supset \{[\mathbb{Z}_2^-]\}$ ;
  - ◊ Otherwise,  $\Delta \supset \{[\mathbb{Z}_2], [\mathbb{D}_2^v]\}$ ;
- If  $d = 1$ , then
  - ◊ If  $m$  and  $n$  are odds, then  $\Delta \supset \{[\mathbb{D}_2^v]\}$ ;
  - ◊ Otherwise  $m$  or  $n$  is even and  $\Delta \supset \{[\mathbb{Z}_2], [\mathbb{Z}_2^-]\}$ ;
- If  $d \neq 1$ , then
  - ◊ If  $\frac{m}{d}$  and  $\frac{n}{d}$  are odds, then  $\Delta \supset \{[\mathbb{Z}_{2d}^-], [\mathbb{D}_{2d}^h]\}$ ;
  - ◊ Otherwise,  $\frac{m}{d}$  or  $\frac{n}{d}$  is even and  $\Delta \supset \{[\mathbb{Z}_d], [\mathbb{D}_d], [\mathbb{D}_d^v]\}$ ;

*Sketch of proof.* We consider decomposition (B.3). If no characteristic axes  $\mathbb{D}_{2n}^h$  and  $g\mathbb{D}_{2m}^h g^{-1}$  match, then their intersection reduces to  $\mathbf{1}$ . Otherwise, from (B.3) it reduces to

$$\begin{aligned} \mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_{2m}^- g^{-1}), \quad \mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_2^{\sigma_{q'_k}} g^{-1}), \quad \mathbb{Z}_2^{p_j} \cap (g\mathbb{Z}_2^{p'_k} g^{-1}), \\ \mathbb{Z}_2^{\sigma_{a_j}} \cap (g\mathbb{Z}_{2m}^- g^{-1}), \quad \mathbb{Z}_2^{\sigma_{a_j}} \cap (g\mathbb{Z}_2^{\sigma_{q'_k}} g^{-1}), \end{aligned}$$

where all  $\mathbb{Z}_2^{\sigma_{a_j}}, \mathbb{Z}_2^{\sigma_{q'_k}}$  are subgroups conjugate to  $\mathbb{Z}_2^-$ .

Now all these intersections have already been studied in the clips operation  $[\mathbb{Z}_{2r}^-] \odot [\mathbb{Z}_{2s}^-]$ . We can thus use Lemma B.3, argue on the characteristic axes and Lemma B.7, to conclude the proof in each case.  $\square$

**B.4.  $\mathbb{O}^-$  subgroup.** Consider the couple  $\mathbb{T} \subset \mathbb{O}$  in Table 3 and the following decompositions

$$\mathbb{O} = \biguplus_{i=1}^3 \mathbb{Z}_4^{\text{fc}_i} \biguplus_{j=1}^4 \mathbb{Z}_3^{\text{vc}_j} \biguplus_{l=1}^6 \mathbb{Z}_2^{\text{ec}_l},$$

and

$$\mathbb{T} = \bigoplus_{j=1}^4 \mathbb{Z}_3^{\mathbf{vt}_j} \bigoplus \mathbb{Z}_2^{\mathbf{et}_1} \bigoplus \mathbb{Z}_2^{\mathbf{et}_2} \bigoplus \mathbb{Z}_2^{\mathbf{et}_3}, \quad \mathbb{Z}_2^{\mathbf{et}_i} \subset \mathbb{Z}_4^{\mathbf{fc}_i}, \quad i = 1, 2, 3.$$

This leads (see [9] for details) to the decomposition

$$(B.5) \quad \mathbb{O}^- := \bigoplus_{i=1}^3 (\mathbb{Z}_4^{\mathbf{fc}_i})^- \bigoplus_{j=1}^4 \mathbb{Z}_3^{\mathbf{vc}_j} \bigoplus_{l=1}^6 \mathbb{Z}_2^{\sigma \mathbf{ec}_l},$$

where  $(\mathbb{Z}_4^{\mathbf{fc}_i})^-$  is the subgroup conjugate to  $\mathbb{Z}_4^-$  with  $\mathbf{fc}_i$  as primary axis. Note also that  $\mathbb{Z}_2^{\sigma \mathbf{ec}_l}$  are subgroups conjugate to  $\mathbb{Z}_2^-$  with  $\mathbf{ec}_l$  as primary axis.

Using this decomposition (B.5), and those of type III closed  $O(3)$  subgroups previously mentioned directly lead to the following corollaries.

**Corollary B.10.** *Let  $n \geq 2$  be an integer. Set  $d_2(n) = \gcd(n, 2)$  and  $d_3(n) = \gcd(3, n)$ . Then,*

- if  $n$  is odd, we have

$$[\mathbb{O}^-] \odot [\mathbb{Z}_{2n}^-] = \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_{d_3(n)}]\};$$

- if  $n = 2 + 4k$  for  $k \in \mathbb{N}$ , we have

$$[\mathbb{O}^-] \odot [\mathbb{Z}_{2n}^-] = \{[\mathbf{1}], [\mathbb{Z}_4^-], [\mathbb{Z}_{d_3(n)}]\};$$

- if  $n$  is even and  $4 \nmid n$ , we have

$$[\mathbb{O}^-] \odot [\mathbb{Z}_{2n}^-] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3(n)}]\}.$$

Moreover in all cases, we have

$$[\mathbb{O}^-] \odot [\mathbb{D}_n^v] = \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_{d_2(n)}^v]\}.$$

**Corollary B.11.** *Let  $n \geq 2$  be an integer and  $d_3(n) := \gcd(n, 3)$ .*

- If  $n$  is even and  $n = 2 + 4k$  for  $k \in \mathbb{N}$ , then we have

$$[\mathbb{O}^-] \odot [\mathbb{D}_{2n}^h] = \{[\mathbf{1}], [\mathbb{Z}_4^-], [\mathbb{D}_4^h], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v]\};$$

- if  $n$  is even and  $4 \mid n$ , then we have

$$[\mathbb{O}^-] \odot [\mathbb{D}_{2n}^h] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{D}_2^v], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v]\}.$$

- if  $n$  is odd, then we have

$$[\mathbb{O}^-] \odot [\mathbb{D}_{2n}^h] = \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{D}_2], [\mathbb{D}_2^v], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v]\}.$$

**Corollary B.12.** *We have*

$$[\mathbb{O}^-] \odot [\mathbb{O}^-] = \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{Z}_3]\}.$$

B.5.  $O(2)^-$  **subgroup.** Consider the couple  $SO(2) \subset O(2)$  in Table 3 and set

$$(B.6) \quad O(2)^- := SO(2) \bigsqcup_{v \subset xy\text{-plane}} \mathbb{Z}_2^{\sigma_v}.$$

As  $\mathbb{Z}_2^{\sigma_v}$  are subgroups conjugate to  $\mathbb{Z}_2^-$ , previous results on clips operation of  $[\mathbb{Z}_2^-]$  and Type III subgroups except  $[O(2)^-]$  leads to the following lemma.

**Lemma B.13.** *Let  $n \geq 2$  be some integer. Set  $d_2(n) := \gcd(2, n)$  and*

$$i(n) := 3 - \gcd(2, n) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Then, we have

$$\begin{aligned} [O(2)^-] \odot [\mathbb{Z}_{2n}^-] &= \{[\mathbf{1}], [\mathbb{Z}_{i(n)}^-], [\mathbb{Z}_n]\}, \\ [O(2)^-] \odot [\mathbb{D}_n^v] &= \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{D}_n^v]\}, \\ [O(2)^-] \odot [\mathbb{D}_{2n}^h] &= \{[\mathbf{1}], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_2^-], [\mathbb{D}_{i(n)}^v], [\mathbb{D}_n^v]\}, \\ [O(2)^-] \odot [O^-] &= \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{D}_3^v], [\mathbb{D}_2^v]\}, \\ [O(2)^-] \odot [O(2)^-] &= \{[\mathbb{Z}_2^-], [O(2)^-]\}. \end{aligned}$$

*Sketch of proof.* We will only focus on the clips operation  $[O(2)^-] \odot [\mathbb{D}_{2n}^h]$  and consider thus intersections

$$O(2)^- \cap \left( g\mathbb{D}_{2n}^h g^{-1} \right), \quad g \in O(3).$$

There are only two non trivial cases to work on, whether characteristic axes match or not.

- (1) If primary axes match, then, by (B.4) and (B.6), we have to consider intersections

$$(B.7) \quad \mathbb{Z}_2^{\sigma_v} \cap \mathbb{Z}_{2n}^-, \quad SO(2) \cap \mathbb{Z}_2^{p_j}, \quad \mathbb{Z}_2^{\sigma_b} \cap \mathbb{Z}_2^{\sigma_{q_j}}$$

which reduce to  $\mathbb{D}_n^v$  (see decomposition (B.2)).

- (2) Suppose, moreover, that  $p_0 = (Oz)$ , in which case  $SO(2) \cap \mathbb{Z}_2^{p_j} = \mathbb{Z}_2$ .
  - (a) For  $n$  odd, there exists some secondary axes  $q_k$  in the  $xy$  plane (see Lemma B.7) and thus  $\mathbb{Z}_2^{\sigma_b} \cap \mathbb{Z}_2^{\sigma_{q_j}}$  reduces to  $\mathbb{Z}_2^{\sigma_{q_k}}$ . Moreover,  $\mathbb{Z}_2^{\sigma_v} \cap \mathbb{Z}_{2n}^-$  reduces to some  $\mathbb{Z}_2^{\sigma_v}$  with  $v$  perpendicular to  $p_0$  and  $q_j$  and the final result is a subgroup conjugate to  $\mathbb{D}_2^v$ .
  - (b) For  $n$  even, we obtain  $\mathbb{Z}_2$ .
- (3) If now the primary axis  $a$  of  $g\mathbb{D}_{2n}^h g^{-1}$  is  $(Ox)$ , and no other characteristic axes correspond to  $(Oz)$  nor  $(Oy)$ , then intersections (B.7) reduce to  $\mathbb{Z}_2^{\sigma_a} \cap (g\mathbb{Z}_{2n}^- g^{-1})$  which is conjugate to  $\mathbb{Z}_2^-$ .

□

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