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Pointwise decay in space and in time for incompressible viscous flow around a rigid body moving with constant velocity.

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Abstract

We present pointwise space-time decay estimates for the velocity part of solutions to the time-dependent Oseen system in 3D, with Dirichlet boundary conditions and vanishing velocity at infinity. In addition, similar estimates are derived for solutions to the time-dependent incompressible Navier-Stokes system with Oseen term, and for solutions to the stability problem associated with the stationary incompressible Navier-Stokes system with Oseen term.

AMS subject classifications. 35Q30, 65N30, 76D05.

Key words. Incompressible Navier-Stokes system, Oseen term, decay.

1. Introduction

Consider the incompressible time-dependent Navier-Stokes system

$$\begin{aligned} \partial_t v - \Delta_x v + \tau \partial_1 v + \tau (v \cdot \nabla_x) v + \nabla_x \pi &= f, \quad \operatorname{div}_x v = 0 \\ \text{for } t \in (0, \infty), \quad x \in \overline{\Omega}^c &:= \mathbb{R}^3 \setminus \overline{\Omega}, \end{aligned} \quad (1.1)$$

with the boundary conditions

$$v(t)|_{\partial\Omega} = b(t), \quad v(x, t) \rightarrow (1, 0, 0) \quad (|x| \rightarrow \infty) \quad \text{for } t \in (0, \infty), \quad (1.2)$$

and the initial condition

$$v(0) = v_0, \quad (1.3)$$

where Ω is an open bounded set in \mathbb{R}^3 with connected Lipschitz boundary. Problem (1.1) – (1.3) is a mathematical model for the flow of a viscous incompressible fluid around a rigid body that moves steadily and without rotation, under the assumption that the underlying reference frame adheres to the body, represented by the set Ω . The “exterior domain” $\overline{\Omega}^c$ is supposed to be filled with the fluid. The function v stands for the unknown velocity field and the function π for the unknown pressure field of the fluid. The real number $\tau > 0$ (Reynolds number) and the functions f (volume force), v_0 (initial velocity) and b (velocity of the fluid particles on the surface of the body) are given.

Problem (1.1) – (1.3) is already normalized in the sense that the flow is characterized by a single parameter – the Reynolds number – and the rigid body moves with the constant velocity $(-1, 0, 0)$ with respect to an observer at rest. This latter feature of the motion of the body, expressed by the boundary condition at infinity stated in (1.2), means in

particular that the negative part of the x_1 -axis corresponds to the upstream and the positive part to the downstream direction of the flow.

We will study the asymptotic behaviour of the fluid far from the rigid body. These asymptotics are of interest because they may be interpreted as features of the flow that may actually be observed. In this respect, particular attention is directed at the wake extending behind the rigid body, in our situation around the positive x_1 -axis. This wake should emerge in the asymptotics provided by theory.

Since nonzero boundary conditions at infinity are inconvenient from a mathematical point of view, we will not deal with equations (1.1) – (1.3) directly, but with an equivalent problem. To this end, we introduce the new velocity $u := v - (1, 0, 0)$, which satisfies the Navier-Stokes system with Oseen term,

$$\partial_t u - \Delta_x u + \tau \partial_1 u + \tau (u \cdot \nabla_x) u + \nabla_x \pi = f, \quad \operatorname{div}_x u = 0 \quad \text{in } \overline{\Omega}^c \times (0, \infty), \quad (1.4)$$

as well as the side conditions

$$u(t)|_{\partial\Omega} = (-1, 0, 0) + b(t), \quad u(x, t) \rightarrow 0 \quad (|x| \rightarrow \infty) \quad \text{for } t \in (0, \infty), \quad (1.5)$$

$$u(0) = a, \quad (1.6)$$

with $a := v_0 - (1, 0, 0)$. In the work at hand, we are interested in temporal and spatial asymptotics of the velocity part u of solutions to (1.4), (1.5), (1.6). But due to the stationary component $(-1, 0, 0)$ of the Dirichlet boundary data in (1.5), the velocity u cannot be expected to decay for $t \rightarrow \infty$, even if $b(t)$ tends to zero. Therefore we modify our problem a second time. To this end, we take a solution (U, Π) of the stationary Navier-Stokes system with Oseen term,

$$-\Delta U + \tau \partial_1 U + \tau (U \cdot \nabla) U + \nabla \Pi = F, \quad \operatorname{div} U = 0 \quad \text{in } \overline{\Omega}^c, \quad (1.7)$$

under Dirichlet boundary conditions

$$U|_{\partial\Omega} = (-1, 0, 0) + B, \quad U(x) \rightarrow 0 \quad (|x| \rightarrow \infty), \quad (1.8)$$

with given functions $F : \overline{\Omega}^c \mapsto \mathbb{R}^3$, $B : \partial\Omega \mapsto \mathbb{R}^3$, and then introduce the new unknowns $\bar{u}(x, t) := u(x, t) - U(x)$, $\bar{\pi}(x, t) := \pi(x, t) - \Pi(x)$, and the new given functions $\bar{f}(x, t) := f(x, t) - F(x)$, $\bar{b}(x, t) := b(x, t) - B(x)$ and $\bar{a} := a - U$. For simplicity denoting these new quantities again by u , π , f , b , a , instead of \bar{u} , $\bar{\pi}$, \bar{f} , \bar{b} , \bar{a} , respectively, we then arrive at the system

$$\begin{aligned} \partial_t u - \Delta_x u + \tau \partial_1 u + \tau (u \cdot \nabla_x) u + \tau (U \cdot \nabla_x) u + \tau (u \cdot \nabla) U + \nabla_x \pi &= f, \\ \operatorname{div}_x u &= 0 \quad \text{in } \overline{\Omega}^c \times (0, \infty), \end{aligned} \quad (1.9)$$

with the boundary conditions

$$u(t)|_{\partial\Omega} = b(t), \quad u(x, t) \rightarrow 0 \quad (|x| \rightarrow \infty) \quad \text{for } t \in (0, \infty), \quad (1.10)$$

and initial condition (1.6) (stability problem associated with (1.7), (1.8)). According to [14, Theorem 1.2], under suitable conditions on the data, the velocity part u of a solution to (1.9), (1.10), (1.6) exhibits an asymptotic behaviour in space described by the estimate

$$|\partial_x^\alpha u(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{-1-|\alpha|/2} \quad \text{for } x \in \mathbb{R}^3 \text{ with } |x| \geq R, \quad t \in (0, \infty), \quad (1.11)$$

$\alpha \in \mathbb{N}_0^3$ with $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \leq 1$, where R is some positive real so large that $\overline{\Omega} \subset B_R$. The same estimate is shown in previous papers [10], [13] for solutions to the linear system ("Oseen system")

$$\partial_t u - \Delta_x u + \tau \partial_1 u + \nabla_x \pi = f, \quad \operatorname{div}_x u = 0 \quad \text{in } \overline{\Omega}^c \times (0, \infty), \quad (1.12)$$

again under side conditions (1.10) and (1.6). The fact that inequality (1.11) holds for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$ means that u and the spatial gradient $\nabla_x u$ of u are evaluated. The factor $\nu(x)$ in (1.11) is defined by

$$\nu(x) := 1 + |x| - x_1 \quad \text{for } x \in \mathbb{R}^3. \quad (1.13)$$

Due to this factor, the right-hand side of (1.11) decays less fast in the wake region around the positive x_1 -axis than it does elsewhere; see [20, section VII.3] for more details. For this reason the presence of the factor $\nu(x)$ in (1.11) may be interpreted as a mathematical manifestation of the wake.

Inequality (1.11) deals only with spatial decay of the velocity. However, for suitable data, the velocity far from the body should decay in space as well as in time. It is the aim of the work at hand to make this idea more precise by determining upper bounds of $|u(x, t)|$ and $|\nabla_x u(x, t)|$ reflecting this type of asymptotics. In certain special situations, such estimates may be deduced immediately from (1.11) and estimates of $\|u(t)\|_\infty$ available in literature, where $\|\cdot\|_\infty$ denotes the norm of $L^\infty(\overline{\Omega}^c)^3$. With respect to the linear problem (1.12), (1.10), (1.6), Enomoto, Shibata [17] showed that if $f = 0$, $b = 0$, Ω smoothly bounded and $a \in L^p(\overline{\Omega}^c)^3$ for some $p \in [1, \infty)$, then $\|u(t)\|_\infty \leq \mathfrak{C} \|a\|_p t^{-3/(2p) - |\alpha|/2}$ for $t \in (0, \infty)$ ([17, Theorem 1.1]). This result combined with (1.11) and the equation $d = d^{1-\epsilon} d^\epsilon$ ($d \geq 0$, $\epsilon \in [0, 1]$) yields the pointwise space-time estimate

$$|\partial_x^\alpha u(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{(-3/(2p)-|\alpha|/2)\epsilon}$$

for x, t, α as in (1.11) and for $\epsilon \in [0, 1]$. In particular, the highest rate of temporal decay that may be attained is $(1+t)^{-3/2-|\alpha|/2}$, arising if $p = 1$.

The theory of the linear problem (1.12), (1.10), (1.6) we present here does not require f or b to vanish, nor Ω to be more regular than Lipschitz bounded. The estimates we derive reflect the asymptotics of the data f, a and b in a rather precise way (Corollary 8.1). In the best possible case, occurring when f and a are both bounded and with compact support and b decays sufficiently rapidly, we show that for $\zeta \in (0, 1)$ arbitrary but fixed, the inequality

$$\begin{aligned} & |\partial_x^\alpha u(x, t)| \\ & \leq \mathfrak{C} \left[(|x| \nu(x))^{-1-|\alpha|/2} (1+t)^{-\zeta} + (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{(-1-|\alpha|/2)\epsilon} \right] \end{aligned} \quad (1.14)$$

holds for x, t, α as in (1.11) and for $\epsilon \in [0, 1]$ (Corollary 8.2). Further below, when we sketch our method of proof, we will indicate why we do not achieve the rate $(1+t)^{-3/2-|\alpha|/2}$ of temporal decay.

Concerning the nonlinear problem (1.9), (1.10), (1.6), decay estimates of $\|u(t)\|_\infty$ were provided by Masuda [26], Heywood [23], Shibata [32] and Enomoto, Shibata [17] for a

non-normalized version of this problem, with [26], [17] and [32] requiring that the viscosity equals 1. Masuda chose initial data in L^2 and assumed $b = 0$, $\partial\Omega$ smooth and U small in a suitable sense. Constructing L^2 -weak solutions that become strong after a certain time, he obtained $\|u(t)\|_\infty \leq \mathfrak{C} t^{-1/8}$ for large t , or instead $\|u(t)\|_\infty \rightarrow 0$ ($t \rightarrow \infty$), depending on the asymptotics of f ([26, p. 297, Theorem; p.298, Remark 1.3]). Heywood [23, p. 674-675], [22, Theorem 4] admitted nonvanishing f and b , improving the decay rate of $\|u(t)\|_\infty$ to $t^{-1/4}$ under various smallness conditions on a , f and b . Enomoto, Shibata [17] worked with initial data in L^3 , constructing mild solutions under the assumptions that f and b vanish, Ω is smoothly bounded and the initial data a and the data of the stationary problem (1.7), (1.8), and hence U , are small. Within this framework, they showed that $\|u(t)\|_\infty \leq \mathfrak{C} t^{-1/2}$ for $t \in (0, \infty)$ ([17, Theorem 1.3]). In a previous article by Shibata [32], a similar but slightly weaker result is derived ([32, Theorem 1.4]).

As in the linear case, these decay estimates of $\|u(t)\|_\infty$ combined with (1.11) yield pointwise space-time decay estimates of u , although not of $\nabla_x u$ because the quantity $\|\nabla_x u(t)\|_\infty$ is not considered in the references in question. It is perhaps not astonishing that the assumptions in these references are restrictive. After all, an algebraic rate of decay of $\|u(t)\|_\infty$ is a rather strong stability result. Too strong for our purposes, we think, because it describes the behaviour of $|u(x, t)|$ for $t \rightarrow \infty$ at any point $x \in \overline{\Omega}^c$, whereas we are interested in the asymptotics of $u(x, t)$ only at points x with $|x|$ large.

We will show that any L^2 -strong solution (see (8.11), (8.12)) to (1.9), (1.10), (1.6) satisfies the estimate

$$|\partial_x^\alpha u(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} X(t)^\epsilon \quad (1.15)$$

for x, t, α as in (1.11) and for $\epsilon \in [0, 1]$, where $X : (0, \infty) \mapsto (0, \infty)$ is a bounded function with $X(t) \downarrow 0$ for $t \uparrow \infty$ (Theorem 8.4). References on existence of L^2 -strong solutions to (1.9), (1.10), (1.6) are listed in the passage following (8.12). We will take the point of view that such a solution is given. Under this assumption, (1.15) may be shown without any smallness condition. A key role in our proofs will be played by the property $\nabla_x u \in L^2(\overline{\Omega}^c \times (0, \infty))^3$ verified by the solutions under consideration.

Concerning algebraic decay of $|\partial_x^\alpha u(x, t)|$ with respect to t , we consider the somewhat simpler system (1.4) (Navier-Stokes system with Oseen term), again with the side conditions (1.10) and (1.6). L^2 -strong solutions to this problem fulfilling the additional assumption

$$\|\nabla_x u(t)\|_2 \leq c t^{-\kappa_1} \quad (t \in (1, \infty)) \quad (1.16)$$

for some constants $c, \kappa_1 > 0$ will be shown to satisfy an inequality that in the best possible case, arising if f and a are bounded and with compact support and b decays sufficiently fast, takes the form

$$|\partial_x^\alpha u(x, t)| \leq \mathfrak{C} \left[(|x| \nu(x))^{-1-|\alpha|/2} (1+t)^{-\zeta} + (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{\tilde{\zeta}\epsilon} \right] \quad (1.17)$$

for x, t, α as in (1.11) and $\epsilon \in [0, 1]$, where ζ and $\tilde{\zeta}$ are constants determined by the data and the exponent κ_1 in (1.16) (Theorem 8.4). As in the case of (1.15), no smallness condition is involved in the proof of this estimate.

The interest of our theory may be illustrated by an existence result due to Neustupa [30]. According to [30, Theorem 4.1] (also see [29, Theorem 1]), if f and b vanish, Ω is smoothly

bounded, a is small with respect to the norm of $H^1(\overline{\Omega}^c)^3$, and all eigenvalues of a certain linear operator have negative real part and stay away from the imaginary axis, then an L^2 -strong solution to (1.9), (1.10), (1.6) in the sense of (8.11), (8.12) exists. Since no smallness of U (velocity part of a solution to (1.7), (1.8)) is required and the viscosity is arbitrary, the results in [26], [23] or [17] cannot be applied to this solution. The theory presented here, however, yields that Neustupa's solution satisfies (1.15) (see Theorem 8.4), provided, of course, that the spatial asymptotics of a are compatible with our assumptions, listed at the beginning of section 8.

Property (1.16) is fulfilled by the solutions to (1.9), (1.10), (1.6) constructed in [26] and [23]; see [26, inequality (7)], [23, p. 675]. It is true that as mentioned above, these references additionally provide algebraic decay of the L^∞ -norm of $u(t)$. However, this latter property is suspended on H^2 -regularity of the Stokes operator ([26, p. 323; p. 299, Proposition 1], [23, p. 675]), and thus on smoothness of $\partial\Omega$. Therefore the L^∞ -estimates of $u(t)$ from [26] or [23] cannot be used in the proof of (1.17) if Ω is supposed to be only Lipschitz bounded. And in any case, they do not yield an access to (1.17) if $|\alpha| = 1$. So, as far as we know, inequality (1.17) is new at least in the case $|\alpha| = 1$, even though it only relates to (1.4) instead of (1.9).

In order to prove our results, we will start from a representation formula established in [11] for solutions to the linear problem (1.12), (1.10), (1.6), and stated as equation (4.9) below, in Theorem 4.10. This formula consists of a sum involving two volume potentials – one on $\mathbb{R}^3 \times (0, \infty)$ and related to f , the other one on \mathbb{R}^3 and linked to the initial data a_- , as well as a single layer potential on $S_\infty := \partial\Omega \times (0, \infty)$ whose weight function solves an integral equation on S_∞ (equation (4.8)). We refer to section 4 for the definition of these potential functions.

In order to solve the integral equation (4.8), we use an L^2 -theory developed by Shen [31] for the Stokes system, and extended to the Oseen system in [9]. In the framework of this theory, the right-hand side of (4.8) must belong to a space whose definition is rather complicated and thus gives rise to much of the technicalities we have to grapple with in what follows. This space, denoted by H_∞ in this work, is introduced in section 3. It is involved in the crucial part of our argument, that is, in determining how the L^2 -norm on $S_{T,\infty} := \partial\Omega \times (T, \infty)$ of the solution to (4.8) is bounded in terms of T . This point is settled in Theorem 7.1.

In each of the sections 5, 6 and 7, we consider one of the three potential functions appearing in the representation formula (4.9), deriving a pointwise decay estimate in space and in time for the function in question, among other results. Theorem 7.1 is applied in this context in order to deal with the single layer potential from (4.9) (Corollary 7.2). Once upper bounds of these potentials are available, the formula in (4.9) yields an estimate of the solution to (1.12), (1.10), (1.6) (Theorem 8.1).

As concerns the nonlinear problems (1.9), (1.10), (1.6) and (1.4), (1.10), (1.6), the idea is, of course, to replace f by $f - \tau(u \cdot \nabla_x)u - \tau(U \cdot \nabla_x)u - \tau(u \cdot \nabla)U$ and $f - \tau(u \cdot \nabla_x)u$, respectively, and then apply our estimates of solutions to the linear problem (1.12), (1.10), (1.6). However, we were not able to shift all difficulties into the theory of this linear problem. In fact, we will need an intermediate result from [14] – stated as the first estimate in Theorem 8.3 – whose proof exploits the interaction between the nonlinearity

and the kernel function of one of the potentials in (4.9).

It is mainly due to the integral equation (4.8) that we cannot deal with weak solutions to (1.9) or (1.4). For this type of solution, we are not able to show that the right-hand side of (4.8) with f replaced in the way just mentioned belongs to the function space H_∞ , as required by Theorem 4.8, on which the resolution theory of (4.8) is based. There are other aspects of our results whose scope is limited by this resolution theory. For example, since we may solve this equation only in an L^2 -framework, but not in L^p with $p \neq 2$, we cannot admit values $\zeta \geq 1$ in (1.14), nor can we obtain an algebraic decay rate of the time variable in (1.15).

Let us mention some further papers related to the work at hand. Knightly [24] considered pointwise decay in space of strong solutions to the nonlinear system (1.9), detecting the wake phenomenon, but he required various smallness conditions on the data and restrictive assumptions on the asymptotics of the solution. Mizumachi [28] studied the spatial asymptotics of strong solutions of (1.4), (1.10), (1.6), but still under rather restrictive assumptions. The results of these two authors were improved in the articles [10], [13] (linear case) and [14] (nonlinear problem (1.9), (1.10), (1.6)), with predecessor papers [6], [7], [8], [9], [11], [12]. As concerns temporal decay of spatial L^p -norms of solutions to the Oseen system (1.12) under side conditions (1.10), (1.6) and with $f = 0$ and $b = 0$, a basic study is due to Kobayashi, Shibata [25]. Their theory was extended in various respects in [16] and [17]. The L^∞ -estimate from [17] mentioned above is an example of such an extension. A different approach was used by Bae, Jin [3], who considered temporal decay of weighted L^p -norms of solutions to (1.12), (1.10), (1.6) with $b = 0$, $f = 0$, where the weight functions take account of the wake phenomenon. This type of result was extended to the nonlinear problem (1.9), (1.10), (1.6) with $b = 0$, $f = 0$ by Bae, Roh [4].

2. Notation. Various auxiliary results.

As we may recall, the bounded open set $\Omega \subset \mathbb{R}^3$ with connected Lipschitz boundary $\partial\Omega$ and the parameter $\tau \in (0, \infty)$ were fixed at the beginning of section 1. We will write $n^{(\Omega)}$ for the outward unit normal to Ω . The notations $S_\infty := \partial\Omega \times (0, \infty)$ for $T \in (0, \infty]$ and $S_{T, \infty} := \partial\Omega \times (T, \infty)$ for $T \in [0, \infty)$ were also already introduced in section 1, as was the function ν (see (1.13)), as well as the abbreviation $|\alpha|$ for the length $\alpha_1 + \alpha_2 + \alpha_3$ of a multiindex $\alpha \in \mathbb{N}_0^3$. The symbol $|\cdot|$ additionally denotes the Euclidean norm in \mathbb{R}^3 .

For $A \subset \mathbb{R}^3$, we set $A^c := \mathbb{R}^3 \setminus A$. Moreover we abbreviate $e_1 := (1, 0, 0)$, and we put $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$ for $x \in \mathbb{R}^3$, and $B_r := B_r(0)$, where $r \in (0, \infty)$. Let A be a nonempty set. If $\varphi : A \mapsto \mathbb{R}$ is a function, we define $|\varphi|_\infty := \sup\{|\varphi(x)| : x \in A\}$. Let $n \in \mathbb{N}$ and B a vector space consisting of functions $f : A \mapsto \mathbb{R}$. Suppose B is equipped with a norm, denoted by $\|\cdot\|_B$. Then we put $B^n := \{F : A \mapsto \mathbb{R}^n : F_j \in B \text{ for } 1 \leq j \leq n\}$, and we equip B^n with the norm $\|F\|_B^{(n)} := (\sum_{j=1}^n \|F_j\|_B^2)^{1/2}$ ($F \in B^n$). But instead of $\|\cdot\|_B^{(n)}$, we will write $\|\cdot\|_B$ again.

Next we introduce a fractional derivative. Let $A \subset \mathbb{R}^3$, $T \in (0, \infty]$ and $\psi : A \times (0, T) \mapsto \mathbb{R}$ a function such that $\psi(x, \cdot)$ is measurable and $\int_0^t (t-r)^{-1/2} |\psi(x, r)| dr < \infty$ for $x \in A$, $t \in (0, T)$. Define $W(x, t) := \int_0^t (t-r)^{-1/2} \psi(x, r) dr$ for these x and t . If the

derivative $\partial_t W(x, t)$ exists for some such x and t , we put $\partial_t^{1/2} V(x, t) := \Gamma(1/2)^{-1} \partial_t W(x, t)$. Here Γ denotes the usual Gamma function. In the case that $\partial_t W(x, t)$ exists for any $x \in A$, $t \in (0, T)$, we define $\partial_4^{1/2} V := \Gamma(1/2)^{-1} \partial_4 W$.

Let $A \subset \mathbb{R}^3$ be open. For $p \in [1, \infty]$, the norm of the Lebesgue space $L^p(A)$, defined with respect to the Lebesgue measure on \mathbb{R}^3 , is denoted by $\|\cdot\|_p$. The same notation is used for the norm of L^p -spaces on $\partial\Omega$ or on subsets of S_∞ . If $T \in (0, \infty]$, put

$$L_n^2(S_T) := \{\psi \in L^2(S_T)^3 : \int_{\partial\Omega} n^{(\Omega)}(x) \cdot \psi(x, t) \, do_x = 0 \text{ for a. e. } t \in (0, T)\}. \quad (2.1)$$

For $a, b \in \mathbb{R} \cup \{\infty\}$, $a < b$, let $L_{loc}^1([a, b])$ stand for the set of all functions $g : (a, b) \mapsto \mathbb{R}$ such that $g|_{(a, c)} \in L^1((a, c))$ for $c \in (a, b)$. For $p \in [1, \infty)$, we write $W^{1,p}(A)$ for the usual Sobolev space of order 1 and exponent p . If $p = 2$, we use the notation $H^1(A)$ instead of $W^{1,2}(A)$. For $s \in (0, 1)$, let $H^s(A)$ be the Sobolev space defined via the intrinsic norm with exponent 2 introduced in [1, section 7.51]. The symbol $H_\sigma^s(A)$ stands for the closure of the set $\{V \in C_0^\infty(A)^3 : \operatorname{div} V = 0\}$ with respect to the norm of $H^s(A)^3$ ($s \in (0, 1)$). Moreover, if A is again an open set in \mathbb{R}^3 , then $W_{loc}^{1,1}(A)$ designates the set of all functions $V : A \mapsto \mathbb{R}^3$ such that $V|_K \in W^{1,1}(K)$ for any open, bounded set $K \subset \mathbb{R}^3$ with $\overline{K} \subset A$. The Sobolev space $H^1(\partial\Omega)$ is to be defined in the standard way (see [19, section III.6], for example). We write $\|\cdot\|_{H^1(\partial\Omega)^3}$ for the norm of $H^1(\partial\Omega)^3$, and $\|\cdot\|_{H^1(\partial\Omega)'} for the usual norm of the canonical dual space $H^1(\partial\Omega)'$ of $H^1(\partial\Omega)$. Frequently we will use the fact that there is $c > 0$ such that for functions $V \in C^1(\mathbb{R}^3)^3$, the inequality $\|V|_{\partial\Omega}\|_{H^1(\partial\Omega)^3} \leq c(\|V|_{\partial\Omega}\|_2 + \sum_{j=1}^3 \|\partial_j V|_{\partial\Omega}\|_2)$ holds.$

If $J \subset \mathbb{R}$ is an interval, $A \subset \mathbb{R}^3$ is open and $v : A \times J \mapsto \mathbb{R}$ is a function with suitable smoothness, then the notation $\Delta_x v$, $\nabla_x v$, $\operatorname{div}_x v$ indicates that the differential operators in question refer to $v(x, t)$ as a function of $x \in A$. It will be convenient to denote this function by $v(t)$, for $t \in J$. For a function $V : A \mapsto \mathbb{R}^3$, we write ΔV , ∇V and $\operatorname{div} V$, respectively.

Let B be a Banach space, $a, b \in \mathbb{R} \cup \{\infty\}$ with $a < b$, and $p \in [1, \infty]$. Then the norm of the space $L^p(a, b, B)$ is denoted by $\|\cdot\|_{L^p(a, b, B)}$. If $B = L^q(A)^n$ for some $q \in [1, \infty]$, $n \in \{1, 3\}$, $A \subset \mathbb{R}^3$ open or $A = \partial\Omega$, then we use the notation $\|\cdot\|_{q,p;b}$ instead of $\|\cdot\|_{L^p(a, b, L^q(A)^n)}$. We consider the spaces $L^p(a, b, L^p(A)^n)$ and $L^p(A \times (a, b))^n$ as identical, and denote their norm by $\|\cdot\|_p$. The term $L^p([a, b], B)$ stands for the set of all functions $F : (a, b) \mapsto \mathbb{R}^3$ such that $F|(a, c) \in L^p(a, c, B)$ for any $c \in (a, b)$.

Let $T \in (0, \infty]$. For any $w \in S_T$, we introduce a mapping $F_w \in L^2(0, T, H^1(\partial\Omega)')$ by setting $F_w(t)(V) := \int_{\partial\Omega} w(x, t) V(x) \, do_x$ for $V \in H^1(\partial\Omega)$ and for a. e. $t \in (0, T)$. However, we will again write w instead of F_w .

We write C for numerical constants, and $C(\gamma_1, \dots, \gamma_n)$ for constants depending exclusively on parameters $\gamma_1, \dots, \gamma_n \in [0, \infty)$, for some $n \in \mathbb{N}$. However, it will not be possible to specify all our constants in such a precise way. So in most cases we will use a different symbol, namely \mathfrak{C} , for generic constants, assuming that their dependencies become clear from context. Occasionally we will use expressions of the form $\mathfrak{C}(\gamma_1, \dots, \gamma_n)$ in order to insist that the constant under consideration depends on $\gamma_1, \dots, \gamma_n \in [0, \infty)$, but it may additionally be a function of other quantities.

We will frequently use Minkowski's inequality for integrals. For the convenience of the reader, we state a suitable version as

Theorem 2.1 ([35, p. 271, Appendix A1]) *Let A_1, A_2 be nonempty sets, \mathfrak{A}_j a measure space on A_j and m_j a σ -finite measure on \mathfrak{A}_j , for $j \in \{1, 2\}$. Let $F : A_1 \times A_2 \mapsto \mathbb{R}$ be an $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ -measurable function, and take $p \in [1, \infty)$. Then*

$$\left(\int_{A_1} \left(\int_{A_2} |F(x, y)| dm_2(y) \right)^p dm_1(x) \right)^{1/p} \leq \int_{A_2} \left(\int_{A_1} |F(x, y)|^p dm_1(x) \right)^{1/p} dm_2(y).$$

Next we state a Sobolev inequality for certain functions in exterior domains.

Theorem 2.2 *Let $V \in W_{loc}^{1,1}(\overline{\Omega}^c)$ with $V \in L^\kappa(\overline{\Omega}^c)$ for some $\kappa \in [1, \infty)$ and $\nabla V \in L^2(\overline{\Omega}^c)^3$. Then $V \in L^6(\overline{\Omega}^c)$ and $\|V\|_6 \leq \mathfrak{C} \|\nabla V\|_2$.*

Proof: [13, Lemma 2.4], which is a consequence of [20, Theorem II.5.1]. \square

We note some technical details whose sense will become clear later on.

Lemma 2.1 *Let $\epsilon \in (0, 1)$. Put $\varphi(\epsilon) := \epsilon - 1/2$ if $\epsilon > 1/2$, and $\varphi(\epsilon) := \min\{1/12, \epsilon/4\}$ if $\epsilon \leq 1/2$. Let $k \in \mathbb{N}$ with $k\epsilon \leq 1 < (k+1)\epsilon$, and let $j \in \{0, \dots, k-1\}$. Then*

$$Z(j) := -1/2 + j/(2k) + 1/k - (j+1)\epsilon/k \leq -\varphi(\epsilon).$$

Proof: Suppose that $\epsilon > 1/2$. Then $k = 1$ so that $j = 0$. It follows that $Z(j) = -1/2 + 1 - \epsilon \leq -\varphi(\epsilon)$. Next suppose that $\epsilon \in (1/3, 1/2]$, so $k = 2$ and $j \in \{0, 1\}$. If $j = 0$, we have $Z(j) = -\epsilon/2 \leq -\varphi(\epsilon)$, and if $j = 1$, $Z(j) = 1/4 - \epsilon \leq -1/12 \leq -\varphi(\epsilon)$. Next consider the case $\epsilon \in (1/4, 1/3]$, so that $k = 3$ and $j \in \{0, 1, 2\}$. If $j = 0$, we find $Z(j) = -1/6 - \epsilon/3 \leq -1/6 \leq -\varphi(\epsilon)$. For $j = 1$, we get $Z(j) = -2\epsilon/3 \leq -1/6 \leq -\varphi(\epsilon)$, and for $j = 2$, $Z(j) = 1/6 - \epsilon \leq -1/12 \leq -\varphi(\epsilon)$.

Now we turn to the case $\epsilon \leq 1/4$, which means that $k \geq 4$. If $j \in [k/2, k-2]$, then $Z(j) \leq -1/2 + (k-2)/(2k) + 1/k - \epsilon/2 = -\epsilon/2 \leq -\varphi(\epsilon)$. The value $j = k-1$ leads to the equation $Z(j) = 1/(2k) - \epsilon$, hence by the choice of k ,

$$Z(j) = \left[(k+1)/k \right] \left(2(k+1) \right)^{-1} - \epsilon \leq \left[(k+1)/k \right] \epsilon/2 - \epsilon = -\epsilon/2 + \epsilon/(2k) \leq -3\epsilon/8,$$

so that $Z(j) \leq -\varphi(\epsilon)$. Next suppose that $k/4 \leq j \leq k/2$ so that

$$Z(j) \leq -1/2 + 1/4 + 1/k - j\epsilon/k \leq -1/4 + 1/k - \epsilon/4 \leq -\epsilon/4 \leq -\varphi(\epsilon),$$

where the second from last inequality holds because $k \geq 4$. If $j \leq k/4$, we get $Z(j) \leq -1/2 + 1/8 + 1/k \leq -1/8 \leq -\varphi(\epsilon)$, with the second from last inequality being a consequence of the relation $k \geq 4$. Thus we have found in any case that $Z(j) \leq -\varphi(\epsilon)$. \square

Lemma 2.2 *Let $T > 0$, $n \in \mathbb{N}$, $n \geq 2$, $\phi \in L^2(\partial\Omega \times (T/2, T))^3$. For $i \in \{0, \dots, n-1\}$, put $I_i := (T(1+i/n)/2, T(1+(i+1)/n)/2)$.*

Then there is $i_0 \in \{0, \dots, n-1\}$ such that $\|\phi|_{\partial\Omega \times I_{i_0}}\|_2 \leq \|\phi\|_2 n^{-1/2}$.

Proof: Suppose that $\|\phi|\partial\Omega \times I_i\|_2 > \|\phi\|_2 n^{-1/2}$ for any $i \in \{0, \dots, n-1\}$. Then

$$\|\phi\|_2^2 = \left(\sum_{i=0}^{n-1} \|\phi|\partial\Omega \times I_i\|_2^2 \right)^{1/2} > \|\phi\|_2 n^{-1/2} \left(\sum_{i=0}^{n-1} 1 \right)^{1/2} = \|\phi\|_2.$$

Since this is a contradiction, the lemma is proved. \square

Lemma 2.3 *Let $T \in (0, \infty)$, $\phi \in L^1_{loc}([0, \infty))$ with $\phi|(T, \infty) \in C^1((T, \infty))$. Define $H(t) := \int_0^t (t-r)^{-1/2} \phi(r) dr$ for $t \in (T, \infty)$. Then $H \in C^1((T, \infty))$ and*

$$\begin{aligned} H'(t) &= \sqrt{2}(t-T)^{-1/2} \phi((t+T)/2) - (1/2) \int_0^{(t+T)/2} (t-r)^{-3/2} \phi(r) dr \quad (2.2) \\ &\quad + \int_{(t+T)/2}^t (t-r)^{-1/2} \phi'(r) dr \quad \text{for } t \in (T, \infty), \end{aligned}$$

with all the preceding integrals existing as Lebesgue integrals.

Proof: Let $T_1, T_2 \in (T, \infty)$ with $T_1 < T_2$, and put $T_0 := (T + T_1)/2$,

$$F(t) := \int_0^{T_0} (t-r)^{-1/2} \phi(r) dr, \quad G(t) := \int_{T_0}^t (t-r)^{-1/2} \phi(r) dr \quad \text{for } t \in [T_1, T_2],$$

hence $H|[T_1, T_2] = F + G$. Lebesgue's theorem and the relation $T_0 < T_1$ yield that $F \in C^1([T_1, T_2])$. Turning to G , we choose a function $\tilde{\phi} \in C^1(\mathbb{R})$ such that $\tilde{\phi}|[T_0, \infty) = \phi|[T_0, \infty)$, and then consider

$$\tilde{G}(\kappa, t) := \int_0^\kappa r^{-1/2} \tilde{\phi}(t-r) dr \quad \text{for } (\kappa, t) \in B := [T_1 - T_0, T_2 - T_0] \times [T_1, T_2].$$

Again by Lebesgue's theorem, the derivative $\partial_2 \tilde{G}$ exists and belongs to $C^0(B)$. Moreover, since for any $t \in [T_1, T_2]$, the function $r \mapsto r^{-1/2} \tilde{\phi}(t-r)$, $r \in [T_1 - T_0, T_2 - T_0]$, is continuous, the derivative $\partial_1 \tilde{G}$ exists, too, and $\partial_1 \tilde{G} \in C^0(B)$. But $\tilde{G}(t - T_0, t) = G(t)$ for $t \in [T_1, T_2]$, so we get $H|[T_1, T_2] = F + G \in C^1([T_1, T_2])$.

Next we differentiate the function $\tilde{G}(t - T_0, t)$, $t \in [T_1, T_2]$, write the result as a sum of $(t - T_0)^{-1/2} \phi(T_0)$ plus an integral from T_0 to t , split off an integral from T_0 to $(t + T)/2$ and integrate by parts in that latter integral. Equation (2.2) follows by a differentiation of F and because $H|[T_1, T_2] = F + G$. \square

Corollary 2.1 *Let $T \in (0, \infty)$, $\Psi \in L^1_{loc}([0, \infty), L^2(\partial\Omega)^3)$, $v \in C^1(\mathbb{R}^3 \times (T, \infty))^3$ with $\Psi|_{S_{T, \infty}} = v|_{S_{T, \infty}}$.*

Then there is a set $N \subset \partial\Omega$ of measure zero such that for any $x \in \partial\Omega \setminus N$, $t \in (T, \infty)$, we have $\int_0^t (t-r)^{-1/2} |\Psi(x, r)| dr < \infty$. Define $W(x, t) := \int_0^t (t-r)^{-1/2} \Psi(x, r) dr$ for such x and t . Then, for $x \in \partial\Omega \setminus N$, we have $W(x, \cdot) \in C^1((T, \infty))^3$, and equation (2.2) holds with $\Psi(x, \cdot)$ in the role of ϕ , with all integrals existing in the Lebesgue sense. In particular, the fractional derivative $\partial_t^{1/2} \Psi(x, t)$ exists for x, t as above.

Proof: Let $T_0 \in (T, \infty)$. Then $\Psi|_{S_{T_0}} \in L^1(0, T_0, L^2(\partial\Omega)^3)$ by our assumptions on Ψ , so there is a set $N \subset \partial\Omega$ of measure zero such that $\Psi(x, \cdot)|(0, T_0) \in L^1((0, T_0))^3$ for

$x \in \partial\Omega \setminus N$. Since $\Psi(x, \cdot)|(T, \infty)$ is a C^1 -function for any $x \in \partial\Omega$, we thus get that $\Psi(x, \cdot) \in L_{loc}^1([0, \infty))^3$ for $x \in \partial\Omega \setminus N$. Now the corollary follows from Lemma 2.3. \square

In order to handle the term $\nu(x)$ defined in (1.13), we need the ensuing three lemmas, all of them well known.

Lemma 2.4 ([15, Lemma 4.8]) $\nu(x - y)^{-1} \leq C|y|\nu(x)^{-1}$ for $x, y \in \mathbb{R}^3$.

Lemma 2.5 ([18, Lemma 2.3]) Let $\beta \in (1, \infty)$. Then $\int_{\partial B_r} \nu(x)^{-\beta} do_x \leq C(\beta)r$ for $r \in (0, \infty)$.

Lemma 2.6 ([6, Lemma 2]) Let $K \in (0, \infty)$. Then for $x \in \mathbb{R}^3$, $t \in (0, \infty)$,

$$|x - \tau t e_1|^2 + t \geq C(K, \tau) \left(\chi_{[0, K]}(|x|) (|x|^2 + t) + \chi_{(K, \infty)}(|x|) (|x|\nu(x) + t) \right).$$

Corollary 2.2 Let $\beta \in (2, \infty)$. Then $\int_{B_R^c} (|x|\nu(x))^{-\beta} do_x \leq C(\beta)R^{-\beta+2}$ for $R \in (0, \infty)$.

Proof: By Lemma 2.5,

$$\int_{B_R^c} (|x|\nu(x))^{-\beta} do_x = \int_R^\infty r^{-\beta} \int_{\partial B_r} \nu(x)^{-\beta} do_x dr \leq C(\beta) \int_R^\infty r^{-\beta+1} dr = C(\beta)R^{-\beta+2}.$$

\square

3. Definition of the space H_∞ and its norm. Some properties of this space.

In this section, we introduce the key function space of our theory, denoted by H_∞ and taken from [31]. We start by fixing $T_0 \in (0, \infty]$ and defining

$$\tilde{H}_{T_0} := \{w|_{S_{T_0}} : w \in C_0^\infty(\mathbb{R}^4)^3, w|\mathbb{R}^3 \times (-\infty, 0) = 0\}.$$

For $\Psi \in \tilde{H}_{T_0}$, we set

$$\|\Psi\|_{H_{T_0}} := \left(\int_0^{T_0} \left(\|\Psi(t)\|_{H^1(\partial\Omega)^3}^2 + \|\partial_4^{1/2}\Psi(t)\|_2^2 + \|n^{(\Omega)} \cdot \partial_4\Psi(t)\|_{H^1(\partial\Omega)'}^2 \right) dt \right)^{1/2}.$$

The mapping $\|\cdot\|_{H_{T_0}}$ is a norm on \tilde{H}_{T_0} . In the case $T_0 = \infty$, we further put

$$\|\Psi|_{S_{T, \infty}}\|_{H_{T, \infty}} := \left(\int_T^\infty \left(\|\Psi(t)\|_{H^1(\partial\Omega)^3}^2 + \|\partial_4^{1/2}\Psi(t)\|_2^2 + \|n^{(\Omega)} \cdot \partial_4\Psi(t)\|_{H^1(\partial\Omega)'}^2 \right) dt \right)^{1/2}$$

for $T \in [0, \infty)$, $\Psi \in \tilde{H}_\infty$. Note that the term $\|\Psi|_{S_{T, \infty}}\|_{H_{T, \infty}}$ depends on $\Psi|_{S_T}$, via the fractional derivative $\partial_4^{1/2}\Psi$. Further note that in the case $T = 0$, we have $\|\Psi|_{S_{T, \infty}}\|_{H_{T, \infty}} = \|\Psi\|_{H_\infty}$.

In order to construct a completion of \tilde{H}_{T_0} , we consider sequences (Ψ_n) in \tilde{H}_{T_0} which are Cauchy sequences with respect to the norm $\|\cdot\|_{H_{T_0}}$, with the additional property that

there is $\Psi \in L^2_n(S_{T_0})$ with $\|\Psi_n - \Psi\|_2 \rightarrow 0$. In such a situation it is not clear a priori how Ψ is related to the limit of $(\partial_4^{1/2}\Psi_n)$ in $L^2(S_{T_0})$ and to the limit of $(n^{(\Omega)} \cdot \partial_4\Psi_n)$ in $L^2(0, T_0, H^1(\partial\Omega)')$. But it turns out that if $\Psi = 0$, these limits vanish as well. This fact and some other, related ones are the subject of

Lemma 3.1 *Let $T_0 \in (0, \infty]$, $\Psi \in L^2(S_{T_0})^3$, $v_n \in C_0^\infty(\mathbb{R}^4)^3$ with $v_n|_{\mathbb{R}^3 \times (-\infty, 0)} = 0$ for $n \in \mathbb{N}$ such that $\|v_n - \Psi\|_2 \rightarrow 0$ and $(v_n|_{S_{T_0}})$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{H_{T_0}}$ on \tilde{H}_{T_0} .*

Then the sequence $(\|v_n|_{S_{T_0}}\|_{H_{T_0}})$ converges.

If $\Psi = 0$, then $\lim_{n \rightarrow \infty} \|v_n|_{S_{T_0}}\|_{H_{T_0}} = 0$.

If $T_0 = \infty$, $T \in (0, \infty)$ and $\Psi|_{S_T} = 0$, then $\lim_{n \rightarrow \infty} \|v_n|_{S_{T,\infty}}\|_{H_{T,\infty}} = \lim_{n \rightarrow \infty} \|v_n|_{S_\infty}\|_{H_\infty}$.

If $T_0 = \infty$, $\tilde{T}, T \in (0, \infty)$ with $\tilde{T} \leq T$, then $\lim_{n \rightarrow \infty} \|v_n|_{S_{T,\infty}}\|_{H_{T,\infty}} \leq \lim_{n \rightarrow \infty} \|v_n|_{S_{\tilde{T},\infty}}\|_{H_{\tilde{T},\infty}}$.

Proof: Since $(v_n|_{S_{T_0}})$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{H_{T_0}}$, the sequence $(\|v_n|_{S_{T_0}}\|_{H_{T_0}})$ is a Cauchy sequence in \mathbb{R} and thus converges. It further follows there are functions $\Psi^{(1)} \in L^2(0, T_0, H^1(\partial\Omega)^3)$, $\Psi^{(2)} \in L^2(S_{T_0})^3$ and $\Psi^{(3)} \in L^2(0, T_0, H^1(\partial\Omega)')$ such that

$$\|v_n - \Psi^{(1)}\|_{L^2(0, T_0, H^1(\partial\Omega)^3)} + \|\partial_4^{1/2}v_n - \Psi^{(2)}\|_2 + \|\bar{v}_n - \Psi^{(3)}\|_{L^2(0, T_0, H^1(\partial\Omega)')} \longrightarrow 0, \quad (3.1)$$

where $\bar{v}_n(x, t) := n^{(\Omega)}(x) \cdot v_n(x, t)$ for $n \in \mathbb{N}$, $x \in \partial\Omega$, $t \in (0, T_0)$. By the definition of the mapping $\|\cdot\|_{H_{T,\infty}}$, this means in the case $T_0 = \infty$ that

$$\lim_{n \rightarrow \infty} \|v_n|_{S_{\tilde{T},\infty}}\|_{H_{\tilde{T},\infty}}^2 = \int_{\tilde{T}}^\infty \left(\|\Psi^{(1)}(t)\|_{H^1(\partial\Omega)^3}^2 + \|\Psi^{(2)}(t)\|_2^2 + \|\Psi^{(3)}(t)\|_{H^1(\partial\Omega)'}^2 \right) dt \quad (3.2)$$

for any $\tilde{T} \in [0, \infty)$. Now suppose that $T_0 = \infty$, $T \in (0, \infty)$, and $\Psi|_{S_T} = 0$. We will show that the limits $\lim_{n \rightarrow \infty} \|v_n|_{S_{T,\infty}}\|_{H_{T,\infty}}$ and $\lim_{n \rightarrow \infty} \|v_n|_{S_\infty}\|_{H_\infty}$ coincide. Since $\|\Psi - v_n\|_2 \rightarrow 0$, we obtain from (3.1) that $\Psi = \Psi^{(1)}$. But $\Psi|_{S_T} = 0$, so we may conclude that $\Psi^{(1)}|_{S_T} = 0$. Moreover, for $V \in L^2(\partial\Omega)^3$, $\phi \in C_0^\infty((0, T))$, we get from (3.1) that

$$\int_{\partial\Omega} \int_0^T \Psi^{(2)}(x, t) \cdot V(x) \phi(t) dt do_x = \lim_{n \rightarrow \infty} \int_{\partial\Omega} \int_0^T \partial_4^{1/2}v_n(x, t) \cdot V(x) \phi(t) dt do_x. \quad (3.3)$$

Put $w_n(x, t) := \int_0^t (t-r)^{-1/2} v_n(x, r) dr$ for $n \in \mathbb{N}$, $x \in \partial\Omega$, $t \in (0, \infty)$. By Lemma 2.3, we have $w_n(x, \cdot) \in C^1((0, \infty))^3$ for any $x \in \partial\Omega$, $n \in \mathbb{N}$. Since $\partial_4^{1/2}v_n(x, t) = \partial_t w_n(x, t)$ for n, x, t as before, we get

$$\int_{\partial\Omega} \int_0^T \partial_4^{1/2}v_n(x, t) \cdot V(x) \phi(t) dt do_x = - \int_{\partial\Omega} \int_0^T w_n(x, t) \cdot V(x) \phi'(t) dt do_x \quad (3.4)$$

for $n \in \mathbb{N}$. But

$$\begin{aligned}
& \left| \int_{\partial\Omega} \int_0^T w_n(x, t) \cdot V(x) \phi'(t) dt do_x \right| \\
&= \left| \int_{\partial\Omega} \int_0^T \int_r^T (t-r)^{-1/2} \phi'(t) dt v_n(x, r) \cdot V(x) dr do_x \right| \\
&\leq \mathfrak{C}(T) |\phi'|_\infty \int_0^T \|v_n(r)|_{\partial\Omega}\|_2 dr \|V\|_2 \leq \mathfrak{C}(T) |\phi'|_\infty \|V\|_2 \|v_n|_{S_T}\|_2 \quad (n \in \mathbb{N}).
\end{aligned} \tag{3.5}$$

Since $\|v_n|_{S_T}\|_2 = \|v_n - \Psi|_{S_T}\|_2$ ($n \in \mathbb{N}$) and $\|\Psi - v_n\|_2 \rightarrow 0$, we may conclude from (3.3) – (3.5) that $\Psi^{(2)}|_{S_T} = 0$. A similar reasoning based on (3.1) yields $\Psi^{(3)}|_{S_T} = 0$. Now we may conclude from (3.2) with $\tilde{T} = 0$ and $\tilde{T} = T$ that $\lim_{n \rightarrow \infty} \|v_n|_{S_{T,\infty}}\|_{H_{T,\infty}} = \lim_{n \rightarrow \infty} \|v_n|_{S_\infty}\|_{H_\infty}$.

If $T_0 \in (0, \infty]$ and $\Psi = 0$, the same reasoning yields $\Psi^{(i)} = 0$ for $i \in \{1, 2, 3\}$. In this way we deduce from (3.1) that $\lim_{n \rightarrow \infty} \|v_n|_{S_{T_0}}\|_{H_{T_0}} = 0$.

The last claim of the lemma is an immediate consequence of (3.2). \square

Corollary 3.1 *Let $T_0 \in (0, \infty]$, $\Psi \in L^2(S_T)^3$, (Ψ_n) and $(\tilde{\Psi}_n)$ sequences in \tilde{H}_{T_0} with $\|\Psi - \Psi_n\|_2 \rightarrow 0$, $\|\Psi - \tilde{\Psi}_n\|_2 \rightarrow 0$ and such that (Ψ_n) and $(\tilde{\Psi}_n)$ are Cauchy sequences with respect to the norm $\|\cdot\|_{H_{T_0}}$ on \tilde{H}_{T_0} . Then $\lim_{n \rightarrow \infty} \|\Psi_n\|_{H_{T_0}} = \lim_{n \rightarrow \infty} \|\tilde{\Psi}_n\|_{H_{T_0}}$. If $T_0 = \infty$ and $T \in (0, \infty)$, it further follows that $\lim_{n \rightarrow \infty} \|\Psi_n|_{S_{T,\infty}}\|_{H_{T,\infty}} = \lim_{n \rightarrow \infty} \|\tilde{\Psi}_n|_{S_{T,\infty}}\|_{H_{T,\infty}}$.*

Proof: Lemma 3.1 \square

Let $T_0 \in (0, \infty]$. We define the space H_{T_0} as the set of all functions $\Psi \in L_n^2(S_{T_0})$ for which there is a sequence (Ψ_n) in \tilde{H}_{T_0} such that $\|\Psi - \Psi_n\|_2 \rightarrow 0$ and (Ψ_n) is a Cauchy sequence with respect to the norm $\|\cdot\|_{H_{T_0}}$. This means in particular that $H_{T_0} \subset L_n^2(S_{T_0})$. (See (2.1) for the definition of $L_n^2(S_{T_0})$.)

For $\Psi \in H_{T_0}$ and a sequence (Ψ_n) as above, we set $\|\Psi\|_{H_{T_0}} := \lim_{n \rightarrow \infty} \|\Psi_n\|_{H_{T_0}}$. Corollary 3.1 implies that the mapping $\|\cdot\|_{H_{T_0}}$ is well defined on H_{T_0} . The space H_{T_0} equipped with this mapping is a Banach space, a fact that we will not need in the following.

Suppose that $T_0 = \infty$. We set $\|\Psi|_{S_{T,\infty}}\|_{H_{T,\infty}} := \lim_{n \rightarrow \infty} \|\Psi_n|_{S_{T,\infty}}\|_{H_{T,\infty}}$ for $\Psi \in H_\infty$, $T \in (0, \infty)$, where $\Psi_n \in \tilde{H}_\infty$ for $n \in \mathbb{N}$ with the properties that $\|\Psi - \Psi_n\|_2 \rightarrow 0$ and (Ψ_n) is a Cauchy sequence with respect to the norm $\|\cdot\|_{H_\infty}$. It again follows from Corollary 3.1 that the mapping $\|\cdot\|_{H_{T,\infty}}$ is well defined.

Corollary 3.2 *Let $T \in (0, \infty)$ and $\Psi \in H_\infty$. If $\Psi|_{S_T} = 0$, then $\|\Psi|_{S_{T,\infty}}\|_{H_{T,\infty}} = \|\Psi\|_{H_\infty}$. If $\tilde{T} \in [0, T]$, then $\|\Psi|_{S_{T,\infty}}\|_{H_{T,\infty}} \leq \|\Psi|_{S_{\tilde{T},\infty}}\|_{H_{\tilde{T},\infty}}$.*

Proof: Lemma 3.1.

Lemma 3.2 *Let $T \in (0, \infty)$, $\Psi \in H_\infty$, and suppose there is $v \in C^1(\mathbb{R}^3 \times (T, \infty))^3$ with $\Psi|_{S_{T,\infty}} = v|_{S_{T,\infty}}$.*

Then $\int_0^t (t-r)^{-1/2} |\Psi(x, r)| dr < \infty$ for $t \in (T, \infty)$ and for a. e. $x \in \partial\Omega$. Put $W(x, t) := \int_0^t (t-r)^{-1/2} \Psi(x, r) dr$ for these t and x . Then $W(x, \cdot) \in C^1((T, \infty))^3$ and equation (2.2)

holds with ϕ replaced by $\Psi(x, \cdot)$, for a. e. $x \in \partial\Omega$. Therefore the fractional derivative $\partial_t^{1/2}\Psi(x, t)$ is well defined and equals $\partial_4 W(x, t)$ for $t \in (T, \infty)$ and a. e. $x \in \partial\Omega$. Moreover

$$\|\Psi|_{S_{T,\infty}}\|_{H_{T,\infty}}^2 = \int_T^\infty \left(\|\Psi(t)\|_{H^1(\partial\Omega)^3}^2 + \|\partial_4 W(t)\|_2^2 + \|n^{(\Omega)} \cdot \partial_4 \Psi(t)\|_{H^1(\partial\Omega)'}^2 \right) dt. \quad (3.6)$$

Proof: Since $\Psi \in H_\infty$, we have in particular $\Psi \in L^2(S_\infty)^3$, so $\Psi \in L_{loc}^1([0, \infty) L^2(\partial\Omega)^3)$. In addition $\Psi|_{S_{T,\infty}} = v|_{S_{T,\infty}}$ with $v \in C^1(\mathbb{R}^3 \times (T, \infty))^3$, so the claims about W are valid according to Corollary 2.1. This leaves us to show (3.6). Again since $\Psi \in H_\infty$, there are functions $v_n \in C^\infty(\mathbb{R}^4)^3$ with $v_n|_{\mathbb{R}^3 \times (-\infty, 0)} = 0$ for $n \in \mathbb{N}$ such that $\|v_n - \Psi\|_2 \rightarrow 0$ and such that $(v_n|_{S_\infty})$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{H_\infty}$ on \tilde{H}_{T_0} . The latter property implies there are mappings $\Psi^{(1)} \in L^2(0, \infty, H^1(\partial\Omega)^3)$, $\Psi^{(2)} \in L^2(S_\infty)^3$ and $\Psi^{(3)} \in L^2(0, \infty, H^1(\partial\Omega)')$ as in (3.1) with $T_0 = \infty$. Thus (3.2) holds, so by the definition of $\|\cdot\|_{H_{T,\infty}}$,

$$\|\Psi|_{S_{T,\infty}}\|_{H_{T,\infty}}^2 = \int_T^\infty \left(\|\Psi^{(1)}(t)\|_{H^1(\partial\Omega)^3}^2 + \|\Psi^{(2)}(t)\|_2^2 + \|\Psi^{(3)}(t)\|_{H^1(\partial\Omega)'}^2 \right) dt \quad (3.7)$$

We further conclude from (3.1) and the choice of the sequence (v_n) that $\Psi^{(1)} = \Psi$. Let us show that $\Psi^{(2)}|_{S_{T,\infty}} = \partial_4 W$. To this end, we proceed as in the proof of Lemma 3.1. We take $V \in L^2(\partial\Omega)^3$, $\phi \in C_0^\infty((T, \infty))$, and we put $w_n(x, t) := \int_0^t (t-r)^{-1/2} v_n(x, r) dr$ for $n \in \mathbb{N}$, $x \in \partial\Omega$, $t \in (0, \infty)$. By Lemma 2.3, we have $w_n(x, \cdot) \in C^1((0, \infty))^3$ for any $x \in \partial\Omega$, and $\partial_4^{1/2} v_n = \partial_4 w_n$ for $n \in \mathbb{N}$. Similarly to (3.3) and (3.4), we get

$$\int_{\partial\Omega} \int_T^\infty \Psi^{(2)}(x, t) \cdot V(x) \phi(t) dt do_x = - \lim_{n \rightarrow \infty} \int_{\partial\Omega} \int_T^\infty w_n(x, t) \cdot V(x) \phi'(t) dt do_x. \quad (3.8)$$

There is $T_1 \in (T, \infty)$ such that $\text{supp}(\phi) \subset (T, T_1)$. Thus, with W introduced in the lemma,

$$\begin{aligned} & \left| \int_{\partial\Omega} \int_T^\infty (w_n - W)(x, t) \cdot V(x) \phi'(t) dt do_x \right| \\ &= \left| \int_{\partial\Omega} \int_0^{T_1} \int_0^t (t-r)^{-1/2} (v_n - \Psi)(x, r) \cdot V(x) \phi'(t) dr dt do_x \right| \\ &= \left| \int_{\partial\Omega} \int_0^{T_1} \int_r^{T_1} (t-r)^{-1/2} \phi'(t) dt (v_n - \Psi)(x, r) \cdot V(x) dr do_x \right| \\ &\leq \mathfrak{C}(T_1) |\phi'|_\infty \int_{\partial\Omega} \|(v_n - \Psi)(x, \cdot)\|_2 |V(x)| do_x \leq \mathfrak{C}(T_1) |\phi'|_\infty \|V\|_2 \|v_n - \Psi\|_2 \end{aligned}$$

Since $\|v_n - \Psi\|_2 \rightarrow 0$, we thus get

$$- \int_{\partial\Omega} \int_T^\infty W(x, t) \cdot V(x) \phi'(t) dt do_x = - \lim_{n \rightarrow \infty} \int_{\partial\Omega} \int_T^\infty w_n(x, t) \cdot V(x) \phi'(t) dt do_x. \quad (3.9)$$

On the other hand, since $W(x, \cdot) \in C^1((t, \infty))$ for a. e. $x \in \partial\Omega$,

$$- \int_{\partial\Omega} \int_T^\infty W(x, t) \cdot V(x) \phi'(t) dt do_x = \int_{\partial\Omega} \int_T^\infty \partial_4 W(x, t) \cdot V(x) \phi(t) dt do_x. \quad (3.10)$$

Combining (3.8) – (3.10), we may conclude that $\Psi^{(2)}|_{S_{T,\infty}} = \partial_4 W$. An analogous but simpler reasoning yields that $\Psi^{(3)}(x, t) = n^{(\Omega)}(x) \cdot \partial_4 \Psi(x, t)$ for $t \in (T, \infty)$ and for a. e. $x \in \partial\Omega$. Now equation (3.6) follows from (3.7). \square

4. Some fundamental solutions and potential functions. A representation formula for the velocity part of solutions to the Oseen system with Dirichlet boundary conditions.

We define three potential functions, denoted by $\mathfrak{R}^{(\tau)}(f)$, $\mathfrak{J}^{(\tau)}(a)$ and $\mathfrak{B}^{(\tau)}(\phi)$, respectively. The first is related to the right-hand side f in (1.12), the second to the initial data a in (1.6), and the third to the Dirichlet boundary data b in (1.10). We begin by introducing fundamental solutions to the heat equation, the time-dependent Stokes system and the Oseen system (1.12), respectively. We write \mathfrak{H} for the fundamental solution of the heat equation in \mathbb{R}^3 , that is,

$$\mathfrak{H}(z, t) := (4\pi t)^{-3/2} e^{-|z|^2/(4t)} \quad \text{for } (z, t) \in \mathbb{R}^3 \times (0, \infty).$$

As a fundamental solution of the time-dependent Stokes system, we choose the same function Γ as in [31], that is,

$$\Gamma_{jk}(z, t) := \delta_{jk} \mathfrak{H}(z, t) + \int_t^\infty \partial_j \partial_k \mathfrak{H}(z, s) ds \quad \text{for } (z, t) \in \mathbb{R}^3 \times (0, \infty), \quad 1 \leq j, k \leq 3.$$

Actually this is the velocity part of the fundamental solution in question; we will not need the pressure part associated with. Finally we introduce the velocity part of the looked-for fundamental solution of the time-dependent Oseen system (1.12), setting

$$\Lambda_{jk}(z, t, \tau) := \Gamma_{jk}(z - t\tau e_1, t) \quad \text{for } z, t, j, k \text{ as before.}$$

Note that in what follows, Γ does not stand for the usual Gamma function. We state some properties of \mathfrak{H} , $\Gamma = (\Gamma_{jk})_{1 \leq j, k \leq 3}$ and $\Lambda = (\Lambda_{jk})_{1 \leq j, k \leq 3}$.

Lemma 4.1 $\mathfrak{H} \in C^\infty(\mathbb{R}^3 \times (0, \infty))$, $\int_{\mathbb{R}^3} \mathfrak{H}(z, t) dz = 1$ for $t \in (0, \infty)$, and $|\partial_t^l \partial_z^\alpha \mathfrak{H}(z, t)| \leq C(|z|^2 + t)^{-3/2 - |\alpha|/2 - l}$ for $z \in \mathbb{R}^3$, $t \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$ with $|\alpha| + l \leq 1$.

Proof: For a proof of the estimate at the end of the lemma, we refer to [33]. An estimate of this kind holds for any $\alpha \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$, but we will need it only in the case $|\alpha| + l \leq 1$. \square

Lemma 4.2 ([31, Proposition 2.19]) $\Gamma_{jk}, \Lambda_{jk}(\cdot, \cdot, \tau) \in C^\infty(\mathbb{R}^3 \times (0, \infty))$,

$$\begin{aligned} |\partial_t^l \partial_z^\alpha \Gamma_{jk}(z, t)| &\leq C(|z|^2 + t)^{-3/2 - |\alpha|/2 - l}, \\ |\partial_t^l \partial_z^\alpha \Lambda_{jk}(z, t)| &\leq C(\tau) [(|z - t\tau e_1|^2 + t)^{-3/2 - |\alpha|/2 - l} + \delta_{l1} (|z - t\tau e_1|^2 + t)^{-2}] \end{aligned}$$

for $z \in \mathbb{R}^3$, $t \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$ with $|\alpha| + l \leq 1$, $j, k \in \{1, 2, 3\}$.

By combining Lemma 2.6 and the preceding lemma, we get

Corollary 4.1 Let $K \in (0, \infty)$ and put $\gamma_K(z) := |z|^2$ for $z \in B_K$, $\gamma_K(z) := |z|\nu(z)$ for $z \in B_K^c$. Then $|\partial_t^l \partial_z^\alpha \Lambda_{jk}(z, t)| \leq C(\tau, K) [(\gamma_K(z) + t)^{-3/2 - |\alpha|/2 - l} + \delta_{l1} (\gamma_K(z) + t)^{-2}]$ for z, t, α, l, j, k as in the preceding lemma.

The ensuing theorem provides an estimate of convolutions of Λ .

Theorem 4.1 *Let $q \in [1, \infty)$, $\varrho \in (1, \infty]$, $s \in [1, \infty]$ with $s \leq \varrho$, $M \in (0, \infty)$, $j, k \in \{1, 2, 3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Take $h \in L^s(0, \infty, L^q(\mathbb{R}^3))$.*

Then the ensuing inequality holds for $W = (0, M)$ if $1 - |\alpha|/2 - 3/(2q) - 1/s + 1/\varrho > 0$, and for $W = (M, \infty)$ if $1 - |\alpha|/2 - 3/(2q) - 1/s + 1/\varrho < 0$:

$$\begin{aligned} & \left(\int_0^\infty \left(\int_0^\infty \int_{\mathbb{R}^3} \chi_W(t - \sigma) |\partial_x^\alpha \Lambda_{jk}(x - y, t - \sigma, \tau)| |h(y, \sigma)| dy d\sigma \right)^\varrho dt \right)^{1/\varrho} \\ & \leq C(\tau, q, \varrho, s) M^{1 - |\alpha|/2 - 3/(2q) - 1/s + 1/\varrho} \|h\|_{q, s; \infty} \end{aligned}$$

for $x \in \mathbb{R}^3$ if $\varrho < \infty$, and

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^3} \chi_W(t - \sigma) |\partial_x^\alpha \Lambda_{jk}(x - y, t - \sigma, \tau)| |h(y, \sigma)| dy d\sigma \\ & \leq C(\tau, q, s) M^{1 - |\alpha|/2 - 3/(2q) - 1/s} \|h\|_{q, s; \infty} \end{aligned}$$

for $x \in \mathbb{R}^3$, $t \in (0, \infty)$ if $\varrho = \infty$.

Proof: Theorem 4.1 follows from [10, Lemma 2.7] and [14, Theorem 2.8] if the parameter p in those references is chosen as $p = \infty$. As becomes apparent from the proof of these references, not only L^∞ -estimates are provided if $p = \infty$ or $\varrho = \infty$, but pointwise estimates not involving any sets of measure zero. \square

The ensuing lemma is the basis of the definition of our first potential function.

Lemma 4.3 *Let $q, s \in [1, \infty)$ and $h \in L^s(0, \infty, L^q(\mathbb{R}^3)^3)$. Then, for a. e. $(x, t) \in \mathbb{R}^3 \times (0, \infty)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, we have $\int_0^t \int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x - y, t - \sigma, \tau) \cdot h(y, \sigma)| dy d\sigma < \infty$.*

Proof: The lemma follows from a more general version of Theorem 4.1 ([10, Lemma 2.7]); see the remarks in [10, p. 898 below]. \square

Due to the preceding lemma, we may define a function $\mathfrak{R}^{(\tau)}(h) : \mathbb{R}^3 \times [0, \infty) \mapsto \mathbb{R}^3$ for any $h \in L^s(0, T, L^q(A)^3)$ with $q, s \in [1, \infty)$, $A \subset \mathbb{R}^3$ measurable, $T \in (0, \infty]$, by setting

$$\mathfrak{R}^{(\tau)}(x, t) := \int_0^t \int_{\mathbb{R}^3} \Lambda(x - y, t - \sigma, \tau) \cdot \tilde{h}(y, \sigma) dy d\sigma \quad \text{for a. e. } (x, t) \in \mathbb{R}^3 \times [0, \infty),$$

where \tilde{h} denotes the zero extension of h to $\mathbb{R}^3 \times (0, \infty)$.

Lemma 4.4 ([10, Lemma 2.11]) *Let $q, s \in [1, \infty)$ and $h \in L^s(0, \infty, L^q(\mathbb{R}^3)^3)$. Then the weak derivative $\partial_l \mathfrak{R}^{(\tau)}(h)$ exists for $1 \leq l \leq 3$, in particular $\mathfrak{R}^{(\tau)}(h)(t) \in W_{loc}^{1,1}(\mathbb{R}^3)^{3 \times 3}$ for a. e. $t \in (0, \infty)$. Moreover $\partial_l \mathfrak{R}^{(\tau)}(x, t) = \int_0^t \int_{\mathbb{R}^3} \partial_l \Lambda(x - y, t - \sigma, \tau) \cdot h(y, \sigma) dy d\sigma$ for a. e. $x \in \mathbb{R}^3$, a. e. $t \in (0, \infty)$, $1 \leq l \leq 3$. In particular the trace of $\mathfrak{R}^{(\tau)}(h)(t)$ on $\partial\Omega$ is well defined for a. e. $t \in (0, \infty)$.*

Lemma 4.5 ([10, Lemma 2.12, Corollary 2.13]) *Let $q, s \in [1, \infty)$ and let h be a function belonging to $L^s(0, \infty, L^q(\mathbb{R}^3)^3)$. Then $\int_0^t \int_{\mathbb{R}^3} |\Lambda(x - y, t - \sigma, \tau) \cdot h(y, \sigma)| dy d\sigma < \infty$ for a. e. $(x, t) \in \partial\Omega \times (0, \infty)$. Thus $\mathfrak{R}^{(\tau)}(h)$ is well defined also as a function on S_∞ . Moreover, $\mathfrak{R}^{(\tau)}(h)(t)$ as a function on $\partial\Omega$ is the trace of $\mathfrak{R}^{(\tau)}(h)(t)$ as a function on \mathbb{R}^3 , for a. e. $t \in (0, \infty)$.*

Lemma 4.6 *Let $T \in (0, \infty)$, $q \in [1, 4]$, $s \in [1, \infty)$, $f \in L^s(0, T, L^q(\mathbb{R}^3)^3)$. Then $\mathfrak{R}^{(\tau)}(f)|_{\mathbb{R}^3 \times (T, \infty)} \in C^1(\mathbb{R}^3 \times (T, \infty))^3$ and*

$$\partial_t^l \partial_x^\alpha \mathfrak{R}^{(\tau)}(f)(x, t) = \int_0^T \int_{\mathbb{R}^3} \partial_t^l \partial_x^\alpha \Lambda(x - y, t - \sigma, \tau) \cdot f(y, \sigma) dy d\sigma \quad (4.1)$$

for $x \in \mathbb{R}^3$, $t \in (T, \infty)$, $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^3$ with $l + |\alpha| \leq 1$, where the preceding integral exists as a Lebesgue integral.

Proof: The lemma follows from Lebesgue's theorem. However, since f is not required to belong to $L^1(\mathbb{R}^3 \times (0, T))^3$, it is perhaps not completely obvious how to apply that theorem. In particular, the reason for the condition $q \leq 4$ may not be clear. So we indicate a proof. Let $R \in (0, \infty)$, $T_0 \in (T, \infty)$. It is enough to show that $\mathfrak{R}^{(\tau)}(f)|_{B_R \times (T_0, \infty)}$ is a C^1 -function and equation (4.1) holds for $(x, t) \in B_R \times (T_0, \infty)$. Put $f^{(1)} := \chi_{B_{\frac{R}{2}} \times (0, T)} f$. Then, by Corollary 4.1 with $K = R$ and Lemma 2.4, we get

$$|\partial_t^l \partial_x^\alpha \Lambda(x - y, t - \sigma, \tau) \cdot f^{(1)}(y, \sigma)| \leq C(R, \tau) g_{l, \alpha}(y) |f(y, \sigma)| \quad (4.2)$$

for $x \in B_R$, $y \in \mathbb{R}^3$, $t \in (T_0, \infty)$, $\sigma \in (0, T)$, α, l as in (4.1), where

$$g_{l, \alpha}(y) := \chi_{(2R, \infty)}(|y|) \left[(|y| \nu(y))^{-3/2 - |\alpha|/2 - l} + \delta_{l1} (|y| \nu(y))^{-2} \right] \quad \text{for } y \in \mathbb{R}^3.$$

Since $q < 4$, we have $3q'/2 > 2$, so by Corollary 2.2,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} g_{l, \alpha}(y) |f(y, \sigma)| dy d\sigma &\leq \int_0^T \left(\int_{\mathbb{R}^3} g_{l, \alpha}(y)^{q'} dy \right)^{1/p'} \|f(\sigma)\|_q d\sigma \\ &\leq C(R, q, s, \tau) T^{1/s'} \|f\|_{q, s; T} < \infty, \end{aligned}$$

for l, α as in (4.1). Therefore from (4.2) and Lebesgue's theorem, we may conclude that $\mathfrak{R}^{(\tau)}(f^{(1)})$ is a C^1 -function on $B_R \times (T_0, \infty)$, and equation (4.1) is valid for $x \in B_R$, $t \in (T_0, \infty)$ and with f replaced by $f^{(1)}$. Put $f^{(2)} := \chi_{B_{2R} \times (0, T)} f$. Then $f^{(2)} \in L^1(\mathbb{R}^3 \times (0, T))^3$, and due to Lemma 4.2, we have

$$|\partial_t^l \partial_x^\alpha \Lambda_{jk}(x - y, t - \sigma, \tau)| \leq C(\tau) \left((T_0 - T)^{-3/2 - |\alpha|/2 - l} + \delta_{l1} (T_0 - T)^{-2} \right)$$

for x, y, t, σ as in (4.2), $1 \leq j, k \leq 3$ and α, l as in (4.1). Therefore Lebesgue's theorem yields that $\mathfrak{R}^{(\tau)}(f^{(2)})|_{B_R \times (T_0, \infty)}$ is a C^1 -function and (4.1) holds on $B_R \times (T_0, \infty)$ if f is replaced by $f^{(2)}$. Since $f = f^{(1)} + f^{(2)}$, the lemma is proved. \square

The next theorem presents a criterion on f implying $\mathfrak{R}^{(\tau)}(f)|_{S_\infty} \in H_\infty$.

Theorem 4.2 ([11, Theorem 8.1]) *Let $f \in L^2(0, \infty, L^\varrho(\mathbb{R}^3)^3)$ for $\varrho = 3/2$ and for some $\varrho \in [1, 3/2)$. Then $\mathfrak{R}^{(\tau)}(f)|_{S_\infty} \in H_\infty$ and $\|\mathfrak{R}^{(\tau)}(f)|_{S_\infty}\|_{H_\infty} \leq \mathfrak{C} (\|f\|_{3/2, 2; \infty} + \|f\|_{q, 2; \infty})$.*

Corollary 4.2 *Let T, q, f be given as in Lemma 4.6, and abbreviate $\Psi := \mathfrak{R}^{(\tau)}(f)|_{S_\infty}$. For a. e. $x \in \partial\Omega$ we then have $\Psi(x, \cdot)|(T, \infty) \in C^1((T, \infty))^3$, $\int_0^t (t-r)^{-1/2} |\Psi(x, r)| dr < \infty$ for $t \in (T, \infty)$, and the function $t \mapsto \int_0^t (t-r)^{-1/2} \Psi(x, r) dr$, $t \in (T, \infty)$, belongs to*

$C^1((T, \infty))^3$, so that $\partial_t^{1/2}\Psi(x, t)$ exists for any $t \in (T, \infty)$. In addition, equation (2.2) holds with $\Psi(x, \cdot)$ in the place of ϕ , for a. e. $x \in \partial\Omega$. Finally,

$$\begin{aligned} & \|\Psi|_{S_{T, \infty}}\|_{H_{T, \infty}}^2 \\ &= \int_T^\infty \left(\|\Psi(t)|_{\partial\Omega}\|_{H^1(\partial\Omega)^3}^2 + \int_{\partial\Omega} |\partial_t^{1/2}\Psi(x, t)|^2 do_x + \int_{\partial\Omega} |n^{(\Omega)}(x) \cdot \partial_t\Psi(x, t)|^2 do_x \right) dt \end{aligned} \quad (4.3)$$

Proof: Theorem 4.2, Lemma 4.6 and 3.2. \square

We will need the following estimate on pointwise spatial decay of $\mathfrak{R}^{(\tau)}(f)$.

Theorem 4.3 ([13, Theorem 3.1]) *Let $A \in (2, \infty)$, $B \in [0, 3/2]$ with $A + \min\{1, B\} > 3$, $A + B \geq 7/2$, $\varrho_0 \in (2, \infty)$, $R_0 \in (0, \infty)$, $\gamma \in L^2((0, \infty)) \cap L^{\varrho_0}((0, \infty))$, $f : \mathbb{R}^3 \times (0, \infty) \mapsto \mathbb{R}^3$ measurable with $|f(y, \sigma)| \leq \gamma(\sigma) |y|^{-A} \nu(y)^{-B}$ for $y \in B_{R_0}^c$, $\sigma \in (0, \infty)$. Further suppose that $f|_{B_{R_0} \times (0, \infty)} \in L^2(B_{R_0} \times (0, \infty))^3$. Let $R \in (R_0, \infty)$.*

Then, for $x \in B_R^c$, $t \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, inequality (1.11) holds.

(Note that by Lemma 4.3 and 4.4 with $q = s = 2$, and because $\gamma \in L^2((0, \infty))$ and $A > 2$, the term $\partial_x^\alpha \mathfrak{R}^{(\tau)}(f)(x, t)$ in (1.11) is well defined.)

The next lemma allows to define the potential $\mathfrak{J}^{(\tau)}(a)$ further below and yields a pointwise temporal estimate of this potential.

Lemma 4.7 *Let $p \in [1, \infty]$, $a \in L^p(\mathbb{R}^3)^3$, $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^3$ with $l + |\alpha| \leq 1$. Then*

$$\int_{\mathbb{R}^3} |\partial_t^l \partial_x^\alpha (\mathfrak{H}(x - y - \tau t e_1, t)) a(y)| dy \leq C(p) \|a\|_p (t^{-3/(2p) - |\alpha|/2 - l} + \delta_{1l} t^{-3/(2p) - 1/2})$$

for $x \in \mathbb{R}^3$, $t \in (0, \infty)$, where $3/(2p) := 0$ if $p = \infty$.

Proof: Take x, t as in the lemma, and let the left-hand side of the estimate in Lemma 4.7 be denoted by \mathfrak{A} . In the case $p \in (1, \infty)$, we get with Lemma 4.1 that

$$\begin{aligned} \mathfrak{A} &= \int_{\mathbb{R}^3} |\partial_4^l \partial_x^\alpha \mathfrak{H}(x - y - \tau t e_1, t) - \delta_{1l} \tau \partial_1 \mathfrak{H}(x - y - \tau t e_1, t)| |a(y)| dy \\ &\leq C \|a\|_p \left(\int_{\mathbb{R}^3} \left[(|x - y - \tau t e_1| + t^{1/2})^{(-3 - |\alpha| - 2l)p'} \right. \right. \\ &\quad \left. \left. + \delta_{1l} (|x - y - \tau t e_1| + t^{1/2})^{-4p'} \right] dy \right)^{1/p'} \\ &\leq C \|a\|_p \left(\int_{\mathbb{R}^3} \left[(|z| + t^{1/2})^{(-3 - |\alpha| - 2l)p'} + \delta_{1l} (|z| + t^{1/2})^{-4p'} \right] dz \right)^{1/p'} \\ &\leq C \|a\|_p (t^{-3/2 - |\alpha|/2 - l + 3/(2p')} + \delta_{1l} t^{-2 + 3/(2p')}) \\ &\leq C \|a\|_p (t^{-3/(2p) - |\alpha|/2 - l} + \delta_{1l} t^{-3/(2p) - 1/2}). \end{aligned}$$

If $p = 1$, the estimate $\mathfrak{A} \leq C \|a\|_p (t^{-3/(2p) - |\alpha|/2 - l} + \delta_{1l} t^{-3/(2p) - 1/2})$ follows almost immediately from Lemma 4.1. Suppose $p = \infty$. Then, in the case $\alpha = 0$, $l = 0$,

$$\mathfrak{A} \leq \|a\|_p \int_{\mathbb{R}^3} \mathfrak{H}(x - y - \tau t e_1, t) dy = \|a\|_p \int_{\mathbb{R}^3} \mathfrak{H}(z, t) dz = \|a\|_p$$

by Lemma 4.1, and in the case $|\alpha| + l > 0$ by the same reference,

$$\begin{aligned} \mathfrak{A} &\leq C \|a\|_p \int_{\mathbb{R}^3} ((|z| + t^{1/2})^{-3-|\alpha|-2l} + \delta_{1l} (|z| + t^{1/2})^{-4}) dz \\ &\leq C \|a\|_p (t^{-|\alpha|/2-l} + \delta_{1l} t^{-1/2}). \end{aligned}$$

This completes the proof of the lemma. \square

In view of Lemma 4.7, we may define

$$\mathfrak{J}^{(\tau)}(a)(x, t) := \int_{\mathbb{R}^3} \mathfrak{H}(x - y - \tau t e_1, t) \tilde{a}(y) dy \quad (x \in \mathbb{R}^3, t \in (0, \infty)),$$

where $p \in [1, \infty]$, $a \in L^p(A)^3$, for some measurable subset A of \mathbb{R}^3 , with \tilde{a} denoting the zero extension of a to \mathbb{R}^3 .

Lemma 4.8 *Let $p \in [1, \infty]$ and $a \in L^p(\mathbb{R}^3)^3$. Then $\mathfrak{J}^{(\tau)}(a) \in C^1(\mathbb{R}^3 \times (0, \infty))^3$ and*

$$\partial_t^l \partial_x^\alpha \mathfrak{J}^{(\tau)}(a)(x, t) = \int_{\mathbb{R}^3} \partial_t^l \partial_x^\alpha (\mathfrak{H}(x - y - \tau t e_1, t)) a(y) dy \quad (4.4)$$

for $x \in \mathbb{R}^3$, $t \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$ with $|\alpha| + l \leq 1$.

Proof: See [12, Lemma 2.3] and its proof.

Actually $\mathfrak{J}^{(\tau)}(a)$ is a C^∞ -function, but we will not need this fact.

Theorem 4.4 ([10, Theorem 3.1]) *Let $\epsilon_0 \in (0, 1/2]$ and $a \in H_\sigma^{1/2+\epsilon_0}(\overline{\Omega}^c)$. Then the function $\mathfrak{J}^{(\tau)}(a)|_{S_\infty}$ belongs to H_∞ .*

Corollary 4.3 *Let ϵ_0 and a be given as in the preceding theorem. Take $T \in (0, \infty)$. Then all the conclusions stated in Corollary 4.2 for the function Ψ introduced there hold for $\Psi := \mathfrak{J}^{(\tau)}(a)|_{S_\infty}$ as well.*

Proof: Theorem 4.4, Lemma 4.8 and 3.2. \square

We will need the following pointwise spatial decay estimate of $\mathfrak{J}^{(\tau)}(a)$.

Theorem 4.5 *Let $R_0, \delta_0 \in (0, \infty)$, $\kappa_0 \in (0, 1]$, $a \in L_{loc}^1(\mathbb{R}^3)^3$ such that $a|_{\overline{B_{R_0}^c}} \in W_{loc}^{1,1}(\overline{B_{R_0}^c})^3$, $|\partial^\alpha a(y)| \leq \delta_0 (|y| \nu(y))^{-1-|\alpha|/2-\kappa_0}$ for $y \in \overline{B_{R_0}^c}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$.*

Let $R \in (R_0, \infty)$ and take α as before. Then, for $x \in B_R^c$ and $t \in (0, \infty)$,

$$|\partial_x^\alpha \mathfrak{J}^{(\tau)}(a)(x, t)| \leq \mathfrak{C} (\delta_0 + \|a|_{B_{R_0}}\|_1) (|x| \nu(x))^{-1-|\alpha|/2}.$$

Proof: See [12, Theorem 1.1]. Note that in [12, inequality (1.10)], it should read $|\beta|/2$ instead of $|\beta|$. \square

We turn to a single layer potential whose definition is based on the ensuing

Lemma 4.9 *Let $\phi \in L^2(S_\infty)^3$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Then, for $x \in \mathbb{R}^3 \setminus \partial\Omega$, $t \in (0, \infty)$, we have $\int_0^t \int_{\partial\Omega} |\partial_x^\alpha \Lambda(x-y, t-\sigma, \tau) \cdot \phi(y, \sigma)| do_y d\sigma < \infty$. In addition, the preceding relation holds for a. e. $x \in \partial\Omega$ and a. e. $t \in (0, \infty)$.*

Proof: If $x \in \mathbb{R}^3 \setminus \partial\Omega$, we have $|x - y| \geq \text{dist}(x, \partial\Omega) > 0$ for $y \in \partial\Omega$, so Corollary 4.1 with $K = \text{dist}(x, \partial\Omega)/2$ yields $|\partial_x^\alpha \Lambda_{jk}(x - y, t - \sigma, \tau)| \leq \mathfrak{C} (\text{dist}(x, \partial\Omega))^{-3/2 - |\alpha|/2} > 0$ for y as before, $t \in (0, \infty)$ and $\sigma \in (0, t)$. This estimate and Hölder's inequality imply the first claim of the lemma. The second holds according to [10, Lemma 2.19]. \square

In view of the preceding lemma, we may define

$$\mathfrak{V}^{(\tau)}(\phi)(x, t) := \int_0^t \int_{\partial\Omega} \Lambda(x - y, t - \sigma, \tau) \cdot \tilde{\phi}(y, \sigma) \, do_y \, d\sigma$$

for $T \in (0, \infty]$, $\phi \in L^2(S_T)^3$, $x \in \mathbb{R}^3 \setminus \partial\Omega$, $t \in (0, \infty)$, and for a. e. $x \in \partial\Omega$ and a. e. $t \in (0, \infty)$, where $\tilde{\phi}$ denotes the zero extension of ϕ to S_∞ .

Note that $\mathfrak{V}^{(\tau)}(\phi)$ is defined as a function on $(\mathbb{R}^3 \setminus \partial\Omega) \times (0, \infty)$ and also as a function on $\partial\Omega \times (0, \infty)$. The second of these functions is the trace of the first:

Theorem 4.6 *Let $\phi \in L^2(S_\infty)^3$ and abbreviate $v := \mathfrak{V}^{(\tau)}(\phi)|_{\overline{\Omega}^c \times (0, \infty)}$. Then $v \in W_{loc}^{1,1}(\overline{\Omega}^c \times (0, \infty))^3 \cap C^0(\overline{\Omega}^c \times [0, \infty))^3$, $v(t) \in C^\infty(\overline{\Omega}^c)^3$ and*

$$\partial_l v(x, t) = \int_0^t \int_{\partial\Omega} \partial_l \Lambda(x - y, t - \sigma, \tau) \cdot \phi(y, \sigma) \, do_y \, d\sigma \quad (4.5)$$

for $t \in (0, \infty)$, $x \in \overline{\Omega}^c$, $1 \leq l \leq 3$. Moreover $v \in L^\infty(0, \infty, L^2(\overline{\Omega}^c)^3)$, $\nabla_x v \in L^2(\overline{\Omega}^c \times (0, \infty))^9$, and for $t \in (0, \infty)$, the trace of $v(t)$ coincides with $\mathfrak{V}^{(\tau)}(\phi)(t)$ considered as a function on $\partial\Omega$.

Proof: For the first part of the lemma, up to and including (4.5), we refer to [10, Lemma 2.21]. The second part holds according to [8, Theorem 2.3, 2.4]. \square

If $\phi \in L^2(S_T)^3$ for some $T < \infty$, then $\mathfrak{V}^{(\tau)}(\phi)$ is smooth on $\mathbb{R}^3 \times (T, \infty)$:

Lemma 4.10 *Let $T \in (0, \infty)$ and $\phi \in L^2(S_T)^3$. Then $\mathfrak{V}^{(\tau)}(\phi)|_{\mathbb{R}^3 \times (T, \infty)}$ belongs to $C^1(\mathbb{R}^3 \times (T, \infty))^3$, and*

$$\partial_t^l \partial_x^\alpha \mathfrak{V}^{(\tau)}(\phi)(x, t) = \int_0^T \int_{\partial\Omega} \partial_t^l \partial_x^\alpha \Lambda(x - y, t - \sigma, \tau) \cdot \phi(y, \sigma) \, do_y \, d\sigma$$

for $x \in \mathbb{R}^3$, $t \in (T, \infty)$, $\alpha \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$ with $|\alpha| + l \leq 1$, where the preceding integral exists as Lebesgue integral.

Proof: If $x \in \mathbb{R}^3$, $t \in (T, \infty)$ in the situation of Lemma 4.10, the time integral in the definition of $\mathfrak{V}^{(\tau)}(\phi)(x, t)$ only extends from 0 to T , and not from 0 to t . Thus the lemma follows from Corollary 4.1 and Lebesgue's theorem by a similar but simpler reasoning as in the proof of Lemma 4.6, simpler in particular because $L^2(S_T)^3 \subset L^1(S_T)^3$. \square

We will need the following, much more deep-lying properties of $\mathfrak{V}^{(\tau)}(\phi)$.

Theorem 4.7 *The relation $\mathfrak{V}^{(\tau)}(\phi)|_{S_\infty} \in H_\infty$ holds for $\phi \in L^2(S_\infty)^3$. There is a constant $c_1 > 0$ such that $\|\mathfrak{V}^{(\tau)}(\phi)|_{S_\infty}\|_{H_\infty} \leq c_1 \|\phi\|_2$ for such ϕ .*

Proof: For $\phi \in L^2(S_\infty)^3$, define $\mathfrak{V}^{(0)}(\phi)$ in the same way as $\mathfrak{V}^{(\tau)}(\phi)$, but with the term $\Lambda(x - y, t - \sigma, \tau)$ replaced by $\Gamma(x - y, t - \sigma)$; compare [10, Lemma 2.19] and use

Lemma 4.2. According to [31, Theorem 2.3.3] and the estimates in [31, p. 362], we have $\mathfrak{V}^{(0)}(\phi)|_{S_\infty} \in H_\infty$ for $\phi \in L^2(S_\infty)^3$, and there is $c > 0$ with $\|\mathfrak{V}^{(0)}(\phi)|_{S_\infty}\|_{H_\infty} \leq c\|\phi\|_2$ for ϕ as before. The theorem follows with [9, Theorem 4]. \square

Theorem 4.8 ([9, Corollary 3]) *There is $c_2 > 0$ such that $\|\phi\|_2 \leq \|\mathfrak{V}^{(\tau)}(\phi)|_{S_\infty}\|_{H_\infty}$ for $\phi \in L_n^2(S_\infty)$.*

For $\tilde{b} \in H_\infty$, the integral equation $\mathfrak{V}^{(\tau)}(\phi)|_{S_\infty} = \tilde{b}$ is solved by a unique function $\phi \in L_n^2(S_\infty)$. (The function space $L_n^2(S_\infty)$ was introduced in (2.1)).

Corollary 4.4 *Let $T \in (0, \infty)$ and $\phi \in L^2(S_T)^3$. Then all the conclusions listed in Corollary 4.2 for the function Ψ introduced there are true for $\Psi = \mathfrak{V}^{(\tau)}(\phi)|_{S_\infty}$ as well.*

Proof: Lemma 4.10, Theorem 4.7 and Lemma 2.3. \square

Corollary 4.5 *Let $\phi \in L^2(S_\infty)$, $T \in (0, \infty)$. Then, with c_1 from Theorem 4.7, we have $\|\mathfrak{V}^{(\tau)}(\phi)|_{S_{T,\infty}}\|_{H_{T,\infty}} \leq \|\mathfrak{V}^{(\tau)}(\phi)\|_{H_\infty} \leq c_1\|\phi\|_2$.*

Proof: Theorem 4.7, Corollary 3.2. \square

Corollary 4.6 *Let $T \in (0, \infty)$ and $\phi \in L_n^2(S_\infty)$ with $\phi|_{S_T} = 0$. Then, with c_2 from Theorem 4.8, $\|\phi\|_2 \leq c_2\|\mathfrak{V}^{(\tau)}(\phi)\|_{H_\infty} = c_2\|\mathfrak{V}^{(\tau)}(\phi)|_{S_{T,\infty}}\|_{H_{T,\infty}}$.*

Proof: The inequality stated in the corollary holds according to Theorem 4.8, whereas the equation follows from Corollary 3.2. \square

In view of a representation formula for solutions to (1.12), (1.10), (1.6), we first state a uniqueness theorem for such solutions.

Theorem 4.9 *Let $b : S_\infty \mapsto \mathbb{R}^3$, $a \in L_{loc}^1(\overline{\Omega}^c)^3$, $f \in L_{loc}^2([0, \infty), [H_\sigma^1(\overline{\Omega}^c)]')$. Then there is at most one function $u \in L_{loc}^2([0, \infty), H^1(\overline{\Omega}^c)^3)$ such that*

$$u(t)|_{\partial\Omega} = b(t) \text{ for } t \in (0, \infty), \quad \operatorname{div}_x u = 0, \quad (4.6)$$

$$\int_0^\infty \left(\int_{\overline{\Omega}^c} \left[-(u(x, t) \cdot V(x)) \varphi'(t) \right. \right. \quad (4.7)$$

$$\left. \left. + (\nabla_x u(x, t) \cdot \nabla V(x)) \varphi(t) + \tau (\partial_1 u(x, t) \cdot V(x)) \varphi(t) \right] dx - f(t)(V) \varphi(t) \right) dt$$

$$= \int_{\overline{\Omega}^c} a(x) \cdot V(x) dx \varphi(0)$$

for $\varphi \in C_0^\infty([0, \infty))$, $V \in C_0^\infty(\overline{\Omega}^c)^3$ with $\operatorname{div} V = 0$.

Proof: This theorem may be shown in the same way as an analogous result for the Stokes system. We refer to [11, Theorem 3.7] and its proof. \square

Now we construct a solution to (1.12), (1.10), (1.6) in the form $u = \mathfrak{A}^{(\tau)}(f) + \mathfrak{J}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\phi)$, with ϕ being the solution to the integral equation (4.8) below. In this way we obtain a representation formula for the velocity as a sum of three potential functions, as announced in section 1. We consider this solution as a weak one in the sense of Theorem 4.9 because we want to range it in a uniqueness class which is as large as possible. If f and a are smooth, then u is the velocity part of a solution satisfying (1.12) and (1.6)

pointwise; see [10, Lemma 2.14, Theorem 2.16, Lemma 2.21]. (In [10, Theorem 2.16], there is a misprint: $\mathfrak{R}^{(\tau)}(f)$ is only a C^∞ - and not a C_0^∞ -function.) The boundary condition on $\partial\Omega$ holds according to (4.6), and the boundary condition at infinity is satisfied in the sense that $u(t) \in H^1(\overline{\Omega}^c)^3$ for $t \in (0, \infty)$. If the functions f , a and b decay sufficiently rapidly, this boundary condition will be fulfilled in a stronger way, on the basis of Theorem 4.3, 4.5 and Lemma 7.3 below. We will come back to this subject in section 8.

Theorem 4.10 *Let $\epsilon_0 \in (0, 1/2]$, $q \in [1, 3/2)$, $b \in H_\infty$, $a \in H_\sigma^{1/2+\epsilon_0}(\overline{\Omega}^c)$, $f \in L^2(0, \infty, L^q(\overline{\Omega}^c)^3) \cap L^2(0, \infty, L^{3/2}(\overline{\Omega}^c)^3)$. Then there is a unique function $\phi \in L_n^2(S_\infty)$ (see (2.1)) with*

$$\mathfrak{V}^{(\tau)}(\phi)|_{S_\infty} = -\mathfrak{R}^{(\tau)}(f) - \mathfrak{J}^{(\tau)}(a) + b. \quad (4.8)$$

Moreover there is a unique function $u \in L_{loc}^2([0, \infty), H^1(\overline{\Omega}^c)^3)$ such that (4.6) and (4.7) hold. This function is given by

$$u = \mathfrak{R}^{(\tau)}(f) + \mathfrak{J}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\phi)|_{\overline{\Omega}^c} \times (0, \infty). \quad (4.9)$$

Proof: Theorem 4.10 holds according to [13, Theorem 2.26]. Note that the unique solvability of (4.8) is a consequence of Theorem 4.8, 4.4 and 4.2. \square

5. Temporal decay of the potential $\mathfrak{R}^{(\tau)}(f)$.

This section has two aims. Firstly, we want to estimate $\|\mathfrak{R}^{(\tau)}(f)|_{S_{T,\infty}}\|_{H_{T,\infty}}$, and secondly, we are going to determine an upper bound of $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f)(x, t)|$ under the assumption $|\alpha| \leq 1$. Our starting point will be to split f into a sum $f^{(1)} + f^{(2)}$, with $f^{(1)} = \chi_{\mathbb{R}^3} \times (t/2, \infty)$ and $f^{(2)} = \chi_{\mathbb{R}^3} \times (0, t/2)$. The decay of $\mathfrak{R}^{(\tau)}(f^{(1)})(x, t)$ is then due to the asymptotic behaviour of f for $t \rightarrow \infty$, whereas $\mathfrak{R}^{(\tau)}(f^{(2)})(x, t)$ becomes small for large t due to the decay properties of the fundamental solution Λ . The next lemma addresses some key technical difficulties in the estimate of $\|\mathfrak{R}^{(\tau)}(f^{(2)})|_{S_{T,\infty}}\|_{H_{T,\infty}}$ and also of $\mathfrak{R}^{(\tau)}(f^{(2)})(x, t)$.

Lemma 5.1 *Let $\alpha \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$ with $|\alpha| + l \leq 1$. Let $q \in [1, 4]$, $s \in [1, \infty)$ with $3/(2q) + 1/s > 1 - |\alpha|/2 - l/2$. Take $T \in (0, \infty)$ and $f \in L^s(0, T, L^q(\mathbb{R}^3)^3)$. Then*

$$\begin{aligned} |\partial_t^l \partial_x^\alpha \mathfrak{R}^{(\tau)}(f)(x, t)| &\leq \mathfrak{C} \|f\|_{q,s;T} \left((t-T)^{-3/(2q)-1/s+1-|\alpha|/2-l} \right. \\ &\quad \left. + \delta_{1l} (t-T)^{-3/(2q)-1/s+1/2} \right) \quad \text{for } x \in \mathbb{R}^3, t \in (T, \infty). \end{aligned} \quad (5.1)$$

(Recall that $\mathfrak{R}^{(\tau)}(f)|_{\mathbb{R}^3 \times (T, \infty)} \in C^1(\mathbb{R}^3 \times (T, \infty))^3$ by Lemma 4.6.)

Proof: Take x and t as in the lemma. In the case $l = 0$, hence $|\alpha| \leq 1$, the exponents q and s verify the condition $3/(2q) + 1/s > 1 - |\alpha|/2$. This and the inequality $t - \sigma > (t - T)/2$ for $\sigma \in (0, T)$ allow us to apply Theorem 4.1 with $M = (t - T)/2$ and $\varrho = \infty$. Lemma 4.6 and Theorem 4.1 then yield (5.1) in the case $l = 0$.

Let us suppose for the rest of this proof that $l = 1$. This means in particular that $\alpha = 0$ and $3/(2q) + 1/s > 1/2$. Consider the case $s > 1$ and $q > 1$. Then by Lemma 4.6, Hölder's

inequality and Lemma 4.2,

$$\begin{aligned}
& |\partial_t \mathfrak{A}^{(\tau)}(f)(x, t)| \tag{5.2} \\
& \leq C \int_0^T \int_{\mathbb{R}^3} \sum_{j \in \{5, 4\}} (|x - y - \tau(t - \sigma) e_1| + (t - \sigma)^{1/2})^{-j} |f(y, \sigma)| dy d\sigma \\
& \leq C \|f\|_{q, s; T} \sum_{j \in \{5, 4\}} \left(\int_0^T \left[\int_{\mathbb{R}^3} (|x - y - \tau(t - \sigma) e_1| + (t - \sigma)^{1/2})^{-j q'} dy \right]^{s'/q'} d\sigma \right)^{1/s'}.
\end{aligned}$$

But for any $\sigma \in (0, T)$, $j \in \{5, 4\}$, the preceding integral with respect to y equals $\int_{\mathbb{R}^3} (|z| + (t - \sigma)^{1/2})^{-j q'} dy$, and is hence bounded by $C(q) (t - \sigma)^{-j q'/2 + 3/2}$. Therefore from (5.2),

$$|\partial_t \mathfrak{A}^{(\tau)}(f)(x, t)| \leq C(q) \|f\|_{q, s; T} \sum_{j \in \{5, 4\}} \left(\int_0^T (t - \sigma)^{[-j/2 + 3/(2q')]} s' d\sigma \right)^{1/s'}.$$

But $-j/2 + 3/(2q') = (-j + 3)/2 - 3/(2q)$ for $j \in \{5, 4\}$. Since $3/(2q) + 1/s > 1/2$, as mentioned above, we have $[(-j + 3)/2 - 3/(2q)] s' < -1$ for j as before, so we may conclude that

$$|\partial_t \mathfrak{A}^{(\tau)}(f)(x, t)| \leq C(q, s) \|f\|_{q, s; T} \sum_{j \in \{5, 4\}} (t - T)^{(-j+3)/2 - 3/(2q) + 1/s'}.$$

Thus we have shown (5.1) in the case $s > 1$, $q > 1$, under the assumption $l = 1$, as we may recall. If $s = 1$, $q > 1$, it follows as in (5.2) that $|\partial_t \mathfrak{A}^{(\tau)}(f)(x, t)|$ is bounded by

$$C \sum_{j \in \{5, 4\}} \int_0^T \left(\int_{\mathbb{R}^3} (|x - y - \tau(t - \sigma) e_1| + (t - \sigma)^{1/2})^{-j q'} dy \right)^{1/q'} \|f(\cdot, \sigma)\|_q d\sigma.$$

Since $(t - \sigma)^{1/2} \geq (t - T)^{1/2}$ for $\sigma \in (0, T)$, we may conclude that the term $|\partial_t \mathfrak{A}^{(\tau)}(f)(x, t)|$ may be estimated by $C(q) \|f\|_{q, 1; \infty} \sum_{j \in \{5, 4\}} (t - T)^{-j/2 + 3/(2q')}$, which is the looked-for result in the case $s = 1$, $q > 1$. If $s > 1$, $q = 1$, inequality (5.2) is replaced by the estimate

$$|\partial_t \mathfrak{A}^{(\tau)}(f)(x, t)| \leq C \sum_{j \in \{5, 4\}} \int_0^T \int_{\mathbb{R}^3} (t - \sigma)^{-j/2} |f(y, \sigma)| dy d\sigma,$$

so that $|\partial_t \mathfrak{A}^{(\tau)}(f)(x, t)| \leq C \|f\|_{1, s; T} \sum_{j \in \{5, 4\}} \left(\int_0^T (t - \sigma)^{-j s'/2} d\sigma \right)^{1/s'}$. Inequality (5.1) with $q = 1$, $s > 1$ follows. It is obvious how to evaluate $|\partial_t \mathfrak{A}^{(\tau)}(f)(x, t)|$ if $s = q = 1$. \square

Further below (Corollary 5.1), we will estimate $\|\mathfrak{A}^{(\tau)}(f)\|_{S_{T+\mu, \infty}} \|S_{T+\mu, \infty}\|$ in terms of negative powers of μ , under the assumption that $f|_{\mathbb{R}^3 \times (T, \infty)} = 0$. In the next three lemmas, we derive this type of bound.

Lemma 5.2 *Let q, s, T, f be given as in Lemma 5.1. Suppose in addition that $3/(2q) + 1/s > 3/2$. Let $\mu \in (0, \infty)$. Then*

$$\begin{aligned}
& \left(\int_{T+\mu}^{\infty} \left[\|\mathfrak{A}^{(\tau)}(f)(t)\|_{\partial\Omega}^2 + \int_{\partial\Omega} |n^{(\Omega)} \cdot \partial_t \mathfrak{A}^{(\tau)}(f)(x, t)|^2 dx \right] dt \right)^{1/2} \\
& \leq \mathfrak{C} \|f\|_{q, s; T} (\mu^{-3/(2q) - 1/s + 3/2} + \mu^{-3/(2q) - 1/s + 1/2}).
\end{aligned}$$

(Recall that $\mathfrak{A}^{(\tau)}(f)|_{\mathbb{R}^3 \times (T, \infty)}$ is a C^1 -function by Lemma 4.6.)

Proof: Since $3/(2q)+1/s > 3/2$, the term $(\int_{T+\mu}^{\infty} [(t-T)^{-3/q-2/s+2} + (t-T)^{-3/q-2/s}] dt)^{1/2}$ is bounded by $\mathfrak{C}(\mu^{-3/(2q)-1/s+3/2} + \mu^{-3/(2q)-1/s+1/2})$. Thus the lemma follows from Lemma 5.1. Note that the largest exponent $-3/(2q) - 1/s + 3/2$ arises in the case $\alpha = 0, l = 0$ in Lemma 5.1, and the smallest one $-3/(2q) - 1/s + 1/2$ if $\alpha = 0, l = 1$. \square

In order to deal with $\partial_t^{1/2} \mathfrak{R}^{(\tau)}(f)(x, t)$, we need a preparatory result:

Lemma 5.3 *Let $q \in (1, 2]$, $F \in L^q(\mathbb{R}^3)^3$ and $d \in (0, 3]$. Then, for $r \in (0, \infty)$,*

$$\left(\int_{\partial\Omega} \left[\int_{\mathbb{R}^3} |\Lambda(x-y, r, \tau) \cdot F(y)| dy \right]^2 do_x \right)^{1/2} \leq \mathfrak{C} \|F\|_q \max\{1, r^{-d/(2q')-3/(2q)+1/2}\}.$$

Proof: Choose $R_0 > 0$ so large that $\bar{\Omega} \subset B_{R_0/2}$. This means that $B_{R_0}^c \subset B_{R_0/2}(x)^c$ and $B_{R_0} \subset B_{2R_0}(x)$ for $x \in \partial\Omega$.

Let \mathfrak{A} denote the left-hand side of the estimate stated in the lemma, but with the integral over \mathbb{R}^3 replaced by one over B_{R_0} . Take $r \in (0, \infty)$. Then, by Hölder's inequality we see that \mathfrak{A}^2 is bounded by

$$\mathfrak{C} \sum_{j,k=1}^3 \int_{\partial\Omega} \left[\left(\int_{B_{R_0}} |\Lambda_{jk}(x-y, r, \tau)| dy \right)^{1/q'} \left(\int_{B_{R_0}} |\Lambda_{jk}(x-y, r, \tau)| |F(y)|^q dy \right)^{1/q} \right]^2 do_x.$$

But for $x \in \partial\Omega$, $1 \leq j, k \leq 3$, by Corollary 4.1 with $K = 2R_0$ and because $B_{R_0} \subset B_{2R_0}(x)$,

$$\begin{aligned} \int_{B_{R_0}} |\Lambda_{jk}(x-y, r, \tau)| dy &\leq C(R_0) \int_{B_{R_0}} (|x-y| + r^{1/2})^{-3} dy \\ &\leq C(R_0) r^{-d/2} \int_{B_{2R_0}(x)} (|x-y| + r^{1/2})^{-3+d} dy \leq C(R_0, d) r^{-d/2}. \end{aligned}$$

Therefore $\mathfrak{A} \leq \mathfrak{C} r^{-d/(2q')} \sum_{j,k=1}^3 \left(\int_{\partial\Omega} \left[\int_{B_{R_0}} |\Lambda_{jk}(x-y, r, \tau)| |F(y)|^q dy \right]^{2/q} do_x \right)^{1/2}$. Since $2/q \geq 1$, we may conclude with Theorem 2.1 that

$$\mathfrak{A} \leq \mathfrak{C} r^{-d/(2q')} \sum_{j,k=1}^3 \left(\int_{B_{R_0}} \left(\int_{\partial\Omega} |\Lambda_{jk}(x-y, r, \tau)|^{2/q} do_x \right)^{q/2} |F(y)|^q dy \right)^{1/q}.$$

On the other hand, again by Corollary 4.1 with $K = 2R_0$, for $y \in B_{R_0}$, $1 \leq j, k \leq 3$,

$$\int_{\partial\Omega} |\Lambda_{jk}(x-y, r, \tau)|^{2/q} do_x \leq \mathfrak{C} \int_{\partial\Omega} (|x-y| + r^{1/2})^{-6/q} do_x \leq \mathfrak{C} r^{-3/q+1}.$$

As a consequence, $\mathfrak{A} \leq \mathfrak{C} r^{-d/(2q')-3/(2q)+1/2} \|F\|_q$. Let \mathfrak{B} be defined also by the left-hand side of the estimate stated in the lemma, but this time with the integral over \mathbb{R}^3 replaced by one over $B_{R_0}^c$. We get

$$\mathfrak{B} \leq \sum_{j,k=1}^3 \left(\int_{\partial\Omega} \left[\left(\int_{B_{R_0}^c} |\Lambda_{jk}(x-y, r, \tau)|^{q'} dy \right)^{1/q'} \|F\|_q \right]^2 do_x \right)^{1/2}. \quad (5.3)$$

In order to estimate the integral over $B_{R_0}^c$ in (5.3), we recall that $B_{R_0}^c \subset B_{R_0/2}(x)^c$ for $x \in \partial\Omega$, so we may use Corollary 4.1 with $K = R_0/2$ to obtain

$$\begin{aligned} \int_{B_{R_0}^c} |\Lambda_{jk}(x-y, r, \tau)|^{q'} dy &\leq C(R_0, q) \int_{B_{R_0/2}(x)^c} (|x-y| \nu(x-y))^{-3q'/2} dy \\ &\leq C(R_0, q) \int_{B_{R_0/2}^c} (|z| \nu(z))^{-3q'/2} dz \quad \text{for } x \in \partial\Omega, 1 \leq j, k \leq 3. \end{aligned}$$

But $q \leq 2$, in particular $q < 4$, so $3q'/2 > 2$, hence with Corollary 2.2 we arrive at the inequality $\int_{B_{R_0}^c} |\Lambda_{jk}(x-y, r, \tau)|^{q'} dy \leq C(R_0, q)$. This estimate is inserted into (5.3). We then obtain $\mathfrak{B} \leq \mathfrak{C} \|F\|_q$. The lemma now follows with the estimate of \mathfrak{A} obtained above. \square

Lemma 5.4 *Let $s \in (1, \infty)$, $q \in (1, 3/2)$ with $3/(2q) + 1/s > 3/2$. Let $T, \mu \in (0, \infty)$, $f \in L^s(0, T, L^q(\mathbb{R}^3)^3)$. Then*

$$\left(\int_{T+\mu}^{\infty} \int_{\partial\Omega} |\partial_t^{1/2} \mathfrak{R}^{(\tau)}(f)(x, t)|^2 do_x dt \right)^{1/2} \leq \mathfrak{C} \|f\|_{q,s;T} \left(\sum_{j \in \{2,3\}} \mu^{-3/(2q)-1/s+j/2} + \mu^{-1/s} \right).$$

(According to Corollary 4.2, the fractional derivative $\partial_t^{1/2} \mathfrak{R}^{(\tau)}(f)(x, t)$ is well defined for any $t \in (T, \infty)$ and a. e. $x \in \partial\Omega$.)

Proof: For brevity we set $\Psi := \mathfrak{R}^{(\tau)}(f)|_{S_\infty}$. According to Corollary 4.2, for a. e. $x \in \partial\Omega$ equation (2.2) is verified with $\Psi(x, \cdot)$ in the role of ϕ . We will estimate each term on the right-hand side of (2.2) in the norm of $L^2(S_{T+\mu, \infty})^3$. To begin with, we use Lemma 5.1 to obtain

$$\begin{aligned} &\left(\int_{T+\mu}^{\infty} \int_{\partial\Omega} |(t-T)^{-1/2} \Psi(x, (t+T)/2)|^2 do_x dt \right)^{1/2} \tag{5.4} \\ &\leq \mathfrak{C} \|f\|_{q,s;T} \left(\int_{T+\mu}^{\infty} \int_{\partial\Omega} [(t-T)^{-1/2} ((t+T)/2 - T)^{-3/(2q)-1/s+1}]^2 do_x dt \right)^{1/2} \\ &\leq \mathfrak{C} \|f\|_{q,s;T} \left(\int_{T+\mu}^{\infty} (t-T)^{-3/q-2/s+1} dt \right)^{1/2} \leq \mathfrak{C} \|f\|_{q,s;T} \mu^{-3/(2q)-1/s+1}, \end{aligned}$$

where the last inequality holds because $-3/q - 2/s + 1 < -1$ due to our assumptions on q and s . Again by referring to Lemma 5.1, we get for $t \in (T + \mu, \infty)$, $x \in \partial\Omega$ that the term $|\int_{(t+T)/2}^t (t-r)^{-1/2} \partial_r v(x, r) dr|$ is bounded by

$$\mathfrak{C} \|f\|_{q,s;T} \int_{(t+T)/2}^t (t-r)^{-1/2} \sum_{j \in \{0,1\}} (t-T)^{-3/(2q)-1/s+j/2} dr,$$

and thus by $\mathfrak{C} \|f\|_{q,s;T} \sum_{j \in \{0,1\}} (t-T)^{-3/(2q)-1/s+1/2+j/2}$. As a consequence, since $-3/q - 2/s + 2 < -1$ in view of our assumptions on q and s ,

$$\begin{aligned} &\left(\int_{T+\mu}^{\infty} \int_{\partial\Omega} \left| \int_{(t+T)/2}^t (t-r)^{-1/2} \partial_r \Psi(x, r) dr \right|^2 do_x dt \right)^{1/2} \tag{5.5} \\ &\leq \mathfrak{C} \|f\|_{q,s;T} \sum_{j \in \{0,1\}} \mu^{-3/(2q)-1/s+1+j/2}. \end{aligned}$$

Moreover,

$$\left(\int_{T+\mu}^{\infty} \int_{\partial\Omega} \left| \int_0^{(t+T)/2} (t-r)^{-3/2} v(x,r) dr \right|^2 do_x dt \right)^{1/2} \leq \mathfrak{A}^{(1)} + \mathfrak{A}^{(2)}, \quad (5.6)$$

where $\mathfrak{A}^{(j)}$ for $j \in \{1, 2\}$ is defined as the left-hand side of (5.6) with $\Psi(x, r)$ replaced by

$$\mathfrak{A}^{(1)}(x, r) := \int_0^{\min\{r, T\}} \int_{\mathbb{R}^3} \chi_{(0,1]}(r-\sigma) \Lambda(x-y, r-\sigma, \tau) \cdot f(y, \sigma) dy d\sigma$$

($x \in \partial\Omega$, $r \in (0, \infty)$) in the case $j = 1$, and by a term $\mathfrak{A}^{(2)}(x, r)$ differing from $\mathfrak{A}^{(1)}(x, r)$ insofar as $\chi_{(1,\infty)}(r-\sigma)$ substitutes for $\chi_{(0,1]}(r-\sigma)$ in the case $j = 2$. In order to estimate $\mathfrak{A}^{(1)}$, we exploit the integration over $\partial\Omega$ (Lemma 5.3) in order to reduce the singularity of $\mathfrak{A}^{(1)}(x, r)$ when r tends to zero. In a first step, we use Theorem 2.1 to obtain $\mathfrak{A}^{(1)} \leq \left(\int_{T+\mu}^{\infty} \mathfrak{B}(t)^2 dt \right)^{1/2}$, with

$$\mathfrak{B}(t) := \int_0^{(t+T)/2} (t-r)^{-3/2} \left(\int_{\partial\Omega} |\mathfrak{A}^{(1)}(x, r)|^2 do_x \right)^{1/2} dr$$

Let d be any number from $(0, 3)$, for example $d = 3/2$. Theorem 2.1 applied once more yields that $\left(\int_{\partial\Omega} |\mathfrak{A}^{(1)}(x, r)|^2 do_x \right)^{1/2}$ is bounded by

$$\int_0^{\min\{r, T\}} \chi_{(0,1]}(r-\sigma) \left(\int_{\partial\Omega} \left[\int_{\mathbb{R}^3} |\Lambda(x-y, t-\sigma, \tau) \cdot f(y, \sigma)| dy \right]^2 do_x \right)^{1/2} d\sigma,$$

and hence by $\mathfrak{C} \int_0^{\min\{r, T\}} \chi_{(0,1]}(r-\sigma) \|f(\sigma)\|_q (r-\sigma)^{-d/(2q')-3/(2q)+1/2} d\sigma$ according to Lemma 5.3. This estimate and Hölder's inequality imply for $t \in (T + \mu, \infty)$ that

$$\begin{aligned} \mathfrak{B}(t) &\leq \left(\int_0^{(t+T)/2} (t-r)^{-3s'/2} dr \right)^{1/s'} \\ &\quad \cdot \left(\int_0^{(t+T)/2} \left[\int_0^{\min\{r, T\}} \chi_{(0,1]}(r-\sigma) (r-\sigma)^{-d/(2q')-3/(2q)+1/2} \|f(\sigma)\|_q d\sigma \right]^s dr \right)^{1/s}, \end{aligned}$$

hence with Young's inequality,

$$\mathfrak{B}(t) \leq \mathfrak{C} \|f\|_{q,s;T} (t-T)^{-3/2+1/s'} \int_{\mathbb{R}} \chi_{(0,1]}(\sigma) \sigma^{-d/(2q')-3/(2q)+1/2} d\sigma.$$

But $-d/(2q') - 3/(2q) + 1/2 > -1$ because $d < 3$ and $q > 1$, so

$$\mathfrak{B}(t) \leq \mathfrak{C} \|f\|_{q,s;T} (t-T)^{-3/2+1/s'} \leq \mathfrak{C} \|f\|_{q,s;T} (t-T)^{-1/2-1/s},$$

for t as before. Recalling that $\mathfrak{A}^{(1)} \leq \left(\int_{T+\mu}^{\infty} \mathfrak{B}(t)^2 dt \right)^{1/2}$, we now obtain

$$\mathfrak{A}^{(1)} \leq \mathfrak{C} \|f\|_{q,s;T} \left(\int_{T+\mu}^{\infty} (t-T)^{-1-2/s} dt \right)^{1/2} \leq \mathfrak{C} \|f\|_{q,s;T} \mu^{-1/s}. \quad (5.7)$$

Turning to $\mathfrak{A}^{(2)}$, we deduce from Hölder's inequality that

$$\begin{aligned} \int_0^{(t+T)/2} (t-r)^{-3/2} |\mathfrak{A}^{(2)}(x, r)| dr &\leq \left(\int_0^{(t+T)/2} (t-r)^{-3s'/2} dr \right)^{1/s'} \|\mathfrak{A}^{(2)}(x, \cdot)\|_s \quad (5.8) \\ &\leq \mathfrak{C} (t-T)^{-3/2+1/s'} \|\mathfrak{A}^{(2)}(x, \cdot)\|_s \leq \mathfrak{C} (t-T)^{-1/2-1/s} \|\mathfrak{A}^{(2)}(x, \cdot)\|_s \end{aligned}$$

for $x \in \partial\Omega$, $t \in (T + \mu, \infty)$. But $q < 3/2$, hence $1 - 3/(2q) < 0$, so we get by Theorem 4.1 with $\alpha = 0$, $M = 1$, $\varrho = s$ that $\|\mathfrak{A}^{(2)}(x, \cdot)\|_s \leq C(q, s) \|f\|_{q, s; T}$ for any $x \in \mathbb{R}^3$. This choice of ϱ is possible since $s > 1$. Thus from (5.8), $\mathfrak{A}^{(2)} \leq \mathfrak{C} \left(\int_{T+\mu}^\infty (t-r)^{-1-2/s} dr \right)^{1/2} \|f\|_{q, s; T} \leq \mathfrak{C} \|f\|_{q, s; T} \mu^{-1/s}$. The lemma follows from the preceding inequality, (5.4) – (5.6), (5.7), and from equation (2.2) as indicated at the beginning of this proof. \square

Corollary 5.1 *Let $q \in (1, 3/2)$, $\hat{q} \in [1, 3/2)$, $s \in (1, \infty)$ with $3/(2q) + 1/s > 3/2$, $f \in L^c(0, \infty, L^d(\mathbb{R}^3)^3)$ for $(c, d) \in \{(s, q), (2, \hat{q}), (2, 3/2)\}$. Let $T \in [1, \infty)$. Then*

$$\begin{aligned} \|\mathfrak{A}^{(\tau)}(f)|_{S_{T, \infty}}\|_{H_{T, \infty}} &\leq \mathfrak{C} \|f\|_{q, s; \infty} (T^{-3/(2q)-1/s+3/2} + T^{-1/s}) \\ &\quad + \mathfrak{C} (\|f|_{\mathbb{R}^3 \times (T/2, \infty)}\|_{\hat{q}, 2; \infty} + \|f|_{\mathbb{R}^3 \times (T/2, \infty)}\|_{3/2, 2; \infty}). \end{aligned}$$

(Recall that by Theorem 4.2, we have $\mathfrak{A}^{(\tau)}(f)|_{S_\infty} \in H_\infty$.)

Proof: Put $f^{(1)} := \chi_{\mathbb{R}^3 \times (T/2, \infty)} f$, $f^{(2)} := \chi_{\mathbb{R}^3 \times (0, T/2)} f$. Then by Corollary 3.2 and Theorem 4.2,

$$\|\mathfrak{A}^{(\tau)}(f^{(1)})|_{S_{T, \infty}}\|_{H_{T, \infty}} \leq \|\mathfrak{A}^{(\tau)}(f^{(1)})\|_{H_\infty} \leq (\|f^{(1)}\|_{\hat{q}, 2; \infty} + \|f^{(1)}\|_{3/2, 2; \infty}). \quad (5.9)$$

Concerning $\mathfrak{A}^{(\tau)}(f^{(2)})$, we use equation (4.3) with $f^{(2)}$ in the place of f , and then Lemma 5.2 and 5.4 with T replaced by $T/2$ and $\mu = T/2$. Taking account of the assumption $T \geq 1$, we find that

$$\|\mathfrak{A}^{(\tau)}(f^{(2)})|_{S_{T, \infty}}\|_{H_{T, \infty}} \leq \mathfrak{C} \|f\|_{q, s; \infty} (T^{-3/(2q)-1/s+3/2} + T^{-1/s}). \quad (5.10)$$

Corollary 5.1 follows from (5.9) and (5.10). \square

In section 8, we will need a pointwise estimate on spatial and temporal decay of $\mathfrak{A}^{(\tau)}(f)$, based on Theorem 4.3 and Lemma 5.1. Contrary to the situation in Theorem 4.3, we want to avoid the assumption $f|_{B_{R_0} \times (0, \infty)} \in L^2(B_{R_0} \times (0, \infty))^3$ because it is inconvenient in the nonlinear case. The ensuing lemma allows us to replace this condition by the weaker one $f|_{B_{R_0} \times (0, \infty)} \in L^2(0, \infty, L^1(B_{R_0})^3)$. We will apply this lemma only for $\varrho = 2$, but still consider the case $\varrho \in [1, \infty)$ because one may hope that in future work, the decay rates obtained in the special situation of Lemma 5.5 may be recovered in a more general setting.

Lemma 5.5 *Let $\varrho \in [1, \infty)$, $R_0 \in (0, \infty)$ and $f \in L^\varrho(0, \infty, L^1(B_{R_0})^3)$. Take $R \in (R_0, \infty)$. Then, for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $\epsilon \in [0, 1]$, $x \in B_R^c$ and $t \in (0, \infty)$,*

$$\begin{aligned} &|\partial_x^\alpha \mathfrak{A}^{(\tau)}(f)(x, t)| \\ &\leq \mathfrak{C} \left[(|x| \nu(x))^{-1/2-1/\varrho-|\alpha|/2} \left((1+t)^{-2} \|f\|_{1, \varrho; \infty} + \|f|_{B_{R_0} \times (t/2, \infty)}\|_{1, \varrho; \infty} \right) \right. \\ &\quad \left. + (|x| \nu(x))^{(-1/2-1/\varrho-|\alpha|/2)(1-\epsilon)} (1+t)^{(-1/2-1/\varrho-|\alpha|/2)\epsilon} \|f\|_{1, \varrho; \infty} \right]. \end{aligned}$$

Proof: Take α, ϵ, x, t as above, and put $f^{(1)} := \chi_{B_{R_0} \times (0, t/2)} f$, $f^{(2)} := \chi_{B_{R_0} \times (t/2, \infty)} f$. For $y \in B_{R_0}$, we get

$$|x - y| \geq |x| - |y| = |x|(1 - R_0/R) + |x|R_0/R - |y| \geq |x|(1 - R/R_0) \geq R - R_0, \quad (5.11)$$

where $R - R_0 > 0$. Thus we may apply Corollary 4.1 with $K = R - R_0$. Suppose that $\varrho > 1$. If $\varrho = 1$, the ensuing argument remains valid with some small modifications. Due Lemma 4.6, Corollary 4.1 used as indicated, (5.11) and Lemma 2.4, we get

$$\begin{aligned} & |\partial_x^\alpha \mathfrak{R}^{(\tau)}(f^{(1)})(x, t)| \tag{5.12} \\ & \leq \mathfrak{C} \int_0^{t/2} \int_{B_{R_0}} (|x - y| \nu(x - y) + t - \sigma)^{-3/2 - |\alpha|/2} |f(y, \sigma)| dy d\sigma \\ & \leq \mathfrak{C} \int_0^{t/2} \int_{B_{R_0}} (|x| \nu(x) + t - \sigma)^{-3/2 - |\alpha|/2} |f(y, \sigma)| dy d\sigma \\ & \leq \mathfrak{C} \left(\int_0^{t/2} (|x| \nu(x) + t - \sigma)^{(-3/2 - |\alpha|/2) \varrho'} d\sigma \right)^{1/\varrho'} \|f\|_{1, \varrho; \infty} \\ & \leq \mathfrak{C} (|x| \nu(x) + t)^{-3/2 - |\alpha|/2 + 1/\varrho'} \|f\|_{1, \varrho; \infty}. \end{aligned}$$

In the case $t \leq 1$, we thus get $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f^{(1)})(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{-3/2 + 1/\varrho' - |\alpha|/2} \|f\|_{1, \varrho; \infty}$, and then multiply the right-hand side by $4(1+t)^{-2}$. If $t \geq 1$, it follows from (5.12) that $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f^{(1)})(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{(-3/2 + 1/\varrho' - |\alpha|/2)(1-\epsilon)} (1+t)^{(-3/2 + 1/\varrho' - |\alpha|/2)\epsilon} \|f\|_{1, \varrho; \infty}$. As in (5.12), we obtain that $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f^{(2)})(x, t)|$ is bounded by

$$\mathfrak{C} \left(\int_{t/2}^t (|x| \nu(x) + t - \sigma)^{(-3/2 - |\alpha|/2) \varrho'} d\sigma \right)^{1/\varrho'} \|f|_{B_{R_0} \times (t/2, \infty)}\|_{1, \varrho; \infty},$$

and hence by $\mathfrak{C} (|x| \nu(x))^{-3/2 + 1/\varrho' - |\alpha|/2} \|f|_{B_{R_0} \times (t/2, \infty)}\|_{1, \varrho; \infty}$. The lemma follows from the preceding estimates and because $-3/2 + 1/\varrho' = -1/2 - 1/\varrho$. \square

Lemma 5.6 *Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $q \in [1, 4]$, $\tilde{q}, \bar{q}, s \in [1, \infty)$, $\tilde{s}, \bar{s} \in [1, \infty]$ with $3/(2d) + 1/c > 1 - |\alpha|/2$ for $(c, d) \in \{(s, q), (\tilde{s}, \tilde{q})\}$, and $3/(2\bar{q}) + 1/\bar{s} < 1 - |\alpha|/2$. Suppose that $R_0 \in (0, \infty)$ and $f \in L^c(0, \infty, L^q(B_{R_0}^c)^3)$ for $(c, d) \in \{(s, q), (\tilde{s}, \tilde{q}), (\bar{s}, \bar{q})\}$. Then, for $x \in \mathbb{R}^3$, $t \in [1, \infty)$,*

$$\begin{aligned} |\partial_x^\alpha \mathfrak{R}^{(\tau)}(f)(x, t)| & \leq \mathfrak{C} (\|f\|_{q, s; \infty} (1+t)^{-3/(2q) - 1/s + 1 - |\alpha|/2} \\ & \quad + \|f|_{B_{R_0}^c \times (t/2, \infty)}\|_{\tilde{q}, \tilde{s}; \infty} + \|f|_{B_{R_0}^c \times (t/2, \infty)}\|_{\bar{q}, \bar{s}; \infty}). \end{aligned}$$

Proof: Let x, t as in the lemma, and put $f^{(1)} := \chi_{B_{R_0}^c \times (0, t/2)} f$, $f^{(2)} := \chi_{B_{R_0}^c \times (t/2, \infty)} f$. Lemma 5.1 with f replaced by $f^{(1)}$ and with $T = t/2$, and the assumptions $t \geq 1$, $3/(2q) + 1/s > 1 - |\alpha|/2$ imply $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f^{(1)})(x, t)| \leq \mathfrak{C} \|f\|_{q, s; \infty} (1+t)^{-3/(2q) - 1/s + 1 - |\alpha|/2}$. We further note that according to Lemma 4.4, we have $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f^{(2)})(x, t)| \leq \mathfrak{A}_1 + \mathfrak{A}_2$, with $\mathfrak{A}_1 := \int_{t/2}^t \int_{B_{R_0}^c} \chi_{(0, 1]}(t - \sigma) |\partial_x^\alpha \Lambda(x - y, t - \sigma, \tau) \cdot f(y, \sigma)| dy d\sigma$, and \mathfrak{A}_2 defined as \mathfrak{A}_1 , but with the term $\chi_{(0, 1]}(t - \sigma)$ replaced by $\chi_{(1, \infty)}(t - \sigma)$. Since $3/(2\bar{q}) + 1/\bar{s} < 1 - |\alpha|/2$, Theorem

4.1 with $\varrho = \infty$, $M = 1$ yields $\mathfrak{A}_1 \leq \mathfrak{C} \|f|_{B_{R_0}^c \times (t/2, \infty)}\|_{\tilde{q}, \tilde{s}, \infty}$. Similarly, the assumption $3/(2\tilde{q}) + 1/\tilde{s} > 1 - |\alpha|/2$ and Theorem 4.1, again with $\varrho = \infty$, $M = 1$, imply $\mathfrak{A}_2 \leq \mathfrak{C} \|f|_{B_{R_0}^c \times (t/2, \infty)}\|_{\tilde{q}, \tilde{s}, \infty}$. Lemma 5.6 follows by combining the preceding estimates. \square

Now we collect what we found in this section about temporal pointwise decay of $\mathfrak{R}^{(\tau)}(f)$, and combine it with a pointwise spatial decay estimate.

Corollary 5.2 *Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $\varrho \in [1, \infty)$, $q \in [1, 4]$, $\tilde{q}, \bar{q}, s \in [1, \infty)$, $\tilde{s}, \bar{s} \in [1, \infty]$ with $3/(2d) + 1/c > 1 - |\alpha|/2$ for $(c, d) \in \{(s, q), (\tilde{s}, \tilde{q})\}$, and $3/(2\bar{q}) + 1/\bar{s} < 1 - |\alpha|/2$. Suppose that $R_0 \in (0, \infty)$ and the function $f : \mathbb{R}^3 \times (0, \infty) \mapsto \mathbb{R}^3$ satisfies $f|_{B_{R_0} \times (0, \infty)} \in L^\varrho(0, \infty, L^1(B_{R_0})^3)$, $f|_{B_{R_0}^c \times (0, \infty)} \in L^c(0, \infty, L^q(B_{R_0}^c)^3)$ with (c, d) being anyone of the pairs (s, q) , (\tilde{s}, \tilde{q}) and (\bar{s}, \bar{q}) . Let $R \in (R_0, \infty)$ and further suppose there is $\mathfrak{D}_0 > 0$ such that*

$$|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f|_{B_{R_0}^c \times (0, \infty)})(x, t)| \leq \mathfrak{D}_0 (|x| \nu(x))^{-1/2-1/\varrho-|\alpha|/2} \text{ for } x \in B_R^c, t > 0. \quad (5.13)$$

(This condition is satisfied if, for example, $\varrho = 2$ and $\chi_{B_{R_0}^c \times (0, \infty)} f$ fulfills the assumptions of Theorem 4.3 in the place of f .) Then, for $x \in B_R^c$, $t \in (0, \infty)$, $\epsilon \in [0, 1]$,

$$\begin{aligned} |\partial_x^\alpha \mathfrak{R}^{(\tau)}(f)(x, t)| &\leq \mathfrak{C} [(|x| \nu(x))^{-1/2-1/\varrho-|\alpha|/2} ((\mathfrak{D}_0 + K_1) (1+t)^{-2} + K_2(t)) \\ &\quad + (\mathfrak{D}_0^{1-\epsilon} + K_1) (|x| \nu(x))^{(-1/2-1/\varrho-|\alpha|/2)(1-\epsilon)} \\ &\quad \left((1+t)^{-1/2-1/\varrho-|\alpha|/2} + K_3 (1+t)^{-3/(2q)-1/s+1-|\alpha|/2} + K_4(t)^\epsilon \right)], \end{aligned} \quad (5.14)$$

with $K_1 := \|f|_{B_{R_0} \times (0, \infty)}\|_{1, \varrho, \infty}$, $K_2(t) := \|f|_{B_{R_0} \times (t/2, \infty)}\|_{1, \varrho, \infty}$, $K_3 := \|f|_{B_{R_0}^c \times (0, \infty)}\|_{q, s, \infty}$, $K_4(t) := \|f|_{B_{R_0}^c \times (t/2, \infty)}\|_{\tilde{q}, \tilde{s}, \infty} + \|f|_{B_{R_0}^c \times (t/2, \infty)}\|_{\bar{q}, \bar{s}, \infty}$.

Proof: Take x, t, ϵ as in the lemma. We apply Lemma 5.5 with f replaced by $f^{(1)} := f|_{B_{R_0} \times (0, \infty)}$. It follows that $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f^{(1)})(x, t)|$ is bounded by the right-hand side of (5.14). Suppose that $t \geq 1$. Starting from the equation $d = d^\epsilon d^{1-\epsilon}$ for $d > 0$, we use (5.13) as well as Lemma 5.6 with f replaced by $f^{(2)} := \chi_{B_{R_0}^c \times (0, \infty)} f$. It follows that $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f^{(2)})(x, t)|$ is also bounded by the right-hand side of (5.14). In the case $t \leq 1$, inequality (5.13) multiplied by $4(1+t)^{-2}$ yields a suitable estimate of $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f^{(2)})(x, t)|$. \square

6. Temporal decay of the potential $\mathfrak{J}^{(\tau)}(a)$.

First we consider the behaviour of $\|\mathfrak{J}^{(\tau)}(a)|_{S_{T, \infty}}\|_{H_{T, \infty}}$ when T tends to infinity.

Theorem 6.1 *Let $\epsilon_0 \in (0, 1/2]$, $p \in (1, 2]$, $a \in H_\sigma^{1/2+\epsilon_0}(\overline{\Omega}^c) \cap L^p(\overline{\Omega}^c)^3$ and $T \in [1, \infty)$. Then $\|\mathfrak{J}^{(\tau)}(a)|_{S_{T, \infty}}\|_{H_{T, \infty}} \leq \mathfrak{C} \|a\|_p T^{-3/(2p)+1/2}$. (Note that $\mathfrak{J}^{(\tau)}(a)|_{S_\infty} \in H_\infty$ due to the assumption $a \in H_\sigma^{1/2+\epsilon_0}(\overline{\Omega}^c)$; see Theorem 4.4.)*

Proof: Recall that $\mathfrak{J}^{(\tau)}(a) \in C^1(\mathbb{R}^3 \times (0, \infty))^3$ according to Lemma 4.8. From (4.4) and Lemma 4.7 we get for $\alpha \in \mathbb{N}_0^3$ and $l \in \mathbb{N}_0$ with $|\alpha| + l \leq 1$ that

$$\begin{aligned} & \left(\int_T^\infty \int_{\partial\Omega} |\partial_t^l \partial_x^\alpha \mathfrak{J}^{(\tau)}(a)(x, t)|^2 do_x dt \right)^{1/2} \leq \mathfrak{C} \|a\|_p \left(\int_T^\infty (t^{-3/p - |\alpha| - 2l} + \delta_{1l} t^{-3/p - 1}) dt \right)^{1/2} \\ & \leq \mathfrak{C} \|a\|_p (T^{-3/(2p) - |\alpha|/2 - l + 1/2} + \delta_{1l} T^{-3/(2p)}). \end{aligned}$$

Since $T \geq 1$, the right-hand side of this estimate is bounded by $\mathfrak{C} \|a\|_p T^{-3/(2p) + 1/2}$, so

$$\int_T^\infty (\|\mathfrak{J}^{(\tau)}(a)(t)|_{\partial\Omega}\|_{H^1(\partial\Omega)^3}^2 + \|n^{(\Omega)} \cdot \partial_4 \mathfrak{J}^{(\tau)}(a)(t)\|_2^2) dt \leq \mathfrak{C} \|a\|_p^2 T^{-3/p + 1}. \quad (6.1)$$

Turning to an estimate of $\partial_4^{1/2}(\mathfrak{J}^{(\tau)}(a)|_{S_\infty})$, we note that by Corollary 4.3, this fractional derivative exists, and equation (2.2) holds with $\mathfrak{J}^{(\tau)}(a)(x, \cdot)$ in the role of ϕ and with T replaced by $T/2$, for a. e. $x \in \partial\Omega$. With this in mind, we find with Lemma 4.7 that

$$\begin{aligned} & \left(\int_T^\infty \int_{\partial\Omega} |(t - T/2)^{-1/2} \mathfrak{J}^{(\tau)}(a)(x, (t + T/2)/2)|^2 do_x dt \right)^{1/2} \\ & \leq C(p) \|a\|_p \left(\int_T^\infty (t^{-1/2} (t + T/2)^{-3/(2p)})^2 dt \right)^{1/2} \leq C(p) \|a\|_p T^{-3/(2p)}. \end{aligned} \quad (6.2)$$

We further find for $t \in (T, \infty)$, $x \in \partial\Omega$ that by (4.4) and Lemma 4.7,

$$\begin{aligned} & \int_{(t+T/2)/2}^t (t-r)^{-1/2} |\partial_4 \mathfrak{J}^{(\tau)}(a)(x, r)| dr \\ & \leq C(p) \|a\|_p \int_{(t+T/2)/2}^t (t-r)^{-1/2} \sum_{j \in \{0, 1\}} r^{-3/(2p) - 1/2 - j/2} dr \\ & \leq C(p) \|a\|_p \sum_{j \in \{0, 1\}} (t + T/2)^{-3/(2p) - 1/2 - j/2} \int_{(t+T/2)/2}^t (t-r)^{-1/2} dr \leq C(p) \|a\|_p t^{-3/(2p)}, \end{aligned}$$

where the last inequality is valid because $t \geq T \geq 1$. As a consequence, since $p < 3$, hence $-3/p < -1$,

$$\left(\int_T^\infty \int_{\partial\Omega} \left| \int_{(t+T/2)/2}^t (t-r)^{-1/2} \partial_4 \mathfrak{J}^{(\tau)}(a)(x, r) dr \right| do_x dt \right)^{1/2} \leq C(p) \|a\|_p T^{-3/(2p) + 1/2}. \quad (6.3)$$

Next we consider the term

$$\mathfrak{A} := \left(\int_T^\infty \int_{\partial\Omega} \left| \int_0^{(t+T/2)/2} (t-r)^{-3/2} \mathfrak{J}^{(\tau)}(a)(x, r) dr \right| do_x dt \right)^{1/2}. \quad (6.4)$$

In the case $p > 3/2$, we have $3/(2p) < 1$, so we get with Lemma 4.7 that

$$\begin{aligned} \mathfrak{A} & \leq C(p) \|a\|_p \left(\int_T^\infty \left[\int_0^{(t+T/2)/2} (t-r)^{-3/2} r^{-3/(2p)} dr \right]^2 dt \right)^{1/2} \\ & \leq C(p) \|a\|_p \left(\int_T^\infty t^{-3} \left[\int_0^{(t+T/2)/2} r^{-3/(2p)} dr \right]^2 dt \right)^{1/2} \leq C(p) \|a\|_p T^{-3/(2p)}. \end{aligned} \quad (6.5)$$

Now suppose that $p \leq 3/2$ so that $3/(2p) \geq 1$. Then we use the splitting $\mathfrak{A} \leq \mathfrak{A}_1 + \mathfrak{A}_2$, where \mathfrak{A}_1 is defined in the same way as \mathfrak{A} (see (6.4)), but with the lower bound 0 of the integral with respect to r replaced by $3/4$. Similarly, the definition of \mathfrak{A}_2 follows that of \mathfrak{A} , but the integral with respect to r is to extend from 0 to $3/4$. If $p < 3/2$, we again use the first two inequalities in (6.5), with the lower bound of the integral with respect to r being $3/4$ instead of 0. Since $3/(2p) > 1$, we have $\int_{3/4}^{(t+T/2)/2} t^{-3/(2p)} dr \leq C(p)$, so we get that $\mathfrak{A}_1 \leq C(p) \|a\|_p \left(\int_T^\infty t^{-3} dt \right)^{1/2} \leq C(p) \|a\|_p T^{-1}$.

In the case $3/(2p) = 1$, we observe that $\int_{3/4}^{(t+T/2)/2} t^{-3/(2p)} dr \leq \int_{3/4}^{3t/4} r^{-1} dr = \ln t$. Therefore, again starting as in (6.5), we obtain $\mathfrak{A}_1 \leq C(p) \|a\|_p T^{-1} (1 + \ln T)$ for $p = 3/2$. By what we have seen further above, the preceding estimate is valid in the case $p < 3/(2p)$ as well, and thus holds if $p \leq 3/(2p)$. In order to deal with \mathfrak{A}_2 , we reduce the singularity of the variable r in $\mathfrak{J}^{(\tau)}(a)(x, r)$ for $r \downarrow 0$ by exploiting the integration on $\partial\Omega$. To this end, we use Hölder's inequality and Theorem 2.1, as well as Lemma 4.1, obtaining with the abbreviation $F(x, y, r) := \mathfrak{H}(x - y - \tau r e_1, r)$ that

$$\begin{aligned} & \int_{\partial\Omega} |\mathfrak{J}^{(\tau)}(a)(x, r)|^2 do_x \\ & \leq \int_{\partial\Omega} \left[\left(\int_{\mathbb{R}^3} F(x, y, r) dy \right)^{1/p'} \left(\int_{\mathbb{R}^3} F(x, y, r) |a(y)|^p dy \right)^{1/p} \right]^2 do_x \\ & = \int_{\partial\Omega} \left(\int_{\mathbb{R}^3} F(x, y, r) |a(y)|^p dy \right)^{2/p} do_x \leq \left(\int_{\mathbb{R}^3} \left[\int_{\partial\Omega} F(x, y, r)^{2/p} do_x \right]^{p/2} |a(y)|^p dy \right)^{2/p} \end{aligned}$$

for $r \in (0, \infty)$. Note that $2/p \geq 1$ because $p \in (1, 2]$. Now we apply Lemma 4.1 again to deduce from the preceding estimate that $\left(\int_{\partial\Omega} |\mathfrak{J}^{(\tau)}(a)(x, r)|^2 do_x \right)^{1/2}$ is bounded by

$$C \left(\int_{\mathbb{R}^3} \left[\int_{\partial\Omega} (|x - y - \tau r e_1| + r^{1/2})^{-6/p} do_x \right]^{p/2} |a(y)|^p dy \right)^{1/p},$$

and hence by $\mathfrak{C} r^{-3/(2p)+1/2} \|a\|_p$ for $r \in (0, \infty)$. Therefore, once more by Theorem 2.1,

$$\begin{aligned} \mathfrak{A}_2 & \leq C \left(\int_T^\infty t^{-3} \int_{\partial\Omega} \left[\int_0^{3/4} |\mathfrak{J}^{(\tau)}(a)(x, r)|^2 dr \right] do_x dt \right)^{1/2} \\ & \leq C \left(\int_T^\infty t^{-3} \left[\int_0^{3/4} \left(\int_{\partial\Omega} |\mathfrak{J}^{(\tau)}(a)(x, r)|^2 do_x \right)^{1/2} dr \right]^2 dt \right)^{1/2} \\ & \leq \mathfrak{C} \|a\|_p \left(\int_T^\infty t^{-3} \left[\int_0^{3/4} r^{-3/(2p)+1/2} dr \right]^2 dt \right)^{1/2}. \end{aligned}$$

Since $p > 1$, we have $-3/(2p) + 1/2 > -1$, so we arrive at the inequality $\mathfrak{A}_2 \leq \mathfrak{C} \|a\|_p T^{-1}$. On combining the definition of \mathfrak{A} in (6.4), the estimate of \mathfrak{A} in the case $p > 3/2$ in (6.5), the inequalities $\mathfrak{A} \leq \mathfrak{A}_1 + \mathfrak{A}_2$ and $\mathfrak{A}_1 \leq \mathfrak{C} \|a\|_p T^{-1} (1 + \ln T)$ (see above), and the preceding estimate of \mathfrak{A}_2 , we find $\mathfrak{A} \leq \mathfrak{C} \|a\|_p T^{-\varrho(p)} \sigma(p, T)$, with $\varrho(p) := 3/(2p)$, $\sigma(p, T) := 1$ if $p > 3/2$, and $\varrho(p) := 1$, $\sigma(p, T) := 1 + \ln T$ in the case $p \leq 3/2$. But $T \geq 1$, so we get in any case that $\mathfrak{A} \leq \mathfrak{C} \|a\|_p T^{-3/(2p)+1/2}$. Since equation (2.2) is valid with $\mathfrak{J}^{(\tau)}(a)(x, \cdot)$ in the role of ϕ and with T replaced by $T/2$, for a. e. $x \in \partial\Omega$ (Corollary 4.3), we may deduce

from the preceding estimate and (6.2) – (6.4) that $(\int_T^\infty \int_{\partial\Omega} |\partial_4^{1/2} \mathfrak{J}^{(\tau)}(a)(x, t)|^2 do_x dt)^{1/2} \leq \mathfrak{C} \|a\|_p T^{-3/(2p)+1/2}$. But by Corollary 4.3, equation (4.3) holds for $\Psi := \mathfrak{J}^{(\tau)}(a)|_{S_\infty}$, so Theorem 6.1 follows from (6.1) and the estimate of $\partial_4^{1/2}(\mathfrak{J}^{(\tau)}(a)|_{S_\infty})$ just proved. \square

A pointwise estimate with respect to the asymptotics of $\mathfrak{J}^{(\tau)}(a)$ in time and in space is readily available, due to Theorem 4.5 and Lemma 4.7.

Corollary 6.1 *Take R_0, δ_0, κ_0 and a as in Theorem 4.5. Let $p \in [1, \infty]$ and suppose that $a \in L^p(\overline{\Omega}^c)^3$. Let $R \in (R_0, \infty)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Then, for $\epsilon \in [0, 1]$, $x \in B_R^c$, $t \in (0, \infty)$,*

$$|\mathfrak{J}^{(\tau)}(a)(x, t)| \leq \mathfrak{C} [(\delta_0 + \|a|_{\Omega_{R_0}}\|_1) (|x| \nu(x))^{(-1-|\alpha|/2)} (1+t)^{-2} + (\delta_0 + \|a|_{\Omega_{R_0}}\|_1)^{1-\epsilon} \|a\|_p^\epsilon (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{(-3/(2p)-|\alpha|/2)\epsilon}].$$

Proof: In the case $t \leq 1$, Theorem 4.5 immediately yields that $|\mathfrak{J}^{(\tau)}(a)(x, t)|$ is bounded by $\mathfrak{C}(\delta_0 + \|a|_{\Omega_{R_0}}\|_1)(|x| \nu(x))^{-1-|\alpha|/2} (1+t)^{-2}$. If $t \geq 1$, the looked-for estimate holds due to Theorem 4.5, (4.4), Lemma 4.7 and because $d = d^\epsilon d^{1-\epsilon}$ for $d > 0$. \square

7. Temporal decay of the potential $\mathfrak{V}^{(\tau)}(\phi)$ and of the solution of the integral equation (4.8).

A key element of our theory is a decay estimate of the solution to the integral equation (4.8). This element will be presented in Theorem 7.1 below. Its proof depends on certain features of the potential $\mathfrak{V}^{(\tau)}(\phi)$, which we establish in the ensuing two lemmas.

Lemma 7.1 *Let $\mu, T \in (0, \infty)$ and $\phi \in L^2(S_T)^3$. Then*

$$\left(\int_{T+\mu}^\infty \int_{\partial\Omega} |\partial_t^{1/2} \mathfrak{V}^{(\tau)}(\phi)(x, t)|^2 do_x dt \right)^{1/2} \leq \mathfrak{C} (\mu^{-3/2} + \mu^{-1}) T^{1/2} \|\phi\|_2.$$

(By Corollary 4.4, the fractional derivative $\partial_t^{1/2} \mathfrak{V}^{(\tau)}(\phi)(x, t)$ exists for a. e. $x \in \partial\Omega$ and for $t \in (T, \infty)$).

Proof: For brevity we set $v := \mathfrak{V}^{(\tau)}(\phi)$. By Corollary 4.4, equation (2.2) holds with $v(x, \cdot)$ in the place of the function ϕ in Lemma 2.3, for a. e. $x \in \partial\Omega$. With this reference in mind, we observe that by Lemma 4.10 and 4.2, for $r \in (T, \infty)$, $x \in \partial\Omega$, $l \in \{0, 1\}$,

$$\begin{aligned} |\partial_r^l v(x, r)| &\leq \int_0^T \int_{\partial\Omega} ((r-\sigma)^{-3/2-l} + (r-\sigma)^{-3/2-l/2}) |\phi(y, \sigma)| do_y d\sigma \\ &\leq \mathfrak{C} ((r-T)^{-3/2-l} + (r-T)^{-3/2-l/2}) T^{1/2} \|\phi\|_2, \end{aligned}$$

where we used that $\|\phi\|_1 \leq CT^{1/2} \|\phi\|_2$. The preceding inequality implies

$$\begin{aligned} & \left(\int_{T+\mu}^{\infty} \int_{\partial\Omega} |(t-T)^{-1/2} v(x, (t+T)/2)|^2 do_x dt \right)^{1/2} \\ & \leq \mathfrak{C} T^{1/2} \|\phi\|_2 \left(\int_{T+\mu}^{\infty} (t-T)^{-4} dt \right)^{1/2} \leq \mathfrak{C} T^{1/2} \|\phi\|_2 \mu^{-3/2}, \end{aligned} \quad (7.1)$$

$$\begin{aligned} & \left(\int_{T+\mu}^{\infty} \int_{\partial\Omega} \left| \int_{(t+T)/2}^t (t-r)^{-1/2} \partial_r v(x, r) dr \right|^2 do_x dt \right)^{1/2} \\ & \leq \mathfrak{C} T^{1/2} \|\phi\|_2 \left(\int_{T+\mu}^{\infty} \left[\int_{(t+T)/2}^t (t-r)^{-1/2} ((t-T)^{-5/2} + (t-T)^{-2}) dr \right]^2 dt \right)^{1/2} \\ & \leq \mathfrak{C} T^{1/2} \|\phi\|_2 (\mu^{-3/2} + \mu^{-2}). \end{aligned} \quad (7.2)$$

For $t \in [T + \mu, \infty)$, $r \in (0, (t+T)/2]$, the inequality $t - r \geq (t - T)/2$ holds. Therefore

$$\begin{aligned} \mathfrak{B} & := \left(\int_{T+\mu}^{\infty} \int_{\partial\Omega} \left| \int_0^{(t+T)/2} (t-r)^{-3/2} v(x, r) dr \right|^2 do_x dt \right)^{1/2} \\ & \leq \mathfrak{C} \left(\int_{T+\mu}^{\infty} (t-T)^{-3} \int_{\partial\Omega} \left[\int_0^{(t+T)/2} |v(x, r)| dr \right]^2 do_x dt \right)^{1/2}. \end{aligned} \quad (7.3)$$

By extending the domain of integration of the variable r to $(0, \infty)$, we may separate the integration with respect to r and t . In this way we get

$$\mathfrak{B} \leq \mathfrak{C} \mu^{-1} \left(\int_{\partial\Omega} \left[\int_0^{\infty} |v(x, r)| dr \right]^2 do_x \right)^{1/2}. \quad (7.4)$$

But by Corollary 4.1 with $K = \text{diam } \Omega$, the term $\int_0^{\infty} |v(x, r)| dr$ for $x \in \partial\Omega$ is bounded by $\mathfrak{C} \int_0^{\infty} \int_0^T \int_{\partial\Omega} (|x-y|^2 + r - \sigma)^{-3/2} |\phi(y, \sigma)| do_y d\sigma dr$. Integrating with respect to r , this triple integral, in turn, may be estimated by $\mathfrak{C} \int_0^T \int_{\partial\Omega} |x-y|^{-1} |\phi(y, \sigma)| do_y d\sigma$, and hence with Hölder's inequality by $\mathfrak{C} T^{1/2} \int_{\partial\Omega} |x-y|^{-1} \|\phi(y, \cdot)\|_2 do_y$. On applying Hölder's inequality once more, we may thus deduce from (7.4) that

$$\begin{aligned} \mathfrak{B} & \leq \mathfrak{C} \mu^{-1} T^{1/2} \left(\int_{\partial\Omega} \left[\int_{\partial\Omega} |x-y|^{-1} do_y \right] \left[\int_{\partial\Omega} |x-y|^{-1} \|\phi(y, \cdot)\|_2^2 do_y \right] do_x \right)^{1/2} \\ & \leq \mathfrak{C} \mu^{-1} T^{1/2} \left(\int_{\partial\Omega} \int_{\partial\Omega} |x-y|^{-1} do_x \|\phi(y, \cdot)\|_2^2 do_y \right)^{1/2} \leq \mathfrak{C} \mu^{-1} T^{1/2} \|\phi\|_2. \end{aligned}$$

The lemma follows from this inequality, (7.1) – (7.3) and, as explained at the beginning of this proof, from (2.2). \square

Lemma 7.2 *As in Lemma 7.1, let $\mu, T \in (0, \infty)$ and $\phi \in L^2(S_T)^3$. Then*

$$\begin{aligned} & \left(\int_{T+\mu}^{\infty} \left[\|\mathfrak{Y}^{(\tau)}(\phi)(t)|_{\partial\Omega}\|_{H^1(\partial\Omega)^3}^2 + \|n^{(\Omega)} \cdot \partial_t \mathfrak{Y}^{(\tau)}(\phi)(t)\|_2^2 \right] dt \right)^{1/2} \\ & \leq \mathfrak{C} (\mu^{-2} + \mu^{-1}) T^{1/2} \|\phi\|_2. \end{aligned} \quad (7.5)$$

(Recall that by Lemma 4.10, we have $\mathfrak{Y}^{(\tau)}(\phi)|_{\mathbb{R}^3 \times (T, \infty)} \in C^1(\mathbb{R}^3 \times (T, \infty))$.)

Proof: We again set $v := \mathfrak{V}^{(\tau)}(\phi)$ for brevity. Lemma 4.2 and 4.10 yield for $x \in \partial\Omega$ and $t \in [T + \mu, \infty)$ that

$$\begin{aligned} & |v(x, t)| + \sum_{j=1}^3 |\partial_j v(x, t)| + |\partial_t v(x, t)| \\ & \leq \mathfrak{C} \int_0^T \int_{\partial\Omega} ((t - \sigma)^{-3/2} + (t - \sigma)^{-2} + (t - \sigma)^{-5/2}) |\phi(y, \sigma)| \, d\sigma_y \, d\sigma \\ & \leq \mathfrak{C} ((t - T)^{-3/2} + (t - T)^{-5/2}) \|\phi\|_1 \leq \mathfrak{C} T^{1/2} ((t - T)^{-3/2} + (t - T)^{-5/2}) \|\phi\|_2. \end{aligned} \quad (7.6)$$

Denote the left-hand side of (7.5) by \mathfrak{B} . We may deduce from (7.6) that \mathfrak{B} is bounded by $\mathfrak{C} T^{1/2} \|\phi\|_2 (\int_{T+\mu}^\infty [(t - T)^{-3} + (t - T)^{-5}] \, dt)^{1/2}$, and hence by $\mathfrak{C} T^{1/2} \|\phi\|_2 (\mu^{-1} + \mu^{-2})$. \square

Corollary 7.1 *There is a constant $c_3 > 0$ such that $\|\mathfrak{V}^{(\tau)}(\phi)|_{S_{T+\mu, \infty}}\|_{H_{T+\mu, \infty}}$ is bounded by $c_3 (1 + T)^{1/2} (1 + \mu)^{-1} \|\phi\|_2$, for $T, \mu \in (0, \infty)$ and $\phi \in L^2(S_T)^3$.*

Proof: First suppose that $\mu \in (0, 1]$. Then Corollary 3.2 and Theorem 4.7 yield that $\|\mathfrak{V}^{(\tau)}(\phi)|_{S_{T+\mu, \infty}}\|_{H_{T+\mu, \infty}} \leq \|\mathfrak{V}^{(\tau)}(\phi)|_{S_\infty}\|_{H_\infty} \leq \mathfrak{C} \|\phi\|_2 \leq \mathfrak{C} \|\phi\|_2 (1 + \mu)^{-1} (1 + T)^{1/2}$. If $\mu \geq 1$, we use Lemma 7.1, 7.2 and Corollary 4.4 to obtain that $\|\mathfrak{V}^{(\tau)}(\phi)|_{S_{T+\mu, \infty}}\|_{H_{T+\mu, \infty}}$ is bounded by $\mathfrak{C} T^{1/2} \|\phi\|_2 (\mu^{-1} + \mu^{-2})$, and thus again by $\mathfrak{C} \|\phi\|_2 (1 + \mu)^{-1} (1 + T)^{1/2}$. \square

Theorem 7.1 *Let $\tilde{b} \in H_\infty$. Suppose there are numbers $\delta \in (0, \infty)$, $\zeta \in (0, 1)$ such that $\|\tilde{b}|_{S_{T, \infty}}\|_{H_{T, \infty}} \leq \delta T^{-\zeta}$ for $T \in (1, \infty)$. Let ϕ be the unique function from $L_n^2(S_\infty)$ such that $\mathfrak{V}^{(\tau)}(\phi)|_{S_\infty} = \tilde{b}$ (Theorem 4.8). Then $\|\phi|_{S_{T, \infty}}\|_2 \leq \mathfrak{C} (1 + T)^{-\zeta}$ for $T \in (0, \infty)$.*

Proof: For brevity, set $\epsilon := 1 - \zeta$. Let $n \in \mathbb{N}$ with $n \geq 400 c_1^2 c_2^2$, where c_1 was introduced in Theorem 4.7, and c_2 in Theorem 4.8. Let k be the unique number from \mathbb{N} such that $k\epsilon < 1 \leq (k + 1)\epsilon$. Define $\varphi(\epsilon)$ as in Lemma 2.1. In view of a later proof by induction, suppose that $T, B \in (0, \infty)$ are such that

$$T \geq \max\{4, (160 c_2 c_3 n)^2, (320 c_2 c_3 k)^{1/\varphi(\epsilon)}\}, \quad (7.7)$$

$$B \geq \max\{40 c_2 c_3 (\|\phi|_{S_1}\|_2 + 1), 20\delta\}, \quad \|\phi|_{S_{t, \infty}}\|_2 \leq B t^{-1+\epsilon} \text{ for } t \in [1, T]. \quad (7.8)$$

We want to show that $\|\phi|_{S_{T, \infty}}\|_2 \leq (B/2) T^{-1+\epsilon}$ in this situation, hence $\|\phi|_{S_{t, \infty}}\|_2 \leq B t^{-1+\epsilon}$ for $t \in [1, 2T]$. (Note that $S_{t, \infty} \subset S_{T, \infty}$ for $t \in [T, 2T]$.) With this aim in mind, we refer to Lemma 2.2, choosing a number $i_0 \in \{0, \dots, n - 1\}$ such that the inequality $\|\phi|_{\partial\Omega \times (t_0 - T/(2n), t_0)}\| \leq \|\phi|_{S_T \setminus S_{T/2}}\|_2 n^{-1/2}$ holds with $t_0 := (T/2) (1 + (i_0 + 1)/n)$. Since $t_0 \leq T$ and $\phi|_{S_{t_0, \infty}} \in L_n^2(S_{t_0, \infty})$, we obtain with Corollary 4.6 that

$$\begin{aligned} \|\phi|_{S_{T, \infty}}\|_2 & \leq \|\phi|_{S_{t_0, \infty}}\|_2 \leq c_2 \|\mathfrak{V}^{(\tau)}(\phi|_{S_{t_0, \infty}})|_{S_{t_0, \infty}}\|_{H_{t_0, \infty}} \\ & \leq c_2 (\|\mathfrak{V}^{(\tau)}(\phi)|_{S_{t_0, \infty}}\|_{H_{t_0, \infty}} + \|\mathfrak{V}^{(\tau)}(\phi|_{S_{t_0}})|_{S_{t_0, \infty}}\|_{H_{t_0, \infty}}). \end{aligned} \quad (7.9)$$

where the last inequality holds because $\mathfrak{V}^{(\tau)}(\phi) = \mathfrak{V}^{(\tau)}(\phi|_{S_{t_0}}) + \mathfrak{V}^{(\tau)}(\phi|_{S_{t_0, \infty}})$. Recalling that $\mathfrak{V}^{(\tau)}(\phi)|_{S_\infty} = \tilde{b}$, and splitting $\mathfrak{V}^{(\tau)}(\phi|_{S_{t_0}})$ into a suitable sum, we may conclude from (7.9) that $\|\phi|_{S_{T, \infty}}\|_2 \leq c_2 (\|\tilde{b}|_{S_{t_0, \infty}}\|_{H_{t_0, \infty}} + \mathfrak{A}_1 + \mathfrak{A}_2 + \sum_{j=0}^{k-1} \mathfrak{B}_j + \mathfrak{A}_3)$, where the terms $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_0, \dots, \mathfrak{B}_{k-1}, \mathfrak{A}_3$ all have the form $\|\mathfrak{V}^{(\tau)}(\phi|_A)|_{S_{t_0, \infty}}\|_{H_{t_0, \infty}}$, with A being defined

by $A = \partial\Omega \times (t_0 - T/(2n), t_0)$, $A = S_{t_0 - T/(2n)} \setminus S_{T/4}$ and $A = S_1$ in the case of \mathfrak{A}_1 , \mathfrak{A}_2 and \mathfrak{A}_3 , respectively, and $A = S_{(T/4)^{1-j/k}} \setminus S_{(T/4)^{1-(j+1)/k}}$ in the case of \mathfrak{B}_j , for $0 \leq j \leq k-1$. The definition of \mathfrak{B}_j makes sense because $T \geq 4$ (see (7.7)), so $(T/4)^{1-j/k} > (T/4)^{1-(j+1)/k}$ for j as before. But $t_0 \geq T/2$, so by Corollary 3.2 and the preceding estimate of $\|\phi|_{S_{T,\infty}}\|_2$,

$$\|\phi|_{S_{T,\infty}}\|_2 \leq c_2 (\|\tilde{b}|_{S_{T/2,\infty}}\|_{H_{T/2,\infty}} + \mathfrak{A}_1 + \mathfrak{A}_2 + \sum_{j=0}^{k-1} \tilde{\mathfrak{B}}_j + \tilde{\mathfrak{A}}_3), \quad (7.10)$$

where $\tilde{\mathfrak{B}}_0, \dots, \tilde{\mathfrak{B}}_{k-1}, \tilde{\mathfrak{A}}_3$ are defined in the same way as $\mathfrak{B}_0, \dots, \mathfrak{B}_{k-1}, \mathfrak{A}_3$, respectively, except that the restriction to $S_{t_0,\infty}$ is replaced by a restriction to $S_{T/2,\infty}$, and the mapping $\|\cdot\|_{H_{t_0,\infty}}$ by $\|\cdot\|_{H_{T/2,\infty}}$. Corollary 4.5 and the choice of t_0 imply

$$\begin{aligned} c_2 \mathfrak{A}_1 &\leq c_1 c_2 \|\phi|\partial\Omega \times (t_0 - T/(2n), t_0)\|_2 \leq c_1 c_2 \|\phi|_{S_T \setminus S_{T/2}}\|_2 n^{-1/2} \\ &\leq c_1 c_2 \|\phi|_{S_{T/2,\infty}}\|_2 n^{-1/2}, \end{aligned}$$

so by (7.8) and the choice of n we get $c_2 \mathfrak{A}_1 \leq c_1 c_2 B (T/2)^{-1+\epsilon} n^{-1/2} \leq (B/10) T^{-1+\epsilon}$. Concerning \mathfrak{A}_2 , we use Corollary 7.1 with T replaced by $t_0 - T/(2n)$ and μ by $T/(2n)$, to obtain

$$\begin{aligned} c_2 \mathfrak{A}_2 &\leq c_2 c_3 (1 + T/(2n))^{-1} (1 + t_0 - T/(2n))^{1/2} \|\phi|_{S_{t_0 - T/(2n)} \setminus S_{T/4}}\|_2 \\ &\leq 2 c_2 c_3 n T^{-1} (1 + T)^{1/2} \|\phi|_{S_{T/4,\infty}}\|_2. \end{aligned}$$

But $T \geq 1$, hence $(1+T)^{1/2} \leq 2T^{1/2}$, so it follows from the preceding estimate of $c_2 \mathfrak{A}_2$ and from (7.8) with $t = T/4$ that $c_2 \mathfrak{A}_2 \leq 4 c_2 c_3 n T^{-1/2} \|\phi|_{S_{T/4,\infty}}\|_2 \leq 16 c_2 c_3 n T^{-3/2+\epsilon} B$. By the choice of T in (7.7), we may conclude that $c_2 \mathfrak{A}_2 \leq (B/10) T^{-1+\epsilon}$. Moreover, by Corollary 7.1 with T , μ replaced by $T/2 - 1$ and 1 , respectively, we get $c_2 \tilde{\mathfrak{A}}_3 \leq 4 c_2 c_3 T^{-1} \|\phi|_{S_1}\|_2$, hence $c_2 \tilde{\mathfrak{A}}_3 \leq (B/10) T^{-1} \leq (B/10) T^{-1+\epsilon}$ by the choice of B in (7.8) and because $T \geq 1$. Due to the assumptions on \tilde{b} , the definition of ϵ and because $T > 2$, we obtain $c_2 \|\tilde{b}|_{S_{T/2,\infty}}\|_{H_{T/2,\infty}} \leq \delta(T/2)^{-1+\epsilon}$, so that $c_2 \|\tilde{b}|_{S_{T/2,\infty}}\|_{H_{T/2,\infty}} \leq (B/10) T^{-1+\epsilon}$ by the choice of B in (7.8). This leaves us to estimate $c_2 \sum_{j=0}^{k-1} \tilde{\mathfrak{B}}_j$. To this end let $j \in \{0, \dots, k-1\}$. Then Corollary 7.1 with $(T/4)^{1-j/k}$ in the role of T and $T/2 - (T/4)^{1-j/k}$ in that of μ implies that $c_2 \tilde{\mathfrak{B}}_j$ is bounded by

$$c_2 c_3 (1 + T/2 - (T/4)^{1-j/k})^{-1} (1 + (T/4)^{1-j/k})^{1/2} \|\phi|_{S_{(T/4)^{1-j/k}} \setminus S_{(T/4)^{1-(j+1)/k}}}\|_2.$$

But $T \geq 4$, so $(1 + (T/4)^{1-j/k})^{1/2} \leq 2^{1/2} (T/4)^{1/2-j/(2k)}$ and $1 + T/2 - (T/4)^{1-j/k} \geq 1 + T/4 \geq T/4$. Therefore we get $c_2 \tilde{\mathfrak{B}}_j \leq 8 c_2 c_3 T^{-1/2-j/(2k)} \|\phi|_{S_{(T/4)^{1-(j+1)/k},\infty}}\|_2$, so it follows with (7.8) that

$$c_2 \tilde{\mathfrak{B}}_j \leq 8 c_2 c_3 T^{-1/2-j/(2k)} B (T/4)^{(1-(j+1)/k)(-1+\epsilon)} \leq 32 c_2 c_3 B T^{Z(j)} T^{-1+\epsilon}.$$

where $Z(j) := -1/2 + j/(2k) + 1/k - \epsilon(j+1)/k$. But $Z(j) \leq -\varphi(\epsilon)$ by Lemma 2.1, with $\varphi(\epsilon)$ defined there. Thus $c_2 \tilde{\mathfrak{B}}_j \leq 32 c_2 c_3 B T^{-\varphi(\epsilon)} T^{-1+\epsilon}$, hence by the choice of T in (7.7), $c_2 \tilde{\mathfrak{B}}_j \leq T^{-1+\epsilon} B/(10k)$. Since this holds for any $j \in \{0, \dots, k-1\}$, we thus get $c_2 \sum_{j=0}^{k-1} \tilde{\mathfrak{B}}_j \leq (B/10) T^{-1+\epsilon}$. Combining (7.10) with the preceding estimates

of $\|\tilde{b}|S_{T/2,\infty}\|_{H_{T/2,\infty}}$, \mathfrak{A}_1 , \mathfrak{A}_2 , $\sum_{j=0}^{k-1} \tilde{\mathfrak{B}}_j$ and $\tilde{\mathfrak{A}}_3$, we get $\|\phi|S_{T,\infty}\|_2 \leq (B/2)T^{-1+\epsilon}$, so we have in fact shown that if $\|\phi|S_{t,\infty}\|_2 \leq Bt^{-1+\epsilon}$ for $t \in [1, T]$ (see (7.8)), then $\|\phi|S_{T,\infty}\|_2 \leq (B/2)T^{-1+\epsilon}$.

Now, in view of a proof by induction, put $T_0 := \max\{4, (160c_2c_3n)^2, (320c_2c_3k)^{1/\varphi(\epsilon)}\}$ and $B_0 := \max\{40c_2c_3(\|\phi|S_1\|_2 + 1), 20\delta, \|\phi\|_2 T_0^{1-\epsilon}, 2\|\phi\|_2\}$. Then for $t \in [1, T_0]$, we find $\|\phi|S_{t,\infty}\|_2 \leq \|\phi\|_2 \leq \|\phi\|_2 T_0^{1-\epsilon} t^{-1+\epsilon} \leq B_0 t^{-1+\epsilon}$. Suppose that $m \in \mathbb{N}_0$ and $\|\phi|S_{t,\infty}\|_2 \leq B_0 t^{-1+\epsilon}$ for $t \in [1, 2^m T_0]$. Since B_0 fulfills the condition on B in (7.8), as does $2^m T_0$ the one on T in (7.7), we may apply the first part of this proof with $2^m T_0$ and B_0 in the place of T and B , respectively. It follows that $\|\phi|S_{2^m T_0,\infty}\|_2 \leq (B_0/2)(2^m T_0)^{-1+\epsilon}$. Thus for $t \in [2^m T_0, 2^{m+1} T_0]$,

$$\|\phi|S_{t,\infty}\|_2 \leq \|\phi|S_{2^m T_0,\infty}\|_2 \leq (B_0/2)(2^m T_0)^{-1+\epsilon} \leq B_0(2^{m+1} T_0)^{-1+\epsilon} \leq B_0 t^{-1+\epsilon}.$$

Thus we have shown by induction that $\|\phi|S_{t,\infty}\|_2 \leq B_0 t^{-1+\epsilon}$ for any $t \in [1, 2^m T_0]$ and any $m \in \mathbb{N}_0$, so the preceding inequality is valid for any $t \in [1, \infty)$. Hence, for such t we find $\|\phi|S_{t,\infty}\|_2 \leq 2B_0(1+t)^{-1+\epsilon}$. But for $t \in (0, 1]$, it is obvious that $\|\phi|S_{t,\infty}\|_2 \leq \|\phi\|_2 \leq 2\|\phi\|_2(1+t)^{-1+\epsilon} \leq B_0(1+t)^{-1+\epsilon}$. This completes the proof of Theorem 7.1. \square

We will combine Theorem 7.1 with the following pointwise temporal and spatial estimate of $\mathfrak{Y}^{(\tau)}(\phi)$.

Lemma 7.3 *Let $R \in (0, \infty)$ with $\bar{\Omega} \subset B_R$, $\phi \in L^2(S_\infty)^3$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Then, for $x \in B_R^c$, $t \in (0, \infty)$,*

$$|\partial_x^\alpha \mathfrak{Y}^{(\tau)}(\phi)(x, t)| \leq \mathfrak{C} \left[(|x|\nu(x))^{-1-|\alpha|/2} \|\phi|S_{t/2,\infty}\|_2 + (|x|\nu(x) + t)^{-1-|\alpha|/2} \|\phi\|_2 \right].$$

Proof: Choose $R_0 \in (0, R)$ with $\bar{\Omega} \subset B_{R_0}$. Take x, t as in the lemma. For $y \in \partial\Omega$, we have $|y| \leq R_0$, so we find as in (5.11) that $|x - y| \geq |x|(1 - R_0/R) \geq R - R_0 > 0$, and by Lemma 2.4, $\nu(x - y)^{-1} \leq C|y|\nu(x)^{-1} \leq CR_0\nu(x)^{-1}$. Hence by (4.5) and Corollary 4.1 with $K = R - R_0$,

$$|\partial_x^\alpha \mathfrak{Y}^{(\tau)}(\phi)(x, t)| \leq C(R, R_0, \tau) \int_0^t \int_{\partial\Omega} (|x|\nu(x) + t - \sigma)^{-3/2-|\alpha|/2} |\phi(y, \sigma)| d\sigma,$$

so $|\partial_x^\alpha \mathfrak{Y}^{(\tau)}(\phi)(x, t)| \leq \mathfrak{C} \int_0^t (|x|\nu(x) + t - \sigma)^{-3/2-|\alpha|/2} \|\phi(\sigma)\|_2 d\sigma$. By Hölder's inequality,

$$\begin{aligned} \int_0^{t/2} (|x|\nu(x) + t - \sigma)^{-3/2-|\alpha|/2} \|\phi(\sigma)\|_2 d\sigma &\leq \left(\int_0^{t/2} (|x|\nu(x) + t - \sigma)^{-3-|\alpha|} d\sigma \right)^{1/2} \|\phi\|_2 \\ &\leq C (|x|\nu(x) + t)^{-1-|\alpha|/2} \|\phi\|_2. \end{aligned}$$

If the integral from 0 to $t/2$ in the preceding estimate is replaced by one from $t/2$ to t , the same type of estimate yields the upper bound $C (|x|\nu(x))^{-1-|\alpha|/2} \|\phi|S_{t/2,\infty}\|_2$ for this modified integral. Lemma 7.3 follows from these estimates. \square

Corollary 7.2 *Let $\tilde{b} \in H_\infty$, $\delta \in (0, \infty)$, $\zeta \in (0, 1)$ with $\|\tilde{b}|S_{T,\infty}\|_{H_{T,\infty}} \leq \delta T^{-\zeta}$ for $T \in (1, \infty)$. Let $\phi \in L_n^2(S_\infty)$ with $\mathfrak{Y}^{(\tau)}(\phi)|S_\infty = \tilde{b}$ (Theorem 4.8), $R_0, R \in (0, \infty)$ with*

$R_0 < R$ and $\overline{\Omega} \subset B_{R_0}$, and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Then, for $x \in B_R^c$, $t \in (0, \infty)$, $\epsilon \in [0, 1]$,

$$\begin{aligned} |\partial_x^\alpha \mathfrak{V}^{(\tau)}(\phi)(x, t)| &\leq \mathfrak{C} \left[(|x| \nu(x))^{(-1-|\alpha|/2)} (1+t)^{-\zeta} \right. \\ &\quad \left. + (|x| \nu(x) + t)^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{(-1-|\alpha|/2)\epsilon} \|\phi\|_2 \right]. \end{aligned}$$

Proof: Lemma 7.3 and Theorem 7.1 yield that the estimate in the corollary holds under the additional assumption $t \geq 1$. In the case $t \in (0, 1]$, we deduce from Lemma 7.3 that $|\partial_x^\alpha \mathfrak{V}^{(\tau)}(\phi)(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{(-1-|\alpha|/2)} \|\phi\|_2$, so we again obtain the estimate stated in the corollary. \square

8. Main results.

We begin by collecting our assumptions on the right-hand side f in the differential equations (1.12), (1.9) and (1.4), the initial data a in (1.6) and the Dirichlet boundary data b in (1.10).

Let $A \in (2, \infty)$, $B \in [0, 3/2]$ with $A + \min\{1, B\} > 3$, $A + B \geq 7/2$, $\varrho_0 \in (2, \infty)$, $R_1 \in (0, \infty)$ with $\overline{\Omega} \subset B_{R_1}$, $\gamma \in L^2((0, \infty)) \cap L^{\varrho_0}((0, \infty))$, $f : \mathbb{R}^3 \times (0, \infty) \mapsto \mathbb{R}^3$ measurable with $f|_{B_{R_1} \times (0, \infty)} \in L^2(0, \infty, L^1(B_{R_0})^3)$ and

$$|f(y, \sigma)| \leq \gamma(\sigma) |y|^{-A} \nu(y)^{-B} \quad \text{for } y \in B_{R_0}^c, \sigma \in (0, \infty). \quad (8.1)$$

Moreover, suppose there are numbers $\hat{q}_0 \in [1, 3/2)$, $q_0 \in (1, 3/2)$, $\bar{q}_0 \in (1, \infty)$, $s_0 \in (1, \infty)$, $\bar{s}_0 \in [1, \infty]$ such that $3/(2q_0) + 1/s_0 > 3/2$, $3/(2\bar{q}) + 1/\bar{s} < 1$,

$$f \in L^c(0, \infty, L^d(\mathbb{R}^3)^3) \quad \text{for } (c, d) \in \{(2, 3/2), (2, \hat{q}_0), (s_0, q_0)\}$$

and $f|_{B_{R_1}^c \times (0, \infty)} \in L^{\bar{s}_0}(0, \infty, L^{\bar{q}_0}(B_{R_1}^c)^3)$. Further suppose there are numbers $\delta_1, \zeta_1 \in (0, \infty)$ such that

$$\|f|_{\mathbb{R}^3 \times (t/2, \infty)}\|_{\hat{q}_0, 2; \infty} + \|f|_{\mathbb{R}^3 \times (t/2, \infty)}\|_{3/2, 2; \infty} \leq \delta_1 t^{-\zeta_1} \quad \text{for } t \in (1, \infty). \quad (8.2)$$

Note that (8.2) implies $\|f|_{B_{R_1} \times (t/2, \infty)}\|_{1, 2; \infty} \leq C(R_1) \delta_1 t^{-\zeta_1}$ for $t \in (1, \infty)$. Moreover, let $\epsilon_0 \in (0, 1/2]$, $p_0 \in (1, 2]$, and assume that $a \in H_\sigma^{1/2+\epsilon_0}(\overline{\Omega}^c) \cap L^{p_0}(\overline{\Omega}^c)^3$. In addition, suppose there are numbers $R_2, \delta_2 \in (0, \infty)$, $\kappa_0 \in (0, 1]$ such that $\overline{\Omega} \subset B_{R_2}$, $a|_{\overline{B_{R_2}^c}} \in W_{loc}^{1,1}(\overline{B_{R_2}^c})^3$,

$$|\partial_y^\alpha a(y)| \leq \delta_2 (|y| \nu(y))^{-1-|\alpha|/2-\kappa_0} \quad \text{for } y \in \overline{B_{R_2}^c}, \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq 1. \quad (8.3)$$

Finally let $b \in H_\infty$, and suppose there are numbers $\delta_3 \in (0, \infty)$, $\zeta_2 \in (0, 1)$ such that

$$\|b|_{S_{T, \infty}}\|_{H_{T, \infty}} \leq \delta_3 T^{-\zeta_2} \quad \text{for } T \in (1, \infty). \quad (8.4)$$

Now we turn to our main result on spatial and temporal pointwise decay of solutions to (1.12), (1.10), (1.6). In view of later applications in the nonlinear case, we state this result in the form of a theorem and a corollary.

Theorem 8.1 *Suppose that f , a and b satisfy the assumptions listed above. Then there is a unique function $\phi \in L_n^2(S_\infty)$ verifying the integral equation (4.8). Put $u := \mathfrak{R}^{(\tau)}(f) + \mathfrak{J}^{(\tau)}(a) + \mathfrak{B}^{(\tau)}(\phi)|\overline{\Omega}^c \times (0, \infty)$ (see section 4 for the definition of the preceding potential functions). Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. In addition to the assumptions above, suppose that $3/(2\bar{q}) + 1/\bar{s} < 1 - |\alpha|/2$. Put*

$$\begin{aligned}\varrho_1 &:= \min\{\zeta_1, \zeta_2, 3/(2q_0) + 1/s_0 - 3/2, 1/s_0, 3/(2p_0) - 1/2\}, \\ \varrho_2 &:= \min\{\zeta_1, 1 + |\alpha|/2, 3/(2q_0) + 1/s_0 - 1 + |\alpha|/2, 3/(2p_0) + |\alpha|/2\}.\end{aligned}$$

Let $R \in (\max\{R_1, R_2\}, \infty)$. (The parameters $\bar{q}_0, \bar{s}_0, q_0, s_0, \zeta_1, \zeta_2, R_1, R_2, p_0$ were introduced at the beginning of this section.) Then, for $x \in B_R^c$, $t \in (0, \infty)$, $\epsilon \in [0, 1]$,

$$\begin{aligned}|\partial_x^\alpha u(x, t)| &\leq \mathfrak{C} \left[(|x|\nu(x))^{-1-|\alpha|/2} (1+t)^{-\varrho_1} \right. \\ &\quad \left. + (|x|\nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} \left((1+t)^{-\varrho_2} + \|f|B_{R_1}^c \times (t/2, \infty)\|_{\bar{q}_0, \bar{s}_0; \infty}^\epsilon \right) \right].\end{aligned}\quad (8.5)$$

Proof: Concerning existence and uniqueness of ϕ , we refer to Theorem 4.10. In order to prove (8.5), take ϵ, x, t as in the theorem. By Corollary 5.1 and assumption (8.2), we have $\|\mathfrak{R}^{(\tau)}(f)|S_{T, \infty}\|_{H_{T, \infty}} \leq \mathfrak{C} T^{-\min\{3/(2q_0)+1/s_0-3/2, 1/s_0, \zeta_1\}}$ for $T \in (1, \infty)$. Moreover, by Theorem 6.1, $\|\mathfrak{J}^{(\tau)}(a)|S_{T, \infty}\|_{H_{T, \infty}} \leq \mathfrak{C} T^{-3/(2p_0)+1/2}$ equally for $T \in (1, \infty)$. These estimates, our assumptions on b , equation (4.8) and Corollary 7.2 yield

$$\begin{aligned}|\partial_x^\alpha \mathfrak{B}^{(\tau)}(\phi)(x, t)| &\leq \mathfrak{C} \left[(|x|\nu(x))^{-1-|\alpha|/2} (1+t)^{-\zeta_1} \right. \\ &\quad \left. + (|x|\nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{(-1-|\alpha|/2)\epsilon} \right],\end{aligned}\quad (8.6)$$

with ϱ_1 defined in the theorem. Theorem 4.3 with R_0, f replaced by $R_1, \chi_{B_{R_1}^c \times (0, \infty)} f$, respectively, and (8.1) imply $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f|B_{R_1}^c \times (0, \infty))(y, r)| \leq \mathfrak{C} (|y|\nu(y))^{-1-|\alpha|/2}$ for $y \in B_R^c$, $r \in (0, \infty)$. This estimate, Corollary 5.2 with $\varrho = 2$, $(\tilde{s}, \tilde{q}) = (2, \hat{q}_0)$ and assumption (8.2) yield

$$\begin{aligned}|\partial_x^\alpha \mathfrak{R}^{(\tau)}(f)(x, t)| &\leq \mathfrak{C} \left[(|x|\nu(x))^{-1-|\alpha|/2} (1+t)^{-\zeta_1} \right. \\ &\quad \left. + (|x|\nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} \left((1+t)^{-\varrho_2} + \|f|B_{R_1}^c \times (t/2, \infty)\|_{\bar{q}_0, \bar{s}_0; \infty}^\epsilon \right) \right].\end{aligned}\quad (8.7)$$

Corollary 6.1 provides the estimate

$$\begin{aligned}|\partial_x^\alpha \mathfrak{J}^{(\tau)}(a)(x, t)| &\leq \mathfrak{C} \left[(|x|\nu(x))^{-1-|\alpha|/2} (1+t)^{-2} \right. \\ &\quad \left. + (|x|\nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{(-3/(2p_0)-|\alpha|/2)\epsilon} \right].\end{aligned}\quad (8.8)$$

Inequality 8.5 follows from (8.6) - (8.8). \square

Theorem 4.9, 4.10 and 8.1 taken together yield the following

Corollary 8.1 *Under the assumptions on f , a and b listed at the beginning of this section, there is a unique function $u \in L_{loc}^2(0, \infty, H^1(\overline{\Omega}^c)^3)$ satisfying (4.6) and (4.7) (u velocity part of a solution to (1.12), (1.10), (1.6)). This function is given by $u = \mathfrak{R}^{(\tau)}(f) + \mathfrak{J}^{(\tau)}(a) + \mathfrak{B}^{(\tau)}(\phi)|\overline{\Omega}^c \times (0, \infty)$, with ϕ from (4.8), and it fulfills (8.5).*

In order to exhibit the best possible rate of temporal decay inherent to inequality (8.5), we consider the case that f and a are bounded functions of compact support:

Corollary 8.2 *Suppose that $f : \mathbb{R}^3 \times (0, \infty) \mapsto \mathbb{R}^3$ is measurable, bounded and with compact support, and $a \in H_\sigma^1(\overline{\Omega}^c)$ also bounded and with compact support. Let $\zeta \in [1/2, 1)$ and suppose that (8.4) is satisfied with $\zeta_2 = \zeta$. Let u be the solution to (4.6), (4.7). Then there is $R \in (0, \infty)$ with $\overline{\Omega} \subset B_R$ such that $|\partial_x^\alpha u(x, t)|$ is bounded by*

$$\mathfrak{C} \left[(|x| \nu(x))^{-1-|\alpha|/2} (1+t)^{-\zeta} + (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{(-1-|\alpha|/2)\epsilon} \right]$$

for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $\epsilon \in [0, 1]$, $x \in B_R^c$ and $t \in (0, \infty)$.

Proof: There are numbers $q \in (1, 3/2)$, $p \in (1, 2]$ and $s \in (1, \infty)$ so close to 1 that $3/(2q) + 1/s - 3/2 \geq \zeta$ (in particular $3/(2q) + 1/s - 1 + |\alpha|/2 \geq 1 + |\alpha|/2$), $1/s \geq \zeta$ and $3/(2p) - 1/2 \geq \zeta$ (in particular $3/(2p) + |\alpha|/2 \geq 1 + |\alpha|/2$). Moreover we may choose $R_0, T_0 \in (0, \infty)$ such that $\overline{\Omega} \subset B_{R_0}$, $\text{supp}(f) \subset B_{R_0} \times [0, T_0]$ and $\text{supp}(a) \subset B_{R_0}$. Then $f|_{\mathbb{R}^3 \times (t/2, \infty)} = 0$ for $t \in [2T_0, \infty)$, so there is $c > 0$ with

$$\|f|_{\mathbb{R}^3 \times (t/2, \infty)}\|_{1,2;\infty} + \|f|_{\mathbb{R}^3 \times (t/2, \infty)}\|_{3/2,2;\infty} \leq c(1+t)^{-2} \quad \text{for } t \in (0, \infty). \quad (8.9)$$

Thus the assumptions on f and a stated at the beginning of the chapter are verified if we suppose that $R_1 = R_0$, $\gamma = 0$, $\hat{q} = 1$, $q_0 = q$, $s_0 = s$, $p_0 = p$, with q, s, p as chosen above, $\delta_1 = c$, $\zeta_1 = 2$, $\epsilon_0 = 1/2$, $R_2 = R_0$. The parameters $A, B, \varrho_0, \bar{q}_0, \bar{s}_0, \delta_2$ and κ_0 are irrelevant due to the choice of R_0, γ and T_0 ; they may be chosen as a matter of form in any way corresponding to the specifications given at the beginning of this section. According to the assumptions in the corollary, inequality (8.4) holds with $\zeta_2 = \zeta$.

Now take $R \in (R_0, \infty)$. Then we may conclude with Theorem 8.1 that (8.5) holds. But due to (8.9) and our choice of q, s, p, ζ_1 and ζ_2 , the parameters ϱ_1 and ϱ_2 in (8.5) equal ζ and $1 + |\alpha|/2$, respectively. This completes the proof. \square

Turning to the nonlinear systems (1.9) and (1.4), we first specify which type of solution to the stationary problem (1.7) will be considered.

Theorem 8.2 *Let $B \in H^{1/2}(\partial\Omega)^3$ with $\int_{\partial\Omega} B \cdot n^{(\Omega)} dx = 0$, $\Psi \in L^{6/5}(\overline{\Omega}^c)^3$, $c, R \in (0, \infty)$, $\sigma \in (4, \infty)$ such that $|\Psi(y)| \leq c|x|^{-\sigma}$ for $x \in B_R^c$.*

Then there is $U \in L^6(\overline{\Omega}^c)^3 \cap W_{loc}^{1,1}(\overline{\Omega}^c)^3$ with $\nabla U \in L^2(\overline{\Omega}^c)^9$, $\text{div} U = 0$, $U|_{\partial\Omega} = -e_1 + B$, $\int_{\overline{\Omega}^c} [\nabla U \cdot \nabla V + (\tau \partial_1 U + \tau(U \cdot \nabla)U - \Psi) \cdot V] dx = 0$ for any $V \in C_0^\infty(\overline{\Omega})^3$ with $\text{div} V = 0$. In addition, there are numbers $R_3, c \in (0, \infty)$ such that $\overline{\Omega} \subset B_{R_3}$ and

$$|\partial_x^\alpha U(x)| \leq c (|x| \nu(x))^{-1-|\alpha|/2} \quad \text{for } x \in B_{R_3}^c, \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq 1. \quad (8.10)$$

Proof: For the first part of this theorem, up to but excluding inequality (8.10), we refer to [21, Theorem IX.4.1], [20, Theorem II.5.1]. As for (8.10), a slightly different version of this estimate was proved in [2] in the case $\Psi = 0$; see [2, p. 658 above and p. 661, (3.8)]. In [5], inequality (8.10) was deduced from the theory in [21]. \square

In order to deal simultaneously with initial-boundary value problem (1.9), (1.10), (1.6) on the one hand and (1.4), (1.10), (1.6) on the other, we introduce the new parameter $\tilde{\tau}$,

which is to take the values 0 or τ : $\tilde{\tau} \in \{0, \tau\}$. Then we consider a function u with the following properties:

$$u \in L^\infty(0, \infty, H^1(\overline{\Omega^c})^3), \nabla_x u \in L^2(\overline{\Omega^c} \times (0, \infty))^9, \quad (8.11)$$

$$u(t)|_{\partial\Omega} = b(t) \text{ for } t \in (0, \infty), \operatorname{div}_x u = 0,$$

$$\begin{aligned} & \int_0^\infty \int_{\overline{\Omega^c}} \left(-u(x, t) \cdot V(x) \varphi'(t) + \nabla_x u(x, t) \cdot \nabla V(x) \varphi(t) \right. \\ & \quad + [\tau \partial_1 u(x, t) + \tau (u(x, t) \cdot \nabla_x) u(x, t) + \tilde{\tau} (U(x) \cdot \nabla_x) u(x, t) + \tilde{\tau} (u(x, t) \cdot \nabla) U(x) \\ & \quad \left. - f(x, t) \right] \cdot V(x) \varphi(t) \, dx \, dt = \int_{\overline{\Omega^c}} a(x) \cdot V(x) \, dx \varphi(0) \end{aligned} \quad (8.12)$$

for $\varphi \in C_0^\infty([0, \infty))$, $V \in C_0^\infty(\overline{\Omega^c})^3$ with $\operatorname{div} V = 0$. This means that u is an L^2 -strong solution to (1.9), (1.10), (1.6) if $\tilde{\tau} = 1$, and an L^2 -strong solution to (1.4), (1.10), (1.6) in the case $\tilde{\tau} = 0$. Of course, strictly speaking, u is merely the velocity part of such a solution. But as usual in this context, the preceding definition involves only this velocity part, which we call “solution” without any further qualification. Results on existence of this type of solutions are due to Heywood [23, p. 674], Solonnikov [34, Theorem 10.1, Remark 10.1], and Neustupa [29, Theorem 1], [30, Theorem 4.1] under smallness conditions on the data. Solutions in L^p -spaces with $p \neq 2$ were constructed by Solonnikov [34], Miyakawa [27], Shibata [32] and Enomoto, Shibata [17]. If a solution as in (8.11), (8.12) is considered as given, and if the data f and a decay in space as specified in (8.1) and (8.3), then it can be shown without any smallness conditions that u , too, decays in space. This is the result whose essential point is stated in (1.11), and which we now formulate in detail, choosing a version which is suitable for what follows.

Theorem 8.3 *Suppose that f , a and b satisfy the assumptions listed above (some of which are not relevant here because they are related to temporal decay). Let U be the function from Theorem 8.2, and assume that u satisfies (8.11) and (8.12). Put $F(x, t) := -\tau (u(x, t) \cdot \nabla_x) u(x, t)$, $G(x, t) := -\tilde{\tau} (U(x) \cdot \nabla_x) u(x, t) - \tilde{\tau} (u(x, t) \cdot \nabla) U(x)$ for $x \in \overline{\Omega^c}$, $t \in (0, \infty)$. Then there are constants c , $R_4 \in (0, \infty)$ such that $\overline{\Omega} \subset B_{R_4}$, $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(F + G)(x, t)| \leq c (|x| \nu(x))^{-1-|\alpha|/2}$ and $|\partial_x^\alpha u(x, t)| \leq c (|x| \nu(x))^{-1-|\alpha|/2}$ for $x \in B_{R_4}^c$, $t \in (0, \infty)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$.*

Proof: First consider the case $\tilde{\tau} = 1$. We refer to [14, Corollary 3.5, Theorem 4.6, 4.8] for the first inequality, which actually is an intermediate result in the proof of the second ([14, Theorem 1.2]). Note that in [14], the function f is supposed to fulfill the relation $f|_{B_{R_1} \times (0, \infty)} \in L^2(B_{R_1} \times (0, \infty))^3$, instead of the weaker condition $f|_{B_{R_1} \times (0, \infty)} \in L^2(0, \infty, L^1(B_{R_1})^3)$ required here. However, the former condition enters into [14] only via [14, Theorem 2.16], which restates [13, Theorem 3.1] and is reproduced as Theorem 4.3 in the work at hand. But the conclusion of this theorem remains valid if its assumptions are modified as indicated above. In fact, Theorem 4.3 as it stands yields (1.11) in the case that f is replaced by $\chi_{B_{R_0}^c \times (0, \infty)} f$, and Lemma 5.5 implies (1.11) if $f \in L^2(0, \infty, L^1(B_{R_0})^3)$. (In these two references, the parameter R_0 is in the role of R_1 here.) Combining these two results, we end up with inequality (1.11) under the modified assumptions on f . We further remark that in [14], we supposed $\Psi = 0$ and $B = 0$ in the analogue of Theorem

8.2 ([14, Theorem 1.1]). However, these relations are not used anywhere in [14]; only the conclusions on U stated in Theorem 8.2 are relevant. Concerning the case $\tilde{\tau} = 0$, we indicate that all the proofs in [14] also carry through, some of them in a much simpler form, if the function U in that reference vanishes. Therefore the preceding arguments remain valid if $\tilde{\tau} = 0$. \square

In the ensuing lemmas, we collect some integrability properties of the functions F (non-linearity) and G defined in Theorem 8.3.

Lemma 8.1 *Let u be a function satisfying (8.11), and define F and G as in Theorem 8.3. Let $H \in \{F, G\}$. Then $H \in L^2(0, \infty, L^\varrho(\overline{\Omega}^c)^3)$ and $\|H|_{\overline{\Omega}^c} \times (t/2, \infty)\|_{\varrho, 2; \infty} \leq \mathfrak{C} \|\nabla_x u|_{\overline{\Omega}^c} \times (t/2, \infty)\|_2$ for $t \in (0, \infty)$, $\varrho \in [1, 3/2]$ in the case of $H = F$, and $\varrho \in [11/10, 3/2]$ if $H = G$.*

Proof: First suppose that $H = F$. Then the statements of the lemma are well known. By the proof of [14, Lemma 3.2, estimate (3.1) with $s_0 = 2$, $r_0 = 6$] for example, we have $\|F(r)\|_\varrho \leq \mathfrak{C} (\|u(r)\|_2 + \|u(r)\|_6) \|\nabla_x u(r)\|_2$ for $r \in (0, \infty)$, $\varrho = 1$ and $\varrho = 3/2$. The claims in Lemma 8.1 with respect to F follow with (8.11) and Theorem 2.2.

Now consider the case $H = G$. Let $r \in (0, \infty)$. By Hölder's inequality and Theorem 2.2, we get $\|(U \cdot \nabla_x)u(r)\|_{3/2} \leq C \|U\|_6 \|\nabla_x u(r)\|_2$ and $\|(u(r) \cdot \nabla)U\|_{3/2} \leq C \|u(r)\|_6 \|\nabla U\|_2 \leq \mathfrak{C}(U) \|\nabla_x u(r)\|_2$, so that $\|G(r)\|_{3/2} \leq \mathfrak{C}(U) \|\nabla_x u(r)\|_2$. Moreover, as in the proof of [14, Lemma 3.2], inequality (8.10) and Corollary 2.2 imply $U \in L^{22/9}(\overline{\Omega}^c)^3$, $\nabla U \in L^{66/49}(\overline{\Omega}^c)^9$, so by Hölder's inequality, $\|(U \cdot \nabla_x)u(r)\|_{11/10} \leq C \|U\|_{22/9} \|\nabla_x u(r)\|_2 \leq \mathfrak{C}(U) \|\nabla_x u(r)\|_2$ and $\|(u(r) \cdot \nabla)U\|_{11/10} \leq \|u(r)\|_6 \|\nabla U\|_{66/49} \leq \mathfrak{C}(U) \|\nabla_x u(r)\|_2$. Therefore $\|G(r)\|_{11/10}$ is bounded by $\mathfrak{C}(U) \|\nabla_x u(r)\|_2$. The statements in the lemma relating to G now follow by interpolation. Note that in [14, (3.4)], it should read $\|u\|_{6, 2; T_0}$ instead of $\|u\|_{2, 6; T_0}$. \square

Corollary 8.3 *In the situation of Theorem 8.3, let $\overline{R} \in [R_4, \infty)$, and take $R \in (\overline{R}, \infty)$. Then the estimate $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(F + G|_{B_R^c} \times (0, \infty))(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{-1-|\alpha|/2}$ holds for $x \in B_R^c$, $t \in (0, \infty)$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$.*

Proof: By Lemma 8.1, we have $F + G \in L^2(0, \infty, L^{11/10}(\overline{\Omega}^c)^3)$, so Lemma 5.5 with $\epsilon = 0$ and $F + G|_{B_R^c} \times (0, \infty)$ in the place of f yields for x, t, α as above that $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(F + G|_{B_R^c} \times (0, \infty))(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{-1-|\alpha|/2}$. Thus the corollary follows from the first inequality in Theorem 8.3. \square

Lemma 8.2 *Let f, a, b, U, u, F, G be given as in Theorem 8.3, and take $p \in [2, 6]$, $\varrho \in [p, \infty)$. Put $\overline{R} := \max\{R_3, R_4\}$, with R_3 from Theorem 8.2 and R_4 from Theorem 8.3. Then, for $H \in \{F, G\}$, the function $H|_{B_{\overline{R}}^c} \times (0, \infty)$ belongs to $L^\varrho(0, \infty, L^p(B_{\overline{R}}^c)^3)$, and $\|H|_{B_{\overline{R}}^c} \times (t, \infty)\|_{p, \varrho; \infty} \leq \mathfrak{C} \|\nabla_x u|_{\overline{\Omega}^c} \times (t, \infty)\|_2^{2/\varrho}$ for $t \in (0, \infty)$.*

Proof: We note that $(1/p - 1/6)^{-1} \in (4/3, \infty]$, so $\|\nabla U|_{B_{\overline{R}}^c}\|_{(1/p - 1/6)^{-1}} < \infty$ by (8.10) and Corollary 2.2. Inequality (8.10) further yields that $\|U|_{B_{\overline{R}}^c}\|_\infty < \infty$. Define $G_1(x, r) := (u(x, r) \cdot \nabla)U(x)$ for $x \in \overline{\Omega}^c$, $r \in (0, \infty)$. Then we may conclude with Theorem 2.2 that $\|G_1(r)|_{B_{\overline{R}}^c}\|_p \leq C \|u(r)\|_6 \|\nabla U|_{B_{\overline{R}}^c}\|_{(1/p - 1/6)^{-1}} \leq \mathfrak{C}(U) \|\nabla u(r)\|_2$ for $r \in (0, \infty)$. Let

$t \in [0, \infty)$. Since $\varrho \geq p \geq 2$, the preceding estimate implies

$$\begin{aligned} \|G_1|_{B_R^c} \times (t, \infty)\|_{p, \varrho; \infty} &\leq \mathfrak{C}(U) \left(\int_t^\infty \|\nabla_x u(r)\|_2^\varrho dr \right)^{1/\varrho} \\ &\leq \mathfrak{C} \|\nabla_x u\|_{2, \infty; \infty}^{1-2/\varrho} \left(\int_t^\infty \|\nabla_x u(r)\|_2^2 dr \right)^{1/\varrho} \leq \mathfrak{C} \|\nabla_x u\|_{2, \infty; \infty}^{1-2/\varrho} \|\nabla_x u|_{\overline{\Omega}^c} \times (t, \infty)\|_2^{2/\varrho}. \end{aligned}$$

Put $G_2(x, r) := (U(x) \cdot \nabla_x)u(x, r)$ for $x \in \overline{\Omega}^c$, $r \in (0, \infty)$. Then Theorem 8.3 yields that

$$\|G_2(r)|_{B_R^c}\|_p \leq C \|U|_{B_R^c}\|_\infty \|\nabla_x u(r)|_{B_R^c}\|_p \leq \mathfrak{C}(U) \left(\int_{B_R^c} |\nabla u(x, r)|^2 |x|^{-(3/2)(p-2)} dx \right)^{1/p}$$

for $r \in (0, \infty)$, so $\|G_2(r)|_{B_R^c}\|_p \leq \mathfrak{C} \|\nabla_x u(r)\|_2^{2/p}$. Since $\varrho \geq p$, hence $2\varrho/p \geq 2$, we thus get as in the estimate of G_1 that

$$\|G_2|_{B_R^c} \times (t, \infty)\|_{p, \varrho; \infty} \leq \mathfrak{C} \|\nabla_x u\|_{2, \infty; \infty}^{2(1/p-1/\varrho)} \|\nabla_x u|_{\overline{\Omega}^c} \times (t, \infty)\|_2^{2/\varrho}.$$

But $\|\nabla_x u\|_{2, \infty; \infty} < \infty$ by (8.11), so the claims of the lemma referring to the case $H = G$ are proved. Since by Theorem 8.3 and Corollary 2.2, there is $c > 0$ with $\|u(r)|_{B_R^c}\|_\infty \leq c$, $\|\nabla_x u(r)|_{B_R^c}\|_{(1/p-1/6)^{-1}} \leq c$ for $r \in (0, \infty)$, the case $H = F$ may be handled in the same way. \square

For a function u which satisfies not only (8.11) and (8.12), but also the pointwise decay estimate of $\|\nabla_x u(t)\|_2$ in (1.16), the following relations hold for the nonlinearity $(u \cdot \nabla_x)u$, that is, for the function F from Theorem 8.3.

Lemma 8.3 *In the situation of Theorem 8.3, suppose in addition that that u satisfies (1.16). Then*

$$\|\nabla_x u(t)\|_2 \leq \mathfrak{C}(1+t)^{-\kappa_1} \quad \text{for } t > 0; \quad (8.13)$$

$$\|F|_{\overline{\Omega}^c} \times (t, \infty)\|_{q, 2; \infty} \leq \mathfrak{C}(1+t)^{3\kappa_1(1-1/q)} \quad \text{for } t > 0, q \in [1, 3/2]; \quad (8.14)$$

$$\|F|_{B_{R_4}^c} \times (t, \infty)\|_{q, \infty; \infty} \leq \mathfrak{C}(1+t)^{-2\kappa_1/q} \quad \text{for } t > 0, q \in [2, \infty). \quad (8.15)$$

Proof: Since $u \in L^\infty(0, \infty, H^1(\overline{\Omega}^c)^3)$ by (8.11), the estimate in (8.13) follows immediately with (1.16). Moreover, with Theorem 2.2 and (8.13), we find that $\|F(r)\|_{3/2} \leq C \|u(r)\|_6 \|\nabla_x u(r)\|_2 \leq \mathfrak{C} \|\nabla_x u(r)\|_2^2 \leq \mathfrak{C} \|\nabla_x u(r)\|_2 (1+r)^{-\kappa_1}$ for $r \in (0, \infty)$. This implies (8.14) in the case $q = 3/2$. From (8.11) we get $\|F(r)\|_1 \leq C \|u(r)\|_2 \|\nabla_x u(r)\|_2 \leq \mathfrak{C} \|\nabla_x u(r)\|_2 \leq \mathfrak{C}$ for $r \in (0, \infty)$, so that (8.14) holds for $q = 1$ as well. That latter estimate for $q \in (1, 3/2)$ now follows by interpolation. Let $q \in [2, \infty)$. Then for r as before, by the second estimate in Theorem 8.3 and (8.13), $\|F(r)|_{B_{R_4}^c}\|_q \leq \mathfrak{C} \left(\int_{B_{R_4}^c} |x|^{-q-(3/2)(q-2)} |\nabla_x u(x, r)|^2 dx \right)^{1/q} \leq \mathfrak{C} \|\nabla_x u(r)\|_2^{2/q} \leq \mathfrak{C}(1+r)^{-2\kappa_1/q}$. This proves (8.15). \square

Now we are in a position to prove our main results about the asymptotics of solutions to the nonlinear problem (1.9), (1.10), (1.6) and (1.4), (1.10), (1.6).

Theorem 8.4 *Suppose that f , a and b satisfy the assumptions listed at the beginning of this section. Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, and let the parameters \bar{q}_0, \bar{s}_0 introduced at the beginning of this section verify the additional condition $3/(2\bar{q}_0) + 1/\bar{s}_0 < 1 - |\alpha|/2$. Take U as in Theorem 8.2, let $\tilde{\tau} \in \{0, 1\}$, and suppose that u is given as in (8.11), (8.12), that is, u is a L^2 -strong solution to (1.9), (1.10), (1.6) if $\tilde{\tau} = 1$, and to (1.4), (1.10), (1.6) in the case $\tilde{\tau} = 0$. Let $R \in (\max\{R_1, \dots, R_4\}, \infty)$, with R_3 from Theorem 8.2, R_4 from Theorem 8.3, and R_1, R_2 fixed at the beginning of this section.*

Then there is a bounded function $X : (0, \infty) \mapsto (0, \infty)$ with $X(t) \downarrow 0$ for $t \rightarrow \infty$ such that

$$|\partial_x^\alpha u(x, t)| \leq (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} X(t)^\epsilon \quad \text{for } x \in B_R^c, t \in (0, \infty), \epsilon \in [0, 1]. \quad (8.16)$$

If $\tilde{\tau} = 0$ and u additionally satisfies (1.16) with $\kappa_1 > 0$, and if $q_1, \hat{q}_1 \in (1, 3/2)$, $\bar{q}_1 \in [2, \infty)$ with $3/(2 - |\alpha|) < \bar{q}_1$, then

$$\begin{aligned} |\partial_x^\alpha u(x, t)| \leq & \mathfrak{C} \left[(|x| \nu(x))^{(-1-|\alpha|/2)} (1+t)^{-\min\{\varrho_1, 3/(2q_1)-1, 3\kappa_1(1-1/\hat{q}_1), \kappa_1\}} \right. \\ & \left. + (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} \right. \\ & \left. \cdot \left((1+t)^{-\min\{\varrho_2, 3/(2q_1)-1/2+|\alpha|/2, 2\kappa_1/\bar{q}_2\}} + \|f|_{B_{R_1}^c} \times (t/2, \infty)\|_{\bar{q}_0, \bar{s}_0; \infty} \right)^\epsilon \right] \end{aligned} \quad (8.17)$$

for x, t, ϵ as in (8.16), with ϱ_1, ϱ_2 from Theorem 8.1.

Assume in addition that f and a are bounded and with compact support, the parameter ζ_2 from (8.4) belongs to $[1/2, 1)$, and $\delta > 0$ is sufficiently small. Then

$$\begin{aligned} |\partial_x^\alpha u(x, t)| \leq & \mathfrak{C} \left[(|x| \nu(x))^{(-1-|\alpha|/2)} (1+t)^{-\min\{1/2, \kappa_1\}+\delta} \right. \\ & \left. + (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{-(\min\{1+|\alpha|/2, k(\alpha)\}+\delta)\epsilon} \right], \end{aligned} \quad (8.18)$$

with $k(\alpha) := \kappa_1$ in the case $\alpha = 0$, and $k(\alpha) := 2\kappa_1/3$ if $|\alpha| = 1$.

Proof: Define F and G as in the proof of Theorem 8.3. By our assumptions on f and by Lemma 8.1, we have $f + F + G \in L^2(0, \infty, L^p(\bar{\Omega}^c)^3)$ for $p \in [\max\{11/10, \hat{q}_0\}, 3/2]$. From (8.11), (8.12), we obtain that u satisfies (4.7) with $f + F + G$ in the place of f , as well as (4.6). Thus Theorem 4.10 yields there is a unique function $\phi \in L_n^2(S_\infty)$ such that

$$\mathfrak{V}^{(\tau)}(\phi)|_{S_\infty} = -\mathfrak{R}^{(\tau)}(f + F + G) - \mathfrak{J}^{(\tau)}(a) + b, \quad (8.19)$$

and

$$u = \mathfrak{R}^{(\tau)}(f + F + G) + \mathfrak{J}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\phi)|_{\bar{\Omega}^c} \times (0, \infty). \quad (8.20)$$

On the other hand, since f belongs to $L^2(0, \infty, L^q(\bar{\Omega}^c)^3)$ for $q \in \{\hat{q}_0, 3/2\}$, we know from Theorem 4.2 that $\mathfrak{R}^{(\tau)}(f)|_{S_\infty} \in H_\infty$. Moreover $\mathfrak{J}^{(\tau)}(a)|_{S_\infty} \in H_\infty$ by Theorem 4.4, and $b \in H_\infty$ by assumption. Therefore Theorem 4.8 states there is a unique function $\phi^{(1)} \in L_n^2(S_\infty)$ such that

$$\mathfrak{V}^{(\tau)}(\phi^{(1)})|_{S_\infty} = -\mathfrak{R}^{(\tau)}(f) - \mathfrak{J}^{(\tau)}(a) + b. \quad (8.21)$$

According to Lemma 8.1, the function $F + G$ belongs to $L^2(0, \infty, L^p(\overline{\Omega}^c)^3)$ for $p \in [11/10, 3/2]$ so that $\mathfrak{R}^{(\tau)}(F + G)|_{S_\infty} \in H_\infty$ (Theorem 4.2). Therefore, once more by Theorem 4.8, there is $\phi^{(2)} \in L_n^2(S_\infty)$ such that

$$\mathfrak{V}^{(\tau)}(\phi^{(2)})|_{S_\infty} = -\mathfrak{R}^{(\tau)}(F + G)|_{S_\infty}. \quad (8.22)$$

In view of the uniqueness statement in Theorem 4.8, and because of (8.19), (8.21) and (8.22), we may conclude that $\phi = \phi^{(1)} + \phi^{(2)}$, so by (8.20) $u = u^{(1)} + u^{(2)}$, where $u^{(1)} := \mathfrak{R}^{(\tau)}(f) + \mathfrak{J}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\phi^{(1)})|_{\overline{\Omega}^c} \times (0, \infty)$ and $u^{(2)} := \mathfrak{R}^{(\tau)}(F + G) + \mathfrak{V}^{(\tau)}(\phi^{(2)})|_{\overline{\Omega}^c} \times (0, \infty)$. By the definition of $u^{(1)}$, inequality (8.5) holds with $u^{(1)}$ in the place of u , hence

$$|\partial_x^\alpha u^{(1)}(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} X_1(t)^\epsilon \quad (8.23)$$

for x, t, ϵ as in (8.16), with

$$X_1(t) := (1+t)^{-\varrho_1} + (1+t)^{-\varrho_2} + \|f|_{B_{R_1}^c} \times (t/2, \infty)\|_{\overline{q}_0, \overline{s}_0; \infty}.$$

Turning to $u^{(2)}$, we may deduce from Lemma 7.3 that

$$|\partial_x^\alpha \mathfrak{V}^{(\tau)}(\phi^{(2)})(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} X_2(t)^\epsilon \quad (8.24)$$

for x, ϵ as in (8.16) and $t \in [1, \infty)$, where $X_2(t)$ is defined by

$$X_2(t) := (1+t)^{-1-|\alpha|/2} + \|\phi^{(2)}|_{S_{t/2, \infty}}\|_2.$$

If $t \in (0, 1)$, we argue as in the proof of Corollary 7.2, referring to Lemma 7.3 to obtain $|\mathfrak{V}^{(\tau)}(\phi^{(2)})(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{-1-|\alpha|/2}$ for x, t as in (8.16). Thus we obtain (8.24) again, this time for $t \in (0, 1)$. Put $\overline{R} := \max\{R_3, R_4\}$. The term $\|F + G|_{B_{\overline{R}}^c} \times (t/2, \infty)\|_{6,8; \infty}$ is bounded by $\mathfrak{C} \|\nabla_x u|_{\overline{\Omega}^c} \times (t/2, \infty)\|_2^{1/4}$ for $t \in (0, \infty)$ according to Lemma 8.2. Moreover Lemma 8.1 yields that $F + G \in L^2(0, \infty, L^p(\overline{\Omega}^c)^3)$ and $\|F + G|_{\overline{\Omega}^c} \times (t/2, \infty)\|_{p,2; \infty} \leq \mathfrak{C} \|\nabla_x u|_{\overline{\Omega}^c} \times (t/2, \infty)\|_2$ for $t \in (0, \infty)$, $p \in [11/10, 3/2]$. From this latter estimate we may conclude that $\|F + G|_{(\overline{\Omega}^c \cap B_{\overline{R}})} \times (t/2, \infty)\|_{1,2; \infty} \leq \mathfrak{C} \|\nabla_x u|_{\overline{\Omega}^c} \times (t/2, \infty)\|_2$. By Corollary 8.3, we know there is $\mathfrak{D}_0 > 0$ with $|\partial_x^\alpha \mathfrak{R}^{(\tau)}(F + G|_{B_{\overline{R}}^c} \times (0, \infty))(x, t)| \leq \mathfrak{D}_0 (|x| \nu(x))^{-1-|\alpha|/2}$ for $x \in B_{\overline{R}}^c$, $t \in (0, \infty)$, hence (5.13) is valid with $F + G, \overline{R}$ in the place of f and R_0 , respectively. Due to these estimates, we may apply Corollary 5.2 with $\varrho = 2$, $q = 4/3$, $s = 2$, $\tilde{q} = 11/10$, $\tilde{s} = 2$, $\overline{q} = 6$, $\overline{s} = 8$ and R_0, f replaced by \overline{R} and $F + G$, respectively, to obtain

$$|\partial_x^\alpha \mathfrak{R}^{(\tau)}(F + G)(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} X_3(t)^\epsilon \quad (8.25)$$

for x, t, ϵ as in (8.16), with $X_3(t) := (1+t)^{-7/8} + \sum_{r \in \{1, 1/4\}} \|\nabla_x u|_{\overline{\Omega}^c} \times (t/2, \infty)\|_2^r$. As a consequence of (8.24) and (8.25) and the definition of $u^{(2)}$, we arrive at the estimate

$$|\partial_x^\alpha u^{(2)}(x, t)| \leq \mathfrak{C} (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (X_2(t) + X_3(t))^\epsilon$$

for x, t, ϵ as in (8.16). Inequality (8.16) now follows with (8.23) and because $u = u^{(1)} + u^{(2)}$.

Now suppose that $\tilde{\tau} = 0$ so that $G = 0$. Further suppose that u satisfies (1.16) with $\kappa_1 > 0$. Let q_1, \hat{q}_1 and \bar{q}_1 be chosen as in the theorem. Note that $F \in L^2(0, \infty, L^{q_1}(\bar{\Omega}^c)^3)$ according to Lemma 8.1. By applying (8.14) with $q = \hat{q}_1$ and $q = 3/2$, and Corollary 5.1 with $q = q_1, s = 2, \hat{q} = \hat{q}_1$, we get $\|\mathfrak{R}^{(\tau)}(F)|_{S_{T,\infty}}\|_{H_{T,\infty}} \leq \mathfrak{C} T^{-\min\{3/(2q_1)-1, 3\kappa_1(1-1/\hat{q}_1), \kappa_1\}}$ for $T \in (1, \infty)$. This inequality, (8.22) and Corollary 7.2 imply

$$|\partial_x^\alpha \mathfrak{V}^{(\tau)}(\phi^{(2)})(x, t)| \leq \mathfrak{C} \left[(|x| \nu(x))^{-1-|\alpha|/2} (1+t)^{-\min\{3/(2q_1)-1, 3\kappa_1(1-1/\hat{q}_1), \kappa_1\}} \right. \\ \left. + (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{(-1-|\alpha|/2)\epsilon} \right] \quad (8.26)$$

for x, t, ϵ as in (8.16). Next we want to apply Corollary 5.2 to $\mathfrak{R}^{(\tau)}(F)$, with f, R_0 replaced by F and \bar{R} , respectively. As indicated further above, inequality (5.13) holds for the preceding choice of f and R_0 in that latter estimate. Recall that by Lemma 8.1, $F \in L^2(0, \infty, L^p(\bar{\Omega}^c)^3)$ for any $p \in [1, 3/2]$. On the other hand, it follows with (8.14) that $\|F|_{(\bar{\Omega}^c \cap B_{\bar{R}})} \times (t/2, \infty)\|_{1,2;\infty} \leq C(\bar{R}) \|F|_{\bar{\Omega}^c} \times (t/2, \infty)\|_{3/2,2;\infty} \leq (1+t)^{-\kappa_1}$. Due to (8.15), we get $\|F|_{B_{\bar{R}}^c} \times (t/2, \infty)\|_{\bar{q}_1,\infty;\infty} \leq \mathfrak{C} (1+t)^{-2\kappa_1/\bar{q}_1}$ for $t \in (0, \infty)$. Thus Corollary 5.2 with $\varrho = 2, q = 1, s = 2, \tilde{q} = 3/2, \tilde{s} = 2, \bar{q} = \bar{q}_1, \bar{s} = \infty$ and \bar{R} in the place of R_0 yields

$$|\partial_x^\alpha \mathfrak{R}^{(\tau)}(F)(x, t)| \leq \mathfrak{C} \left[(|x| \nu(x))^{-1-|\alpha|/2} (1+t)^{-\min\{2, \kappa_1\}} \right. \\ \left. + (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{-\min\{1+|\alpha|/2, 2\kappa_1/\bar{q}_1\}\epsilon} \right]$$

for x, t, ϵ as in (8.16). The preceding estimate, (8.26) and the definition of $u^{(2)}$ imply

$$|\partial_x^\alpha u^{(2)}(x, t)| \leq \mathfrak{C} \left[(|x| \nu(x))^{-1-|\alpha|/2} (1+t)^{-\min\{3/(2q_1)-1, 3\kappa_1(1-1/\hat{q}_1), \kappa_1\}} \right. \\ \left. + (|x| \nu(x))^{(-1-|\alpha|/2)(1-\epsilon)} (1+t)^{-\min\{1+|\alpha|/2, 2\kappa_1/\bar{q}_1\}\epsilon} \right],$$

again for x, t, ϵ as in (8.16). This result combined with (8.5) with $u^{(1)}$ in the place of u and the equation $u = u^{(1)} + u^{(2)}$ lead to (8.17).

Suppose in addition that f and a are bounded with compact support, and $\zeta_2 \in [1/2, 1)$. Then ϱ_1 may be taken as ζ_2 , and ϱ_2 as $1 + |\alpha|/2$; see the proof of Corollary 8.2. Moreover $3/(2q_1) \uparrow 3/2$ for $q_1 \downarrow 1$, and $3\kappa_1(1-1/\hat{q}_1) \rightarrow \kappa_1$ for $\hat{q}_1 \uparrow 3/2$. If $\alpha = 0$, we may take $\bar{q}_1 = 2$. In the case $|\alpha| = 1$, the relation $2\kappa_1/\bar{q}_1 \uparrow 2\kappa_1/3$ holds for $\bar{q}_1 \downarrow 3$. The term $\|f|_{B_{R_1}^c} \times (t/2, \infty)\|_{\bar{q}_0, \bar{s}_0;\infty}$ may be estimated by $\mathfrak{C}(1+t)^{-2}$. These remarks and (8.17) imply (8.18). \square

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