The stochastic $p(\omega, t, x)$-Laplace equation with cylindrical Wiener process

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We propose the analysis of a non-linear parabolic problem of $p(\omega, t, x)$-Laplace type in the framework of Orlicz Lebesgue and Sobolev spaces with variable random exponents and a stochastic forcing by a cylindrical Wiener process. We give a result of well-posedness: existence, uniqueness and stability of the solution, for additive and multiplicative problems.

1. Introduction

We are interested in a result of existence and uniqueness of the solution to the problem:

$$(P,H)\begin{cases} du - \Delta_{p(\cdot)}u\ dt = H(\cdot,u)\ dW & \text{in } \Omega\times(0,T)\times D,\\ u=0 & \text{on } \Omega\times(0,T)\times\partial D,\\ u(0,\cdot)=u_0 & \text{in } L^2(D),\end{cases}$$

where $T > 0$, $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain, $Q := (0, T) \times D$ and $(\Omega, \mathcal{F}, P)$ is a classical Wiener space endowed with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$.

$$H : \Omega \times (0, T) \times \mathbb{R} \to HS(L^2(D)), \quad (\omega, t, \lambda) \mapsto H(\omega, t, \lambda)$$

is a Carathéodory function, continuous with respect to λ, progressively measurable with respect to $\mathcal{F}_t$ and square integrable with respect to $dP \otimes dt$, where $HS(L^2(D))$ denotes the space of Hilbert–Schmidt operators on $L^2(D)$ with values in $L^2(D)$. We will give the precise assumptions on H below. $\Delta_{p(\cdot)}u =$

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$\operatorname{div}(|\nabla u|^{p(\omega,t,x)-2}\nabla u)$ denotes the p-Laplace operator with a variable exponent $p : \Omega\times Q \to (1,\infty)$ satisfying the following conditions:

($p1$) $1 < p^- := \operatorname{ess\,inf}_{(\omega,t,x)} p(\omega,t,x) \le p^+ := \operatorname{ess\,sup}_{(\omega,t,x)} p(\omega,t,x) < \infty$,

($p2$) ω a.s. in Ω, $(t,x)\mapsto p(\omega,t,x)$, is log-Hölder continuous, *i.e.* there exists $C\ge 0$ (which may depend on ω) such that, for all (t,x), $(s,y)\in Q$,

$$|p(\omega,t,x)-p(\omega,s,y)| \le \frac{C}{\ln(e+\frac{1}{|(t,x)-(s,y)|})} \tag{1}$$

($p3$) progressive measurability of the variable exponent, *i.e.*

$$\Omega\times[0,t]\times D \ni (\omega,s,x)\mapsto p(\omega,s,x)$$

is $\mathcal{F}_t\times\mathcal{B}(0,t)\times\mathcal{B}(D)$-measurable for all $0\le t\le T$.

For an orthonormal basis (e_k) of $L^2(D)$ and $(\beta_k(t))$ a family of independent, real-valued Brownian motions adapted to $(\mathcal{F}_t)$, we (formally) define the cylindrical Wiener process,

$$W(t) := \sum_{k=1}^{\infty} e_k\beta_k(t). \tag{2}$$

It is well-known that the sum on the right-hand side of (2) does not converge in $L^2(D)$, therefore we have to give a meaning to (2) following the ideas of [2] and [7]: For $u=\sum_{k=1}^{\infty}u_ke_k$ and $v=\sum_{k=1}^{\infty}v_ke_k$

$$(u,v)_U := \sum_{k=1}^{\infty}\frac{u_kv_k}{k^2}$$

is a scalar product on $L^2(D)$. Now we define the (bigger) Hilbert space U as the completion of $L^2(D)$ with respect to the norm $\|\cdot\|_U$ induced by $(\cdot,\cdot)_U$. It is then easy to see that (ke_k) is an orthonormal basis of U. Note that

$$W(t) = \sum_{k=1}^{\infty} e_k\beta_k(t) = \sum_{k=1}^{\infty}\frac{1}{k}ke_k\beta_k(t) \tag{3}$$

and therefore $W(t)$ can be interpreted as a Q-Wiener process with covariance Matrix $Q=\operatorname{diag}(\frac{1}{k^2})$ and values in U. Since $Q^{\frac{1}{2}}(U)=L^2(D)$, for all square integrable and predictable $\Phi : \Omega\times(0,T)\to HS(L^2(D))$ the stochastic integral with respect to the cylindrical Wiener process $W(t)$ can be defined by

$$\int_0^t \Phi\, dW = \sum_{k=1}^{\infty}\int_0^t \frac{1}{k}\Phi(ke_k)\, d\beta_k = \sum_{k=1}^{\infty}\int_0^t \Phi(e_k)\, d\beta_k.$$

In particular, $\Phi(e_k)\in N^2_w(0,T;L^2(D))$ for all $k\in\mathbb{N}^*$.

Assume that $H:\Omega\times(0,T)\times L^2(D)\to HS(L^2(D))$ is defined by

$$H(\omega,t,u)(e_k) = \{x\mapsto h_k(\omega,t,x,u(x))\},$$

where, for any $k\in\mathbb{N}^*$, $h_k:\Omega\times(0,T)\times\mathbb{R}^{d+1}\to\mathbb{R}$ is a Carathéodory function such that for all $\lambda\in\mathbb{R}$, $h_k(\cdot,\lambda)\in N^2_w(0,T;L^2(D))$ and $\lambda\mapsto h_k(\omega,t,\lambda)$ is continuous $P_T\otimes\mathcal{L}^d$-a.e. where P_T denotes the predictable σ-algebra and $\mathcal{L}$ the Lebesgue measure (see [2] for example). Moreover, for technical reasons,

$(H1)$ There exist $C_1, C_2 \geq 0$ and $C_3 \in L^1(D)$ such that a.e. in (ω, t, x),

$$\sum_{k=1}^{\infty} |h_k(\cdot, \lambda)|^2 \leq C_1|\lambda|^2 + C_3, \sum_{k=1}^{\infty} |h_k(\cdot, \lambda) - h_k(\cdot, \mu)|^2 \leq C_2|\lambda - \mu|^2.$$

In particular for any $u \in L^2(D)$ and for a.e. $(\omega, t) \in \Omega \times (0, T)$ thanks to $(H1)$ we have

$$\begin{aligned} \|H(\omega, t, u)\|^2_{HS(L^2(D))} &= \sum_{k=1}^{\infty} \|H(\omega, t, u)(e_k)\|^2_{L^2(D)} \\ &= \int_D \sum_{k=1}^{\infty} |h_k(\omega, t, x, u(x))|^2 \, dx \leq \|C_3\|_{L^1(D)} + C_1\|u\|^2_{L^2(D)} \end{aligned}$$

and therefore $H(\omega, t, u)$ is a Hilbert–Schmidt operator.

Our aim in this paper is to extend the previous result published in [1] to the case of a random variable exponent and to a more general noise, here a cylindrical Wiener process. At the beginning, the methodology is close to the one presented in [1], then it has to be adapted to the new situation. The result is first proved in the additive case for a finite-dimensional Wiener process: as in [1], one considers a singular perturbation of the $p(\cdot)$-Laplace operator by a q-Laplace one (q being a big enough constant) with very regular additive integrands H before passing to the limits on the perturbation, then on the regularization of the integrands. The result is then proved in the additive case for a general infinite-dimensional Wiener process, then in the multiplicative case by using a fixed point argument.

The organization of the paper is the following one: the next section presents the functional framework and the one after introduces the main result. The last section is dedicated to the proof of the main result.

2. Function spaces with variable exponent

The following function space serves as the variable exponent version of the classical Bochner space setting: there exists a full-measure set $\tilde{\Omega} \subset \Omega$ such that we can define

$$X_\omega(Q) := \{u \in L^2(Q) \cap L^1(0, T; W_0^{1,1}(D)) \mid \nabla u \in (L^{p(\omega,\cdot)}(Q))^d\}$$

which is a separable, reflexive Banach space for all $\omega \in \tilde{\Omega}$ with respect to the norm

$$\|u\|_{X_\omega(Q)} = \|u\|_{L^2(Q)} + \|\nabla u\|_{L^{p(\omega,\cdot)}(Q)}.$$

$X_\omega(Q)$ is a parametrization by ω of the space

$$X(Q) := \{u \in L^2(Q) \cap L^1(0, T; W_0^{1,1}(D)) \mid \nabla u \in (L^{p(t,x)}(Q))^d\}$$

which has been introduced in [3] for the case of a variable exponent depending on (t, x). For the basic properties of $X(Q)$, we refer to [3]. For $u \in X_\omega(Q)$, it follows directly from the definition that $u(t) \in L^2(D) \cap W_0^{1,1}(D)$ for almost every $t \in (0, T)$. Moreover, from $\nabla u \in L^{p(\omega,\cdot)}(Q)$ and Fubini's theorem it follows that $\nabla u(t, \cdot)$ is in $L^{p(\omega,t,\cdot)}(D)$ a.e. in $(0, T)$.

Let us introduce the space

$$\mathcal{E} := \{u \in L^2(\Omega \times Q) \cap L^{p^-}(\Omega \times (0, T); W_0^{1,p^-}(D)) \mid \nabla u \in L^{p(\cdot)}(\Omega \times Q)\}$$

which is a separable, reflexive Banach space with respect to the norm

$$u \in \mathcal{E} \mapsto \|u\|_{\mathcal{E}} = \|u\|_{L^2(\Omega\times Q)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega\times Q)}.$$

Thanks to Fubini's theorem, $u \in \mathcal{E}$ implies that $u(\omega) \in X_\omega(Q)$ a.s. in Ω and, since Poincaré's inequality is available with respect to (t,x), independently of ω, $u \in \mathcal{E}$ implies also $u(\omega,t) \in L^2(D) \cap W_0^{1,p(\omega,t,\cdot)}(D)$ for almost all $(\omega,t) \in \Omega \times (0,T)$.

3. Main result

Definition 3.1. A solution to (P,H) is a function $u \in L^2(\Omega; C([0,T]; L^2(D))) \cap N_w^2(0,T; L^2(D)) \cap \mathcal{E}$, such that, for almost every $\omega \in \Omega$, $u(0,\cdot) = u_0$, a.e. in D and for all $t \in [0,T]$,

$$u(t) - u_0 - \int_0^t \Delta_{p(\cdot)} u \, ds = \int_0^t H(\cdot, u) \, dW,$$

holds a.s. in D; or, equivalently, in the weak sense:

$$\partial_t [u(t) - \int_0^t H(\cdot, u) \, dW] - \Delta_{p(\cdot)} u = 0 \text{ in } X'_\omega(Q).$$

Theorem 3.1. *There exists a unique solution to (P,H). Moreover, if u_1, u_2 are the solutions to (P,H_1), (P,H_2) respectively, then:*

$$E\left(\sup_{t\in[0,T]} \|(u_1-u_2)(t)\|^2_{L^2(D)}\right) + E\int_Q \left(|\nabla u_1|^{p(\cdot)-2}\nabla u_1 - |\nabla u_2|^{p(\cdot)-2}\nabla u_2\right)\cdot \nabla(u_1-u_2) \, d(t,x)$$

$$\leq CE\int_0^T \|H_1(\cdot,u_1) - H_2(\cdot,u_2)\|^2_{HS(L^2(D))} \, dt. \tag{4}$$

4. Proof of the main result

Notation: for a square integrable and predictable process $\Phi : \Omega \times (0,T) \to HS(L^2(D))$ and $N \in \mathbb{N}^*$, we define the predictable and square integrable process $\Phi_N : \Omega \times (0,T) \to HS(L^2(D))$ by $\Phi_N(e_k) = \Phi(e_k)$ for $k \leq N$ and $\phi_N(e_k) = 0$ for $k > N$. Consequently

$$\int_0^T \Phi_N \, dW = \sum_{k=1}^N \int_0^T \Phi(e_k) \, d\beta_k.$$

Let us remark that this corresponds to the case of the finite-dimensional Wiener process: $W_N(t) = \sum_{k=1}^N \beta_k(t) e_k$.

4.1. The result for nice processes

Let $S_w^2(0,T; H_0^j(D))$ be the subset of simple, predictable processes with values in $H_0^j(D)$ for sufficiently large values of j. Note that $S_w^2(0,T; H_0^j(D))$ is densely imbedded into $N_w^2(0,T; L^2(D))$. We will first prove the result when $\Phi_N(e_k) \in S_w^2(0,T; H_0^j(D))$ for all $k = 1, \dots N$. We will call such Φ_N a nice process in the sequel.

Proposition 4.1. *For $q \geq \max(2, p^+)$, $0 < \varepsilon \leq 1$, $N \in \mathbb{N}^*$ and a nice process Φ_N there exists*

$$u^\varepsilon \in L^2(\Omega; C([0,T]; L^2(D))) \cap N^2_w(0,T; L^2(D)) \cap L^q(\Omega \times (0,T); W^{1,q}_0(D))$$

and a set $\tilde{\Omega} \subset \Omega$ of total probability 1 on which $u(0,\cdot) = u_0$ a.e. in D and

$$u^\varepsilon(t) - u_0 - \int_0^t [\varepsilon \Delta_q u^\varepsilon + \Delta_{p(\cdot)} u^\varepsilon] \, ds = \int_0^t \Phi_N \, dW \tag{5}$$

in $W^{-1,q'}(D)$ for all $t \in [0,T]$.

Proof. For $q \geq \max(2, p^+)$ and $\varepsilon > 0$, the operator

$$A : \Omega \times (0,T) \times W^{1,q}_0(D) \to W^{-1,q'}(D), \quad A(\omega, t, u) = -\varepsilon \Delta_q u - \Delta_{p(\omega,t,x)} u,$$

is monotone with respect to u for a.e. $(\omega, t) \in \Omega \times (0,T)$ and progressively measurable, *i.e.* for every $t \in [0,T]$ the mapping

$$A : \Omega \times (0,t) \times W^{1,q}_0(D) \to W^{-1,q'}(D), \quad (\omega, s, u) \mapsto A(\omega, s, u)$$

is $\mathcal{F}_t \times \mathcal{B}(0,t) \times \mathcal{B}(W^{1,q}_0(D))$-measurable. In particular, $-A$ satisfies the hypotheses of [5, Theorem 2.1, p. 1253], therefore for any $\varepsilon > 0$ there exists a continuous process with values in $L^2(D)$ solution to the problem (5). Then, [2, Prop. 3.17 p. 84] and [5, Theorem 2.3, p. 1254] yield $u^\varepsilon \in L^2(\Omega; C([0,T]; L^2(D)))$.

Proposition 4.2. *For any nice process Φ_N, there exist a unique function $u \in \mathcal{E} \cap L^2(\Omega; C([0,T]; L^2(D)))$ and a full-measure set $\tilde{\Omega} \in \mathcal{F}$ such that for all $\omega \in \tilde{\Omega}$ we have $u(0,\cdot) = u_0$ a.e. in D and*

$$u(t) - u_0 - \int_0^t \Delta_{p(\cdot)} u \, ds = \int_0^t \Phi_N \, dW \tag{6}$$

holds a.e. in D for all $t \in [0,T]$. In particular u is a solution to (P, Φ_N) in the sense of Definition 3.1.

Proof. For the first part of the proof, mainly based on deterministic arguments, we can repeat the arguments of [1]: If we set $v^\varepsilon := u^\varepsilon - \int_0^t \Phi_N \, dW$, such that $v^\varepsilon(0) = u_0$, then u^ε satisfies (5), iff there exists a full-measure set $\tilde{\Omega} \in \mathcal{F}$ such that

$$\partial_t v^\varepsilon - \varepsilon \Delta_q (v^\varepsilon + \int_0^t \Phi_N \, dW) - \Delta_{p(\cdot)} (v^\varepsilon + \int_0^t \Phi_N \, dW) = 0 \tag{7}$$

in $L^{q'}(0,T; W^{-1,q'}(D))$ for all $\omega \in \tilde{\Omega}$. Testing (7) with v^ε to get *a priori* estimates, we can use classical (monotonicity) arguments to conclude that pointwise for every $\omega \in \tilde{\Omega}$ we have the following convergence results, passing to a (not relabeled) subsequence if necessary when ϵ tends to 0:

1.) $v^\varepsilon \rightharpoonup v$ in $X_\omega(Q)$ and $L^\infty(0,T; L^2(D))$ weak-*,
2.) for any t, $v^\varepsilon(t) \to v(t)$ in $L^2(D)$,
3.) $\int_Q |\nabla v^\varepsilon - \nabla v|^{p(\omega,t,x)} \, dx dt \to 0$.

Then, passing to the limit in the singular perturbation, v satisfies the problem

$$\partial_t v - \Delta_{p(\cdot)}(v + \int_0^t \Phi_N \, dW) = 0.$$

In particular, $\partial_t v \in X'_\omega(Q)$ (see [3]) and $v \in W_\omega(Q)$ where one denotes by

$$W_\omega(Q) := \{v \in X_\omega(Q) \mid \partial_t v \in X'_\omega(Q)\}.$$

Thanks to [3], $W_\omega(Q) \hookrightarrow C([0,T]; L^2(D))$ with a continuity constant depending only on T and the time-integration by parts formula is available. Thus, $v \in C([0,T]; L^2(D))$ and v is a solution of the above problem in $W_\omega(Q)$, for the initial condition u_0. Since this solution is unique, no subsequence is needed in the above limits. Then, the above convergence yields for all $\omega \in \tilde{\Omega}$:

1.) $u^\varepsilon \to u$ in $L^2(0,T; L^2(D))$ with $\partial_t[u - \int_0^\cdot \Phi_N \, dW] \in X'_\omega(Q)$,
2.) for any t, $u^\varepsilon(t) \to u(t)$ in $L^2(D)$,
3.) $\Delta_{p(\omega,t,x)} u^\varepsilon \rightharpoonup \Delta_{p(\omega,t,x)} u$ in $X'_\omega(Q)$,
4.) $\int_Q |\nabla u^\varepsilon - \nabla u|^{p(\omega,t,x)} \, dxdt \to 0$.

We continue with the argumentation as in [1]: from the previous convergence results, the *a priori* estimates and since $\nabla \Phi_N$ is bounded, we get uniform estimates that allow us to use Lebesgue Dominated Convergence theorem and therefore it follows that

$$\forall t, \ u^\varepsilon(t) \to u(t) \ \text{ in } \ L^2(\Omega; L^2(D)) \quad \text{and} \quad u^\varepsilon \to u \ \text{ in } \ \mathcal{E}. \tag{8}$$

Note that the above limits in $L^2(\Omega; L^2(D))$ and $L^2(\Omega; L^2(Q))$ are standard results obtained in classical Bochner spaces, but the measurability of ∇u with respect to $d(t,x) \otimes dP$ deserves our attention. Since ∇u^ε and $\nabla u^{\epsilon'}$ are globally measurable functions, Lebesgue Dominated Convergence theorem, together with *a priori* estimates yield

$$E \int_Q |\nabla u^\varepsilon - \nabla u^{\varepsilon'}|^{p(\omega,t,x)} \, dxdt \to 0$$

and thus, (∇u^ε) is a Cauchy sequence in $L^{p(\cdot)}(\Omega \times Q)$ and therefore a converging sequence. It is then a direct consequence to see that ∇u is the limit in $L^{p(\cdot)}(\Omega \times Q)$ of ∇u^ε.

Then, passing to a (not relabeled) subsequence if needed, it follows that $u^\varepsilon \to u$ a.e. in $\Omega \times Q$.

Hence u satisfies (6), or, in other words, $\partial_t[u - \int_0^t \Phi_N \, dW] - \Delta_{p(\cdot)} u = 0$.

In particular, since Φ_N is regular, one gets that $u - \int_0^t \Phi_N \, dW \in \mathcal{E}$ with $\partial_t[u - \int_0^t \Phi_N \, dW] \in \mathcal{E}'$.

We need now to prove that $u \in L^2(\Omega; C([0,T]; L^2(D)))$. We already know that $u : \Omega \times Q \to L^2(D)$ is a stochastic process. Since $u(\omega, \cdot) \in W_\omega(Q) \hookrightarrow C([0,T]; L^2(D))$ for a.e. $\omega \in \Omega$, the measurability follows from [2, Prop. 3.17 p. 84] with arguments as in [4, Cor. 1.1.2, p. 8]. Then, a.s. in Ω, the equation satisfied by u yields $\partial_t v - \Delta_{p(\cdot)} u = 0$, so that, for almost every $t \in [0,T]$,

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|^2_{L^2(D)} + \int_D |\nabla u|^{p(\omega,t,x)-2} \nabla u \cdot \nabla v \, dx = 0.$$

Since, ω a.s.,

$$\sup_{t\in[0,T]} \|v(\omega,t,\cdot)\|^2_{L^2(D)} \leq \|u_0\|^2_{L^2(D)} + 2\int_0^T\int_D \frac{1}{p^-}|\nabla u|^{p(\omega,s,x)} + \frac{1}{(p')^-}\left|\int_0^s \nabla\Phi_N\ dW\right|^{p'(\omega,s,x)} dx\ ds$$

with a right-hand side in $L^1(\Omega)$, one gets that $u, v \in L^2(\Omega; C([0,T]; L^2(D))$.

Lemma 4.1. *For any $m, n \in \mathbb{N}$ and nice processes $\Phi_{N,n}, \Phi_{N,m}$ let u_n be the solution to $(P, \Phi_{N,n})$ and u_m be the solution to $(P, \Phi_{N,m})$. There exist constants K_1, $K_2 \geq 0$ depending on the Burkholder–Davies–Gundy inequality such that*

$$E\left(\|u_n\|^2_{C([0,T];L^2(D))} + \int_Q |\nabla u_n|^{p(\cdot)}\ d(t,x)\right) \leq K_1 E\left(\int_0^T \|\Phi_{N,n}\|^2_{HS(L^2(D))}\ dt + \|u_0\|^2_{L^2(D)}\right), \tag{9}$$

$$E\left(\|(u_n-u_m)\|^2_{C([0,T];L^2(D))}\right) + E\int_Q (|\nabla u_n|^{p(\cdot)-2}\nabla u_n - |\nabla u_m|^{p(\cdot)-2}\nabla u_m)\cdot\nabla(u_n-u_m)\ d(t,x)$$

$$\leq K_2 E\int_0^T \|\Phi_{N,n} - \Phi_{N,m}\|^2_{HS(L^2(D))}\ dt. \tag{10}$$

Proof. Using the Itô formula in (5), it follows that for all $t \in [0,T]$, a.s. in Ω, we have

$$\|u_n^\varepsilon(t)\|^2_{L^2(D)} + 2\int_0^t\int_D |\nabla u_n^\varepsilon|^{p(\cdot)}\ dx\ ds$$

$$\leq 2\int_0^t (u_n^\varepsilon, \Phi_{N,n}(\cdot))_{L^2(D)}\ dW + \int_0^t \|\Phi_{N,n}\|^2_{HS(L^2(D))}\ ds + \|u_0\|^2_{L^2(D)} \tag{11}$$

or, by subtracting (5) with $\Phi_{N,m}$ from (5) with $\Phi_{N,n}$,

$$\|(u_n^\varepsilon - u_m^\varepsilon)(t)\|^2_{L^2(D)} + 2\int_0^t\int_D (|\nabla u_n^\varepsilon|^{p(\cdot)-2}\nabla u_n^\varepsilon - |\nabla u_m^\varepsilon|^{p(\cdot)-2}\nabla u_m^\varepsilon)\cdot\nabla(u_n^\varepsilon - u_m^\varepsilon)\ dx\ ds$$

$$\leq 2\int_0^t (u_n^\varepsilon - u_m^\varepsilon, [\Phi_{N,n} - \Phi_{N,m}](\cdot))_{L^2(D)}\ dW + \int_0^t \|\Phi_{N,n} - \Phi_{N,m}\|^2_{HS(L^2(D))}\ ds. \tag{12}$$

Thus, by passing to the limit with $\varepsilon \to 0$, to the supremum over t and then taking the expectation, it follows that ($c \geq 0$ being a constant)

$$E(\sup_{t\in[0,T]} \|u_n(t)\|^2_{L^2(D)}) + E\int_0^T\int_D |\nabla u_n|^{p(\cdot)}\ dx\ ds$$

$$\leq cE\left(\sup_{t\in[0,T]}\int_0^t (u_n, \Phi_{N,n}(\cdot))_{L^2(D)}\ dW\right) + c\|\Phi_{N,n}\|^2_{L^2(\Omega\times(0,T);HS(L^2(D)))} + c\|u_0\|^2_{L^2(D)}, \tag{13}$$

$$E(\sup_{t\in[0,T]}\|(u_n-u_m)(t)\|^2_{L^2(D)})+E\int_0^T\int_D(|\nabla u_n|^{p(\cdot)-2}\nabla u_n-|\nabla u_m|^{p(\cdot)-2}\nabla u_m)\cdot\nabla(u_n-u_m)\;dx\;ds$$

$$\leq cE\left(\sup_{t\in[0,T]}\int_0^t(u_n-u_m,[\Phi_{N,n}-\Phi_{N,n}](\cdot))_{L^2(D)}\;dW\right)+c\|\Phi_{N,n}-\Phi_{N,m}\|^2_{L^2(\Omega\times(0,T);HS(L^2(D)))}. \tag{14}$$

Using Burkholder, Hölder and Young inequalities on (13) we get for any $\gamma>0$

$$E\left(\sup_{t\in[0,T]}\int_0^t(u_n,\Phi_{N,n}(\cdot))_{L^2(D)}\;dW\right)\leq 3E\left(\int_0^T\|(u_n,\Phi_{N,n}(\cdot))_{L^2(D)}\|^2_{HS(L^2(D),\mathbb{R})}\;ds\right)^{1/2}, \tag{15}$$

where

$$\begin{aligned}\|(u_n,\Phi_{N,n}(\cdot))_{L^2(D)}\|^2_{HS(L^2(D),\mathbb{R})}&=\sum_{k=1}^{\infty}|(u_n,\Phi_{N,n}(e_k))_{L^2(D)}|^2\\&=\sum_{k=1}^{\infty}|(\Phi^*_{N,n}(u_n),e_k)_{L^2(D)}|^2\leq\|\Phi_{N,n}\|^2_{HS(L^2(D))}\|u_n\|^2_{L^2(D)}\end{aligned} \tag{16}$$

and therefore

$$\begin{aligned}E\left(\sup_{t\in[0,T]}\int_0^t(u_n,\Phi_{N,n})_{L^2(D)}\;dW\right)&\leq 3E\left(\int_0^T\|\Phi_{N,n}\|^2_{HS(L^2(D))}\|u_n\|^2_{L^2(D)}\;dt\right)^{1/2}\\&\leq 3E\left[\left(\sup_{t\in[0,T]}\|u_n\|^2_{L^2(D)}\right)^{1/2}\left(\int_0^T\|\Phi_{N,n}\|^2_{HS(L^2(D))}\right)^{1/2}\right]\\&\leq 3\gamma E\left(\sup_{t\in[0,T]}\|u_n\|^2_{L^2(D)}\right)+\frac{3}{\gamma}\|\Phi_{N,n}\|^2_{L^2(\Omega\times(0,T);HS(L^2(D)))}\end{aligned}$$

and similarly on (14),

$$E\left(\sup_{t\in[0,T]}\int_0^t(u_n-u_m,[\Phi_{N,n}-\Phi_{N,m}](\cdot))_{L^2(D)}\;dW\right)$$

$$\leq 3\gamma E\left(\sup_{t\in[0,T]}\|u_n-u_m\|^2_{L^2(D)}\right)+\frac{3}{\gamma}\|\Phi_{N,n}-\Phi_{N,m}\|^2_{L^2(\Omega\times(0,T);HS(L^2(D)))}. \tag{17}$$

Plugging (15) into (13), (17) into (14) and choosing $\gamma>0$ small enough yield Lemma 4.1.

Remark 4.1. It is an open question if the Itô formula is directly available for a solution of (6) since we are not in Bochner spaces: The stochastic energy has to be defined in different Banach spaces depending on $t\in[0,T]$ and $\omega\in\Omega$. That is why we need to apply the Itô formula to u^{ε}, and then pass to the limit. But then, only an inequality is obtained.

4.2. Existence for arbitrary Φ_N

Proposition 4.3. *For any $N \in \mathbb{N}^*$ and any Φ_N defined as in the beginning of the section, there exists a unique solution u to (P, Φ_N), i.e., $u \in \mathcal{E} \cap L^2(\Omega; C([0,T]; L^2(D)) \cap N_w^2(0,T; L^2(D))$ such that a.s.*

$$u(t) - u_0 - \int_0^t \Delta_{p(\cdot)} u \, ds = \int_0^t \Phi_N \, dW \tag{18}$$

for all $t \in [0,T]$, a.e. in D.

Moreover, if u_1, u_2 are the solutions to $(P, \Phi_{N,1})$, $(P, \Phi_{N,2})$ respectively, then:

$$E\left(\sup_t \|(u_1 - u_2)(t)\|^2_{L^2(D)}\right) + E\int_Q (|\nabla u_1|^{p(\cdot)-2}\nabla u_1 - |\nabla u_2|^{p(\cdot)-2}\nabla u_2) \cdot \nabla(u_1 - u_2) \, d(t,x)$$

$$\leq CE \int_0^T \|\Phi_{N,1} - \Phi_{N,2}\|^2_{HS(L^2(D))} \, dt. \tag{19}$$

Proof. For any $k \in \mathbb{N}$ there exists a sequence $(\Phi_{n,k,N})_n \subset S_w^2(0,T; H_0^j(D))$ converging to $\Phi_N(e_k)$ in $N_w^2(0,T; L^2(D))$ when n goes to ∞. If we define $\Phi_{n,N} : \Omega \times (0,T) \to HS(L^2(D))$ by $\Phi_{n,N}(e_k) = \Phi_{n,k,N}$ for $k \leq N$, then $\Phi_{n,N}$ is a nice process such that $\Phi_{n,N} \to \Phi_N$ in $L^2(\Omega \times (0,T); HS(L^2(D)))$ when $n \to \infty$. Let $(u_n) \in \mathcal{E} \cap L^2(\Omega; C([0,T]; L^2(D)))$ be the sequence of corresponding solutions to $(P, \Phi_{n,N})$, then from (9) it follows that (u_n) is a bounded sequence in $\mathcal{E} \cap L^2(\Omega; C([0,T]; L^2(D)))$ and (10) ensures that (u_n) is a Cauchy sequence in $L^2(\Omega; C([0,T]; L^2(D))$.

Hence there exists $u \in \mathcal{E} \cap L^2(\Omega; C([0,T]; L^2(D)))$ such that $u_n \rightharpoonup u$ in $\mathcal{E}$ and $u_n \to u$ in $L^2(\Omega; C([0,T]; L^2(D)))$.

Moreover there exists a full-measure set $\tilde{\Omega} \in \mathcal{F}$ such that, passing to a (not relabeled) subsequence if necessary, $u_n \to u$ in $C([0,T]; L^2(D))$ for all $\omega \in \tilde{\Omega}$. In particular, $u(0, \cdot) = u_0$ a.e. in D for all $\omega \in \tilde{\Omega}$.

For $\mu = d(t,x) \otimes dP$ we have

$$\int_{\Omega \times Q} |\nabla u_n - \nabla u_m|^{p(\cdot)} d\mu = \int_{1<p<2} |\nabla u_n - \nabla u_m|^{p(\cdot)} d\mu + \int_{p \geq 2} |\nabla u_n - \nabla u_m|^{p(\cdot)} d\mu.$$

Then, from (10) and the fundamental inequality ([6, Section 10]), for any $\xi, \eta \in \mathbb{R}^d$,

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq \begin{cases} 2^{2-p}|\xi - \eta|^p, \ p \geq 2 \\ (p-1)|\xi - \eta|^2(1 + |\eta|^2 + |\xi|^2)^{\frac{p-2}{2}}, \ 1 \leq p < 2 \end{cases}.$$

it follows first that

$$\int_{p \geq 2} |\nabla u_n - \nabla u_m|^{p(\cdot)} d\mu \leq 2^{p^+ - 2} K_2 \|\Phi_{n,N} - \Phi_{m,N}\|^2_{L^2(\Omega \times (0,T); HS(L^2(D)))}. \tag{20}$$

Then, from the generalized Young inequality it follows secondly that, for any $0 < \epsilon < 1$,

$$\int_{1<p<2} |\nabla u_n - \nabla u_m|^{p(\cdot)} d\mu = \int_{1<p<2} \frac{|\nabla u_n - \nabla u_m|^{p(\cdot)}}{(1 + |\nabla u_n|^2 + |\nabla u_m|^2)^{p(\cdot)\frac{2-p(\cdot)}{4}}} (1 + |\nabla u_n|^2 + |\nabla u_m|^2)^{p(\cdot)\frac{2-p(\cdot)}{4}} d\mu$$

$$\leq \int\limits_{1<p<2} \epsilon^{\frac{p(\cdot)-2}{p(\cdot)}} \frac{|\nabla u_n - \nabla u_m|^2}{(1+|\nabla u_n|^2+|\nabla u_m|^2)^{\frac{2-p(\cdot)}{2}}} d\mu$$

$$+ \epsilon \int\limits_{1<p<2} (1+|\nabla u_n|^2+|\nabla u_m|^2)^{\frac{p(\cdot)}{2}} d\mu$$

$$\leq \frac{1}{\epsilon(p^- - 1)} \int\limits_{1<p<2} (p-1) \frac{|\nabla u_n - \nabla u_m|^2}{(1+|\nabla u_n|^2+|\nabla u_m|^2)^{\frac{2-p(\cdot)}{2}}} d\mu + K_3 \epsilon$$

$$\leq \frac{1}{\epsilon(p^- - 1)} K_2 \|\Phi_{n,N} - \Phi_{m,N}\|^2_{L^2(\Omega\times(0,T);HS(L^2(D)))} + K_3\epsilon, \tag{21}$$

since the sequence (u_n) is bounded in $L^{p(\cdot)}(\Omega\times Q)$ and μ is a finite measure.

From (20), (21) and $\lim_{n,m\to\infty} \|\Phi_{n,N} - \Phi_{m,N}\|^2_{L^2(\Omega\times(0,T);HS(L^2(D)))} = 0$ one gets that ∇u_n is a Cauchy sequence in $L^{p(\cdot)}(\Omega\times Q)$, thus a converging sequence.

In conclusion, u_n converges to u in $\mathcal{E}\cap L^2(\Omega;C([0,T];L^2(D)))\cap N^2_w(0,T;L^2(D))$ and, by a standard argument based on the Nemytskii operator induced by the Carathéodory function $G:(\omega,t,x,\xi)\in\Omega\times Q\times\mathbb{R}^d \mapsto |\xi|^{p(\omega,t,x)-2}\xi\in\mathbb{R}^d$, $|\nabla u_n|^{p(\cdot)-2}\nabla u_n$ converges to $|\nabla u|^{p(\cdot)-2}\nabla u$ in $L^{p'(\cdot)}(\Omega\times Q)$ since $|G(\omega,t,x,\xi)|^{p'(\omega,t,x)} = |\xi|^{p(\omega,t,x)}$.

Let us recall that, for any $n\in\mathbb{N}$, u_n satisfies

$$\partial_t\left(u_n - \int_0^t \Phi_{n,N}\, dW\right) - \Delta_{p(\cdot)} u_n = 0 \tag{22}$$

in $\mathcal{E}'$. Now we can choose a (not relabeled) subsequence of (u_n) such that all previous convergence results hold true. For any test function $\phi(\omega,t,x)=\rho(\omega)\gamma(t)\nu(x)$ with $\rho\in L^\infty(\Omega)$, $\gamma\in\mathcal{D}([0,T))$ and $\nu\in\mathcal{D}(D)$ we have

$$\left\langle \partial_t\left(u_n - \int_0^t \Phi_{n,N}\, dW\right), \phi\right\rangle_{\mathcal{E}',\mathcal{E}} = \int_\Omega \left\langle \partial_t\left(u_n - \int_0^t \Phi_{n,N}\, dW\right), \phi\right\rangle_{X'_\omega, X_\omega} dP$$

$$= -\int_\Omega \left\langle \left(u_n - \int_0^t \Phi_{n,N}\, dW\right), \partial_t\phi\right\rangle_{X'_\omega, X_\omega} dP - \int_{\Omega\times D} u_0\varphi(\omega,0,x)\, dx\, dP. \tag{23}$$

In particular u_n satisfies

$$-\int_{\Omega\times Q} \left(u_n - \int_0^t \Phi_{n,N}\, dW\right)\cdot\partial_t\phi + |\nabla u_n|^{p(\cdot)-2}\nabla u_n\cdot\nabla\phi\, d\mu - \int_{\Omega\times D} u_0\varphi(\omega,0,x)\, dx\, dP = 0 \tag{24}$$

for all $n\in\mathbb{N}$. When $n\to\infty$, from Itô isometry it follows that

$$\int_\Omega\int_0^T \left\|\int_0^t \Phi_{n,N} - \Phi_N\, dW\right\|^2_{L^2(D)} dt\, dP \leq \sup_{t\in[0,T]} E\left\|\int_0^t \Phi_{n,N} - \Phi_N\, dW\right\|^2_{L^2(D)}$$

$$= E\int_0^T \|\Phi_{n,N} - \Phi_N\|^2_{HS(L^2(D))}\, dt \to 0, \tag{25}$$

hence $\int_0^\cdot \Phi_{n,N}\ dW \to \int_0^\cdot \Phi_N\ dW$ in $L^2(\Omega \times (0,T); L^2(D))$ for $n \to \infty$. Therefore, using our convergence results, we are able to pass to the limit in (24) and obtain

$$\partial_t \left(u - \int_0^t \Phi_N\ dW \right) - \Delta_{p(\cdot)} u = 0 \tag{26}$$

in $\mathcal{E}'$. (26), and a classical argument of separability, imply that a.s.

$$\partial_t \left(u - \int_0^t \Phi_N\ dW \right) = \Delta_{p(\cdot)} u, \text{ in } X'_\omega(Q) \hookrightarrow L^{\alpha'}(0,T; W^{-1,\alpha'}(D)) \tag{27}$$

with $\alpha \geq p^+ + 2$. Moreover, a.s.

$$u - \int_0^t \Phi_N\ dW \in C([0,T]; L^2(D)).$$

Thus we can integrate (27) to obtain a.s.

$$u(t) - u_0 - \int_0^t \Delta_{p(\cdot)} u\ ds = \int_0^t \Phi_N\ dW \tag{28}$$

in $L^2(D)$ for all $t \in [0,T]$.

To finish the proof, note that passing to the limit in (10) yields the stability inequality (19), and if we assume that $u_1, u_2 \in \mathcal{E} \cap L^2(\Omega; C([0,T]; L^2(D))) \cap N_W^2(0,T; L^2(D))$ are both satisfying (18), it follows that a.s. in Ω

$$\partial_t(u_1 - u_2) - (\Delta_{p(\cdot)} u_1 - \Delta_{p(\cdot)} u_2) = 0 \text{ in } (X_\omega(Q))'. \tag{29}$$

Using $u_1 - u_2$ as a test function in (29) and the integration by parts formula in $W_\omega(Q)$, we obtain the result of uniqueness.

4.3. Existence for cylindrical Wiener process

Now, our aim is to pass to the limit when $N \to \infty$ in Proposition 4.3. For $M, N \in \mathbb{N}^*$, let u_N and u_M be the solutions obtained by Proposition 4.3 corresponding to right-hand sides Φ_N and Φ_M respectively. Thanks to the stability result of the proposition, one has that

$$\begin{aligned} E\left(\|(u_N - u_M)\|^2_{C([0,T];L^2(D))} \right) + E \int_Q (|\nabla u_N|^{p(\cdot)-2} \nabla u_N - |\nabla u_M|^{p(\cdot)-2} \nabla u_M) \cdot \nabla(u_N - u_M)\ d(t,x) \\ \leq K_2 E \int_0^T \|\Phi_N - \Phi_M\|^2_{HS(L^2(D))}\ dt. \end{aligned} \tag{30}$$

Since Φ_N converges to Φ in $L^2(\Omega \times (0,T); HS(L^2(D)))$, the right-hand side of (30) converges to 0 when $M, N \to \infty$. Therefore (u_N) is a Cauchy sequence in $L^2(\Omega; C([0,T]; L^2(D)))$. Using (20) and (21), we find that (∇u_N) is a Cauchy sequence in $L^{p(\cdot)}(\Omega \times Q)^d$. Thus, there exists u in $\mathcal{E} \cap L^2(\Omega; C([0,T]; L^2(D))) \cap$

$N_w^2(0,T;L^2(D))$ such that $u_N \to u$ in $\mathcal{E}$. Moreover, $|\nabla u_N|^{p(\cdot)-2}\nabla u_N$ converges to $|\nabla u|^{p(\cdot)-2}\nabla u$ in $L^{p'(\cdot)}(\Omega\times Q)$ by a standard argument based again on the Nemytskii operator induced by the Carathéodory function $G:(\omega,t,x,\xi)\in\Omega\times Q\times\mathbb{R}^d\mapsto|\xi|^{p(\omega,t,x)-2}\xi\in\mathbb{R}^d$, since $|G(\omega,t,x,\xi)|^{p'(\omega,t,x)}=|\xi|^{p(\omega,t,x)}$. In addition, from Itô isometry and the convergence of Φ_N to Φ in $L^2(\Omega\times(0,T);HS(L^2(D)))$ when $N\to\infty$ it follows that

$$\int_0^{\cdot}\Phi_N\,dW\to\int_0^{\cdot}\Phi\,dW$$

in $L^2(\Omega\times(0,T);L^2(D))$ when $N\to\infty$. From

$$-\int_{\Omega\times Q}\left(u_N-\int_0^t\Phi_N\,dW\right)\cdot\partial_t\phi+|\nabla u_N|^{p(\cdot)-2}\nabla u_N\cdot\nabla\phi\,d\mu-\int_{\Omega\times D}u_0\varphi(\omega,0,x)\,dx\,dP=0 \tag{31}$$

and with analogous arguments as in (26) and (27) it follows that

$$\partial_t\left(u-\int_0^t\Phi\,dW\right)\in L^{\alpha'}(0,T;W^{-1,p'}(D)),\quad u-\int_0^t\Phi\,dW\in C([0,T];L^2(D))$$

a.s. in Ω for $\alpha\geq 2+p^+$. Thus we can integrate (27) to obtain a.s.

$$u(t)-u_0-\int_0^t\Delta_{p(\cdot)}u\,ds=\int_0^t\Phi\,dW \tag{32}$$

in $L^2(D)$ for all $t\in[0,T]$.

As a conclusion, on has that

Proposition 4.4. *Proposition 4.3 still holds for any $\Phi\in N_w^2(0,T;HS(L^2(D)))$.*

4.4. The multiplicative case

We want to apply Banach's fixed point theorem to the map

$$\Psi:S\in N_w^2(0,T;L^2(D))\to u_S\in N_w^2(0,T;L^2(D))$$

(see also [1]) where u_S is the solution to $(P,H(\cdot,S))$ to deduce the existence of a unique solution u of (P,H) in the sense of Definition 3.1.

Let us first note that thanks to the assumptions on (h_k) and by classical arguments based on Nemytskii operators, one has that $H(\cdot,S)\in N_w^2(0,T;HS(L^2(D)))$ when $S\in N_w^2(0,T;L^2(D))$.

Thus, thanks to Proposition 4.4, the mapping Ψ is well-defined and u is a solution to the multiplicative problem iff it is a fixed point for Ψ.

Set $S_1,S_2\in N_w^2(0,T;L^2(D))$ and denote by $u_1=\Psi(S_1)$ and $u_2=\Psi(S_2)$. Thanks to the stability inequality for the additive case and to $(H1)$ it follows that

$$E\|(u_1-u_2)\|^2_{C([0,T];L^2(D))}\leq C\|H(\cdot,S_1)-H(\cdot,S_2)\|^2_{L^2(\Omega\times(0,T);HS(L^2(D)))}$$

$$
\begin{aligned}
&= CE \int_0^T \int_D \sum_{k=1}^{\infty} |h_k(\omega, t, S_1(\omega, t, x)) - h_k(\omega, t, S_2(\omega, t, x))|^2 \; dx \; dt \\
&\leq C \|S_1 - S_2\|^2_{N_w^2(0,T;L^2(D))},
\end{aligned} \tag{33}
$$

where $C \geq 0$ is a constant that may change from line to line. From (33) it follows that

$$
\int_0^T E\|(u_1 - u_1)(t)\|^2_{L^2(D)} e^{-\alpha t} \; dt \leq \frac{C}{\alpha}(1 - e^{-\alpha T}) \int_0^T E\|S_1 - S_2\|^2_{L^2(D)} e^{-\alpha t} \; dt \tag{34}
$$

for $\alpha > 0$ and therefore the mapping S has a fixed point in $N_w^2(0, T; L^2(D))$ for a suitable value of α.

This finishes the proof of Theorem 3.1.

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References

[1] C. Bauzet, G. Vallet, P. Wittbold, A. Zimmermann, On a $p(t, x)$-Laplace evolution equation with a stochastic force, Stoch. Partial Differ. Equ., Anal. Comput. 1 (3) (2013) 552–570.
[2] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia Math. Appl., vol. 44, Cambridge University Press, Cambridge, 1992.
[3] L. Diening, P. Nägele, M. Ruzicka, Operator theory for unsteady problems in variable exponent spaces, Complex Var. Elliptic Equ. 57 (11) (2012) 1209–1231.
[4] J. Droniou, Intégration et espaces de Sobolev à valeurs vectorielles, Tech. Report, University of Marseille, 2011, Gm3-02.
[5] N.V. Krylov, B.L. Rozovskii, Stochastic evolution equations, J. Sov. Math. 16 (4) (1981) 1233–1277.
[6] P. Lindqvist, Notes on the p-Laplace equation, Tech. Report, Univ. Jyväskylä Dep. Math. Stat., vol. 102, 2006.
[7] C. Prévôt, M. Röckner, A Concise Course on Stochastic Partial Differential Equations, Lecture Notes in Math., vol. 1905, Springer, Berlin, 2007.