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On the double points of operator stable Lévy processes

Tomasz Luks · Yimin Xiao

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Abstract We determine the Hausdorff dimension of the set of double points for a symmetric operator stable Lévy process $X = \{X(t), t \in \mathbb{R}_+\}$ in terms of the eigenvalues of its stability exponent.

Keywords Multiple points · Hausdorff dimension · Operator stable process · Lévy process

Mathematics Subject Classification (2010) 60J25 · 60J30 · 60G51 · 60G17

1 Introduction

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a stochastic process with values in \mathbb{R}^d , $d \geq 1$, and let $k \geq 2$ be an integer. A point $x \in \mathbb{R}^d$ is called a k -multiple point of X if there exist k distinct times $t_1, \dots, t_k \in \mathbb{R}_+$ such that

$$X(t_1) = \dots = X(t_k) = x.$$

Denote by M_k the set of k -multiple points of X . If $k = 2$, then x is also called a *double point* of X . The existence of multiple points (or intersections) has been intensely studied for Brownian motion and more general Lévy processes in the literature (see [2, 3, 4, 5, 7, 8, 15, 17, 19] and the references therein).

In the present paper we focus on the Hausdorff dimension of the set of double points for a symmetric operator stable Lévy process in \mathbb{R}^d , where $d \geq 2$. A Lévy process $X = \{X(t), t \in \mathbb{R}_+\}$ with values in \mathbb{R}^d is called *operator stable* if the distribution ν of $X(1)$ is full (i.e., not supported on any $(d-1)$ -dimensional hyperplane) and there exists a linear operator B on \mathbb{R}^d such that $\nu^t = t^B \nu$ for all $t > 0$, where ν^t denotes the t -fold convolution power of the infinitely divisible law ν and $t^B \nu(dx) = \nu(t^{-B} dx)$ is the image measure of ν under the linear operator t^B . The operator B is called a *stability exponent* of X . We refer to [16] for more information on operator stable laws.

Our first result gives a general formula for the Hausdorff dimension of the set of k -multiple points for a symmetric Lévy process $X = \{X(t), t \in \mathbb{R}_+\}$ in terms of its characteristic exponent Ψ . See Section 2 for the terminology.

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Theorem 1 *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a symmetric, absolutely continuous Lévy process with values in \mathbb{R}^d ($d \geq 1$) and Lévy exponent Ψ . Then for any $k \geq 2$ we have*

$$\dim_{\mathbb{H}} M_k = d - \inf \left\{ \beta \in (0, d) : \int_{\mathbb{R}^{kd}} \left[\frac{1}{1 + \|\sum_{l=1}^k \xi_l\|^\beta} \prod_{j=1}^k \frac{1}{1 + \Psi(\xi_j)} \right] d\bar{\xi} < \infty \right\}$$

almost surely, where $\bar{\xi} := (\xi_1, \dots, \xi_k) \in \mathbb{R}^{kd}$ and $\xi_i \in \mathbb{R}^d$ for $i = 1, \dots, k$. If the integral above is infinite for all $\beta \in (0, d)$, then $\dim_{\mathbb{H}} M_k = 0$ a.s.

Next we apply Theorem 1 to a symmetric operator stable Lévy process with stability exponent B . To this end, we factor the minimal polynomial of B into $q_1(x) \cdots q_p(x)$, where all the roots of $q_i(x)$ have real parts a_i and $a_i < a_j$ for $i < j$. Define $V_i = \text{Ker}(q_i(B))$ and $d_i = \dim(V_i)$. Then $d_1 + \cdots + d_p = d$ and $V_1 \oplus \cdots \oplus V_p$ is a direct sum decomposition of \mathbb{R}^d into B -invariant subspaces. We may write $B = B_1 \oplus \cdots \oplus B_p$, where $B_i: V_i \rightarrow V_i$ and every eigenvalue of B_i has real part equal to a_i . For $j = 1, \dots, d$ and $l = 1, \dots, p$ denote $\alpha_j = a_l^{-1}$ whenever $\sum_{i=0}^{l-1} d_i < j \leq \sum_{i=0}^l d_i$, where $d_0 := 0$. We then have $\alpha_1 \geq \dots \geq \alpha_d$, and note that $0 < \alpha_j \leq 2$ in view of [16, Theorem 7.2.1]. Our second theorem, which is the main result of the paper, provides an explicit formula for the Hausdorff dimension of M_2 in terms of the exponents α_j .

Theorem 2 *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a symmetric operator stable Lévy process in \mathbb{R}^d with stability exponent B and let M_2 be the set of double points of X .*

(a) *If $d = 2$ then*

$$\dim_{\mathbb{H}} M_2 = \min \left\{ \alpha_1 \left(2 - \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right), 2\alpha_2 - \frac{2\alpha_2}{\alpha_1} \right\} \quad a.s.$$

(b) *If $d = 3$ then*

$$\dim_{\mathbb{H}} M_2 = \alpha_1 \left(2 - \frac{1}{\alpha_1} - \frac{1}{\alpha_2} - \frac{1}{\alpha_3} \right) \quad a.s.,$$

where a negative dimension means that $M_2 = \emptyset$. Furthermore, $M_2 = \emptyset$ for all $d \geq 4$.

Theorem 2 is more general than Theorem 1 in [19], where B is assumed to be a diagonal matrix with entries on the diagonal $\alpha_j \in (1, 2)$ ($1 \leq j \leq d$). Also, the methods used in [19] are probabilistic in nature and they provide formulas for all $k \geq 2$. Our approach is analytical and it extends the results of [19] to the whole family of symmetric operator stable Lévy processes in the case $k = 2$. In addition, we provide a necessary and sufficient condition for the existence of double points of symmetric operator stable Lévy processes in terms of the exponents α_i , see Theorem 5 and Theorem 7 below. The latter also reveals some subtle behavior when B is not a diagonal matrix. Adapting our techniques to the case $k \geq 3$ is more involved and will be dealt with separately.

The paper is organized as follows. In Section 2 we give definitions and some facts concerning operator stable Lévy processes and we prove Theorem 1. In Sections 3 and 4 we focus on the double points problems for $d = 2$ and $d = 3$, respectively.

Throughout the rest of the paper, C will denote a positive constant, whose value may change in each appearance.

2 Preliminaries

A stochastic process $X = \{X(t), t \in \mathbb{R}_+\}$ with values in \mathbb{R}^d is called a Lévy process if X has stationary and independent increments, $X(0) = 0$ a.s. and $t \mapsto X(t)$ is continuous in probability. We refer to the books [1, 18] for systematic accounts on Lévy processes.

It is known that the finite-dimensional distributions of X are determined by the characteristic function

$$\mathbb{E}[e^{i\langle \xi, X(t) \rangle}] = e^{-t\Psi(\xi)}, \quad \forall t \geq 0,$$

where $\Psi: \mathbb{R}^d \mapsto \mathbb{C}$ is given by the Lévy-Khintchine formula and is called the *characteristic or Lévy exponent* of X .

A Lévy process X is said to be *symmetric* if $-X$ and X have the same finite-dimensional distributions. In such a case, $\Psi(\xi) \geq 0$ for all $\xi \in \mathbb{R}^d$. Using the terminology in [10,11], we say that X is *absolutely continuous*, if for all $t > 0$, the function $\xi \mapsto e^{-t\Psi(\xi)}$ is in $L^1(\mathbb{R}^d)$.

The following theorem gives a necessary and sufficient condition for the existence of k -multiple points for a symmetric, absolutely continuous Lévy process in terms of its characteristic exponent. See [5,15,6,14] for appropriate conditions in terms of the potential density.

Theorem 3 *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a symmetric, absolutely continuous Lévy process with values in \mathbb{R}^d ($d \geq 1$) and characteristic exponent Ψ . Then X has k -multiple points if and only if*

$$\int_{\mathbb{R}^{d(k-1)}} \prod_{j=1}^k \frac{1}{1 + \Psi(\xi_{j-1} - \xi_j)} d\bar{\xi} < \infty,$$

where $\bar{\xi} := (\xi_1, \dots, \xi_{k-1}) \in \mathbb{R}^{d(k-1)}$, $\xi_i \in \mathbb{R}^d$ for $i = 1, \dots, k-1$, and $\xi_0 = \xi_k := 0$.

Proof. The existence of k -multiple points of X is equivalent with the existence of intersections of k independent copies of X , see [15, Proof of Theorem 1]. Furthermore, since X is symmetric, by [11, Theorem 2.1] X is also weakly unimodal. Hence, by [10, Remark 6.6], k independent copies of X intersect if and only if

$$\int_{\mathbb{R}^{d(k-1)}} \frac{1}{1 + \Psi(\sum_{j=1}^{k-1} v_j)} \prod_{j=1}^{k-1} \frac{1}{1 + \Psi(v_j)} d\bar{v} < \infty.$$

Applying the change of variables $v_j = \xi_j - \xi_{j+1}$, $j = 1, \dots, k-1$, where $\xi_k := 0$, we obtain the desired result. \square

According to Theorem 3, a symmetric, absolutely continuous Lévy process X in \mathbb{R}^d with Lévy exponent Ψ has double points if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1 + \Psi(\xi)} \right)^2 d\xi < \infty.$$

Since $\Psi(\xi) \leq \|\xi\|^2$ for all $\|\xi\|$ large enough, the above condition does not hold when $d \geq 4$, which immediately implies the last statement of Theorem 2.

Proof of Theorem 1. Let X be a symmetric, absolutely continuous Lévy process in \mathbb{R}^d and let X_1, \dots, X_k be k independent copies of X . For any $x_1, x_2, \dots, x_k \in \mathbb{R}^d$, let $\tilde{X}_j(t) = x_j + X_j(t)$ for all $t \geq 0$ and $1 \leq j \leq k$. Thus, $\tilde{X}_j = \{\tilde{X}_j(t), t \geq 0\}$ is a Lévy process starting from x_j .

Denote by \tilde{M}_k the set of intersections of $\tilde{X}_1, \dots, \tilde{X}_k$, i.e.,

$$\tilde{M}_k = \bigcap_{i=1}^k \tilde{X}_i([0, \infty)).$$

Each X_i has a one-potential density $u: \mathbb{R}^d \mapsto \overline{\mathbb{R}}_+$ satisfying $u(0) > 0$. Indeed, by the symmetry, the transition density of X_i satisfies

$$p_t(0) = \int_{\mathbb{R}^d} (p_{t/2}(x))^2 dx > 0.$$

By [14, Theorem 1.5], we can derive that, for x_1, \dots, x_k in a small neighborhood of the origin, the formula of Theorem 1 holds for $\dim_{\mathbb{H}} \widetilde{M}_k$. Therefore, it is enough to show that $\dim_{\mathbb{H}} M_k = \dim_{\mathbb{H}} \widetilde{M}_k$ almost surely. This last property is part of the folklore for Lévy processes and has already been applied by e.g., [7, p. 85] or [19, Section 3]. However, as pointed out in [15, p. 510], the equality needs some rigorous justification in our general context.

Since the idea is similar to the argument in the proof of [15, Theorem 1], we only provide the main steps of our proof. For a positive integer N and $\alpha \in (0, 2)$ let $Y_{\alpha, N}$ be an N -parameter additive α -stable process with values in \mathbb{R}^d (cf. [10, 14]), independent of X, X_1, \dots, X_k . By [12, Corollary 3.4, see also Section 4], for any Borel set $F \subset \mathbb{R}^d$ we have

$$\dim_{\mathbb{H}} F = d - \inf \{ N\alpha > 0 : \mathbb{P}(F \cap Y_{\alpha, N}(\mathbb{R}_+^N)) > 0 \}. \quad (1)$$

For $0 \leq s < t$, let $X([s, t])$ denotes the path of X on the time interval $[s, t]$. The following conditions are equivalent.

- (i) $\mathbb{P}(\widetilde{M}_k \cap Y_{\alpha, N}(\mathbb{R}_+^N)) > 0$.
- (ii) There exist a constant $s > 0$ and a neighborhood U of 0 in \mathbb{R}^d such that for any (initial states of \widetilde{X}_i) $x_1, \dots, x_k \in U$ and $r > 0$ we have

$$\mathbb{P}\left(Y_{\alpha, N}(\mathbb{R}_+^N) \cap \bigcap_{i=1}^k \widetilde{X}_i([0, s]) \neq \emptyset, \widetilde{X}_i(s) \in B(0, r), i = 1, \dots, k\right) > 0.$$

- (iii) There exist a constant $s > 0$ and positive real numbers a_1, \dots, a_{k-1} satisfying $a_i \geq a_{i-1} + 2, a_0 = 0$, such that

$$\mathbb{P}\left(Y_{\alpha, N}(\mathbb{R}_+^N) \cap \bigcap_{i=0}^{k-1} X([a_i s, (a_i + 1)s]) \neq \emptyset\right) > 0.$$

The equivalence between (i) and (ii) follows from the fact that $p_t(0) > 0$ for all $t > 0$. Indeed, by the lower semicontinuity of p_t , for any $t > 0$ there exists a neighborhood U of 0 such that for all $x, y \in U$ we have $p_t(x - y) > 0$, see also [15, Proof of Theorem 3]. The equivalence between (ii) and (iii) can be proved using similar techniques as in [15, Proof of Theorem 1]. Finally, Condition (iii) is equivalent with $\mathbb{P}(M_k \cap Y_{\alpha, N}(\mathbb{R}_+^N)) > 0$. This, (1) and the Borel-Cantelli argument in [15, p. 511] yield Theorem 1. \square

Let X be an operator stable Lévy process in \mathbb{R}^d with stability exponent B , where $d \geq 2$. It is well-known that there exists a real invertible $d \times d$ matrix P such that

$$B = PDP^{-1},$$

where D is a real $d \times d$ matrix of the form

$$D = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_p \end{pmatrix}, \quad (2)$$

and each block $J_i, i = 1, \dots, p$, is of the form

$$\begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 1 & a & 0 & \dots & 0 \\ 0 & 1 & a & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} C & 0 & 0 & \dots & 0 \\ I & C & 0 & \dots & 0 \\ 0 & I & C & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & C \end{pmatrix}, \quad (3)$$

where a is a real eigenvalue of B in the first case, and in the second case

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4)$$

where $a \pm ib$ is a complex conjugate pair of eigenvalues of B . Using the notation from the Introduction, the size of J_i is equal to d_i and a_i is the real part of the corresponding eigenvalue of J_i . Recall also that $\alpha_j = a_l^{-1}$ whenever $\sum_{i=0}^{l-1} d_i < j \leq \sum_{i=0}^l d_i$, where $d_0 := 0$, $j = 1, \dots, d$ and $l = 1, \dots, p$.

If X is symmetric, then the Lévy exponent Ψ of X is nonnegative, and by [17, Theorem 4.2] the following condition holds: for every $\varepsilon > 0$, there exists a constant $\tau > 1$ such that

$$\frac{K^{-1}}{\|\xi\|^\varepsilon \sum_{j=1}^d |\xi_j|^{\alpha_j}} \leq \frac{1}{1 + \Psi(\xi)} \leq \frac{K \|\xi\|^\varepsilon}{\sum_{j=1}^d |\xi_j|^{\alpha_j}} \quad (5)$$

for all $\xi \in \mathbb{R}^d$ with $\|\xi\| \geq \tau$, where $K \geq 1$ is a constant which depends on ε and τ only.

3 Double points problem for $d = 2$

Throughout this section, X is a symmetric operator stable Lévy process in \mathbb{R}^2 with Lévy exponent Ψ and stability exponent B whose eigenvalues have real parts $\alpha_1^{-1}, \alpha_2^{-1}$, as explained in Section 1.

To prove Part (a) of Theorem 2, we apply Theorem 1 to X using the estimate (5). Let

$$A = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x| > 1, |y| > 1\},$$

and let

$$I_\beta = \iint_A \frac{dx dy}{(1 + |x_1 + y_1|^\beta + |x_2 + y_2|^\beta)(|x_1|^{\alpha_1} + |x_2|^{\alpha_2})(|y_1|^{\alpha_1} + |y_2|^{\alpha_2})}.$$

By Theorem 1, (5) and by Lemma 2 below, we have

$$\dim_{\mathbb{H}} M_2 = 2 - \inf \{\beta \in (0, 2) : I_\beta < \infty\} \quad \text{a.s.} \quad (6)$$

Therefore, Part (a) of Theorem 2 is an immediate consequence of (6) and Theorem 4 below.

Theorem 4 *If $2 - 1/\alpha_1 - 1/\alpha_2 > 0$ then*

$$\inf \{\beta > 0 : I_\beta < \infty\} = \max \left\{ 3 + \frac{\alpha_1}{\alpha_2} - 2\alpha_1, 2 + \frac{2\alpha_2}{\alpha_1} - 2\alpha_2 \right\}. \quad (7)$$

If $2 - 1/\alpha_1 - 1/\alpha_2 \leq 0$, then $I_\beta = \infty$ for any $\beta > 0$. Thus $\dim_{\mathbb{H}} M_2 = 0$ a.s.

Proof. We divide the proof into two parts. Part (i) considers the case when both α_1 and α_2 are not integers. Part (ii) deals with the remaining cases [i.e., $\alpha_1 = \alpha_2 = 2$ and $\alpha_1 = 2, \alpha_2 = 1$], where an extra factor of $\ln k$ or $\ln n$ may appear in upper bounds for the integrals in (11)–(14) below. Since only slight modifications will be needed to prove (7), we will not provide all the details.

Part (i): For $k, n \in \mathbb{N}$ denote by $A_{k,n}$ the set

$$\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x| > 1, |y| > 1, k - 1 \leq |x_1 + y_1| < k, n - 1 \leq |x_2 + y_2| < n\}.$$

We have $I_\beta < \infty$ if and only if

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^\beta + n^\beta} \int \int_{A_{k,n}} \frac{dx dy}{(|x_1|^{\alpha_1} + |x_2|^{\alpha_2})(|y_1|^{\alpha_1} + |y_2|^{\alpha_2})} < \infty. \quad (8)$$

Furthermore, we can assume that $|x_1|, |x_2| \geq 1$ and $y_1, y_2 \geq 1$. We then have

$$|y_1 - k| \wedge |y_1 - k + 1| \leq |x_1| \leq y_1 + k,$$

$$|y_2 - n| \wedge |y_2 - n + 1| \leq |x_2| \leq y_2 + n.$$

The first estimate follows from the inequalities

$$-y_1 - k \leq x_1 \leq -y_1 - k + 1,$$

or

$$-y_1 + k - 1 \leq x_1 \leq -y_1 + k,$$

for the second one the argument is similar. Furthermore, we may assume that $|x_1| \geq |y_1 - k|$, $|x_2| \geq |y_2 - n|$ and $|y_1 - k| + |y_2 - n| \geq C > 0$. Therefore, the double series in (8) is bounded from below by

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^{\beta} + n^{\beta}} \int_1^{\infty} \int_1^{\infty} \frac{dy_1 dy_2}{((y_1 + k)^{\alpha_1} + (y_2 + n)^{\alpha_2})(y_1^{\alpha_1} + y_2^{\alpha_2})}, \quad (9)$$

and from above by

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^{\beta} + n^{\beta}} \int_{B_{k,n}} \frac{dy}{(|y_1 - k|^{\alpha_1} + |y_2 - n|^{\alpha_2})(y_1^{\alpha_1} + y_2^{\alpha_2})}, \quad (10)$$

where

$$B_{k,n} = \{y \in \mathbb{R}^2 : \min\{y_1, y_2, |y_1 - k|, |y_2 - n|\} \geq 1\}.$$

To estimate (10), we may assume that $k, n \geq 3$ and note that

$$\begin{aligned} & \int_{B_{k,n}} \frac{dy}{(|y_1 - k|^{\alpha_1} + |y_2 - n|^{\alpha_2})(y_1^{\alpha_1} + y_2^{\alpha_2})} \\ &= \int_1^{n-1} \int_1^{k-1} \frac{dy_1 dy_2}{((k - y_1)^{\alpha_1} + (n - y_2)^{\alpha_2})(y_1^{\alpha_1} + y_2^{\alpha_2})} \end{aligned} \quad (11)$$

$$+ \int_{n+1}^{\infty} \int_1^{k-1} \frac{dy_1 dy_2}{((k - y_1)^{\alpha_1} + (y_2 - n)^{\alpha_2})(y_1^{\alpha_1} + y_2^{\alpha_2})} \quad (12)$$

$$+ \int_1^{n-1} \int_{k+1}^{\infty} \frac{dy_1 dy_2}{((y_1 - k)^{\alpha_1} + (n - y_2)^{\alpha_2})(y_1^{\alpha_1} + y_2^{\alpha_2})} \quad (13)$$

$$+ \int_{n+1}^{\infty} \int_{k+1}^{\infty} \frac{dy_1 dy_2}{((y_1 - k)^{\alpha_1} + (y_2 - n)^{\alpha_2})(y_1^{\alpha_1} + y_2^{\alpha_2})}. \quad (14)$$

We start with the integral (14). After a change of variables, (14) is equal to

$$\int_1^{\infty} \int_1^{\infty} \frac{dtds}{(t^{\alpha_1} + s^{\alpha_2})((t+k)^{\alpha_1} + (s+n)^{\alpha_2})}.$$

Observe that the same double integral appears in the lower estimate (9). We have

$$\frac{1}{(t+k)^{\alpha_1} + (s+n)^{\alpha_2}} \asymp \frac{1}{t^{\alpha_1} + k^{\alpha_1} + s^{\alpha_2} + n^{\alpha_2}} \asymp \frac{1}{k^{\alpha_1} + n^{\alpha_2}} \wedge \frac{1}{t^{\alpha_1}} \wedge \frac{1}{s^{\alpha_2}}.$$

Here \asymp means that there are two sided estimates with a constant depending only on α_1 and α_2 . Hence, (14) is comparable with

$$\begin{aligned} & \int_1^{\infty} \int_1^{\infty} \left(\frac{1}{t^{\alpha_1}} \wedge \frac{1}{s^{\alpha_2}} \right) \left(\frac{1}{k^{\alpha_1} + n^{\alpha_2}} \wedge \frac{1}{t^{\alpha_1}} \wedge \frac{1}{s^{\alpha_2}} \right) dtds \\ &= \int_1^{\infty} \int_1^{s^{\alpha_2/\alpha_1}} \frac{1}{s^{\alpha_2}} \left(\frac{1}{k^{\alpha_1} + n^{\alpha_2}} \wedge \frac{1}{s^{\alpha_2}} \right) dtds \\ &+ \int_1^{\infty} \int_{s^{\alpha_2/\alpha_1}}^{\infty} \frac{1}{t^{\alpha_1}} \left(\frac{1}{k^{\alpha_1} + n^{\alpha_2}} \wedge \frac{1}{t^{\alpha_1}} \right) dtds =: I_1 + I_2. \end{aligned}$$

We write

$$\begin{aligned} I_2 &= \int_1^\infty \int_{s^{\alpha_2/\alpha_1}}^\infty \frac{1}{t^{\alpha_1}} \left(\frac{1}{k^{\alpha_1} + n^{\alpha_2}} \wedge \frac{1}{t^{\alpha_1}} \right) dt ds \\ &= \int_1^{(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_2}} \int_{s^{\alpha_2/\alpha_1}}^\infty \frac{1}{t^{\alpha_1}} \left(\frac{1}{k^{\alpha_1} + n^{\alpha_2}} \wedge \frac{1}{t^{\alpha_1}} \right) dt ds \\ &\quad + \int_{(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_2}}^\infty \int_{s^{\alpha_2/\alpha_1}}^\infty \frac{1}{t^{2\alpha_1}} dt ds =: I_2^{(1)} + I_2^{(2)}. \end{aligned}$$

It can be seen that if $2 - 1/\alpha_1 - 1/\alpha_2 > 0$, then

$$I_2^{(2)} = C \int_{(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_2}}^\infty s^{\alpha_2/\alpha_1 - 2\alpha_2} ds = C (k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1 + 1/\alpha_2 - 2}.$$

On the other hand, if $2 - 1/\alpha_1 - 1/\alpha_2 \leq 0$, then the integral (14) is infinite and so is the series (9). So $I_\beta = \infty$ for any $\beta > 0$, which proves the second part of the theorem.

Next we consider the case when $2 - 1/\alpha_1 - 1/\alpha_2 > 0$. This implies that $\alpha_1 > 1$. We have

$$\begin{aligned} I_2^{(1)} &= \frac{1}{k^{\alpha_1} + n^{\alpha_2}} \int_1^{(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_2}} \int_{s^{\alpha_2/\alpha_1}}^{(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1}} t^{-\alpha_1} dt ds \\ &\quad + \int_1^{(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_2}} ds \int_{(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1}}^\infty t^{-2\alpha_1} dt \\ &\leq \frac{C}{k^{\alpha_1} + n^{\alpha_2}} \int_1^{(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_2}} s^{\alpha_2/\alpha_1 - \alpha_2} ds + C (k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1 + 1/\alpha_2 - 2} \\ &\leq C (k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1 + 1/\alpha_2 - 2}. \end{aligned}$$

In the last inequality we have used the assumption $\alpha_2 < 2$. Similarly, we verify that

$$\begin{aligned} I_1 &\leq \int_1^\infty s^{\alpha_2/\alpha_1 - \alpha_2} \left(\frac{1}{k^{\alpha_1} + n^{\alpha_2}} \wedge \frac{1}{s^{\alpha_2}} \right) ds \\ &= \int_1^{(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_2}} \frac{s^{\alpha_2/\alpha_1 - \alpha_2}}{k^{\alpha_1} + n^{\alpha_2}} ds + \int_{(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_2}}^\infty s^{\alpha_2/\alpha_1 - 2\alpha_2} ds \\ &\leq C (k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1 + 1/\alpha_2 - 2}. \end{aligned}$$

Hence we have

$$(14) \asymp \left(\frac{1}{k^{\alpha_1} + n^{\alpha_2}} \right)^{2 - 1/\alpha_1 - 1/\alpha_2}. \quad (15)$$

Now we consider the integral (11). After the change of variables $t = k - y_1$, $s = n - y_2$, we get

$$\begin{aligned} &\int_{\frac{n}{2}}^{\frac{n}{2}} \int_1^{k-1} \frac{dy_1 dy_2}{[(k - y_1)^{\alpha_1} + (n - y_2)^{\alpha_2}] (y_1^{\alpha_1} + y_2^{\alpha_2})} \\ &= \int_{\frac{n}{2}}^{n-1} \int_1^{k-1} \frac{dt ds}{(t^{\alpha_1} + s^{\alpha_2}) [(k - t)^{\alpha_1} + (n - s)^{\alpha_2}]}. \end{aligned}$$

Hence, the integral (11) is equal to

$$\begin{aligned} &2 \int_1^{\frac{n}{2}} \int_1^{k-1} \frac{dt ds}{(t^{\alpha_1} + s^{\alpha_2}) [(k - t)^{\alpha_1} + (n - s)^{\alpha_2}]} \leq \frac{C}{n^{\alpha_2}} \int_1^{\frac{n}{2}} \int_1^{k-1} \frac{dt ds}{(t + s^{\alpha_2/\alpha_1})^{\alpha_1}} \\ &= \frac{C}{n^{\alpha_2}} \int_1^{\frac{n}{2}} \int_{1+s^{\alpha_2/\alpha_1}}^{k+s^{\alpha_2/\alpha_1}-1} u^{-\alpha_1} du ds \leq \frac{C}{n^{\alpha_2}} \int_1^{\frac{n}{2}} s^{\alpha_2/\alpha_1 - \alpha_2} ds, \end{aligned} \quad (16)$$

where the last inequality follows from the fact that $\alpha_1 > 1$. Since $\alpha_2 < 2$, we get

$$\frac{C}{n^{\alpha_2}} \int_1^{\frac{n}{2}} s^{\alpha_2/\alpha_1 - \alpha_2} ds \leq C \left(\frac{1}{n^{\alpha_2}} \right)^{2-1/\alpha_1-1/\alpha_2}. \quad (17)$$

Interchanging the roles of n and k , the same argument as in (16) and (17) shows that the integral (11) is at most

$$C \left(\frac{1}{k^{\alpha_1}} \wedge \frac{1}{n^{\alpha_2}} \right)^{2-1/\alpha_1-1/\alpha_2} \leq C \left(\frac{1}{k^{\alpha_1} + n^{\alpha_2}} \right)^{2-1/\alpha_1-1/\alpha_2}.$$

For the integral (12), a change of variables $k - y_1 = t$, $y_2 - n = s$ implies that it is equal to

$$\begin{aligned} & \int_1^\infty \int_1^{k-1} \frac{dt ds}{(t^{\alpha_1} + s^{\alpha_2})[(k-t)^{\alpha_1} + (s+n)^{\alpha_2}]} \\ & \leq C \int_1^\infty \frac{ds}{(s+n)^{\alpha_2}} \int_1^{k-1} \frac{dt}{(t+s^{\alpha_2/\alpha_1})^{\alpha_1}} \\ & \leq C \int_1^\infty \frac{ds}{(s+n)^{\alpha_2} s^{\alpha_2 - \alpha_2/\alpha_1}} \leq C \left(\frac{1}{n^{\alpha_2}} \right)^{2-1/\alpha_1-1/\alpha_2}. \end{aligned}$$

In deriving the last inequality we have used the assumption that $\alpha_2 < 2$.

To bound the integral (12) in terms of k , we note that

$$\int_1^\infty \int_1^{\frac{k}{2}} \frac{dt ds}{(t^{\alpha_1} + s^{\alpha_2})[(k-t)^{\alpha_1} + (s+n)^{\alpha_2}]} \leq \int_1^\infty \int_1^{\frac{k}{2}} \frac{C dt ds}{(t^{\alpha_1} + s^{\alpha_2})(k^{\alpha_1} + s^{\alpha_2})}$$

and

$$\int_1^\infty \int_{\frac{k}{2}}^{k-1} \frac{dt ds}{(t^{\alpha_1} + s^{\alpha_2})[(k-t)^{\alpha_1} + (s+n)^{\alpha_2}]} \leq \int_1^\infty \int_1^{\frac{k}{2}} \frac{C du ds}{(k^{\alpha_1} + s^{\alpha_2})(u^{\alpha_1} + s^{\alpha_2})}.$$

Thus,

$$\int_1^\infty \int_1^{k-1} \frac{dt ds}{(t^{\alpha_1} + s^{\alpha_2})[(k-t)^{\alpha_1} + (s+n)^{\alpha_2}]} \leq \int_1^{\frac{k}{2}} \int_1^\infty \frac{C ds dt}{(t^{\alpha_1} + s^{\alpha_2})(k^{\alpha_1} + s^{\alpha_2})}.$$

Assume first that $\alpha_2 > 1$. We can verify that the integral (12) is less than

$$\frac{C}{k^{\alpha_1}} \int_1^{\frac{k}{2}} \int_1^\infty \frac{ds dt}{(t^{\alpha_1/\alpha_2} + s)^{\alpha_2}} \leq C \left(\frac{1}{k^{\alpha_1}} \right)^{2-1/\alpha_1-1/\alpha_2},$$

Combining the above yields that (12) is bounded from above by

$$C (k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1+1/\alpha_2-2}.$$

By symmetry we also get that (13) is less than $C (k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1+1/\alpha_2-2}$ (indeed, this case is even easier since $\alpha_1 > 1$). Therefore, we have proved that

$$(11), (12), (13) \leq C \left(\frac{1}{k^{\alpha_1} + n^{\alpha_2}} \right)^{2-1/\alpha_1-1/\alpha_2}.$$

This and (15) imply that, except the cases $\alpha_1 = \alpha_2 = 2$ or $\alpha_1 = 2$ and $\alpha_2 = 1$, the series (9) and (10) (and hence I_β) are finite if and only if $2 - 1/\alpha_1 - 1/\alpha_2 > 0$ and

$$\sum_{k=1}^\infty \sum_{n=1}^\infty \frac{1}{k^\beta + n^\beta} \left(\frac{1}{k^{\alpha_1} + n^{\alpha_2}} \right)^{2-1/\alpha_1-1/\alpha_2} < \infty.$$

Hence the theorem follows from Lemma 1 below.

Part (ii): If $\alpha_2 = \alpha_1 = 2$ then the methods used in Part (i) still apply. In this case, the left hand side of (17) is less than $Cn^{-2} \ln n$, and by symmetry, the integral (11) can be estimated by

$$\frac{\ln k}{k^2} \wedge \frac{\ln n}{n^2} \leq C \left(\frac{1}{k^{2-\varepsilon} + n^{2-\varepsilon}} \right)^{2-2/(2-\varepsilon)},$$

for any small $\varepsilon > 0$ and $k, n \geq N_\varepsilon > 0$, provided N_ε is sufficiently large. Therefore, Lemma 1 can be applied with $\alpha_1 = \alpha_2 = 2 - \varepsilon$ for the upper estimate and with $\alpha_1 = \alpha_2 = 2$ for the lower bound. Since $\varepsilon > 0$ is arbitrary, we obtain the desired result. In the case $\alpha_1 = 2$ and $\alpha_2 = 1$, the reasoning is similar and is omitted. \square

Lemma 1 *We have*

$$\begin{aligned} & \inf \left\{ \beta \in (0, 2) : \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^\beta + n^\beta} \left(\frac{1}{k^{\alpha_1} + n^{\alpha_2}} \right)^{2-1/\alpha_1-1/\alpha_2} < \infty \right\} \\ & = \max \left\{ 3 + \frac{\alpha_1}{\alpha_2} - 2\alpha_1, 2 + \frac{2\alpha_2}{\alpha_1} - 2\alpha_2 \right\}. \end{aligned}$$

Proof. The convergence of the series is equivalent with the convergence of the integral

$$\begin{aligned} & \int_1^\infty \int_1^\infty \left(\frac{1}{x} \wedge \frac{1}{y} \right)^\beta \left(\frac{1}{x^{\alpha_1}} \wedge \frac{1}{y^{\alpha_2}} \right)^{2-1/\alpha_1-1/\alpha_2} dx dy \\ & = \int_1^\infty \int_1^{y^{\alpha_2/\alpha_1}} \frac{1}{y^\beta} \left(\frac{1}{y^{\alpha_2}} \right)^{2-1/\alpha_1-1/\alpha_2} dx dy \\ & \quad + \int_1^\infty \int_{y^{\alpha_2/\alpha_1}}^y \frac{1}{y^\beta} \left(\frac{1}{x^{\alpha_1}} \right)^{2-1/\alpha_1-1/\alpha_2} dx dy \\ & \quad + \int_1^\infty \int_y^\infty \frac{1}{x^\beta} \left(\frac{1}{x^{\alpha_1}} \right)^{2-1/\alpha_1-1/\alpha_2} dx dy =: I_1 + I_2 + I_3. \end{aligned}$$

It can be seen that $I_1 < \infty$ if and only if

$$\int_1^\infty y^{-\beta-2\alpha_2+2\alpha_2/\alpha_1+1} dy < \infty,$$

and the last condition is equivalent with $\beta > 2 + 2\alpha_2/\alpha_1 - 2\alpha_2$.

Next we consider I_2 . If $\alpha_1 = \alpha_2$, then $I_2 = 0$, so assume that $\alpha_1 \neq \alpha_2$. If $-2\alpha_1 + \alpha_1/\alpha_2 = -2$, then

$$I_2 = (1 - \alpha_2/\alpha_1) \int_1^\infty y^{-\beta} \ln y dy.$$

So $I_2 < \infty$ if and only if $\beta > 1 = 3 + \alpha_1/\alpha_2 - 2\alpha_1$. Suppose now $-2\alpha_1 + \alpha_1/\alpha_2 \neq -2$. Then we have

$$I_2 = \int_1^\infty y^{-\beta} \left(\frac{y^{-2\alpha_1+\alpha_1/\alpha_2+2} - (y^{\alpha_2/\alpha_1})^{-2\alpha_1+\alpha_1/\alpha_2+2}}{-2\alpha_1 + \alpha_1/\alpha_2 + 2} \right) dx dy.$$

We consider two cases:

(a) If $-2\alpha_1 + \alpha_1/\alpha_2 + 2 > 0$, then

$$I_2 \leq \int_1^\infty \frac{y^{-\beta-2\alpha_1+\alpha_1/\alpha_2+2}}{-2\alpha_1 + \alpha_1/\alpha_2 + 2} dx dy,$$

and the last integral is finite if $\beta > 3 + \alpha_1/\alpha_2 - 2\alpha_1$.

(b) If $-2\alpha_1 + \alpha_1/\alpha_2 + 2 < 0$, then

$$I_2 \leq \int_1^\infty y^{-\beta} \left(\frac{(y^{\alpha_2/\alpha_1})^{-2\alpha_1 + \alpha_1/\alpha_2 + 2}}{-2\alpha_1 + \alpha_1/\alpha_2 + 2} \right) dx dy = \int_1^\infty \frac{y^{-\beta + 2\alpha_2/\alpha_1 - 2\alpha_2 + 1}}{-2\alpha_1 + \alpha_1/\alpha_2 + 2} dy,$$

which is finite if $\beta > 2 + 2\alpha_2/\alpha_1 - 2\alpha_2$.

Therefore, the condition $\beta > \max\{3 + \alpha_1/\alpha_2 - 2\alpha_1, 2 + \alpha_2/\alpha_1 - 2\alpha_2\}$ implies $I_2 < \infty$.

Finally, we consider I_3 . A necessary condition for $I_3 < \infty$ is $-\beta - 2\alpha_1 + \alpha_1/\alpha_2 + 1 < -1$. Assuming this we get

$$I_3 = \int_1^\infty \frac{y^{-\beta - 2\alpha_1 + \alpha_1/\alpha_2 + 2}}{\beta + 2\alpha_1 - \alpha_1/\alpha_2 - 2} dy.$$

Thus $I_3 < \infty$ if and only if $\beta > 3 + \alpha_1/\alpha_2 - 2\alpha_1$.

Therefore, we have proved that

$$\beta > \max\{3 + \alpha_1/\alpha_2 - 2\alpha_1, 2 + \alpha_2/\alpha_1 - 2\alpha_2\} \Rightarrow I_1, I_2, I_3 < \infty,$$

and

$$I_1, I_3 < \infty \Rightarrow \beta > \max\{3 + \alpha_1/\alpha_2 - 2\alpha_1, 2 + \alpha_2/\alpha_1 - 2\alpha_2\}.$$

This yields the conclusion of the lemma. \square

Lemma 2 *We have*

$$\begin{aligned} & \inf\{\beta \in (0, 2) : I_\beta < \infty\} \\ &= \inf\left\{\beta \in (0, 2) : \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dx dy}{(1 + \|x + y\|^\beta)(1 + \Psi(x))(1 + \Psi(y))} < \infty\right\}. \end{aligned} \quad (18)$$

Proof. Denote the first and the second term in (18) by γ and γ' , respectively. For any fixed $\beta > \gamma$, we show that if $\varepsilon > 0$ is small enough, then

$$\iint_A \frac{\|x\|^\varepsilon \|y\|^\varepsilon}{(1 + \|x + y\|^\beta)(|x_1|^{\alpha_1} + |x_2|^{\alpha_2})(|y_1|^{\alpha_1} + |y_2|^{\alpha_2})} dx dy < \infty, \quad (19)$$

where $A = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x| > 1, |y| > 1\}$. By the upper bound in (5), this implies $\gamma' \leq \gamma$.

To prove (19), it is enough to show that

$$\iint_A \frac{(|x_1|^{\varepsilon\alpha_1} + |x_2|^{\varepsilon\alpha_2})(|y_1|^{\varepsilon\alpha_1} + |y_2|^{\varepsilon\alpha_2})}{(1 + |x_1 + y_1|^\beta + |x_2 + y_2|^\beta)(|x_1|^{\alpha_1} + |x_2|^{\alpha_2})(|y_1|^{\alpha_1} + |y_2|^{\alpha_2})} dx dy < \infty$$

for sufficiently small $\varepsilon > 0$. Furthermore, the above integral is comparable to

$$\iint_A \frac{dx dy}{(1 + |x_1 + y_1|^\beta + |x_2 + y_2|^\beta)(|x_1|^{\alpha'_1} + |x_2|^{\alpha'_2})(|y_1|^{\alpha'_1} + |y_2|^{\alpha'_2})} =: I'_\beta,$$

where $\alpha'_i := (1 - \varepsilon)\alpha_i$. By Theorem 4,

$$\gamma = \max\left\{3 + \frac{\alpha_1}{\alpha_2} - 2\alpha_1, 2 + \frac{2\alpha_2}{\alpha_1} - 2\alpha_2\right\},$$

and since $\beta > \gamma$, we may choose $\varepsilon > 0$ such that

$$\max\left\{3 + \frac{\alpha'_1}{\alpha'_2} - 2\alpha'_1, 2 + \frac{2\alpha'_2}{\alpha'_1} - 2\alpha'_2\right\} < \beta.$$

Theorem 4 implies $I'_\beta < \infty$ and thus (19) holds. In order to show that $\gamma' \geq \gamma$ we use similar arguments and the lower estimate of (5). \square

According to Part (a) of Theorem 2, the set of double points of X has positive Hausdorff dimension if and only if $2 - 1/\alpha_1 - 1/\alpha_2 > 0$. The next theorem shows that this is also a necessary condition for the existence of double points of X .

Theorem 5 M_2 is nonempty if and only if $2 - 1/\alpha_1 - 1/\alpha_2 > 0$.

Proof. By Theorem 3 we have $M_2 \neq \emptyset$ if and only if

$$\int_{\mathbb{R}^2} \left(\frac{1}{1 + \Psi(\xi)} \right)^2 d\xi < \infty. \quad (20)$$

According to the decomposition described in Section 2, the stability exponent of X satisfies $B = PDP^{-1}$. Since we consider the case $d = 2$, the matrix D can have the following forms

- (a) $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$;
 (b) $\begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}$ or $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

By [17, (4.9),(4.14),(4.15),(4.16)], we have the following estimates of the Lévy exponent $\Psi(\xi)$ when $\|\xi\| \rightarrow \infty$, depending on the cases (a) and (b):

- (a) $\alpha_1 = 1/a_1$, $\alpha_2 = 1/a_2$, and

$$\Psi(\xi) \asymp |\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_2};$$

- (b) $\alpha_1 = \alpha_2 = 1/a$, and

$$\Psi(\xi) \asymp |\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_1} (\ln \|\xi\|)^{\alpha_1}.$$

In the case (a), it follows from the proof of Theorem 4 and the estimates of the integral (14) that (20) holds if and only if $2 - 1/\alpha_1 - 1/\alpha_2 > 0$. In the case (b) we have $\alpha_1 = \alpha_2$ and the inequality $2 - 1/\alpha_1 - 1/\alpha_2 > 0$ is equivalent with $\alpha_1 > 1$. Hence, it is enough to show that (20) does not hold for $\alpha_1 = 1$. Under this assumption we have

$$\begin{aligned} \int_{\mathbb{R}^2} \left(\frac{1}{1 + \Psi(\xi)} \right)^2 d\xi &\geq C \int_1^\infty \int_1^\infty \frac{dxdy}{(x + y \ln \sqrt{x^2 + y^2})^2} \\ &\geq C \int_1^\infty \int_1^x \frac{dydx}{(x + y \ln x)^2} \\ &= \int_1^\infty \left(\frac{1}{x + \ln x} - \frac{1}{x + x \ln x} \right) \frac{dx}{\ln x}. \end{aligned}$$

Since the last integral diverges, the theorem is proved. \square

4 Double points problem for $d = 3$

We will now focus on the proof of Part (b) of Theorem 2. Denote

$$D = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| > 1, |y| > 1\},$$

and for $\beta > 0$ let

$$\begin{aligned} J_\beta &= \iint_D \frac{1}{(1 + |x_1 + y_1|^\beta + |x_2 + y_2|^\beta + |x_3 + y_3|^\beta)} \\ &\quad \times \frac{dxdy}{(|x_1|^{\alpha_1} + |x_2|^{\alpha_2} + |x_3|^{\alpha_3})(|y_1|^{\alpha_1} + |y_2|^{\alpha_2} + |y_3|^{\alpha_3})}. \end{aligned}$$

As in Section 3, we use (5) and Theorem 1 to conclude that

$$\dim_{\text{H}} M_2 = 3 - \inf \{\beta \in (0, 3) : J_\beta < \infty\}. \quad (21)$$

Hence, Part (b) of Theorem 2 follows from (21) and Theorem 6 below.

Theorem 6 When $2 - \sum_{j=1}^3 \frac{1}{\alpha_j} > 0$, we have

$$\inf \{ \beta > 0 : J_\beta < \infty \} = 4 + \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_3} - 2\alpha_1.$$

If $2 - \sum_{j=1}^3 \frac{1}{\alpha_j} \leq 0$, then $J_\beta = \infty$ for all $\beta > 0$.

Proof. In the proof we assume that $\alpha_3 \leq \alpha_2 < 2$. When $\alpha_3 \leq \alpha_2 = \alpha_1 = 2$, the reasoning is similar to Part (ii) of the proof of Theorem 4. For $k, n, m \in \mathbb{N}$, denote

$$D_{k,n,m} = \{ (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| > 1, |y| > 1, k-1 \leq |x_1 + y_1| < k, \\ n-1 \leq |x_2 + y_2| < n, m-1 \leq |x_3 + y_3| < m \}.$$

We have $J_\beta < \infty$ if and only if

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{k^\beta + n^\beta + m^\beta} \iint_{D_{k,n,m}} \frac{1}{(|x_1|^{\alpha_1} + |x_2|^{\alpha_2} + |x_3|^{\alpha_3})} \\ \times \frac{dxdy}{(|y_1|^{\alpha_1} + |y_2|^{\alpha_2} + |y_3|^{\alpha_3})} < \infty. \quad (22)$$

Our goal is to prove that if $2 - 1/\alpha_1 - 1/\alpha_2 - 1/\alpha_3 > 0$ then

$$\iint_{D_{k,n,m}} \frac{dxdy}{(|x_1|^{\alpha_1} + |x_2|^{\alpha_2} + |x_3|^{\alpha_3})(|y_1|^{\alpha_1} + |y_2|^{\alpha_2} + |y_3|^{\alpha_3})} \\ \asymp (k^{\alpha_1} + n^{\alpha_2} + m^{\alpha_3})^{1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 - 2} \quad (23)$$

for sufficiently large k, n, m , whereas the integral in (23) is infinite if $2 - 1/\alpha_1 - 1/\alpha_2 - 1/\alpha_3 \leq 0$. The theorem will then follow from Lemma 4 below.

For this purpose, we may and will assume that $|x_1|, |x_2|, |x_3| \geq 1$ and $y_1, y_2, y_3 \geq 1$. We then have

$$|y_1 - k| \wedge |y_1 - k + 1| \leq |x_1| \leq y_1 + k, \\ |y_2 - n| \wedge |y_2 - n + 1| \leq |x_2| \leq y_2 + n, \\ |y_3 - m| \wedge |y_3 - m + 1| \leq |x_3| \leq y_3 + m.$$

Furthermore, we may also assume that $|x_1| \geq |y_1 - k|$, $|x_2| \geq |y_2 - n|$, $|x_3| \geq |y_3 - m|$ and $|y_1 - k| + |y_2 - n| + |y_3 - m| \geq C > 0$. Therefore, the integral in (23) can be estimated from below by

$$\int_1^\infty \int_1^\infty \int_1^\infty \frac{dy_1 dy_2 dy_3}{(y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3})(y_1 + k)^{\alpha_1} + (y_2 + n)^{\alpha_2} + (y_3 + m)^{\alpha_3}} \quad (24)$$

and from above by

$$\int_{E_{k,n,m}} \frac{dy}{(y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3})(|y_1 - k|^{\alpha_1} + |y_2 - n|^{\alpha_2} + |y_3 - m|^{\alpha_3})}, \quad (25)$$

where

$$E_{k,n,m} = \{ y \in \mathbb{R}^3 : \min \{ y_1, y_2, y_3, |y_1 - k|, |y_2 - n|, |y_3 - m| \} \geq 1 \}.$$

The integral (25) can be written as

$$\left(\int_1^{m-1} + \int_{m+1}^\infty \right) \left(\int_1^{n-1} + \int_{n+1}^\infty \right) \left(\int_1^{k-1} + \int_{k+1}^\infty \right) \frac{1}{y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3}} \\ \times \frac{dy_1 dy_2 dy_3}{|y_1 - k|^{\alpha_1} + |y_2 - n|^{\alpha_2} + |y_3 - m|^{\alpha_3}},$$

which is equal to the sum of the following integrals

$$\begin{aligned}
1) & \int_1^{m-1} \int_1^{n-1} \int_1^{k-1}, \\
2) & \int_1^{m-1} \int_{n+1}^{\infty} \int_{k+1}^{\infty}, \int_{m+1}^{\infty} \int_1^{n-1} \int_{k+1}^{\infty}, \int_{m+1}^{\infty} \int_{n+1}^{\infty} \int_1^{k-1}, \\
3) & \int_1^{m-1} \int_1^{n-1} \int_{k+1}^{\infty}, \int_1^{m-1} \int_{n+1}^{\infty} \int_1^{k-1}, \int_{m+1}^{\infty} \int_1^{n-1} \int_1^{k-1}, \\
4) & \int_{m+1}^{\infty} \int_{n+1}^{\infty} \int_{k+1}^{\infty}.
\end{aligned}$$

Since the integral in 4) is the same as the one in the lower bound (24), we start by establishing desired upper and lower bounds as in (23) for the integral in 4). To simplify the notation, denote $\eta := k^{\alpha_1} + n^{\alpha_2} + m^{\alpha_3}$. One can verify that

$$\begin{aligned}
& \int_{m+1}^{\infty} \int_{n+1}^{\infty} \int_{k+1}^{\infty} \frac{dy_1 dy_2 dy_3}{(y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3})(|y_1 - k|^{\alpha_1} + |y_2 - n|^{\alpha_2} + |y_3 - m|^{\alpha_3})} \\
& \asymp \int_1^{\infty} \int_1^{\infty} \int_1^{\infty} \frac{dx_1 dx_2 dx_3}{(x_1^{\alpha_1} + x_2^{\alpha_2} + x_3^{\alpha_3})(x_1^{\alpha_1} + x_2^{\alpha_2} + x_3^{\alpha_3} + \eta)} \\
& \asymp \int_1^{\infty} \int_1^{\infty} \int_1^{\infty} \left(\frac{1}{x_1^{\alpha_1}} \wedge \frac{1}{x_2^{\alpha_2}} \wedge \frac{1}{x_3^{\alpha_3}} \right) \left(\frac{1}{x_1^{\alpha_1}} \wedge \frac{1}{x_2^{\alpha_2}} \wedge \frac{1}{x_3^{\alpha_3}} \wedge \frac{1}{\eta} \right) dx_1 dx_2 dx_3 \\
& = \int_1^{\infty} \int_1^{x_3^{\alpha_3/\alpha_2}} \int_1^{x_2^{\alpha_2/\alpha_1}} \frac{1}{x_3^{\alpha_3}} \left[\frac{1}{x_3^{\alpha_3}} \wedge \frac{1}{\eta} \right] dx_1 dx_2 dx_3 \\
& \quad + \int_1^{\infty} \int_{x_3^{\alpha_3/\alpha_2}}^{\infty} \int_1^{x_2^{\alpha_2/\alpha_1}} \frac{1}{x_2^{\alpha_2}} \left[\frac{1}{x_2^{\alpha_2}} \wedge \frac{1}{\eta} \right] dx_1 dx_2 dx_3 \\
& \quad + \int_1^{\infty} \int_{x_2^{\alpha_2/\alpha_1}}^{\infty} \int_1^{x_1^{\alpha_1/\alpha_3}} \frac{1}{x_1^{\alpha_1}} \left[\frac{1}{x_1^{\alpha_1}} \wedge \frac{1}{\eta} \right] dx_3 dx_1 dx_2 \\
& \quad + \int_1^{\infty} \int_{x_2^{\alpha_2/\alpha_1}}^{\infty} \int_{x_1^{\alpha_1/\alpha_3}}^{\infty} \frac{1}{x_3^{\alpha_3}} \left[\frac{1}{x_3^{\alpha_3}} \wedge \frac{1}{\eta} \right] dx_3 dx_1 dx_2 =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For I_1 , by breaking the integral according to $x_3 \leq \eta^{1/\alpha_3}$ and $x_3 > \eta^{1/\alpha_3}$, we can verify that I_1 is convergent if and only if $2 - 1/\alpha_1 - 1/\alpha_2 - 1/\alpha_3 > 0$ and in the later case,

$$I_1 \asymp \eta^{1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 - 2}.$$

This also proves the second part of the theorem.

Next we assume $2 - 1/\alpha_1 - 1/\alpha_2 - 1/\alpha_3 > 0$. Then it is elementary to verify that

$$I_2, I_3, I_4 \leq C\eta^{1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 - 2}.$$

Hence the integral in 4) satisfies the same bounds.

Next, consider the integral in 1). Noticing that

$$\int_1^{m-1} \int_1^{n-1} \int_1^{k/2} (\dots) = \int_1^{m-1} \int_1^{n-1} \int_{k/2}^{k-1} (\dots),$$

we have

$$\begin{aligned}
& \int_1^{m-1} \int_1^{n-1} \int_1^{k-1} \frac{dy_1 dy_2 dy_3}{(y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3})(|y_1 - k|^{\alpha_1} + |y_2 - n|^{\alpha_2} + |y_3 - m|^{\alpha_3})} \\
& \leq C \int_1^{k/2} \frac{dy_1}{(k - y_1)^{\alpha_1}} \int_1^{n-1} \int_1^{m-1} \frac{dy_3 dy_2}{(y_1^{\alpha_1/\alpha_3} + y_2^{\alpha_2/\alpha_3} + y_3)^{\alpha_3}}.
\end{aligned}$$

Integrating out dy_3 and dy_2 we see that the last integral is at most

$$Ck^{-\alpha_1} \int_1^{k/2} (n + y_1^{\alpha_1/\alpha_2})^{1+\alpha_2/\alpha_3-\alpha_2} dy_1 \leq Ck^{-\alpha_1} (k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-1}.$$

By symmetry, the integral in 1) is also less than

$$Cn^{-\alpha_2} (k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-1}.$$

Combining two terms we see that the integral in 1) is at most

$$\begin{aligned} & C(k^{-\alpha_1} \wedge n^{-\alpha_2})(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-1} \\ & \leq C(k^{\alpha_1} + n^{\alpha_2})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2}. \end{aligned}$$

Since similar estimates work with pairs $k^{\alpha_1}, m^{\alpha_3}$ and $n^{\alpha_2}, m^{\alpha_3}$, we obtain the following majorant for the integral in 1):

$$\begin{aligned} & C \min \{k^{\alpha_1} + n^{\alpha_2}, (k^{\alpha_1} + m^{\alpha_3}), (n^{\alpha_2} + m^{\alpha_3})\}^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2} \\ & \leq C(k^{\alpha_1} + n^{\alpha_2} + m^{\alpha_3})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2}. \end{aligned}$$

Next, by the symmetry, it is enough to consider only one integral of type 2) and one of type 3). Consider first an integral of type 3) as follows.

$$\begin{aligned} & \int_1^{m-1} \int_1^{n-1} \int_{k+1}^{\infty} \frac{dy_1 dy_2 dy_3}{(y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3})(|y_1 - k|^{\alpha_1} + |y_2 - n|^{\alpha_2} + |y_3 - m|^{\alpha_3})} \quad (26) \\ & = \left(\int_1^{m/2} \int_1^{n/2} \int_1^{\infty} + \int_{m/2}^{m-1} \int_1^{n/2} \int_1^{\infty} + \int_1^{m/2} \int_{n/2}^{n-1} \int_1^{\infty} + \int_{m/2}^{m-1} \int_{n/2}^{n-1} \int_1^{\infty} \right) \\ & \times \frac{dy_1 dy_2 dy_3}{(y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3})((y_1 + k)^{\alpha_1} + (n - y_2)^{\alpha_2} + (m - y_3)^{\alpha_3})} =: J_1 + J_2 + J_3 + J_4. \end{aligned}$$

We obtain

$$\begin{aligned} J_1 & = \int_1^{m/2} \int_1^{n/2} \int_1^{\infty} \frac{dy_1 dy_2 dy_3}{(y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3})((y_1 + k)^{\alpha_1} + (n - y_2)^{\alpha_2} + (m - y_3)^{\alpha_3})} \\ & \leq C \int_1^{m/2} \int_1^{n/2} \int_1^{\infty} \frac{dy_1 dy_2 dy_3}{(y_2^{\alpha_2} + y_3^{\alpha_3})(y_1^{\alpha_1} + m^{\alpha_3})} \\ & \leq C \int_1^{\infty} \frac{dy_1}{(y_1 + m^{\alpha_3/\alpha_1})^{\alpha_1}} \int_1^{m/2} \int_1^{n/2} \frac{dy_2 dy_3}{(y_2 + y_3^{\alpha_3/\alpha_2})^{\alpha_2}} \\ & \leq C (m^{\alpha_3})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2}. \end{aligned}$$

For J_2 , a simple change of variable yields

$$\begin{aligned} J_2 & = \int_1^{m/2} \int_1^{n/2} \int_1^{\infty} \frac{dy_1 dy_2 dr}{(y_1^{\alpha_1} + y_2^{\alpha_2} + (m - r)^{\alpha_3})((y_1 + k)^{\alpha_1} + (n - y_2)^{\alpha_2} + r^{\alpha_3})} \\ & \leq C \int_1^{m/2} \int_1^{n/2} \int_1^{\infty} \frac{dy_1 dy_2 dr}{(y_2^{\alpha_2} + m^{\alpha_3})(y_1^{\alpha_1} + r^{\alpha_3})} \\ & \leq C (m^{\alpha_3})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2}. \end{aligned}$$

In a similar manner we obtain

$$\begin{aligned} J_3 &= \int_1^{m/2} \int_1^{n/2} \int_1^\infty \frac{dy_1 ds dy_3}{(y_1^{\alpha_1} + (n-s)^{\alpha_2} + y_3^{\alpha_3})((y_1+k)^{\alpha_1} + s^{\alpha_2} + (m-y_3)^{\alpha_3})} \\ &\leq C \int_1^{m/2} \int_1^{n/2} \int_1^\infty \frac{dy_1 ds dy_3}{(y_1^{\alpha_1} + y_3^{\alpha_3})(s^{\alpha_2} + m^{\alpha_3})}, \end{aligned}$$

so the estimate is the same as for J_2 . Finally,

$$\begin{aligned} J_4 &= \int_1^{m/2} \int_1^{n/2} \int_1^\infty \frac{dy_1 ds dr}{(y_1^{\alpha_1} + (n-s)^{\alpha_2} + (m-r)^{\alpha_3})((y_1+k)^{\alpha_1} + s^{\alpha_2} + r^{\alpha_3})} \\ &\leq C \int_1^{m/2} \int_1^{n/2} \int_1^\infty \frac{dy_1 ds dr}{(y_1^{\alpha_1} + m^{\alpha_3})(s^{\alpha_2} + r^{\alpha_3})}, \end{aligned}$$

which is the same integral appeared in the estimation of J_1 . Hence we have proved that the integral (26) is less than $C(m^{\alpha_3})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2}$. By symmetry, it is also less than $C(n^{\alpha_2})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2}$.

In order to obtain a similar upper bound in terms of k^{α_1} instead of m^{α_3} we observe that

$$\begin{aligned} &\int_1^{m-1} \int_1^{n-1} \int_{k+1}^\infty \frac{dy_1 dy_2 dy_3}{(y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3})(|y_1-k|^{\alpha_1} + |y_2-n|^{\alpha_2} + |y_3-m|^{\alpha_3})} \\ &\leq n \int_1^{m-1} \int_1^\infty \frac{dy_1 dy_3}{(y_1^{\alpha_1} + y_3^{\alpha_3})((y_1+k)^{\alpha_1} + (m-y_3)^{\alpha_3})}. \end{aligned}$$

The last double integral is of the same type as the integral (13) in the proof of Theorem 4, and therefore, it is less than

$$Cn(k^{\alpha_1})^{1/\alpha_1+1/\alpha_3-2} = C\left(\frac{n^{\alpha_2}}{k^{\alpha_1}}\right)^{1/\alpha_2} (k^{\alpha_1})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2}.$$

If $k^{\alpha_1} > n^{\alpha_2}$, then we get the desired estimate. On the other hand, when $k^{\alpha_1} \leq n^{\alpha_2}$, then

$$(n^{\alpha_2})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2} \leq (k^{\alpha_1})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2},$$

since $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 - 2 < 0$ by our assumption, and the upper bound follows from the previous part of the proof. Therefore, the minimum of obtained upper bounds gives $C(k^{\alpha_1} + n^{\alpha_2} + m^{\alpha_3})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2}$ as a majorant for the integrals of type 3).

Finally, we consider an integral of type 2). After a change of variables we have

$$\begin{aligned} &\int_{m+1}^\infty \int_{n+1}^\infty \int_1^{k-1} \frac{dy_1 dy_2 dy_3}{(y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3})(|y_1-k|^{\alpha_1} + |y_2-n|^{\alpha_2} + |y_3-m|^{\alpha_3})} \\ &= \int_1^\infty \int_1^\infty \int_1^{k-1} \frac{dt ds dr}{(t^{\alpha_1} + s^{\alpha_2} + r^{\alpha_3})((k-t)^{\alpha_1} + (n+s)^{\alpha_2} + (m+r)^{\alpha_3})}. \end{aligned}$$

Furthermore, by breaking the integration interval $[1, k-1]$ in dt into $[1, k/2]$ and $[k/2, k-1]$, we derive

$$\begin{aligned} &\int_1^\infty \int_1^\infty \int_1^{k-1} \frac{dt ds dr}{(t^{\alpha_1} + s^{\alpha_2} + r^{\alpha_3})((k-t)^{\alpha_1} + (n+s)^{\alpha_2} + (m+r)^{\alpha_3})} \\ &\leq C \int_1^\infty \int_1^\infty \int_1^{k/2} \frac{dt ds dr}{(t^{\alpha_1} + s^{\alpha_2} + r^{\alpha_3})(k^{\alpha_1} + s^{\alpha_2} + r^{\alpha_3})}. \end{aligned}$$

One can show that the last term is less than $C(k^{\alpha_1})^{1/\alpha_1+1/\alpha_2+1/\alpha_3-2}$, which implies the same estimate for the initial integral of type 2). Since the method is similar to the case of the integral of type 4), we omit the details. Therefore, given Lemma 4, the proof of Theorem 6 is finished. \square

In order to prove Lemma 4 we will make use of the following inequality, whose proof is elementary and is omitted.

Lemma 3 For $2 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 > 1$ we have

$$\max \left\{ 2 + \frac{2\alpha_3}{\alpha_1} + \frac{2\alpha_3}{\alpha_2} - 2\alpha_3, 3 + \frac{2\alpha_2}{\alpha_1} + \frac{\alpha_2}{\alpha_3} - 2\alpha_2 \right\} \leq 4 + \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_3} - 2\alpha_1.$$

Lemma 4 Let $\gamma := 2 - 1/\alpha_1 - 1/\alpha_2 - 1/\alpha_3$. We have

$$\inf \left\{ \beta \in (0, 3) : \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(k^{\alpha_1} + n^{\alpha_2} + m^{\alpha_3})^{-\gamma}}{k^{\beta} + n^{\beta} + m^{\beta}} < \infty \right\} = 4 + \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_3} - 2\alpha_1.$$

Proof. The convergence of the series is equivalent to the convergence of the integral

$$\begin{aligned} & \int_1^{\infty} \int_1^{\infty} \int_1^{\infty} [x \vee y \vee z]^{-\beta} [x^{\alpha_1} \vee y^{\alpha_2} \vee z^{\alpha_3}]^{-\gamma} dx dy dz \\ &= \int_1^{\infty} \int_1^{\infty} \int_1^{y^{\alpha_2/\alpha_1}} (y \vee z)^{-\beta} (y^{\alpha_2} \vee z^{\alpha_3})^{-\gamma} dx dy dz \\ &+ \int_1^{\infty} \int_1^{\infty} \int_{y^{\alpha_2/\alpha_1}}^y (y \vee z)^{-\beta} (x^{\alpha_1} \vee z^{\alpha_3})^{-\gamma} dx dy dz \\ &+ \int_1^{\infty} \int_1^{\infty} \int_y^{\infty} (x \vee z)^{-\beta} (x^{\alpha_1} \vee z^{\alpha_3})^{-\gamma} dx dy dz =: I_1 + I_2 + I_3. \end{aligned}$$

We have $I_1 < \infty$ if and only if

$$\begin{aligned} \infty &> \int_1^{\infty} \int_1^{\infty} y^{\alpha_2/\alpha_1} (y \vee z)^{-\beta} (y^{\alpha_2} \vee z^{\alpha_3})^{-\gamma} dy dz \\ &= \int_1^{\infty} \int_1^{z^{\alpha_3/\alpha_2}} y^{\alpha_2/\alpha_1} z^{-\beta - \alpha_3\gamma} dy dz + \int_1^{\infty} \int_{z^{\alpha_3/\alpha_2}}^z y^{\alpha_2/\alpha_1 - \alpha_2\gamma} z^{-\beta} dy dz \\ &+ \int_1^{\infty} \int_z^{\infty} y^{\alpha_2/\alpha_1 - \alpha_2\gamma - \beta} dy dz =: I_1^{(1)} + I_1^{(2)} + I_1^{(3)}. \end{aligned}$$

First, it can be verified that $I_1^{(1)} < \infty$ if and only if $\alpha_3/\alpha_2(\alpha_2/\alpha_1 + 1) - \beta - \alpha_3\gamma < -1$. This gives the inequality $\beta > 2 + 2\alpha_3/\alpha_1 + 2\alpha_3/\alpha_2 - 2\alpha_3$. Furthermore, a necessary condition for $I_1^{(3)} < \infty$ is $\alpha_2/\alpha_1 - \alpha_2\gamma - \beta < -1$. Assuming this we get

$$I_1^{(3)} = \int_1^{\infty} \frac{z^{\alpha_2/\alpha_1 - \alpha_2\gamma - \beta + 1}}{\alpha_2\gamma + \beta - \alpha_2/\alpha_1 - 1} dz.$$

Hence, $I_1^{(3)} < \infty$ if and only if $\alpha_2/\alpha_1 - \alpha_2\gamma - \beta + 1 < -1$, which gives $\beta > 3 + 2\alpha_2/\alpha_1 + \alpha_2/\alpha_3 - 2\alpha_2$.

In order to estimate $I_1^{(2)}$, we observe that $\alpha_2/\alpha_1 - \alpha_2\gamma > -1$. Hence

$$I_1^{(2)} \asymp \int_1^{\infty} \frac{z^{1 + \alpha_2/\alpha_1 - \alpha_2\gamma - \beta}}{\alpha_2/\alpha_1 - \alpha_2\gamma + 1} dz,$$

and the last integral is finite if and only if $\beta > 3 + 2\alpha_2/\alpha_1 + \alpha_2/\alpha_3 - 2\alpha_2$. Therefore, we have proved that $I_1 < \infty$ if and only if

$$\beta > \max \left\{ 2 + \frac{2\alpha_3}{\alpha_1} + \frac{2\alpha_3}{\alpha_2} - 2\alpha_3, 3 + \frac{2\alpha_2}{\alpha_1} + \frac{\alpha_2}{\alpha_3} - 2\alpha_2 \right\}.$$

Next we rewrite I_2 as

$$\begin{aligned} I_2 &= \int_1^\infty \int_1^{z^{\alpha_3/\alpha_1}} \int_{y^{\alpha_2/\alpha_1}}^y z^{-\beta-\gamma\alpha_3} dx dy dz + \int_1^\infty \int_{z^{\alpha_3/\alpha_1}}^{z^{\alpha_3/\alpha_2}} \int_{y^{\alpha_2/\alpha_1}}^{z^{\alpha_3/\alpha_1}} z^{-\beta-\gamma\alpha_3} dx dy dz \\ &+ \int_1^\infty \int_{z^{\alpha_3/\alpha_1}}^{z^{\alpha_3/\alpha_2}} \int_{z^{\alpha_3/\alpha_1}}^y z^{-\beta} x^{-\gamma\alpha_1} dx dy dz + \int_1^\infty \int_{z^{\alpha_3/\alpha_2}}^z \int_{y^{\alpha_2/\alpha_1}}^y z^{-\beta} x^{-\gamma\alpha_1} dx dy dz \\ &+ \int_1^\infty \int_z^\infty \int_{y^{\alpha_2/\alpha_1}}^y y^{-\beta} x^{-\gamma\alpha_1} dx dy dz =: I_2^{(1)} + I_2^{(2)} + I_2^{(3)} + I_2^{(4)} + I_2^{(5)}. \end{aligned}$$

We get

$$I_2^{(1)} \leq \int_1^\infty \int_1^{z^{\alpha_3/\alpha_1}} y z^{-\beta-\gamma\alpha_3} dy dz \leq \int_1^\infty z^{-\beta-\gamma\alpha_3+2\alpha_3/\alpha_1} dz,$$

and the last integral is finite if and only if $\beta > 2+3\alpha_3/\alpha_1+\alpha_3/\alpha_2+2\alpha_3$. However, since $\alpha_3/\alpha_1 \leq \alpha_3/\alpha_2$, the condition $\beta > 2 + 2\alpha_3/\alpha_1 + 2\alpha_3/\alpha_2 + 2\alpha_3$ implies $I_2^{(1)} < \infty$. Secondly,

$$I_2^{(2)} \leq \int_1^\infty \int_{z^{\alpha_3/\alpha_1}}^{z^{\alpha_3/\alpha_2}} z^{-\beta-\gamma\alpha_3+\alpha_3/\alpha_1} dy dz \leq \int_1^\infty z^{-\beta-\gamma\alpha_3+\alpha_3/\alpha_1+\alpha_3/\alpha_2} dz.$$

Hence the condition $\beta > 2 + 2\alpha_3/\alpha_1 + 2\alpha_3/\alpha_2 + 2\alpha_3$ implies $I_2^{(2)} < \infty$. Next we have

$$I_2^{(3)} \vee I_2^{(4)} \leq \int_1^\infty \int_1^z \int_1^y z^{-\beta} x^{-\gamma\alpha_1} dx dy dz.$$

When $\alpha_3 < 2$, then $-\gamma\alpha_1 > -1$ and the last term is less than

$$C \int_1^\infty \int_1^z z^{-\beta} y^{1-\gamma\alpha_1} dy dz \leq C \int_1^\infty z^{2-\beta-\gamma\alpha_1} dz.$$

The last integral is finite if and only if $2 - \beta - \gamma\alpha_1 < -1$, which is equivalent to $\beta > 4 + \alpha_1/\alpha_2 + \alpha_1/\alpha_3 - 2\alpha_1$. If $\alpha_3 = 2$, then

$$\int_1^\infty \int_1^z \int_1^y z^{-\beta} x^{-\gamma\alpha_1} dx dy dz = \int_1^\infty \int_1^z z^{-\beta} \ln(y) dy dz \leq \int_1^\infty z^{1-\beta} \ln(z) dz,$$

which again gives the condition $\beta > 2 = 4 + \alpha_1/\alpha_2 + \alpha_1/\alpha_3 - 2\alpha_1$. This then implies $I_2^{(3)}, I_2^{(4)} < \infty$. Finally,

$$I_2^{(5)} \leq C \int_1^\infty \int_z^\infty y^{1-\beta-\gamma\alpha_1} dy dz.$$

For $1 - \beta - \gamma\alpha_1 < -1$ the last term is less than

$$C \int_1^\infty z^{2-\beta-\gamma\alpha_1} dz.$$

Therefore, the inequality $\beta > 4 + \alpha_1/\alpha_2 + \alpha_1/\alpha_3 - 2\alpha_1$ implies $I_2^{(5)} < \infty$. Hence, we have proved $I_2 < \infty$ provided

$$\beta > \max \left\{ 2 + \frac{2\alpha_3}{\alpha_1} + \frac{2\alpha_3}{\alpha_2} - 2\alpha_3, 4 + \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_3} - 2\alpha_1 \right\}.$$

Consider I_3 . We have

$$\begin{aligned}
I_3 &= \int_1^\infty \int_1^{z^{\alpha_3/\alpha_1}} \int_y^{z^{\alpha_3/\alpha_1}} z^{-\beta-\gamma\alpha_3} dx dy dz + \int_1^\infty \int_1^{z^{\alpha_3/\alpha_1}} \int_{z^{\alpha_3/\alpha_1}}^z z^{-\beta} x^{-\gamma\alpha_1} dx dy dz \\
&\quad + \int_1^\infty \int_1^{z^{\alpha_3/\alpha_1}} \int_z^\infty x^{-\beta-\gamma\alpha_1} dx dy dz + \int_1^\infty \int_{z^{\alpha_3/\alpha_1}}^z \int_y^z z^{-\beta} x^{-\gamma\alpha_1} dx dy dz \\
&\quad + \int_1^\infty \int_{z^{\alpha_3/\alpha_1}}^z \int_z^\infty x^{-\beta-\gamma\alpha_1} dx dy dz + \int_1^\infty \int_z^\infty \int_y^\infty x^{-\beta-\gamma\alpha_1} dx dy dz \\
&=: I_3^{(1)} + I_3^{(2)} + I_3^{(3)} + I_3^{(4)} + I_3^{(5)} + I_3^{(6)}.
\end{aligned}$$

We obtain

$$I_3^{(1)} \leq \int_1^\infty \int_1^{z^{\alpha_3/\alpha_1}} z^{-\beta-\gamma\alpha_3+\alpha_3/\alpha_1} dy dz \leq \int_1^\infty z^{-\beta-\gamma\alpha_3+2\alpha_3/\alpha_1} dz.$$

The last integral is finite if and only if $-\beta-\gamma\alpha_3+2\alpha_3/\alpha_1 < -1$ which gives $\beta > 2+3\alpha_3/\alpha_1+\alpha_3/\alpha_2-2\alpha_3$. Since

$$2+3\alpha_3/\alpha_1+\alpha_3/\alpha_2-2\alpha_3 < 2+2\alpha_3/\alpha_1+2\alpha_3/\alpha_2-2\alpha_3,$$

the condition $\beta > 2+2\alpha_3/\alpha_1+2\alpha_3/\alpha_2-2\alpha_3$ implies $I_3^{(1)} < \infty$. Next, when $\alpha_3 < 2$ then we get $-\gamma\alpha_1 > -1$ and

$$I_3^{(2)} \leq C \int_1^\infty \int_1^{z^{\alpha_3/\alpha_1}} z^{1-\beta-\gamma\alpha_1} dy dz \leq C \int_1^\infty z^{1+\alpha_3/\alpha_1-\beta-\gamma\alpha_1} dz.$$

The last integral is finite if and only if $1+\alpha_3/\alpha_1-\beta-\gamma\alpha_1 < -1$, which gives $\beta > 3+\alpha_3/\alpha_1+\alpha_1/\alpha_2+\alpha_1/\alpha_3-2\alpha_1$.

Proceeding in a similar manner we can show that $\beta > 4+\alpha_1/\alpha_2+\alpha_1/\alpha_3-2\alpha_1$ implies $I_3^{(3)}, I_3^{(4)}, I_3^{(5)} < \infty$.

Finally,

$$I_3^{(6)} = \int_1^\infty \int_z^\infty \frac{y^{1-\beta-\gamma\alpha_1}}{\beta+\gamma\alpha_1-1} dy dz = \int_1^\infty \frac{z^{2-\beta-\gamma\alpha_1}}{(\beta+\gamma\alpha_1-1)(\beta+\gamma\alpha_1-2)} dz.$$

Hence, $I_3^{(6)} < \infty$ if and only if $2-\beta-\gamma\alpha_1 < -1$, which gives $\beta > 4+\alpha_1/\alpha_2+\alpha_1/\alpha_3-2\alpha_1$.

In summary, we have proved the following

$$\begin{aligned}
I_1 < \infty &\Leftrightarrow \beta > \max \left\{ 2 + \frac{2\alpha_3}{\alpha_1} + \frac{2\alpha_3}{\alpha_2} - 2\alpha_3, 3 + \frac{2\alpha_2}{\alpha_1} + \frac{\alpha_2}{\alpha_3} - 2\alpha_2 \right\}, \\
\beta > \max \left\{ 2 + \frac{2\alpha_3}{\alpha_1} + \frac{2\alpha_3}{\alpha_2} - 2\alpha_3, 4 + \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_3} - 2\alpha_1 \right\} &\Rightarrow I_2, I_3 < \infty,
\end{aligned}$$

and $I_3 < \infty \Rightarrow I_3^{(6)} < \infty \Rightarrow \beta > 4+\alpha_1/\alpha_2+\alpha_1/\alpha_3-2\alpha_1$. The final conclusion follows from Lemma [B](#)

Our last result gives a necessary and sufficient condition for the existence of double points of X in \mathbb{R}^3 . It differs slightly from the case $d = 2$, i.e., X may possess double points even if the Hausdorff dimension of M_2 is 0.

According to the decomposition described in Section 2, the stability exponent of X satisfies $B = PDP^{-1}$. In the present case, the matrix D can have the following forms

$$(a) \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix};$$

$$(b) \begin{pmatrix} a_1 & 0 & 0 \\ 1 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix} \text{ or } \begin{pmatrix} a_1 - b_1 & 0 \\ b_1 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix};$$

$$(c) \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 1 & a_2 \end{pmatrix} \text{ or } \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & -b_2 \\ 0 & b_2 & a_2 \end{pmatrix};$$

$$(d) \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}.$$

Theorem 7 *The existence of double points of X depends on the cases (a)-(d) as follows:*

- In Cases (a), (b) and (c), $M_2 \neq \emptyset$ if and only if $2 - 1/\alpha_1 - 1/\alpha_2 - 1/\alpha_3 > 0$.
- In Case (d), $M_2 \neq \emptyset$ if and only if $\alpha_1 \geq 3/2$.

Proof. By Theorem 3 we have $M_2 \neq \emptyset$ if and only if

$$\int_{\mathbb{R}^3} \left(\frac{1}{1 + \Psi(\xi)} \right)^2 d\xi < \infty. \quad (27)$$

By [17, (4.9), (4.14), (4.15), (4.16)], we have the following estimates of the Lévy exponent $\Psi(\xi)$ when $\|\xi\| \rightarrow \infty$, depending on Cases (a)-(d):

(a) $\alpha_1 = 1/a_1$, $\alpha_2 = 1/a_2$, $\alpha_3 = 1/a_3$, and

$$\Psi(\xi) \asymp |\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_2} + |\xi_3|^{\alpha_3};$$

(b) $\alpha_1 = \alpha_2 = 1/a_1$, $\alpha_3 = 1/a_2$, and

$$\Psi(\xi) \asymp |\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_1} (\ln \|\xi\|)^{\alpha_1} + |\xi_3|^{\alpha_2};$$

(c) $\alpha_1 = 1/a_1$, $\alpha_2 = \alpha_3 = 1/a_2$, and

$$\Psi(\xi) \asymp |\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_2} + |\xi_3|^{\alpha_2} (\ln \|\xi\|)^{\alpha_2};$$

(d) $\alpha_1 = \alpha_2 = \alpha_3 = 1/a$, and

$$\Psi(\xi) \asymp |\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_1} (\ln \|\xi\|)^{\alpha_1} + |\xi_3|^{\alpha_1} (\ln \|\xi\|)^{2\alpha_1}.$$

In Case (a), it follows from the proof of Theorem 6 and the estimates of the integral (24) that (27) holds if and only if $2 - 1/\alpha_1 - 1/\alpha_2 - 1/\alpha_3 > 0$. This proves the theorem for Case (a).

For Case (d), (27) holds if and only if

$$\int_1^\infty \int_1^\infty \int_1^\infty \frac{dx dy dz}{\left(x + y \ln \sqrt{x^2 + y^2 + z^2} + z \ln^2 \sqrt{x^2 + y^2 + z^2} \right)^{2\alpha_1}} < \infty,$$

which, in turn, is equivalent to

$$\begin{aligned} & \int_1^\infty \int_1^x \int_1^x \frac{dz dy dx}{(x + y \ln x + z \ln^2 x)^{2\alpha_1}} + \int_1^\infty \int_1^y \int_1^y \frac{dz dx dy}{(x + y \ln y + z \ln^2 y)^{2\alpha_1}} \\ & + \int_1^\infty \int_1^z \int_1^z \frac{dx dy dz}{(x + y \ln z + z \ln^2 z)^{2\alpha_1}} < \infty. \end{aligned} \quad (28)$$

For the first integral in (28), we have

$$\begin{aligned} \int_1^\infty \int_1^x \int_1^x \frac{dzdydx}{(x+y\ln x+z\ln^2 x)^{2\alpha_1}} &\geq \int_1^\infty \int_1^x \int_1^y \frac{dzdydx}{(x+y\ln x+z\ln^2 x)^{2\alpha_1}} \\ &= \sum_{n=0}^\infty \int_{2^n}^{2^{n+1}} dx \int_1^x \int_1^y \frac{dzdy}{(x+y\ln x+z\ln^2 x)^{2\alpha_1}} \\ &\geq \sum_{n=0}^\infty \int_1^{2^n} \int_1^y \frac{C2^n dzdy}{(2^n+ny+n^2z)^{2\alpha_1}}. \end{aligned}$$

When $\alpha_1 < 3/2$, one can verify that the last series, thus (28), is infinite. This proves that $\alpha_1 \geq 3/2$ is a necessary condition for (27) in the case (d).

To prove sufficiency, it is enough to show that the three integrals in (28) are finite for $\alpha_1 = 3/2$. Since the method is similar, we only consider the first integral.

$$\begin{aligned} \int_1^\infty \int_1^x \int_1^x \frac{dzdydx}{(x+y\ln x+z\ln^2 x)^3} &= \sum_{n=0}^\infty \int_{2^n}^{2^{n+1}} dx \int_1^x \int_1^x \frac{dzdy}{(x+y\ln x+z\ln^2 x)^3} \\ &\leq C \sum_{n=0}^\infty \int_1^{2^{n+1}} \int_1^{2^{n+1}} \frac{2^n dzdy}{(2^n+ny+n^2z)^3} \\ &\leq C \sum_{n=0}^\infty \frac{1}{n^2} < \infty. \end{aligned}$$

This proves the second part of the theorem.

Next we consider Case (b). Since $\alpha_3 \leq \alpha_1$, for $|\xi_3| \geq 1$ and $\|\xi\| \geq e$ we have

$$|\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_1} (\ln \|\xi\|)^{\alpha_1} + |\xi_3|^{\alpha_3} \leq |\xi_1|^{\alpha_1} + |\xi_2|^{\alpha_1} (\ln \|\xi\|)^{\alpha_1} + |\xi_3|^{\alpha_1} (\ln \|\xi\|)^{2\alpha_1}.$$

Hence, if (27) does not hold in the case (d), then it does not hold in (b) either. Therefore, in what follows we may assume that $\alpha_1 \geq 3/2$. Since $\alpha_1 = \alpha_2 = 2$, the initial condition $2 - 1/\alpha_1 - 1/\alpha_2 - 1/\alpha_3 > 0$ translates into $2 - 2/\alpha_1 - 1/\alpha_3 > 0$. By (21) and Theorem 6, the Hausdorff dimension of M_2 is strictly positive (and hence $M_2 \neq \emptyset$) if $2 - 2/\alpha_1 - 1/\alpha_3 > 0$, so it is enough to show that (27) does not hold if $2 - 2/\alpha_1 - 1/\alpha_3 \leq 0$. Under this assumption, we need to prove that

$$\int_1^\infty \int_1^\infty \int_1^\infty \frac{dxdydz}{\left(x^{\alpha_1} + y^{\alpha_1} \ln^{\alpha_1} \sqrt{x^2 + y^2 + z^2} + z^{\alpha_3}\right)^2} = \infty.$$

The integral above can be estimated from below by (up to a constant factor),

$$\begin{aligned} \int_2^\infty \int_1^z \int_1^z \frac{dxdydz}{(x^{\alpha_1} + y^{\alpha_1} \ln^{\alpha_1} z + z^{\alpha_3})^2} &= \sum_{n=1}^\infty \int_{2^n}^{2^{n+1}} \int_1^z \int_1^z \frac{dxdydz}{(x^{\alpha_1} + y^{\alpha_1} \ln^{\alpha_1} z + z^{\alpha_3})^2} \\ &\geq \sum_{n=1}^\infty \int_1^{2^n} \int_1^{2^n} \frac{C2^n dxdy}{(x+ny+2^{n\alpha_3/\alpha_1})^{2\alpha_1}}. \end{aligned}$$

Since $\alpha_1 \geq 3/2$, the last term above is equal to

$$C \sum_{n=1}^\infty 2^n \int_1^{2^n} \left[\left(1+ny+2^{n\alpha_3/\alpha_1}\right)^{1-2\alpha_1} - \left(2^n+ny+2^{n\alpha_3/\alpha_1}\right)^{1-2\alpha_1} \right] dy. \quad (29)$$

It is not hard to verify that the last series diverges if $2 - 2/\alpha_1 - 1/\alpha_3 \leq 0$. This proves the theorem in Case (b).

Finally, we consider Case (c). As in the previous part, we may assume that $\alpha_1 \geq 3/2$. Also, the initial condition $2 - 1/\alpha_1 - 1/\alpha_2 - 1/\alpha_3 > 0$ becomes $2 - 1/\alpha_1 - 2/\alpha_2 > 0$, and by (21) and Theorem 6, it is enough to show that (27) does not hold if $2 - 1/\alpha_1 - 2/\alpha_2 \leq 0$. Furthermore, since $\alpha_2 \leq \alpha_1$, we may assume that $2 - 1/\alpha_1 - 1/\alpha_2 > 0$. Indeed, if $2 - 1/\alpha_1 - 1/\alpha_2 \leq 0$, then the integral in (27) is infinite in Case (b), which implies the same for Case (c). Furthermore it is enough to consider the case $\alpha_2 < \alpha_1$, since for $\alpha_2 = \alpha_1$ the theorem in Case (c) follows from that for Case (b), too. We have

$$\begin{aligned} \int_{\mathbb{R}^3} \left(\frac{1}{1 + \Psi(\xi)} \right)^2 d\xi &\geq \int_2^\infty \int_1^y \int_1^y \frac{C dx dz dy}{(x^{\alpha_1} + y^{\alpha_2} + z^{\alpha_2} \ln^{\alpha_2} y)^2} \\ &\geq \sum_{n=1}^\infty \int_1^{2^n} \int_1^{2^n} \frac{C 2^n dx dz}{(x + 2^{n\alpha_2/\alpha_1} + n^{\alpha_2/\alpha_1} z^{\alpha_2/\alpha_1})^{2\alpha_1}}. \end{aligned}$$

We can verify that the last series is infinite whenever $2 - 1/\alpha_1 - 2/\alpha_2 \leq 0$. This completes the proof of the theorem. \square

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