Kirillov's orbit method: the case of discrete series representations

Paul-Emile Paradan

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Abstract

Let π be a discrete series representation of a real semi-simple Lie group G' and let G be a semi-simple subgroup of G'. In this paper, we give a geometric expression of the G-multiplicities in $\pi|_G$ when the representation π is G-admissible.

Contents

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1 Introduction

This paper is concerned by a central problem of non-commutative harmonic analysis : given a unitary irreducible representation π of a Lie group G', how does π decomposes when restricted to a closed subgroup $G \subset G'$? We analyse this problem for Harish-Chandra discrete series representations of a connected real semi-simple Lie group G' with finite center, relatively to a connected real semi-simple subgroup G (also with finite center).

We start with Harish-Chandra parametrization of the discrete series representations. We can attach an unitary irreducible representation $\pi^{G'}_{\mathcal{O}'}$ of the group G' to any regular admissible elliptic coadjoint orbit $\mathcal{O}' \subset (\mathfrak{g}')^*$, and Schmid proved that the representation $\pi^{G'}_{\mathcal{O}'}$ could be realize as the quantization of the orbit $\mathcal{O}'$ [34, 35]. This is a vast generalization of Borel-Weil-Bott's construction of finite dimensional representations of compact Lie groups. In the following, we denote $\widehat{G}_d$ and $\widehat{G}'_d$ the sets of regular admissible elliptic coadjoint orbits of our connected real semi-simple Lie groups G and G'.

One of the rule of Kirillov's orbit method [13] is concerned with the functoriality relatively to inclusion $G \hookrightarrow G'$ of closed subgroups. It means that, starting with discrete series representations representations $\pi^G_{\mathcal{O}}$ and $\pi^{G'}_{\mathcal{O}'}$ attached to regular admissible elliptic orbits $\mathcal{O} \subset \mathfrak{g}^*$ and $\mathcal{O}' \subset (\mathfrak{g}')^*$, one expects that the multiplicity of $\pi^G_{\mathcal{O}}$ in the restriction $\pi^{G'}_{\mathcal{O}'}|_G$ can be computed geometrically in terms of the space

$$\mathcal{O}' /\!\!/ \mathcal{O} := \mathcal{O}' \cap \mathrm{p}_{\mathfrak{g},\mathfrak{g}'}^{-1}(\mathcal{O})/G, \tag{1.1}$$

where $\mathrm{p}_{\mathfrak{g},\mathfrak{g}'} : (\mathfrak{g}')^* \to \mathfrak{g}^*$ denotes the canonical projection. One recognises that (1.1) is a symplectic reduced space in the sense of Marsden-Weinstein, since $\mathrm{p}_{\mathfrak{g},\mathfrak{g}'} : \mathcal{O}' \to \mathfrak{g}^*$ is the moment map relative to the Hamiltonian action of G on $\mathcal{O}'$.

In other words, Kirillov's orbit method tells us that the branching laws $[\pi^G_{\mathcal{O}} : \pi^{G'}_{\mathcal{O}'}]$ should be compute geometrically. So far, the following special cases have been achieved :

1. $G \subset G'$ are compact. In the 1980s, Guillemin and Sternberg [8] studied the geometric quantization of general G-equivariant compact Kähler

manifolds. They proved the ground-breaking result that the multiplicities of this G-representation are calculated in terms of geometric quantizations of the symplectic reduced spaces. This phenomenon, which has been the center of many research and generalisations [22, 23, 37, 24, 21, 26, 33, 31, 10], is called nowaday "quantization commutes with reduction" (in short, "[Q,R]=0").

2. G is a compact subgroup of G'. In [25], we used the Blattner formula to see that the [Q,R]=0 phenomenon holds in this context when G is a maximal compact subgroup. Duflo-Vergne have generalized this result for any compact subgroup [7]. Recently, Hochs-Song-Wu have shown that the [Q,R]=0 phenomenon holds for any tempered representation of G' relatively to a maximal compact subgroup [11].

3. $\pi^{G'}_{\mathcal{O}'}$ is an holomorphic discrete series. We prove that the [Q,R]=0 phenomenon holds with some assumption on G [29].

However, one can observe that the restriction of $\pi^{G'}_{\mathcal{O}'}$ with respect to G may have a wild behavior in general, even if G is a maximal reductive subgroup in G' (see [15]).

In [15, 16, 17] T. Kobayashi singles out a nice class of branching problems where each G-irreducible summand of $\pi|_G$ occurs discretely with finite multiplicity : the restriction $\pi|_G$ is called G-admissible.

So we focus our attention to a discrete series $\pi^{G'}_{\mathcal{O}'}$ that admit an admissible restriction relatively to G. It is well-known that we have then an Hilbertian direct sum decomposition

$$\pi^{G'}_{\mathcal{O}'}|_G = \sum_{\mathcal{O} \in \widehat{G}_d} m^{\mathcal{O}}_{\mathcal{O}'}\, \pi^G_{\mathcal{O}}$$

where the multiplicities $m^{\mathcal{O}}_{\mathcal{O}'}$ are finite.

We will use the following geometrical characterization of the G-admissibility obtained by Duflo and Vargas [5, 6].

Proposition 1.1 *The representation $\pi^{G'}_{\mathcal{O}'}$ is G-admissible if and only if the restriction of the map $\mathrm{p}_{\mathfrak{g},\mathfrak{g}'}$ to the coadjoint orbit $\mathcal{O}'$ is a proper map.*

Let $(\mathcal{O}', \mathcal{O}) \in \widehat{G}'_d \times \widehat{G}_d$. Let us explain how we can quantize the compact symplectic reduced space $\mathcal{O}' /\!\!/ \mathcal{O}$ when the map $\mathrm{p}_{\mathfrak{g},\mathfrak{g}'} : \mathcal{O}' \to \mathfrak{g}^*$ is proper.

If $\mathcal{O}$ belongs to the set of regular values of $\mathrm{p}_{\mathfrak{g},\mathfrak{g}'} : \mathcal{O}' \to \mathfrak{g}^*$, then $\mathcal{O}' /\!\!/ \mathcal{O}$ is a compact symplectic orbifold equipped with a spin^c structure. We denote $\mathcal{Q}^{\mathrm{spin}}(\mathcal{O}' /\!\!/ \mathcal{O}) \in \mathbb{Z}$ the index of the corresponding spin^c Dirac operator.

In general, we consider an elliptic coadjoint $\mathcal{O}_\epsilon$ closed enough[1] to $\mathcal{O}$, so that $\mathcal{O}'/\!\!/\mathcal{O}_\epsilon$ is a compact symplectic orbifold equipped with a spin^c structure. Let $\mathcal{Q}^{\rm spin}(\mathcal{O}'/\!\!/\mathcal{O}_\epsilon) \in \mathbb{Z}$ be the index of the corresponding spin^c Dirac operator. The crucial fact is that the quantity $\mathcal{Q}^{\rm spin}(\mathcal{O}'/\!\!/\mathcal{O}_\epsilon)$ does not depends on the choice of generic and small enough ϵ. Then we take

$$\mathcal{Q}^{\rm spin}(\mathcal{O}'/\!\!/\mathcal{O}) := \mathcal{Q}^{\rm spin}(\mathcal{O}'/\!\!/\mathcal{O}_\epsilon)$$

for generic and small enough ϵ.

The main result of this article is the following

Theorem 1.2 *Let $\pi^{G'}_{\mathcal{O}'}$ be a discrete series representation of G' attached to a regular admissible elliptic coadjoint orbits $\mathcal{O}'$. If $\pi^{G'}_{\mathcal{O}'}$ is G-admissible we have the Hilbertian direct sum*

$$\pi^{G'}_{\mathcal{O}'}|_G = \sum_{\mathcal{O}\in\widehat{G}_d} \mathcal{Q}^{\rm spin}(\mathcal{O}'/\!\!/\mathcal{O})\, \pi^G_{\mathcal{O}}. \tag{1.2}$$

In other words the multiplicity $[\pi^G_{\mathcal{O}} : \pi^{G'}_{\mathcal{O}'}]$ is equal to $\mathcal{Q}^{\rm spin}(\mathcal{O}'/\!\!/\mathcal{O})$.

In a forthcoming paper we will study Equality (1.2) in further details when G is a symmetric subgroup of G'.

Theorem 1.2 give a positive answer to a conjecture of Duflo-Vargas.

Theorem 1.3 *Let $\pi^{G'}_{\mathcal{O}'}$ be a discrete series representation of G' that is G-admissible. Then all the representations $\pi^G_{\mathcal{O}}$ which occurs in $\pi^{G'}_{\mathcal{O}'}$ belongs to a* unique *family of discrete series representations of G.*

2 Restriction of discrete series representations

Let G be a connected real semi-simple Lie group G with finite center. A discrete series representation of G is an irreducible unitary representation that is isomorphic to a sub-representation of the left regular representation in $L^2(G)$. We denote $\widehat{G}_d$ the set of isomorphism class of discrete series representation of G.

We know after Harish-Chandra that $\widehat{G}_d$ is non-empty only if G has a compact Cartan subgroup. We denote $K \subset G$ a maximal compact subgroup and we suppose that G admits a compact Cartan subgroup $T \subset K$. The Lie algebras of the groups T, K, G are denoted respectively $\mathfrak{t}$, $\mathfrak{k}$ and $\mathfrak{g}$.

In this section we recall well-know facts concerning restriction of discrete series representations.

[1]The precise meaning will be explain in Section 5.2.

2.1 Admissible coadjoint orbits

Here we recall the parametrization of $\widehat{G}_d$ in terms of regular admissible elliptic coadjoint orbits. Let us fix some notations. We denote $\Lambda \subset \mathfrak{t}^*$ the weight lattice: any $\mu \in \Lambda$ defines a 1-dimensional representation $\mathbb{C}_\mu$ of the torus T.

Let $\mathfrak{R}_c \subset \mathfrak{R} \subset \Lambda$ be respectively the set of (real) roots for the action of T on $\mathfrak{k} \otimes \mathbb{C}$ and $\mathfrak{g} \otimes \mathbb{C}$. The non-compact roots are those belonging to the set $\mathfrak{R}_n := \mathfrak{R} \backslash \mathfrak{R}_c$. We choose a system of positive roots $\mathfrak{R}_c^+$ for $\mathfrak{R}_c$, we denote by $\mathfrak{t}^*_+$ the corresponding Weyl chamber. Recall that $\Lambda \cap \mathfrak{t}^*_+$ is the set of dominant weights.

We denote by B the Killing form on $\mathfrak{g}$. It induces a scalar product (denoted by $(-,-)$) on $\mathfrak{t}$, and then on $\mathfrak{t}^*$. An element $\lambda \in \mathfrak{t}^*$ is called *G-regular* if $(\lambda, \alpha) \neq 0$ for every $\alpha \in \mathfrak{R}$, or equivalently, if the stabilizer subgroup of λ in G is T. For any $\lambda \in \mathfrak{t}^*$ we denote

$$\rho(\lambda) := \frac{1}{2} \sum_{\alpha \in \mathfrak{R}, (\alpha,\lambda)>0} \alpha.$$

We denote also $\rho_c := \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_c^+} \alpha$.

Definition 2.1 *1. A coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ is elliptic if $\mathcal{O} \cap \mathfrak{t}^* \neq \emptyset$.*

2. An elliptic coadjoint orbit $\mathcal{O}$ is admissible[2] *when $\lambda - \rho(\lambda) \in \Lambda$ for any $\lambda \in \mathcal{O} \cap \mathfrak{t}^*$.*

Harish-Chandra has parametrized $\widehat{G}_d$ by the set of regular admissible elliptic coadjoint orbits of G. In order to simplify our notation, we denote $\widehat{G}_d$ the set of regular admissible elliptic coadjoint orbits. For an orbit $\mathcal{O} \in \widehat{G}_d$ we denote $\pi^G_{\mathcal{O}}$ the corresponding discrete series representation of G.

Consider the subset $(\mathfrak{t}^*_+)_{se} := \{\xi \in \mathfrak{t}^*_+, (\xi, \alpha) \neq 0, \ \forall \alpha \in \mathfrak{R}_n\}$ of the Weyl chamber. The subscript means *strongly elliptic*, see Section 5.1. By definition any $\mathcal{O} \in \widehat{G}_d$ intersects $(\mathfrak{t}^*_+)_{se}$ in a unique point.

Definition 2.2 *The connected component $(\mathfrak{t}^*_+)_{se}$ are called chambers. If $\mathcal{C}$ is a chamber, we denote $\widehat{G}_d(\mathcal{C}) \subset \widehat{G}_d$ the subset of regular admissible elliptic orbits intersecting $\mathcal{C}$.*

[2] Duflo has defined a notion of admissible coadjoint orbits in a much broader context [4].

Notice that the Harish-Chandra parametrization has still a meaning when $G = K$ is a compact connected Lie group. In this case $\widehat{K}$ corresponds to the set of regular admissible coadjoint orbits $\mathcal{P} \subset \mathfrak{k}^*$, i.e. those of the form $\mathcal{P} = K\mu$ where $\mu - \rho_c \in \Lambda \cap \mathfrak{t}^*_+$: the corresponding representation $\pi^K_{\mathcal{P}}$ is the irreducible representation of K with highest weight $\mu - \rho_c$.

2.2 Spinor representation

Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$: the Killing form of $\mathfrak{g}$ defines a K-invariant Euclidean structure on it. Note that $\mathfrak{p}$ is even dimensional since the groups G and K have the same rank.

We consider the two-fold cover $\mathrm{Spin}(\mathfrak{p}) \to \mathrm{SO}(\mathfrak{p})$ and the morphism $K \to \mathrm{SO}(\mathfrak{p})$. We recall the following basic fact.

Lemma 2.3 *There exists a unique covering $\tilde{K} \to K$ such that*

1. *$\tilde{K}$ is a compact connected Lie group,*
2. *the morphism $K \to \mathrm{SO}(\mathfrak{p})$ lifts to a morphism $\tilde{K} \to \mathrm{Spin}(\mathfrak{p})$.*

Let $\xi \in \mathfrak{t}^*$ be a regular element and consider

$$\rho_n(\xi) := \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_n, (\alpha,\xi)>0} \alpha. \tag{2.3}$$

Note that

$$\tilde{\Lambda} = \Lambda \bigcup \{\rho_n(\xi) + \Lambda\} \tag{2.4}$$

is a lattice that does not depends on the choice of ξ.

Let $T \subset K$ be a maximal torus and $\tilde{T} \subset \tilde{K}$ be the pull-back of T relatively to the covering $\tilde{K} \to K$. We can now precise Lemma 2.3.

Lemma 2.4 *Two situations occur:*

1. *if $\rho_n(\xi) \in \Lambda$ then $\tilde{K} \to K$ and $\tilde{T} \to T$ are isomorphisms, and $\tilde{\Lambda} = \Lambda$.*
2. *if $\rho_n(\xi) \notin \Lambda$ then $\tilde{K} \to K$ and $\tilde{T} \to T$ are two-fold covers, and $\tilde{\Lambda}$ is the lattice of weights for $\tilde{T}$.*

Let $\mathcal{S}_\mathfrak{p}$ the spinor representation of the group $\mathrm{Spin}(\mathfrak{p})$. Let $\mathbf{c} : \mathrm{Cl}(\mathfrak{p}) \to \mathrm{End}_\mathbb{C}(\mathcal{S}_\mathfrak{p})$ be the Clifford action. Let o be an orientation on $\mathfrak{p}$. If $e_1, e_2, \cdots, e_{\dim \mathfrak{p}}$ is an oriented orthonormal base of $\mathfrak{p}$ we define the element

$$\epsilon_o := (i)^{\dim \mathfrak{p}/2} e_1 e_2 \cdots e_{\dim \mathfrak{p}} \in \mathrm{Cl}(\mathfrak{p}) \otimes \mathbb{C}.$$

that depends only of the orientation. We have $\epsilon_o^2 = -1$ and $\epsilon_o v = -v\epsilon$ for any $v \in \mathfrak{p}$. The element $\mathbf{c}(\epsilon_o)$ determines a decomposition $\mathcal{S}_\mathfrak{p} = \mathcal{S}_\mathfrak{p}^{+,o} \oplus \mathcal{S}_\mathfrak{p}^{-,o}$ into irreducible representations $\mathcal{S}_\mathfrak{p}^{\pm,o} = \ker(\mathbf{c}(\epsilon_o) \mp Id)$ of $\mathrm{Spin}(\mathfrak{p})$. We denote

$$\mathcal{S}_\mathfrak{p}^o := \mathcal{S}_\mathfrak{p}^{+,o} \ominus \mathcal{S}_\mathfrak{p}^{-,o}$$

the corresponding virtual representation of $\tilde{K}$.

Remark 2.5 *If o and o' are two orientations on $\mathfrak{p}$, we have $\mathcal{S}_\mathfrak{p}^o = \pm \mathcal{S}_\mathfrak{p}^{o'}$, where the sign $\pm$ is the ratio between o and o'.*

Example 2.6 *Let $\lambda \in \mathfrak{k}$ such that the map $\mathrm{ad}(\lambda) : \mathfrak{p} \to \mathfrak{p}$ is one to one. We get a symplectic form Ω_λ on $\mathfrak{p}$ defined by the relations $\Omega_\lambda(X,Y) = \langle \lambda, [X,Y] \rangle$ for $X, Y \in \mathfrak{p}$. We denote $o(\lambda)$ be the orientation of $\mathfrak{p}$ defined by the top form $\Omega_\lambda^{\dim \mathfrak{p}/2}$.*

2.3 Restriction to the maximal compact subgroup

We start with a definition.

Definition 2.7 • *We denote $\widehat{R}(G,d)$ the group formed by the formal (possibly infinite) sums*

$$\sum_{\mathcal{O} \in \widehat{G}_d} a_\mathcal{O}\, \pi_\mathcal{O}^G$$

where $a_\mathcal{O} \in \mathbb{Z}$.

• *Similarly we denote $\widehat{R}(K)$ the group formed by the formal (possibly infinite) sums $\sum_{\mathcal{P} \in \widehat{K}} a_\mathcal{P}\, \pi_\mathcal{P}^K$ where $a_\mathcal{P} \in \mathbb{Z}$.*

The following technical fact will be used in the proof of Theorem 1.2.

Proposition 2.8 *Let o be an orientation on $\mathfrak{p}$.*

• *The restriction morphism $V \mapsto V|_K$ defines a map $\widehat{R}(G,d) \to \widehat{R}(K)$.*

• *The map $\mathbf{r}^o : \widehat{R}(G,d) \to \widehat{R}(\tilde{K})$ defined by $\mathbf{r}^o(V) := V|_K \otimes \mathcal{S}_\mathfrak{p}^o$ is one to one.*

Proof. When $\mathcal{O} = G\lambda \in \widehat{G}_d$, with $\lambda \in \mathfrak{t}^*$, we denote $c_\mathcal{O}^G = \|\lambda + \rho(\lambda)\|$. Similarly when $\mathcal{P} = K\mu \in \widehat{K}$, with $\mu - \rho_c \in \Lambda \cap \mathfrak{t}^*_+$, we denote $c_\mathcal{P}^K = \|\mu + \rho_c\|$. Note that for each $r > 0$ the set $\{\mathcal{O} \in \widehat{G}_d,\, c_\mathcal{O}^G \leqslant r\}$ is finite.

Consider now the restriction of a discrete series representation $\pi^G_{\mathcal{O}}$ relatively to K. The Blattner's formula [9] tells us that the restriction $\pi^G_{\mathcal{O}}|_K$ admits a decomposition

$$\pi^G_{\mathcal{O}}|_K = \sum_{\mathcal{P}\in\widehat{K}} m_{\mathcal{O}}(\mathcal{P})\, \pi^K_{\mathcal{P}}$$

where the (finite) multiplicities $m_{\mathcal{O}}(\mathcal{P})$ are non-zero only if $c^K_{\mathcal{P}} \geqslant c^G_{\mathcal{O}}$.

Consider now an element $V = \sum_{\mathcal{O}\in\widehat{G}_d} a_{\mathcal{O}}\, \pi^G_{\mathcal{O}} \in \widehat{R}(G,d)$. The multiplicity of $\pi^K_{\mathcal{P}}$ in $V|_K$ is equal to

$$\sum_{\mathcal{O}\in\widehat{G}_d} a_{\mathcal{O}}\, m_{\mathcal{O}}(\mathcal{P}).$$

Here the sum admits a finite number of non zero terms since $m_{\mathcal{O}}(\mathcal{P}) = 0$ if $c^G_{\mathcal{O}} > c^K_{\mathcal{P}}$. So we have proved that the K-multiplicities of $V|_K := \sum_{\mathcal{O}\in\widehat{G}_d} a_{\mathcal{O}}\pi^G_{\mathcal{O}}|_K$ are finite. The first point is proved.

The irreducible representation of $\tilde{K}$ are parametrized by the set $\widehat{\tilde{K}}$ of regular $\tilde{K}$-admissible coadjoint orbits $\mathcal{P} \subset \mathfrak{k}^*$, i.e. those of the form $\mathcal{P} = K\mu$ where $\mu - \rho_c \in \tilde{\Lambda} \cap \mathfrak{t}^*_+$. It contains the set $\widehat{K}$ of regular K-admissible coadjoint orbits. We define

$$\widehat{K}_{out} \subset \widehat{\tilde{K}}$$

as the set of coadjoint orbits $\mathcal{P} = K\mu$ where[3] $\mu - \rho_c \in \{\rho_n(\xi) + \Lambda\} \cap \mathfrak{t}^*_+$. Here ξ is any regular element of $\mathfrak{t}^*$ and $\rho_n(\xi)$ is defined by (2.3).

We notice that $\widehat{K}_{out} = \widehat{K}$ when $\tilde{K} \simeq K$ and that $\widehat{\tilde{K}} = \widehat{K} \cup \widehat{K}_{out}$ when $\tilde{K} \to K$ is a two-fold cover.

We will use the following basic facts.

Lemma 2.9

1. *$\mathcal{O} \mapsto \mathcal{O}_K := \mathcal{O} \cap \mathfrak{k}^*$ defines an injective map between $\widehat{G}_d$ and $\widehat{K}_{out}$.*
2. *We have $\pi^G_{\mathcal{O}}|_K \otimes \mathcal{S}^o_{\mathfrak{p}} = \pm\pi^{\tilde{K}}_{\mathcal{O}_K}$ for all $\mathcal{O} \in \widehat{G}_d$.*

Proof. Let $\mathcal{O} := G\lambda \in \widehat{G}_d$ where λ is a regular element of the Weyl chamber $\mathfrak{t}^*_+$. Then $\mathcal{O}_K = K\lambda$ and the term $\lambda - \rho_c$ is equal to the sum $\lambda - \rho(\lambda) + \rho_n(\lambda)$ where $\lambda - \rho(\lambda) \in \Lambda$ and $\rho_n(\lambda) \in \tilde{\Lambda}$ (see (2.4)), so $\lambda - \rho_c \in \{\rho_n(\xi) + \Lambda\}$. The element $\lambda \in \mathfrak{t}^*_+$ is regular and admissible for $\tilde{K}$: this implies that $\lambda - \rho_c \in \mathfrak{t}^*_+$. We have proved that $\mathcal{O}_K \in \widehat{K}_{out}$.

[3] The set $\{\rho_n(\xi) + \Lambda\} \cap \mathfrak{t}^*_+$ does not depend on the choice of ξ.

The second point is a classical result (a generalisation is given in Theorem 5.7). Let us explain the sign $\pm$ in the relation. Let $\mathcal{O} \in \widehat{G}_d$ and $\lambda \in \mathcal{O} \cap \mathfrak{k}^*$. Then the sign $\pm$ is the ratio between the orientations o and $o(-\lambda)$ of the vector space $\mathfrak{p}$ (see Example 2.6).

We can now finish the proof of the second point of Proposition 2.8. If $V = \sum_{\mathcal{O} \in \widehat{G}_d} a_{\mathcal{O}} \pi^G_{\mathcal{O}} \in \widehat{R}(G, d)$, then $\mathbf{r}^o(V) = \sum_{\mathcal{O} \in \widehat{G}_d} \pm a_{\mathcal{O}} \pi^{\tilde{K}}_{\mathcal{O}_K}$. Hence $\mathbf{r}^o(V) = 0$ only if $V = 0$. □

2.4 Admissibility

Let $\pi^{G'}_{\mathcal{O}'}$ be a discrete series representation of G attached to a regular admissible elliptic orbit $\mathcal{O}' \subset (\mathfrak{g}')^*$.

We denote $\mathrm{As}(\mathcal{O}') \subset (\mathfrak{g}')^*$ the asymptotic support of the coadjoint orbit $\mathcal{O}'$: by definition $\xi \in \mathrm{As}(\mathcal{O}')$ if $\xi = \lim_{n \to \infty} t_n \xi_n$ with $\xi_n \in \mathcal{O}'$ and (t_n) is a sequence of positive number tending to 0.

We consider here a closed connected semi-simple Lie subgroup $G \subset G'$. We choose maximal compact subgroups $K \subset G$ and $K' \subset G'$ such that $K \subset K'$. We denote $\mathfrak{k}^\perp \subset (\mathfrak{k}')^*$ the orthogonal (for the duality) of $\mathfrak{k} \subset \mathfrak{k}'$.

The moment map relative to the G-action on $\mathcal{O}'$ is by definition the map $\Phi_G : \mathcal{O}' \to \mathfrak{g}^*$ which is the composition of the inclusion $\mathcal{O}' \hookrightarrow (\mathfrak{g}')^*$ with the projection $(\mathfrak{g}')^* \to \mathfrak{g}^*$. We use also the moment map $\Phi_K : \mathcal{O}' \to \mathfrak{k}^*$ which the composition of Φ_G with the projection $\mathfrak{g}^* \to \mathfrak{k}^*$.

Let $\mathrm{p}_{\mathfrak{k}',\mathfrak{g}'} : (\mathfrak{g}')^* \to (\mathfrak{k}')^*$ be the canonical projection. The main objective of this section is the proof of the following result that refines Proposition 1.1.

Theorem 2.10 *The following facts are equivalent :*

1. *The representation $\pi^{G'}_{\mathcal{O}'}$ is G-admissible.*
2. *The moment map $\Phi_G : \mathcal{O}' \to \mathfrak{g}^*$ is proper.*
3. $\mathrm{p}_{\mathfrak{k}',\mathfrak{g}'}(\mathrm{As}(\mathcal{O}')) \cap \mathfrak{k}^\perp = \{0\}$.

Theorem 2.10 is a consequence of different equivalences. We start with the following result that is proved in [5, 29].

Lemma 2.11 *The map $\Phi_G : \mathcal{O}' \to \mathfrak{g}^*$ is proper if and only if the map $\Phi_K : \mathcal{O}' \to \mathfrak{k}^*$ is proper.*

We have the same kind of equivalence for the admissibility.

Lemma 2.12 *The representation $\pi^{G'}_{\mathcal{O}'}$ is G-admissible if and only if it is K-admissible.*

Proof. The fact that K-admissibilty implies G-admissibility is proved by T. Kobayashi in [15]. The opposite implication is a consequence of the first point of Proposition 2.8.

At this stage, the proof of Theorem 2.10 is complete if we show that the following facts are equivalent :

(a) The representation $\pi^{G'}_{\mathcal{O}'}$ is K-admissible.

(b) The moment map $\Phi_K : \mathcal{O}' \to \mathfrak{k}^*$ is proper.

(c) $\mathrm{p}_{\mathfrak{k}',\mathfrak{g}'}(\mathrm{As}(\mathcal{O}')) \cap \mathfrak{k}^\perp = \{0\}$.

We start by proving the equivalence $(b) \Longleftrightarrow (c)$.

Proposition 2.13 ([29]) *The map $\Phi_K : \mathcal{O}' \to \mathfrak{k}^*$ is proper if and only*

$$\mathrm{p}_{\mathfrak{k}',\mathfrak{g}'}\left(\mathrm{As}(\mathcal{O}')\right) \cap \mathfrak{k}^\perp = \{0\}.$$

Proof. The moment map $\Phi_{K'} : \mathcal{O}' \to (\mathfrak{k}')^*$ relative to the action of K' on $\mathcal{O}'$ is a proper map that corresponds to the restriction of the projection $\mathrm{p}_{\mathfrak{k}',\mathfrak{g}'}$ to $\mathcal{O}'$.

Let T' be a maximal torus in K' and let $(\mathfrak{t}')^*_+ \subset (\mathfrak{t}')^*$ be a Weyl chamber. The convexity theorem [14, 20] tells us that $\Delta_{K'}(\mathcal{O}') = \mathrm{p}_{\mathfrak{k}',\mathfrak{g}'}(\mathcal{O}') \cap (\mathfrak{t}')^*_+$ is a closed convex polyedral subset. We have proved in [29][Proposition 2.10], that $\Phi_K : \mathcal{O}' \to \mathfrak{k}^*$ is proper if and only

$$K' \cdot \mathrm{As}(\Delta_{K'}(\mathcal{O}')) \cap \mathfrak{k}^\perp = \{0\}.$$

A small computation shows that $K' \cdot \mathrm{As}(\Delta_{K'}(\mathcal{O}')) = \mathrm{p}_{\mathfrak{k}',\mathfrak{g}'}(\mathrm{As}(\mathcal{O}'))$ since $K' \cdot \Delta_{K'}(\mathcal{O}') = \mathrm{p}_{\mathfrak{k}',\mathfrak{g}'}(\mathcal{O}')$. The proof of Proposition 2.13 is completed. $\square$

We denote $\mathrm{AS}_{K'}(\pi^{G'}_{\mathcal{O}'}) \subset (\mathfrak{k}')^*$ the asymptotic support of the following subset of $(\mathfrak{k}')^*$:

$$\{\mathcal{P}' \in \widehat{K'}, [\pi^{K'}_{\mathcal{P}'} : \pi^{G'}_{\mathcal{O}'}] \neq 0\}.$$

The following important fact is proved by T. Kobayashi (see Section 6.3 in [18]).

Proposition 2.14 *The representation $\pi^{G'}_{\mathcal{O}'}$ is K-admissible if and only if*

$$\mathrm{AS}_{K'}(\pi^{G'}_{\mathcal{O}'}) \cap \mathfrak{k}^\perp = \{0\}.$$

We will use also the following result proved by Barbasch and Vogan (see Propositions 3.5 and 3.6 in [2]).

Proposition 2.15 *Let $\pi^{G'}_{\mathcal{O}'}$ be a representation of the discrete series of G' attached to the regular admissible elliptic orbit $\mathcal{O}'$. We have*

$$\mathrm{AS}_{K'}(\pi^{G'}_{\mathcal{O}'}) = \mathrm{p}_{\mathfrak{k}',\mathfrak{g}'}\left(\mathrm{As}(\mathcal{O}')\right).$$

Propositions 2.14 and 2.15 give the equivalence $(a) \Longleftrightarrow (c)$. The proof of Theorem 2.10 is completed. □

In fact Barbasch and Vogan proved also in [2] that the set $\mathrm{As}(\mathcal{O}')$ does not depends on $\mathcal{O}'$ but only on the chamber $\mathcal{C}'$ such that $\mathcal{O}' \in \widehat{G'}_d(\mathcal{C}')$. We obtain the following corollary.

Corollary 2.16 *The G-admissibility of a discrete series representation $\pi^{G'}_{\mathcal{O}'}$ does not depends on $\mathcal{O}'$ but only on the chamber $\mathcal{C}'$ such that $\mathcal{O}' \in \widehat{G'}_d(\mathcal{C}')$.*

3 Spin^c quantization of compact Hamiltonian manifolds

3.1 Spin^c structures

Let N be an even dimensional Riemannian manifold, and let $\mathrm{Cl}(N)$ be its Clifford algebra bundle. A complex vector bundle $\mathcal{E} \to N$ is a $\mathrm{Cl}(N)$-module if there is a bundle algebra morphism $\mathbf{c}_{\mathcal{E}} : \mathrm{Cl}(N) \longrightarrow \mathrm{End}(\mathcal{E})$.

Definition 3.1 *Let $\mathcal{S} \to M$ be a $\mathrm{Cl}(N)$-module such that the map $\mathbf{c}_{\mathcal{S}}$ induces an isomorphism $\mathrm{Cl}(N) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathrm{End}(\mathcal{S})$. Then we say that $\mathcal{S}$ is a* spin^c*-bundle for N.*

Definition 3.2 *The determinant line bundle of a* spin^c*-bundle $\mathcal{S}$ on N is the line bundle $\det(\mathcal{S}) \to M$ defined by the relation*

$$\det(\mathcal{S}) := \hom_{\mathrm{Cl}(N)}(\overline{\mathcal{S}}, \mathcal{S})$$

where $\overline{\mathcal{S}}$ is the $\mathrm{Cl}(N)$-module with opposite complex structure.

Basic examples of spin^c-bundles are those coming from manifolds N equipped with an almost complex structure J. We consider the tangent bundle $\mathbf{T}N$ as a complex vector bundle and we define

$$\mathcal{S}_J := \bigwedge_{\mathbb{C}} \mathbf{T}N.$$

It is not difficult to see that $\mathcal{S}_J$ is a spin^c-bundle on N with determinant line bundle $\det(\mathcal{S}_J) = \bigwedge_{\mathbb{C}}^{max} \mathbf{T}N$. If L is a complex line bundle on N, then $\mathcal{S}_J \otimes L$ is another spin^c-bundle with determinant line bundle equal to $\bigwedge_{\mathbb{C}}^{max} \mathbf{T}N \otimes L^{\otimes 2}$.

3.2 Spin^c-prequantization

In this section G is a semi-simple connected real Lie group.

Let M be an Hamiltonian G-manifold with symplectic form Ω and moment map $\Phi_G : M \to \mathfrak{g}^*$ characterized by the relation

$$\iota(X_M)\omega = -d\langle \Phi_G, X\rangle, \quad X \in \mathfrak{g}, \tag{3.5}$$

where $X_M(m) := \frac{d}{dt}|_{t=0} e^{-tX} \cdot m$ is the vector field on M generated by $X \in \mathfrak{g}$.

In the Kostant-Souriau framework [19, 36], a G-equivariant Hermitian line bundle L_Ω with an invariant Hermitian connection ∇ is a prequantum line bundle over (M, Ω, Φ_G) if

$$\mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi_G, X\rangle \quad \text{and} \quad \nabla^2 = -i\Omega, \tag{3.6}$$

for every $X \in \mathfrak{g}$. Here $\mathcal{L}(X)$ is the infinitesimal action of $X \in \mathfrak{k}$ on the sections of $L_\Omega \to M$. The data (L_Ω, ∇) is also called a Kostant-Souriau line bundle.

Definition 3.3 ([28]) *A G-Hamiltonian manifold (M, Ω, Φ_G) is* spin^c *prequantized if there exists an equivariant* spin^c *bundle $\mathcal{S}$ such that its determinant line bundle* $\det(\mathcal{S})$ *is a prequantum line bundle over $(M, 2\Omega, 2\Phi_G)$.*

Consider the case of a regular elliptic coadjoint orbit $\mathcal{O} = G\lambda$: here $\lambda \in \mathfrak{t}^*$ has a stabilizer subgroup equal to T. The tangent space $\mathbf{T}_\lambda\mathcal{O} \simeq \mathfrak{g}/\mathfrak{t}$ is an even dimensional Euclidean space, equipped with a linear action of T and an T-invariant antisymmetric endomorphism[4] $\mathrm{ad}(\lambda)$. Let $J_\lambda := \mathrm{ad}(\lambda)(-\mathrm{ad}(\lambda)^2)^{-1/2}$ be the corresponding T-invariant complex structure on $\mathfrak{g}/\mathfrak{t}$: we denote V the corresponding T-module. It defines an integrable G-invariant complex structure on $\mathcal{O} \simeq G/T$.

As we have explained in the previous section, the complex structure on $\mathcal{O}$ defines the spin^c-bundle $\mathcal{S}_o := \bigwedge_{\mathbb{C}} \mathbf{T}\mathcal{O}$ with determinant line bundle

$$\det(\mathcal{S}_o) = \bigwedge\nolimits_{\mathbb{C}}^{\max} \mathbf{T}\mathcal{O} \simeq G \times_T \bigwedge_{\mathbb{C}}^{\max} V.$$

[4]Here we see λ has an element of $\mathfrak{t}$, through the identification $\mathfrak{g}^* \simeq \mathfrak{g}$.

A small computation gives that the differential of the T-character $\bigwedge_{\mathbb{C}}^{\max} V$ is equal to i times $2\rho(\lambda)$. In other words, $\bigwedge_{\mathbb{C}}^{\max} V = \mathbb{C}_{2\rho(\lambda)}$.

In the next Lemma we see that for the regular elliptic orbits, the notion of *admissible* orbits is equivalent to the notion of spin^c-*prequantized* orbits.

Lemma 3.4 *Let $\mathcal{O} = G\lambda$ be a regular elliptic coadjoint orbit. Then $\mathcal{O}$ is* spin^c*-prequantized if and only if $\lambda - \rho(\lambda) \in \Lambda$.*

Proof. Any G-equivariant spin^c-bundle on $\mathcal{O}$ is of the form $\mathcal{S}_\phi = \mathcal{S}_o \otimes L_\phi$ where $L_\phi = G \times_T \mathbb{C}_\phi$ is a line bundle associated to a character $e^X \mapsto e^{i\langle\phi, X\rangle}$ of the group T. Then we have

$$\det(\mathcal{S}_\phi) = \det(\mathcal{S}_o) \otimes L_\phi^{\otimes 2} = G \times_T \mathbb{C}_{2\phi + 2\rho(\lambda)}.$$

By G-invariance we know that the only Kostant-Souriau line bundle on $(G\lambda, 2\Omega_{G\lambda})$ is the line bundle $G \times_T \mathbb{C}_{2\lambda}$. Finally we see that $G\lambda$ is spin^c-prequantized by $\mathcal{S}_\phi$ if and only if $\phi = \lambda - \rho(\lambda)$. □

If $\mathcal{O}$ is a regular admissible elliptic coadjoint orbit, we denote $\mathcal{S}_\mathcal{O} := \mathcal{S}_o \otimes L_{\lambda - \rho(\lambda)}$ the corresponding spin^c bundle. Here we use the grading $\mathcal{S}_\mathcal{O} = \mathcal{S}_\mathcal{O}^+ \oplus \mathcal{S}_\mathcal{O}^-$ induced by the symplectic orientation.

3.3 Spin^c quantization of compact manifolds

Let us consider a compact Hamiltonian K-manifold (M, Ω, Φ_K) which is spin^c-prequantized by a spin^c-bundle $\mathcal{S}$. The (symplectic) orientation induces a decomposition $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, and the corresponding spin^c Dirac operator is a first order elliptic operator [3]

$$\mathcal{D}_\mathcal{S} : \Gamma(M, \mathcal{S}^+) \to \Gamma(M, \mathcal{S}^-).$$

Its principal symbol is the bundle map $\sigma(M, \mathcal{S}) \in \Gamma(\mathbf{T}^* M, \hom(p^*\mathcal{S}^+, p^*\mathcal{S}^-))$ defined by the relation

$$\sigma(M, \mathcal{S})(m, \nu) = \mathbf{c}_{\mathcal{S}|_m}(\tilde{\nu}) : \mathcal{S}|_m^+ \longrightarrow \mathcal{S}|_m^-.$$

Here $\nu \in \mathbf{T}^* M \mapsto \tilde{\nu} \in \mathbf{T} M$ is the identification defined by an invariant Riemannian structure.

Definition 3.5 *The* spin^c *quantization of a compact Hamiltonian K-manifold (M, Ω, Φ_K) is the equivariant index of the elliptic operator $\mathcal{D}_\mathcal{S}$ and is denoted*

$$\mathcal{Q}_K^{\mathrm{spin}}(M) \in R(K).$$

3.4 Quantization commutes with reduction

Now we will explain how the multiplicities of $\mathcal{Q}_K^{\rm spin}(M) \in R(K)$ can be computed geometrically.

Recall that the dual $\widehat{K}$ is parametrized by the regular admissible coadjoint orbits. They are those of the form $\mathcal{P} = K\mu$ where $\mu - \rho_c \in \Lambda \cap \mathfrak{t}^*_+$. After Lemma 3.4, we know that any regular admissible coadjoint orbit $\mathcal{P} \in \widehat{K}$ is spin^c-prequantized by a spin^c bundle $\mathcal{S}_\mathcal{P}$ and a small computation shows that $\mathcal{Q}_K^{\rm spin}(\mathcal{P}) = \pi_\mathcal{P}^K$ (see [32]).

For any $\mathcal{P} \in \widehat{K}$, we define the symplectic reduced space

$$M/\!\!/\mathcal{P} := \Phi_K^{-1}(\mathcal{P})/K.$$

If $M/\!\!/\mathcal{P} \neq \emptyset$, then any $m \in \Phi_K^{-1}(\mathcal{P})$ has abelian infinitesimal stabilizer. It implies then that the generic infinitesimal stabilizer for the K-action on M is *abelian*.

Let us explain how we can quantize these symplectic reduced spaces (for more details see [25, 28, 33]).

Proposition 3.6 *Suppose that the generic infinitesimal stabilizer for the K-action on M is abelian.*

- *If $\mathcal{P} \in \widehat{K}$ belongs to the set of regular values of $\Phi_K : M \to \mathfrak{k}^*$, then $M/\!\!/\mathcal{P}$ is a compact symplectic orbifold which is* spin^c*-prequantized. We denote $\mathcal{Q}^{\rm spin}(M/\!\!/\mathcal{P}) \in \mathbb{Z}$ the index of the corresponding* spin^c *Dirac operator [12].*
- *In general, if $\mathcal{P} = K\lambda$ with $\lambda \in \mathfrak{t}^*$, we consider the orbits $\mathcal{P}_\epsilon = K(\lambda+\epsilon)$ for generic small elements $\epsilon \in \mathfrak{t}^*$ so that $M/\!\!/\mathcal{P}_\epsilon$ is a compact symplectic orbifold with a peculiar* spin^c*-structure. Let $\mathcal{Q}^{\rm spin}(M/\!\!/\mathcal{P}_\epsilon) \in \mathbb{Z}$ be the index of the corresponding* spin^c *Dirac operator. The crucial fact is that the quantity $\mathcal{Q}^{\rm spin}(M/\!\!/\mathcal{P}_\epsilon)$ does not depends on the choice of generic and small enough ϵ. Then we take*

$$\mathcal{Q}^{\rm spin}(M/\!\!/\mathcal{P}) := \mathcal{Q}^{\rm spin}(M/\!\!/\mathcal{P}_\epsilon)$$

for generic and small enough ϵ.

The following theorem is proved in [25].

Theorem 3.7 *Let (M, Ω, Φ_K) be a* spin^c*-prequantized compact Hamiltonian K-manifold. Suppose that the generic infinitesimal stabilizer for the K-action on M is* abelian*. Then the following relation holds in $R(K)$:*

$$\mathcal{Q}_K^{\rm spin}(M) = \sum_{\mathcal{P}\in\widehat{K}} \mathcal{Q}^{\rm spin}(M/\!\!/\mathcal{P})\, \pi_\mathcal{P}^K. \tag{3.7}$$

Remark 3.8 *Identity 3.7 admits generalisations when we do not have conditions on the generic stabilizer [28] and also when we allow the 2-form Ω to be degenerate [33]. In this article, we do not need such generalizations.*

For $\mathcal{P} \in \widehat{K}$, we denote $\mathcal{P}^-$ the coadjoint orbit with $\mathcal{P}$ with opposite symplectic structure. The corresponding spin^c bundle is $\mathcal{S}_{\mathcal{P}^-}$. It is not difficult to see that $\mathcal{Q}_K^{\rm spin}(\mathcal{P}^-) = (\pi_{\mathcal{P}}^K)^*$ (see [32]). The shifting trick tell us then that the multiplicity of $\pi_{\mathcal{P}}^K$ in $\mathcal{Q}_K^{\rm spin}(M)$ is equal to $[\mathcal{Q}_K^{\rm spin}(M \times \mathcal{P}^-)]^K$. If we suppose furthermore that the generic infinitesimal stabilizer is abelian we obtain the useful relation

$$\mathcal{Q}^{\rm spin}(M/\!\!/\mathcal{P}) := \left[\mathcal{Q}_K^{\rm spin}(M \times \mathcal{P}^-)\right]^K. \tag{3.8}$$

Let γ that belongs to the center of K: it acts trivially on the orbits $\mathcal{P} \in \widehat{K}$. Suppose now that γ acts also trivially on the manifolds M. We are interested by the action of γ on the fibers of the spin^c-bundle $\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}$. We denote $[\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}]^\gamma$ the subbundle where γ acts trivially.

Lemma 3.9 *If* $[\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}]^\gamma = 0$ *then* $\mathcal{Q}^{\rm spin}(M/\!\!/\mathcal{P}) = 0$.

Proof. Let D be the Dirac operator on $M \times \mathcal{P}^-$ associated to the spin^c bundle $\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}$. Then

$$\left[\mathcal{Q}_K^{\rm spin}(M \times \mathcal{P}^-)\right]^K = [\ker(D)]^K - [\mathrm{coker}(D)]^K.$$

Obviously $[\ker(D)]^K \subset [\ker(D)]^\gamma$ and $[\ker(D)]^\gamma$ is contained in the set of smooth section of the bundle $[\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}]^\gamma$. The same result holds for $[\mathrm{coker}(D)]^K$. Finally, if $[\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}]^\gamma = 0$, then $[\ker(D)]^K$ and $[\mathrm{coker}(D)]^K$ are reduced to 0. □

4 Spin^c quantization of non-compact Hamiltonian manifolds

In this section our Hamiltonian K-manifold (M, Ω, Φ_K) is not necessarily compact, but the moment map Φ_K is supposed to be proper. We assume that (M, Ω, Φ_K) is spin^c-prequantized by a spin^c-bundle $\mathcal{S}$.

In the next section, we will explain how to quantize the data $(M, \Omega, \Phi_K, \mathcal{S})$.

4.1 Formal geometric quantization : definition

We choose an invariant scalar product in $\mathfrak{k}^*$ that provides an identification $\mathfrak{k} \simeq \mathfrak{k}^*$.

Definition 4.1 • *The* Kirwan vector field *associated to* Φ_K *is defined by*

$$\kappa(m) = -\Phi_K(m) \cdot m, \quad m \in M. \tag{4.9}$$

We denote by Z_M the set of zeroes of κ. It is not difficult to see that Z_M corresponds to the set of critical points of the function $\|\Phi_K\|^2 : M \to \mathbb{R}$.

The set Z_M, which is not necessarily smooth, admits the following description. Choose a Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ in the dual of the Lie algebra of a maximal torus T of K. We see that

$$Z_M = \coprod_{\beta \in \mathcal{B}} Z_\beta \tag{4.10}$$

where Z_β corresponds to the compact set $K(M^\beta \cap \Phi_K^{-1}(\beta))$, and $\mathcal{B} = \Phi_K(Z_M) \cap \mathfrak{t}^*_+$. The properness of Φ_K insures that for any compact subset $C \subset \mathfrak{t}^*$ the intersection $\mathcal{B} \cap C$ is finite.

The principal symbol of the Dirac operator $D_{\mathcal{S}}$ is the bundle map $\sigma(M, \mathcal{S}) \in \Gamma(\mathbf{T}^*M, \hom(\mathcal{S}^+, \mathcal{S}^-))$ defined by the Clifford action

$$\sigma(M, \mathcal{S})(m, \nu) = \mathbf{c}_m(\tilde{\nu}) : \mathcal{S}|_m^+ \to \mathcal{S}|_m^-.$$

where $\nu \in \mathbf{T}^*M \simeq \tilde{\nu} \in \mathbf{T}M$ is an identification associated to an invariant Riemannian metric on M.

Definition 4.2 *The symbol* $\sigma(M, \mathcal{S}, \Phi_K)$ *shifted by the vector field* κ *is the symbol on* M *defined by*

$$\sigma(M, \mathcal{S}, \Phi_K)(m, \nu) = \sigma(M, \mathcal{S})(m, \tilde{\nu} - \kappa(m))$$

for any $(m, \nu) \in \mathbf{T}^*M$.

For any K-invariant open subset $\mathcal{U} \subset M$ such that $\mathcal{U} \cap Z_M$ is compact in M, we see that the restriction $\sigma(M, \mathcal{S}, \Phi_K)|_{\mathcal{U}}$ is a transversally elliptic symbol on $\mathcal{U}$, and so its equivariant index is a well defined element in $\widehat{R}(K)$ (see [1, 31]).

Thus we can define the following localized equivariant indices.

Definition 4.3 • *A closed invariant subset* $Z \subset Z_M$ *is called a* component *of* Z_M *if it is a union of connected components of* Z_M*.*

- *If Z is a* compact component *of Z_M, we denote by*

$$\mathcal{Q}_K^{\mathrm{spin}}(M,Z) \ \in \ \widehat{R}(K)$$

the equivariant index of $\sigma(M,\mathcal{S},\Phi_K)|_{\mathcal{U}}$ where $\mathcal{U}$ is an invariant neighbourhood of Z so that $\mathcal{U}\cap Z_M = Z$.

By definition, $Z=\emptyset$ is a component of Z_M and $\mathcal{Q}_K^{\mathrm{spin}}(M,\emptyset)=0$. For any $\beta\in\mathcal{B}$, Z_β is a compact component of Z_M.

When the manifold M is compact, the set $\mathcal{B}$ is finite and we have the decomposition

$$\mathcal{Q}_K^{\mathrm{spin}}(M) = \sum_{\beta\in\mathcal{B}} \mathcal{Q}_K^{\mathrm{spin}}(M,Z_\beta) \quad \in \ \widehat{R}(K).$$

See [24, 31]. When the manifold M is not compact, but the moment map Φ_K is proper, we can define

$$\widehat{\mathrm{Q}}_K^{\mathrm{spin}}(M) := \sum_{\beta\in\mathcal{B}} \mathcal{Q}_K^{\mathrm{spin}}(M,Z_\beta) \ \in \ \widehat{R}(K).$$

The sum of the right hand side is not necessarily finite but it converges in $\widehat{R}(K)$ (see [27, 21, 10]).

Definition 4.4 *We call $\widehat{\mathrm{Q}}_K^{\mathrm{spin}}(M)\in\widehat{R}(K)$ the* spin^c *formal geometric quantization of the Hamiltonian manifold (M,Ω,Φ_K).*

We end up this section with the example of the coadjoint orbits that parametrize the discrete series representations. We have seen in Lemma 3.4 that any $\mathcal{O}\in\widehat{G}_d$ is spin^c-prequantized. Moreover, if we look at the K-action on $\mathcal{O}$, we know also that the moment map $\Phi_K:\mathcal{O}\to\mathfrak{k}^*$ is proper. The element $\widehat{\mathrm{Q}}_K^{\mathrm{spin}}(\mathcal{O})\in\widehat{R}(K)$ is then well-defined.

The following result can be understood as a geometric interpretation of the Blattner formula.

Proposition 4.5 ([25]) *For any $\mathcal{O}\in\widehat{G}_d$ we have the following equality in $\widehat{R}(K)$:*

$$\widehat{\mathrm{Q}}_K^{\mathrm{spin}}(\mathcal{O}) = \pi_{\mathcal{O}}^G|_K.$$

4.2 Formal geometric quantization: main properties

In this section, we recall two important functorial properties of the formal geometric quantization process $\widehat{\mathrm{Q}}^{\mathrm{spin}}$.

We start with the following result of Hochs and Song.

Theorem 4.6 ([10]) *Let (M, Ω, Φ_K) be a* spin^c *prequantized Hamiltonian K-manifold. Assume that the moment map Φ_K is proper and that the generic infinitesimal stabilizer for the K-action on M is* abelian. *Then the following relation holds in $\widehat{R}(K)$:*

$$\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M) = \sum_{\mathcal{P}\in\widehat{K}} \mathcal{Q}^{\mathrm{spin}}(M/\!\!/\mathcal{P})\, \pi^K_{\mathcal{P}}. \tag{4.11}$$

Remark 4.7 *Identity (4.11) admits generalizations when we do not have conditions on the generic stabilizer and also when we allow the 2-form Ω to be degenerate (see [10]).*

Like in the compact setting, consider an element γ belonging to the center of K that acts trivially on the manifold M. Let $\mathcal{P} \in \widehat{K}$ and let $\mathcal{P}^-$ be the orbit $\mathcal{P}$ with opposite symplectic structure. We are interested by the action of γ on the fibers of the spin^c-bundle $\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}$. We denote $[\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}]^\gamma$ the subbundle where γ acts trivially.

Lemma 3.9 extends to the non-compact setting.

Lemma 4.8 *If $[\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}]^\gamma = 0$ then $\mathcal{Q}^{\mathrm{spin}}(M/\!\!/\mathcal{P}) = 0$.*

Proof. The multiplicative property proved by Hochs and Song [10] tells us that the shifting trick still holds in the non compact setting: the multiplicity of $\pi^K_{\mathcal{P}}$ in $\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M)$ is equal to $[\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M \times \mathcal{P}^-)]^K$. If we suppose furthermore that the generic infinitesimal stabilizer is abelian we obtain

$$\begin{aligned}
\mathcal{Q}^{\mathrm{spin}}(M/\!\!/\mathcal{P}) &= \left[\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M \times \mathcal{P}^-)\right]^K \\
&= \left[\mathcal{Q}^{\mathrm{spin}}_K(M \times \mathcal{P}^-, Z_0)\right]^K
\end{aligned}$$

where $Z_0 \subset M \times \mathcal{P}^-$ is the compact set $\{(m, \xi) \in M \times \mathcal{P}^-,\ \Phi_K(m) = \xi\}$.

The quantity $\mathcal{Q}^{\mathrm{spin}}_K(M \times \mathcal{P}^-, Z_0) \in \widehat{R}(K)$ is computed as an index of a K-transversally elliptic operator D_0 acting on the sections of $\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}$. The argument used in the compact setting still work (see Lemma 1.3 in [31]): if $[\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}]^\gamma = 0$ then $[\ker(D_0)]^K$ and $[\mathrm{coker}(D_0)]^K$ are reduced to 0. □

Another important property of the formal geometric quantization procedure is the functoriality relatively to restriction to subgroup. Let $H \subset K$ be a closed connected subgroup. We denote $\Phi_H : M \to \mathfrak{h}^*$ the moment map relative to the H-action: it is equal to the composition of Φ_K with the projection $\mathfrak{k}^* \to \mathfrak{h}^*$.

Theorem 4.9 ([30]) *Let (M, Ω, Φ_K) be a* spin^c *prequantized Hamiltonian K-manifold. Assume that the moment map Φ_H is a proper. Then the element $\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M) \in \widehat{R}(K)$ is H-admissible and we have*

$$\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M)|_H = \widehat{\mathrm{Q}}^{\mathrm{spin}}_H(M).$$

If we apply the previous Theorem to the spin^c-prequantized coadjoint orbits $\mathcal{O} \in \widehat{G}_d$, we obtain the following extension of Proposition 4.5. This result was obtained by other means by Duflo-Vergne [7].

Corollary 4.10 *Let $\mathcal{O} \in \widehat{G}_d$, and $H \subset K$ a closed connected subgroup such that $\Phi_H : \mathcal{O} \to \mathfrak{h}^*$ is proper. Then $\pi^G_{\mathcal{O}}$ is H-admissible and*

$$\widehat{\mathrm{Q}}^{\mathrm{spin}}_H(\mathcal{O}) = \pi^G_{\mathcal{O}}|_H.$$

5 Spin^c quantization of G-Hamiltonian manifolds

In this section G denotes a connected semi-simple Lie group, and we consider a symplectic manifold (M, Ω) equipped with an Hamiltonian action of G: we denote $\Phi_G : M \to \mathfrak{g}^*$ the corresponding moment map.

5.1 Proper^2 Hamiltonian G-manifolds

In this section we suppose that:

1. the moment map Φ_G is *proper*,
2. the G-action on M is *proper*.

For simplicity, we says that (M, Ω, Φ_G) is a proper^2 Hamiltonian G-manifold.

Following Weinstein [38], we consider the G-invariant open subset

$$\mathfrak{g}^*_{se} = \{\xi \in \mathfrak{g}^* \,|\, G_\xi \text{ is compact}\} \tag{5.12}$$

of strongly elliptic elements. It is non-empty if and only if the groups G and K have the same rank : real semi-simple Lie groups with this property are the ones admitting discrete series. If we denote $\mathfrak{t}^*_{se} := \mathfrak{g}^*_{se} \cap \mathfrak{t}^*$, we see that $\mathfrak{g}^*_{se} = G \cdot \mathfrak{t}^*_{se}$. In other words, any coadjoint orbit contained in $\mathfrak{g}^*_{se}$ is elliptic.

First we recall the geometric properties associated to proper[2] Hamiltonian G-manifolds. We denote K a maximal compact subgroup of G and we denote $\Phi_K : M \to \mathfrak{k}^*$ the moment map relative to the K-action on (M, Ω).

Proposition 5.1 ([29]) *Let (M, Ω, Φ_G) be a proper[2] Hamiltonian G-manifold. Then:*

1. *the map Φ_K is proper,*
2. *the set $\mathfrak{g}^*_{se}$ is non-empty,*
3. *the image of Φ_G is contained in $\mathfrak{g}^*_{se}$,*
4. *the set $N := \Phi_G^{-1}(\mathfrak{k}^*)$ is a smooth K-submanifold of M,*
5. *the restriction of Ω on N defines a symplectic form Ω_N,*
6. *the map $[g, n] \mapsto gn$ defines a diffeomorphism $G \times_K N \simeq M$.*

Let T be a maximal torus in K, and let $\mathfrak{t}^*_+$ be a Weyl chamber. Since any coadjoint orbit in $\mathfrak{g}^*_{se}$ is elliptic, the coadjoint orbits belonging to the image of $\Phi_G : N \to \mathfrak{g}^*$ are parametrized by the set

$$\Delta_G(M) = \Phi_G(M) \cap \mathfrak{t}^*_+. \tag{5.13}$$

We remark that $\mathfrak{t}^*_+ \cap \mathfrak{g}^*_{se}$ is equal to $(\mathfrak{t}^*_+)_{se} := \{\xi \in \mathfrak{t}^*_+, (\xi, \alpha) \neq 0, \ \forall \alpha \in \mathfrak{R}_n\}$. The connected component $(\mathfrak{t}^*_+)_{se}$ are called chambers and if $\mathcal{C}$ is a chamber, we denote $\widehat{G}_d(\mathcal{C})$ the set of regular admissible elliptic orbits intersecting $\mathcal{C}$ (see Definition 2.2).

The following fact was first noticed by Weinstein [38].

Proposition 5.2 *$\Delta_G(M)$ is a convex polyhedral set contained in a unique chamber $\mathcal{C}_M \subset (\mathfrak{t}^*_+)_{se}$.*

Proof. We denote $\Phi_K^N : N \to \mathfrak{k}^*$ the restriction of the map Φ_G on the sub-manifold N. It corresponds to the moment map relative to the K-action on (N, Ω_N): notice that Φ_K^N is a proper map.

The diffeomorphism $G \times_K N \simeq M$ shows that the set $\Delta_G(M)$ is equal to $\Delta_K(N) := \text{Image}(\Phi_K^N) \cap \mathfrak{t}^*_+$, and the Convexity Theorem [14, 20] asserts that $\Delta_K(N)$ is a convex polyhedral subset of the Weyl chamber. Finally since $\Delta_K(N)$ is connected and contained in $(\mathfrak{t}^*_+)_{se}$, it must belongs to a unique chamber $\mathcal{C}_M$. $\square$

5.2 Spin^c-quantization of proper^2 Hamiltonian G-manifolds

Now we assume that our proper^2 Hamiltonian G-manifold (M, Ω, Φ_G) is spin^c-prequantized by a G-equivariant spin^c-bundle $\mathcal{S}$.

Note that $\mathfrak{p}$ is even dimensional since the groups G and K have the same rank. Recall that the morphism $K \to \mathrm{SO}(\mathfrak{p})$ lifts to a morphism $\tilde{K} \to \mathrm{Spin}(\mathfrak{p})$, where $\tilde{K} \to K$ is either an isomorphism or a two-fold cover (see Section 2.2). We start with the

Lemma 5.3

- *The G-equivariant* spin^c *bundle $\mathcal{S}$ on M induces a $\tilde{K}$-equivariant* spin^c *bundle $\mathcal{S}_N$ on N such that* $\det(\mathcal{S}_N) = \det(\mathcal{S})|_N$.
- *The $\tilde{K}$-Hamiltonian manifold (N, Ω_N, Φ_K^N) is* spin^c*-prequantized by $\mathcal{S}_N$.*

Proof. By definition we have $\mathbf{T}M|_N = \mathfrak{p} \oplus \mathbf{T}N$. The manifolds M and N are oriented by their symplectic forms. The vector space $\mathfrak{p}$ inherits an orientation $o(\mathfrak{p}, N)$ satisfying the relation $o(M) = o(\mathfrak{p}, N)o(N)$. The orientation $o(\mathfrak{p}, N)$ can be computed also as follows: takes any $\xi \in \mathrm{Image}(\Phi_K^N)$, then $o(\mathfrak{p}, N) = o(\xi)$ (see Example 2.6).

Let $\mathcal{S}_\mathfrak{p}$ be the spinor representation that we see as a $\tilde{K}$-module. The orientation $o(\mathfrak{p}) := o(\mathfrak{p}, N)$ determines a decomposition $\mathcal{S}_\mathfrak{p} = \mathcal{S}_\mathfrak{p}^{+,o(\mathfrak{p})} \oplus \mathcal{S}_\mathfrak{p}^{-,o(\mathfrak{p})}$ and we denote

$$\mathcal{S}_\mathfrak{p}^{o(\mathfrak{p})} := \mathcal{S}_\mathfrak{p}^{+,o(\mathfrak{p})} \ominus \mathcal{S}_\mathfrak{p}^{-,o(\mathfrak{p})} \in R(\tilde{K}).$$

Let $\mathcal{S}_N$ be the unique spin^c-bundle, $\tilde{K}$-equivariant on N defined by the relation

$$\mathcal{S}|_N = \mathcal{S}_\mathfrak{p}^{o(\mathfrak{p})} \boxtimes \mathcal{S}_N. \tag{5.14}$$

Since $\det(\mathcal{S}_\mathfrak{p}^{o(\mathfrak{p})})$ is trivial (as $\tilde{K}$-module), we have the relation $\det(\mathcal{S}_N) = \det(\mathcal{S})|_N$ that implies the second point. □

For $\mathcal{O} \in \widehat{G}_d$, we consider the symplectic reduced space

$$M /\!\!/ \mathcal{O} := \Phi_G^{-1}(\mathcal{O})/G.$$

Notice that $M /\!\!/ \mathcal{O} = \emptyset$ when $\mathcal{O}$ does not belongs to $\widehat{G}_d(\mathcal{C}_M)$. Moreover the diffeomorphism $G \times_K N \simeq M$ shows that $M /\!\!/ \mathcal{O}$ is equal to the reduced space

$$N /\!\!/ \mathcal{O}_K := (\Phi_K^N)^{-1}(\mathcal{O}_K)/K.$$

with $\mathcal{O}_K = \mathcal{O} \cap \mathfrak{k}^*$. Here $N/\!\!/\mathcal{O}_K$ should be understood as the symplectic reduction of the $\tilde{K}$-manifold N relative to the $\tilde{K}$-admissible coadjoint orbit $\mathcal{O}_K \in \widehat{\tilde{K}}$. Hence the quantization $\mathcal{Q}^{\mathrm{spin}}(N/\!\!/\mathcal{O}_K) \in \mathbb{Z}$ of the reduced space $N/\!\!/\mathcal{O}_K$ is well defined (see Proposition 3.6).

Definition 5.4 *For any $\mathcal{O} \in \widehat{G}_d$, we take $\mathcal{Q}^{\mathrm{spin}}(M/\!\!/\mathcal{O}) := \mathcal{Q}^{\mathrm{spin}}(N/\!\!/\mathcal{O}_K)$.*

The main tool to prove Theorem 1.2 is the comparison of the formal geometric quantization of three different geometric data: we work here in the setting where the G-action on M has *abelian infinitesimal stabilizers*.

1. The formal geometric quantization of the G-action on $(M, \Omega, \Phi_G, \mathcal{S})$ is the element $\widehat{\mathrm{Q}}^{\mathrm{spin}}_G(M) \in \widehat{R}(G, d)$ defined by the relation
$$\mathcal{Q}^{\mathrm{spin}}_G(M) := \sum_{\mathcal{O} \in \widehat{G}} \mathcal{Q}^{\mathrm{spin}}(M/\!\!/\mathcal{O})\, \pi^G_{\mathcal{O}}.$$

2. The formal geometric quantization of the K-action on $(M, \Omega, \Phi_K, \mathcal{S})$ is the element $\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M) \in \widehat{R}(K)$ (see Definition 4.4). As the K-action on M has *abelian infinitesimal stabilizers*, we have the decomposition
$$\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M) = \sum_{\mathcal{P} \in \widehat{K}} \mathcal{Q}^{\mathrm{spin}}(M/\!\!/\mathcal{P})\, \pi^K_{\mathcal{P}}.$$

3. The formal geometric quantization of the $\tilde{K}$-action on $(N, \Omega_N, \Phi^N_K, \mathcal{S}_N)$ is the element $\widehat{\mathrm{Q}}^{\mathrm{spin}}_{\tilde{K}}(N) \in \widehat{R}(\tilde{K})$. As the $\tilde{K}$-action on N has *abelian infinitesimal stabilizers*, we have the decomposition
$$\widehat{\mathrm{Q}}^{\mathrm{spin}}_{\tilde{K}}(N) = \sum_{\tilde{\mathcal{P}} \in \widehat{\tilde{K}}} \mathcal{Q}^{\mathrm{spin}}(N/\!\!/\tilde{\mathcal{P}})\, \pi^{\tilde{K}}_{\tilde{\mathcal{P}}}.$$

In the next section we explain the link between these three elements.

5.3 Spin^c-quantization: main results

Let $\mathcal{C}_M \subset \mathfrak{t}^*_+$ be the chamber containing $\Phi_G(M) \cap \mathfrak{t}^*_+$.

Definition 5.5 *We defines the orientation o^+ and o^- on $\mathfrak{p}$ as follows. Take $\lambda \in \mathcal{C}_M$, then $o^+ := o(\lambda)$ and $o^- := o(-\lambda)$ (see Example 2.6).*

We denote $\mathcal{S}_{\mathfrak{p}}^{o^+}, \mathcal{S}_{\mathfrak{p}}^{o^-}$ the virtual representations of $\tilde{K}$ associated to the spinor representation of $\mathrm{Spin}(\mathfrak{p})$ and the orientations o^+ and o^-. We denote $\overline{\mathcal{S}_{\mathfrak{p}}^{o^+}}$ the $\tilde{K}$-module with opposite complex structure. Remark that $\overline{\mathcal{S}_{\mathfrak{p}}^{o^+}} \simeq \mathcal{S}_{\mathfrak{p}}^{o^-}$.

Recall that the map $V \mapsto V|_K$ defines a morphism $\widehat{R}(G,d) \to \widehat{R}(K)$. We have also the morphism $\mathbf{r}^o = \widehat{R}(G,d) \to \widehat{R}(\tilde{K})$ defined by $\mathbf{r}^o(V) = V|_K \otimes \mathcal{S}_{\mathfrak{p}}^o$.

We start with the following

Theorem 5.6 *If the G-action on M has* abelian infinitesimal stabilizers *then*

$$\mathbf{r}^o\left(\widehat{\mathrm{Q}}_G^{\mathrm{spin}}(M)\right) = \epsilon_M^o\, \widehat{\mathrm{Q}}_{\tilde{K}}^{\mathrm{spin}}(N). \tag{5.15}$$

Here $\epsilon_M^o = \pm$ is equal to the ratio between o and o^-.

Proof. If the G-action on M has abelian infinitesimal stabilizers, then the $\tilde{K}$-action on N has also abelian infinitesimal stabilizers. It implies the following relation:

$$\widehat{\mathrm{Q}}_{\tilde{K}}^{\mathrm{spin}}(N) = \sum_{\tilde{\mathcal{P}} \in \widehat{\tilde{K}}} \mathcal{Q}^{\mathrm{spin}}(N /\!\!/ \tilde{\mathcal{P}})\, \pi_{\tilde{\mathcal{P}}}^{\tilde{K}} \quad \in \widehat{R}(\tilde{K}).$$

Following the first point of Lemma 2.9, we consider the following subset $\Gamma := \{\mathcal{O}_K := \mathcal{O} \cap \mathfrak{k}^*,\, \mathcal{O} \in \widehat{G}_d\} \subset \widehat{K}_{out} \subset \widehat{\tilde{K}}$.

Thanks to the second point of Lemma 2.9 we have

$$\begin{aligned}
\mathbf{r}^o\left(\widehat{\mathrm{Q}}_G^{\mathrm{spin}}(M)\right) &= \sum_{\mathcal{O} \in \widehat{G}_d} \mathcal{Q}^{\mathrm{spin}}(M /\!\!/ \mathcal{O})\, \pi_{\mathcal{O}}^G|_K \otimes \mathcal{S}_{\mathfrak{p}}^o. \\
&= \epsilon_M^o \sum_{\mathcal{O} \in \widehat{G}_d} \mathcal{Q}^{\mathrm{spin}}(N /\!\!/ \mathcal{O}_K)\, \pi_{\mathcal{O}_K}^{\tilde{K}} \\
&= \epsilon_M^o \sum_{\tilde{\mathcal{P}} \in \Gamma} \mathcal{Q}^{\mathrm{spin}}(N /\!\!/ \tilde{\mathcal{P}})\, \pi_{\tilde{\mathcal{P}}}^{\tilde{K}}.
\end{aligned}$$

Identity (5.15) is proved if we check that $\mathcal{Q}^{\mathrm{spin}}(N /\!\!/ \tilde{\mathcal{P}}) = 0$ for any $\tilde{\mathcal{P}} \in \widehat{\tilde{K}}$ which does not belong to Γ.

Suppose first that $\tilde{K} \simeq K$. In this case we have $\widehat{\tilde{K}} = \widehat{K}_{out} = \widehat{K}$ and a coadjoint orbit $\tilde{P} = K\mu \in \widehat{K}$ does not belong to Γ if and only if μ is not contained in $\mathfrak{g}_{se}^*$. But the image of Φ_G is contained in $\mathfrak{g}_{se}^*$, so $N /\!\!/ \tilde{\mathcal{P}} = \emptyset$ and then $\mathcal{Q}^{\mathrm{spin}}(N /\!\!/ \tilde{\mathcal{P}}) = 0$ if $\tilde{P} \notin \Gamma$.

Suppose now that $\tilde{K} \to K$ is a two-fold cover and let us denote by $\{\pm 1_{\tilde{K}}\}$ the kernel of this morphism. Here $\gamma := -1_{\tilde{K}}$ acts trivially on N and (5.14) shows that γ acts by multiplication by -1 on the fibers of the spin^c bundle $\mathcal{S}_N$. The element γ acts also trivially on the orbits $\tilde{\mathcal{P}} \in \widehat{\tilde{K}}$:

- if $\tilde{P} \in \widehat{K}_{out}$, then γ acts by multiplication by -1 on the fibers of the spin^c bundle $\mathcal{S}_{\tilde{\mathcal{P}}}$,
- if $\tilde{P} \notin \widehat{K}_{out}$, then γ acts trivially on the fibers of the spin^c bundle $\mathcal{S}_{\tilde{\mathcal{P}}}$.

Our considerations show that $[\mathcal{S}_N \boxtimes \mathcal{S}_{\tilde{\mathcal{P}}^-}]^\gamma = 0$ when $\tilde{P} \in \widehat{\tilde{K}} \backslash \widehat{K}_{out}$. Thanks to Lemma 4.8, it implies the vanishing of $\mathcal{Q}^{\mathrm{spin}}(N /\!\!/ \tilde{P})$ for any $\tilde{P} \in \widehat{\tilde{K}} \backslash \widehat{K}_{out}$.

Like in the previous case, when $\tilde{P} \in \widehat{K}_{out} \backslash \Gamma$, we have $\mathcal{Q}^{\mathrm{spin}}(N /\!\!/ \tilde{P}) = 0$ because $N /\!\!/ \tilde{\mathcal{P}} = \emptyset$. □

We compare now the formal geometric quantizations of the K-manifolds M and N.

Theorem 5.7 *We have the following relation*

$$\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M) \otimes \overline{\mathcal{S}^{o^+}_{\mathfrak{p}}} = \widehat{\mathrm{Q}}^{\mathrm{spin}}_{\tilde{K}}(N) \quad \in R(\tilde{K}). \tag{5.16}$$

When $M = \mathcal{O} \in \widehat{G}_d$ the manifold N is equal to $\mathcal{O}_K := \mathcal{O} \cap \mathfrak{k}^*$. We have $\widehat{\mathrm{Q}}^{\mathrm{spin}}_{\tilde{K}}(N) = \pi^{\tilde{K}}_{\mathcal{O}_K}$ and we know also that $\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(\mathcal{O}) = \pi^G_{\mathcal{O}}|_K$ (see Proposition 4.5). Here (5.16) becomes

$$\pi^G_{\mathcal{O}}|_K \otimes \mathcal{S}^o_{\mathfrak{p}} = \pm \pi^{\tilde{K}}_{\mathcal{O}_K} \tag{5.17}$$

where the sign $\pm$ is the ratio between the orientations o and o^- of the vector space $\mathfrak{p}$.

If we use Theorems 5.6 and 5.7 we get the following

Corollary 5.8 *If the G-action on M has* abelian infinitesimal stabilizers, *we have* $\mathbf{r}^o\left(\widehat{\mathrm{Q}}^{\mathrm{spin}}_G(M)\right) = \widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M) \otimes \mathcal{S}^o_{\mathfrak{p}}$.

The following conjecture says that the functorial property of $\widehat{\mathrm{Q}}^{\mathrm{spin}}$ relative to restrictions (see Theorem 4.9) should also holds for non-compact groups.

Conjecture 5.9 *If the G-action on M has* abelian infinitesimal stabilizers *then the following relation*

$$\widehat{\mathrm{Q}}^{\mathrm{spin}}_G(M)|_K = \widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M)$$

holds in $\widehat{R}(K)$.

The remaining part of this section is devoted to the proof of Theorem 5.7.

We work with the manifold $M := G \times_K N$. We denote $\Phi^N_K : N \to \mathfrak{k}^*$ the restriction of $\Phi_G : M \to \mathfrak{g}^*$ to the submanifold N. We will use the K-equivariant isomorphism $\mathfrak{p} \times N \simeq M$ defined by $(X, n) \mapsto [e^X, n]$.

The maps Φ_G, Φ_K, Φ^N_K are related through the relations $\Phi_G(X, n) = e^X \cdot \Phi^N_K(n)$ and[5] $\Phi_K(X, n) = \mathrm{p}_{\mathfrak{k},\mathfrak{g}}(e^X \cdot \Phi^N_K(n))$.

We consider the Kirwan vector fields on N and M

$$\kappa_N(n) = -\Phi^N_K(n) \cdot n \quad , \quad \kappa_M(m) = -\Phi_K(m) \cdot m.$$

The following result is proved in [29][Section 2.2].

Lemma 5.10 *An element $(X, n) \in \mathfrak{p} \times N$ belongs to $Z_M := \{\kappa_M = 0\}$ if and only if $X = 0$ and $n \in Z_N := \{\kappa_N = 0\}$.*

Let us recall how are defined the characters $\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M)$ and $\widehat{\mathrm{Q}}^{\mathrm{spin}}_{\tilde{K}}(N)$. We start with the decomposition $Z_N = \coprod_{\beta\in\mathcal{B}} Z_\beta$ where $Z_\beta = K(N^\beta \cap (\Phi^N_K)^{-1}(\beta))$, and $\mathcal{B} = \Phi^N_K(Z_N) \cap \mathfrak{t}^*_+$. Thanks to Lemma 5.10 the corresponding decomposition on M is $Z_M := \coprod_{\beta\in\mathcal{B}} \{0\} \times Z_\beta$.

By definiton we have

$$\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(N) := \sum_{\beta\in\mathcal{B}} \mathcal{Q}^{\mathrm{spin}}_K(N, Z_\beta) \ \in \ \widehat{R}(\tilde{K})$$

and $\widehat{\mathrm{Q}}^{\mathrm{spin}}_K(M) = \widehat{\mathrm{Q}}^{\mathrm{spin}}_K(\mathfrak{p} \times N) := \sum_{\beta\in\mathcal{B}} \mathcal{Q}^{\mathrm{spin}}_K(\mathfrak{p} \times N, \{0\} \times Z_\beta) \ \in \ \widehat{R}(K)$. The proof of Theorem 5.7 is completed if we show that for any $\beta \in \mathcal{B}$ we have

$$\mathcal{Q}^{\mathrm{spin}}_K(\mathfrak{p} \times N, \{0\} \times Z_\beta) \otimes \overline{\mathcal{S}^{o+}_{\mathfrak{p}}} \ = \ \mathcal{Q}^{\mathrm{spin}}_{\tilde{K}}(N, Z_\beta) \quad \in R(\tilde{K}). \tag{5.18}$$

Let $\mathcal{S}$ be the G-equivariant spin^c-bundle on M. The K-equivariant diffeomorphism $M \simeq \mathfrak{p} \times N$ induces a $\tilde{K}$-equivariant isomorphism at the level of spin^c bundles:

$$\mathcal{S} \simeq \mathcal{S}^{o+}_{\mathfrak{p}} \otimes \mathcal{S}_N.$$

[5] $\mathrm{p}_{\mathfrak{k},\mathfrak{g}} : \mathfrak{g}^* \to \mathfrak{k}^*$ is the canonical projection.

We denote $\mathrm{cl}_{\mathfrak{p}} : \mathfrak{p} \to \mathrm{End}(\mathcal{S}_{\mathfrak{p}})$ the Clifford action associated to the Clifford module $\mathcal{S}_{\mathfrak{p}}$. Any $X \in \mathfrak{p}$ determines an odd linear map $\mathrm{cl}_{\mathfrak{p}}(X) : \mathcal{S}_{\mathfrak{p}} \to \mathcal{S}_{\mathfrak{p}}$.

For $n \in N$, we denote $\mathrm{cl}_n : \mathbf{T}_n N \to \mathrm{End}(\mathcal{S}_N|_n)$ the Clifford action associated to the spin^c bundle $\mathcal{S}_N$. Any $v \in \mathbf{T}_n N$ determines an odd linear map $\mathrm{cl}_n(v) : \mathcal{S}_N|_n \to \mathcal{S}_N|_n$.

Lemma 5.11 *Let $U_\beta \subset N$ be a small invariant neighborhood of Z_β such that $Z_N \cap \overline{U_\beta} = Z_\beta$.*

• *The character $\mathcal{Q}^{\mathrm{spin}}_{\tilde{K}}(N, Z_\beta)$ is equal to the index of the $\tilde{K}$-transversally elliptic symbol*

$$\sigma^1_n(v) : \mathcal{S}^+_N|_n \longrightarrow \mathcal{S}^-_N|_n, \quad v \in \mathbf{T}_n U_\beta$$

defined by $\sigma^1_n(v) = \mathrm{cl}_n(v + \Phi^N_K(n) \cdot n)$.

• *The character $\mathcal{Q}^{\mathrm{spin}}_K(\mathfrak{p} \times N, \{0\} \times Z_\beta)$ is equal to the index of the K-transversally elliptic symbol*

$$\sigma^2_{(A,n)}(X, v) : (\mathcal{S}^{o^+}_{\mathfrak{p}} \otimes \mathcal{S}_N|_n)^+ \longrightarrow (\mathcal{S}^{o^+}_{\mathfrak{p}} \otimes \mathcal{S}_N|_n)^-$$

defined by $\sigma^2_{(A,n)}(X, v) = \mathrm{cl}_{\mathfrak{p}}(X + [\Phi^N_K(n), A]) \otimes \mathrm{cl}_n(v + \Phi^N_K(n) \cdot n)$ for $(X, v) \in \mathbf{T}_{(A,n)}(\mathfrak{p} \times U_\beta)$.

Proof. The first point corresponds to the definition of the character $\mathcal{Q}^{\mathrm{spin}}_{\tilde{K}}(N, Z_\beta)$.

By definition, $\mathcal{Q}^{\mathrm{spin}}_K(\mathfrak{p} \times N, \{0\} \times Z_\beta)$ is equal to the index of the K-transversally elliptic symbol

$$\tau_{(A,n)}(X, v) = \mathrm{cl}_{\mathfrak{p}}(X + [\Phi_K(X, n), A]) \otimes \mathrm{cl}_n(v + \Phi_K(X, n) \cdot n).$$

It is not difficult to see that

$$\tau^t_{(A,n)}(X, v) = \mathrm{cl}_{\mathfrak{p}}(X + [\Phi_K(tX, n), A]) \otimes \mathrm{cl}_n(v + \Phi_K(tX, n) \cdot n), \qquad 0 \leqslant t \leqslant 1,$$

defines an homotopy of transversally elliptic symbols between $\sigma^2 = \tau^0$ and $\tau = \tau^1$: like in Lemma 5.10, we use the fact that $[\Phi_K(0, n), A] = 0$ only if $A = 0$. It proves the second point. □

We can now finish the proof of (5.18). We use here the following isomorphism of Clifford modules for the vector space $\mathfrak{p} \times \mathfrak{p}$:

$$\mathcal{S}^{o^+}_{\mathfrak{p}} \otimes \overline{\mathcal{S}^{o^+}_{\mathfrak{p}}} \simeq \bigwedge_{\mathbb{C}} \mathfrak{p}_{\mathbb{C}},$$

where the Clifford action $(X,Y)\in\mathfrak{p}\times\mathfrak{p}$ on the left is $\mathrm{cl}_{\mathfrak{p}}(X)\otimes\mathrm{cl}_{\mathfrak{p}}(Y)$ and on the right is $\mathrm{cl}_{\mathfrak{p}_{\mathbb{C}}}(X+iY)$.

The product $\sigma^2\otimes\overline{\mathcal{S}^{o+}_{\mathfrak{p}}}$ corresponds to the symbol

$$\mathrm{cl}_{\mathfrak{p}}(X+[\Phi_K(X,n),A])\otimes\mathrm{cl}_{\mathfrak{p}}(0)\otimes\mathrm{cl}_n(v+\Phi_K^N(n)\cdot n)$$

which is homotopic to

$$\mathrm{cl}_{\mathfrak{p}}(X+[\Phi_K(X,n),A])\otimes\mathrm{cl}_{\mathfrak{p}}(A)\otimes\mathrm{cl}_n(v+\Phi_K^N(n)\cdot n),$$

and is also homotopic to

$$\sigma^3:=\mathrm{cl}_{\mathfrak{p}}(X)\otimes\mathrm{cl}_{\mathfrak{p}}(A)\otimes\mathrm{cl}_n(v+\Phi_K^N(n)\cdot n).$$

We have then proved that the K-equivariant index of σ^2 times $\overline{\mathcal{S}^{o+}_{\mathfrak{p}}}\in R(\tilde{K})$ is equal to the $\tilde{K}$-equivariant index of σ^3 (that we denote $\mathrm{Index}^{\mathfrak{p}\times U_\beta}_{\tilde{K}}(\sigma^3)$). The multiplicative property of the equivariant index [1] tells us that

$$\mathrm{Index}^{\mathfrak{p}\times U_\beta}_{\tilde{K}}(\sigma^3)=\mathrm{Index}^{\mathfrak{p}}_{\tilde{K}}(\mathrm{cl}_{\mathfrak{p}_{\mathbb{C}}}(X+iA))\cdot\mathrm{Index}^{U_\beta}_{\tilde{K}}(\sigma^1).$$

But $\mathrm{cl}_{\mathfrak{p}_{\mathbb{C}}}(X+iA):\bigwedge^+_{\mathbb{C}}\mathfrak{p}_{\mathbb{C}}\to\bigwedge^-_{\mathbb{C}}\mathfrak{p}_{\mathbb{C}}$, $(X,A)\in\mathbf{T}\mathfrak{p}$, is the Bott symbol and its index is equal to the trivial 1-dimensional representation of $\tilde{K}$. We have finally proved that the K-equivariant index of σ^2 times $\overline{\mathcal{S}^{o+}_{\mathfrak{p}}}$ is equal to the $\tilde{K}$-equivariant index of σ^1. The proof of (5.18) is complete. □

5.4 Proof of the main Theorem

Let G be a connected semi-simple subgroup of G' with finite center, and let $\mathcal{O}'\in\widehat{G'}_d$. We suppose that the representation $\pi^{G'}_{\mathcal{O}'}$ is G-admissible. Then we have a decomposition

$$\pi^{G'}_{\mathcal{O}'}|_G=\sum_{\mathcal{O}\in\widehat{G}_d}m_{\mathcal{O}}\,\pi^G_{\mathcal{O}}.$$

Let $\Phi_G:\mathcal{O}'\to\mathfrak{g}^*$ be the moment map relative to the G-action on $\mathcal{O}'$. We have proved in Theorem 2.10, that the G-admissibility of $\pi^{G'}_{\mathcal{O}'}$ implies the properness of Φ_G. Moreover, since $\mathcal{O}'$ is a regular orbit, the G-action on it is proper. Finally we see that $\mathcal{O}'$ is a spin^c prequantized proper[2] Hamiltonian G-manifold. We can consider its formal spin^c quantization $\widehat{\mathrm{Q}}^{\mathrm{spin}}_G(\mathcal{O}')\in\widehat{R}(G,d)$, which is defined by the relation

$$\widehat{\mathrm{Q}}^{\mathrm{spin}}_G(\mathcal{O}'):=\sum_{\mathcal{O}\in\widehat{G}_d}\mathcal{Q}^{\mathrm{spin}}(\mathcal{O}'/\!\!/\mathcal{O})\,\pi^G_{\mathcal{O}}.$$

Theorem 1.2 is proved if we show that $\pi_{\mathcal{O}'}^{G'}|_G$ and $\widehat{\mathrm{Q}}_G^{\mathrm{spin}}(\mathcal{O}')$ are equal in $\widehat{R}(G,d)$. Since the morphism $\mathbf{r}^o : \widehat{R}(G,d) \to \widehat{R}(\tilde{K})$ is one to one, it is sufficient to prove that

$$\mathbf{r}^o\left(\pi_{\mathcal{O}'}^{G'}|_G\right) = \mathbf{r}^o\left(\widehat{\mathrm{Q}}_G^{\mathrm{spin}}(\mathcal{O}')\right). \tag{5.19}$$

On one hand, the element $\mathbf{r}^o\left(\pi_{\mathcal{O}'}^{G'}|_G\right)$ is equal to $\pi_{\mathcal{O}'}^{G'}|_K \otimes \mathcal{S}_{\mathfrak{p}}^o$. The restriction $\pi_{\mathcal{O}'}^{G'}|_K \in \widehat{R}(K)$, which is well defined since the moment map $\Phi_K : \mathcal{O}' \to \mathfrak{k}^*$ is proper, is equal to $\widehat{\mathrm{Q}}_K^{\mathrm{spin}}(\mathcal{O}')$ (see Corollary 4.10). So we get

$$\mathbf{r}^o\left(\pi_{\mathcal{O}'}^{G'}|_G\right) = \widehat{\mathrm{Q}}_K^{\mathrm{spin}}(\mathcal{O}') \otimes \mathcal{S}_{\mathfrak{p}}^o.$$

On the other hand, Corollary 4.10 tells us that

$$\mathbf{r}^o\left(\widehat{\mathrm{Q}}_G^{\mathrm{spin}}(\mathcal{O}')\right) = \widehat{\mathrm{Q}}_K^{\mathrm{spin}}(\mathcal{O}') \otimes \mathcal{S}_{\mathfrak{p}}^o.$$

Hence we obtain Equality (5.19). The proof of Theorem 1.2 is completed.

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