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# ASYMPTOTIC EXPANSION OF THE MEAN-FIELD APPROXIMATION

THIERRY PAUL AND MARIO PULVIRENTI

ABSTRACT. We consider the  $N$ -body quantum evolution of a particle system in the mean-field approximation. We show that the  $j$ th order marginals  $F_j^N(t)$ , for factorized initial data  $F(0)^{\otimes N}$ , are explicitly expressed, modulo  $N^{-\infty}$ , out of the solution  $F(t)$  of the corresponding non-linear mean-field equation and the solution of its linearization around  $F(t)$ . The result is valid for all times  $t$ , uniformly in  $j = O(N^{\frac{1}{2}-\alpha})$  for any  $\alpha > 0$ . We establish and estimate the full asymptotic expansion in integer powers of  $\frac{1}{N}$  of  $F_j^N(t)$ ,  $j = O(\sqrt{N})$ , whose computation at order  $n$  involves a finite number of operations depending on  $j$  and  $n$  but not on  $N$ . Our results are also valid for more general models including Kac models. As a by-product we get that the rate of convergence to the mean-field limit in  $\frac{1}{N}$  is optimal in the sense that the first correction to the mean-field limit does not vanish.

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## 1. INTRODUCTION

The mean-field limit concerns systems of interacting (classical or quantum) particles whose number diverges in a way linked with a rescaling of the interaction insuring an equilibrium between interaction and kinetic energy. In the case of an additive one-body kinetic energy part and a two-body interaction, and without taking in consideration

1 quantum statistics, this equilibrium is reached by putting in front of the interaction a  
 2 coupling constant proportional to the inverse of the number of particles.

3 The system is then described by isolating the evolution of one (or  $j$ ) particle(s) and  
 4 averaging over all the other. This leads to a partial information on the system driven  
 5 by the so-called  $j$ -marginals. The mean-field theory ensures that the  $j$ -marginals tend,  
 6 as the number of particles diverges, to the  $j$ -tensor power of the solution of a non-linear  
 7 one-body mean-field equation (Vlasov, Hartree,...) issued from the 1-marginal on the  
 8 initial  $N$ -body state. This program has been achieved in many different situations,  
 9 and the literature concerning the mean-field approach is enormous. We refer to [30]  
 10 for a review and recent references.

11 As regards the fluctuations around this limit, namely the correction to be added to  
 12 the factorized limit in order to get better approximations of the true evolution of the  
 13  $j$ -marginals, there are some results.

14 The identification of the leading order of these fluctuations with a Gaussian sto-  
 15 chastic process has been established in the quantum context first in [16] and in the  
 16 classical one in [5]. For the classical dynamics of hard spheres, the fluctuations around  
 17 the Boltzmann equation have been computed at leading order in [29], generalizing to  
 18 non-equilibrium states the results of [3]. More recently, for the quantum case, fluctu-  
 19 ations near the Hartree dynamics have been derived in [23] (after [22]) and in [2] also  
 20 for the grand canonical ensemble formalism (number of particles non fixed), using in  
 21 both cases the methods of second quantization (Fock space) (see also [25] for a proof  
 22 using the usual quantization formalism). In the case of pure states, the  $N$ -body wave  
 23 function is shown to be  $\frac{1}{\sqrt{N}}$ -close in  $L^2$  norm to a sum of partially factorized states  
 24 constructed out of the so-called Bogoliubov hierarchy. Note that these results rise a  
 25 problem fundamentally different from the one treated in the present paper, whose goal  
 26 is to compute mean-field approximation of the  $N$ -body problem with an accuracy of  
 27 any order in powers of  $\frac{1}{N}$ .

28 Nevertheless the basic object of our analysis, namely the kinetic error  $E_j$  (see below  
 29 for the definition), is strictly related to the expectation of the fluctuations of intensive  
 30 observables (see e.g. [5]). However the analysis of the fluctuations problem from  
 31 the present point of view requires an additional analysis which goes beyond the main  
 32 purposes of the present paper.

1 Recently, we developed (together with S. Simonella) in [26] a method to derive mean-  
 2 field limits, alternative to the ones using empirical measures or direct estimates on the  
 3 “BBGKY-type” hierarchies (systems of coupled equations satisfied by the set of the  
 4  $j$ -marginals). This method rather uses the hierarchy followed by the “kinetic errors”  
 5  $E_{j-k}$  (defined below), already used (under the name “ $v$ -functions”) to deal with kinetic  
 6 limits of stochastic models [10, 7, 4, 11, 12, 6, 8, 13] and recently investigated in the  
 7 more singular low density limit of hard spheres [27] (note that error terms are also used  
 8 in [23, 22, 2, 25] for the total (pure state) wave function with a quite different point  
 9 of view). These quantities are, roughly speaking, the coefficient of the decomposition  
 10 of the  $j$ -marginal as a linear combination of the  $k$ -th tensor powers,  $k = 1, \dots, j$ , of the  
 11 solution of the mean-field equation issued from the 1-marginal of the initial full state.  
 12 We developed in [26] a strategy suitable in particular for Kac models (homogeneous  
 13 original one [17, 18] and non-homogeneous [9]) and quantum mean-field theory. This  
 14 strategy allowed us to derive the limiting factorization property of the  $j$ -marginals up  
 15 to, roughly speaking,  $j \lesssim \sqrt{N}$ . This threshold is, on the other side, the one obtained  
 16 by heuristic arguments as shown in [26] and rigorously in [15] for the Kac’s model. At  
 17 the contrary, let us recall that the quantum mean-field limit was obtained in [19] for  
 18 marginals of order  $j = o(N)$ , for pure states initial data.

19 Here and in all this article,  $N$  denotes the number of particles of the system under  
 20 consideration.

21 In the present paper we provide and estimate a full asymptotic expansion in powers of  
 22  $\frac{1}{N}$  of the difference between the evolution of  $j$ -marginals and its factorized leading order  
 23 form (Theorem 3.2), following a similar result for the kinetic errors  $E_j(t)$  (Theorem  
 24 3.1). Our results are valid for  $j \leq C\sqrt{N}$  for some explicit constant  $C$  and are valid for  
 25 quantum, Kac’s models and in the framework of the abstract formalism, slightly more  
 26 general than the one developed in [26], described in Appendix A.

27 The non-vanishing of the first correction is established, showing therefore that the  
 28 rate of the mean-field convergence is at most of order  $\frac{1}{N}$  (Corollary 3.4) : we will  
 29 prove that the first marginals (of order  $j = 1, 2$ ) are kept away from their mean-field  
 30 factorized limits at a distance bounded from below by  $CN^{-1}$ ,  $C > 0$  (and not, e.g.  $N^{-2}$ )  
 31 as  $N \rightarrow \infty$ <sup>1</sup>

---

<sup>1</sup>This has not to be mistaken with the problem of optimality in  $j \rightarrow \infty$  versus  $N \rightarrow \infty$  for which it has been established in [19] a rate of convergence in  $\sqrt{\frac{j}{N}}$ , for pure states initial data.

1 Moreover, as the mean-field solution issued from the first marginal of the  $N$  body  
 2 symmetrical factorized initial data determines the leading order of the  $j$ -marginal, we  
 3 show that the additional knowledge of the linearization of the mean-field flow around  
 4 it gives an explicit construction of the full asymptotic expansion of the  $j$ -marginals in  
 5 powers of  $\frac{1}{N}$  uniformly in  $j, N$  satisfying  $j \leq CN^{\frac{1}{2}-\alpha}$  for any  $C, \alpha > 0$  (Theorem 3.5).  
 6 Let us note the analogy with the quantum propagation of semiclassical observables,  
 7 driven by the classical underlying flow at leading order in the Planck constant, and  
 8 whose full asymptotic expansion is explicitly computable by the only knowledge of the  
 9 linearized flow.

10 Let us summarize in words our main result:

11 The knowledge of the mean-field flow  $F(t)$  and its linearization around  $F(t)$  determines  
 12 explicitly, modulo  $N^{-\infty}$ , uniformly for  $j = O(N^{\frac{1}{2}-\alpha})$ ,  $\alpha > 0$ , the  $j$ -marginals of the  $N$ -  
 13 body flow issued from  $F(0)^{\otimes N}$ .

## 14 2. QUANTUM MEAN-FIELD

15 Let  $\mathcal{L}^1(L^2(\mathbf{R}^d))$  be the space of trace class operators on  $L^2(\mathbf{R}^d)$ , with their associated  
 16 norms.

We consider the evolution of a system of  $N$  quantum particles interacting through a  
 (real-valued) two-body, even potential  $V$ , described for any value of the Planck constant  
 $\hbar > 0$  by the Schrödinger equation

$$i\hbar\partial_t\psi = H_N\psi, \quad \psi|_{t=0} = \psi_{in} \in \mathfrak{H}_N := L^2(\mathbf{R}^d)^{\otimes N},$$

where

$$H_N := -\frac{\hbar^2}{2} \sum_{k=1}^N \Delta_{x_k} + \frac{1}{2N} \sum_{\substack{1 \leq k, l \leq N \\ k \neq l}} V(x_k - x_l).$$

17 We assume in this paper that  $V$  is bounded, which implies that the  $N$ -body Hamil-  
 18 tonian  $H_N$  is self-adjoint on  $H^2(\mathbf{R}^d)$ , the second Sobolev space.

Instead of the Schrödinger equation written in terms of wave functions, we shall  
 rather consider the quantum evolution of density matrices. An  $N$ -body density matrix  
 is an operator  $F^N$  such that

$$0 \leq F^N = (F^N)^*, \quad \text{trace}_{\mathfrak{H}_N}(F^N) = 1.$$

1 The evolution of the density matrix  $F^N \mapsto F^N(t)$  of a  $N$ -particle system is governed  
 2 for any value of the Planck constant  $\hbar > 0$  by the von Neumann equation

$$(1) \quad \partial_t F^N = \frac{1}{i\hbar} [H_N, F^N],$$

3 equivalent to the Schrödinger equation when  $F^N(0)$  is a rank one projector, modulo a  
 4 global phase.

5 Positivity, norm and trace are obviously preserved by (1) since  $H_N$  is self-adjoint.

For each  $j = 1, \dots, N$ , the  $j$ -particle marginal  $F_j^N(t)$  of  $F^N(t)$  is the unique trace class operator on  $\mathfrak{H}_j$  such that

$$\text{trace}_{\mathfrak{H}_N} [F^N(t)(A_1 \otimes \dots \otimes A_j \otimes I_{\mathfrak{H}_{N-j}})] = \text{trace}_{\mathfrak{H}_j} [F_j^N(t)(A_1 \otimes \dots \otimes A_j)].$$

6 for all  $A_1, \dots, A_j$  bounded operators on  $\mathfrak{H}$ . Alternatively and equivalently, the  $F_j^N$  can  
 7 be defined by the partial trace of  $F^N$  on the  $N-j$  last “particles”: defining  $F^N$  through  
 8 its integral kernel  $F^N(x_1, x'_1; \dots; x_N, x'_N)$ , the integral kernel of  $F_j^N$  is defined as (see  
 9 [1])

$$(2) \quad \begin{aligned} F_j^N(x_1, x'_1; \dots; x_j, x'_j) &:= (\text{Tr}^{j+1} \dots \text{Tr}^N F^N)(x_1, x'_1; \dots; x_j, x'_j) \\ &:= \int_{\mathbf{R}^{d(N-j)}} F^N(x_1, x'_1; \dots; x_j, x'_j; x_{j+1}, x_{j+1}; \dots; x_N, x_N) dx_{j+1} \dots dx_N. \end{aligned}$$

10 It will be convenient for the sequel to rewrite (1) in the following operator form

$$(3) \quad \partial_t F^N = (K^N + V^N) F^N$$

11 where  $K^N, V^N$  are operators on  $\mathcal{L}^1(L^2(\mathbf{R}^{Nd}))$  defined by

$$(4) \quad K^N = \frac{1}{i\hbar} \left[ -\frac{\hbar^2}{2} \Delta_{\mathbf{R}^{dN}}, \cdot \right], \quad V^N = \frac{1}{2N} \sum_{k,l} V_{k,l} \text{ with } V_{k,l} := \frac{1}{i\hbar} [V(x_k - x_l), \cdot].$$

12 The self-adjointness of  $H_N$  implies that

$$(5) \quad \|e^{t(K^N + V^N)}\|_{\mathcal{L}^1(L^2(\mathbf{R}^d)) \rightarrow \mathcal{L}^1(L^2(\mathbf{R}^d))} = \|e^{tK^N}\|_{\mathcal{L}^1(L^2(\mathbf{R}^{Nd})) \rightarrow \mathcal{L}^1(L^2(\mathbf{R}^{Nd}))} = 1, \quad t \in \mathbf{R}.$$

13 We will denote

$$(6) \quad \mathbb{L} := \mathcal{L}^1(L^2(\mathbf{R}^d)) \text{ so that } \mathbb{L}^{\otimes n} = \mathcal{L}^1(L^2(\mathbf{R}^{nd})), \quad n = 1, \dots, N,$$

14 and, with a slight abuse of notation,

$$(7) \quad \begin{cases} \|\cdot\|_1 & \text{the trace norm on any } \mathbb{L}^{\otimes j}, \\ \|\cdot\| & \text{the operator norm on any } \mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j}) \end{cases}$$

15 for  $i, j = 1, \dots, N$  (here  $\mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j})$  is the set of bounded operators from  $\mathbb{L}^{\otimes i}$  to  $\mathbb{L}^{\otimes j}$ ).

A density matrix  $F^n \in \mathbb{L}^{\otimes n}$  is called symmetric if its integral kernel  $F^n(x_1, x'_1; \dots; x_n, x'_n)$  is invariant by any permutation

$$(x_i, x'_i) \leftrightarrow (x_j, x'_j), \quad i, j = 1, \dots, n.$$

1 Note that the symmetry of  $F^N$  is preserved by the equation (1) due to the particular  
2 form of the potential.

3 We define, for  $n = 1, \dots, N$ ,

$$(8) \quad \mathcal{D}_n = \{F \in \mathbb{L}^{\otimes n} \mid F > 0, \quad \|F\|_1 = 1 \quad \text{and } F \text{ is symmetric}\}.$$

Note that  $F_j^N \in \mathbb{L}^{\otimes j}$  ( $F_0^N = 1 \in \mathbb{L}^{\otimes 0} := \mathbf{C}$ ) and  $F_j^N > 0, \|F_j^N\|_1 = \|F^N\|_1$ , and obviously  $F_j^N$  is symmetric as  $F^N$ . That is to say:

$$F_j^N \in \mathcal{D}_j.$$

4 The family of  $j$ -marginals,  $j = 1, \dots, N$ , is solution of the BBGKY hierarchy of  
5 equations (see [28] and also [1])

$$(9) \quad \partial_t F_j^N = \left( K^j + \frac{T_j}{N} \right) F_j^N + \frac{(N-j)}{N} C_{j+1} F_{j+1}^N$$

6 where:

$$(10) \quad K^j = \frac{1}{i\hbar} \left[ -\frac{\hbar^2}{2} \Delta_{\mathbf{R}^{jd}}, \cdot \right]$$

$$(11) \quad T_j = \sum_{1 \leq i < r \leq j} T_{i,r} \quad \text{with } T_{i,r} = V_{i,r}$$

8 and

$$(12) \quad C_{j+1} F_{j+1}^N = \sum_{i=1}^j C_{i,j+1} F_{j+1}^N$$

9 with

$$(13) \quad \begin{aligned} C_{i,j} : \quad \mathbb{L}^{\otimes(j+1)} &\rightarrow \mathbb{L}^{\otimes j} \\ C_{i,j+1} F_{j+1}^N &= \text{Tr}^{j+1} (V_{i,j+1} F_{j+1}^N), \end{aligned}$$

10 where  $\text{Tr}^{j+1}$  is the partial trace with respect to the  $(j+1)$ th variable, as in (2).

11 Note that, for all  $i \leq j = 1, \dots, N$ ,

$$(14) \quad \|T_j\| \leq j^2 \frac{\|V\|_{L^\infty}}{\hbar}, \quad \text{and } \|C_{i,j+1}\| \leq \frac{\|V\|_{L^\infty}}{\hbar}.$$

12 (meant for  $\|T_j\|_{\mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j}}$  and  $\|C_{i,j+1}\|_{\mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}}$  in accordance with (7)).

1 The Hartree equation is

$$(15) \quad i\hbar\partial_t F = \left[-\frac{\hbar^2}{2}\Delta + V_F(x), F\right], \quad F(0) \in \mathcal{D}_1,$$

2 where  $V_F(x) = \int_{\mathbf{R}^d} V(x-y)F(y, y)dy$ ,  $F(y, y')$  being the integral kernel of  $F$ .

3 Note that (15) reads also

$$(16) \quad \partial_t F = K^1 F + Q(F, F),$$

4 with

$$(17) \quad Q(F, F) = \text{Tr}^2(V_{1,2}(F \otimes F)).$$

5 Since  $V$  is bounded, (15) has for all time a unique solution  $F(t) > 0$  and  $\|F(t)\|_1 = 1$   
6 (see again [28] and [1]).

7 In order to define the correlation error in an easy way, we need a bit more of notations  
8 concerning the variables of integral kernels.

9 For  $i \leq j = 1, \dots, N$ , we define the variables  $z_i = (x_i, x'_i)$ , and  $Z_j = (z_1, \dots, z_j)$ . For  
10  $\{i_1, \dots, i_k\} \subset \{1, \dots, j\}$ , we denote by  $Z_j^{\{i_1, \dots, i_k\}} \in \mathbf{R}^{2(j-k)d}$ , the vector  $Z_j := (z_1, \dots, z_j)$   
11 after removing the components  $z_{i_1}, \dots, z_{i_k}$ .

12 **Definition 2.1.** For any  $j = 1, \dots, N$ , we define the correlation error  $E_j \in \mathbb{L}^{\otimes j}$  by its  
13 integral kernel

$$(18) \quad E_j(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} (-1)^k F(z_{i_1}) \dots F(z_{i_k}) F_{j-k}^N(Z_j^{\{i_1, \dots, i_k\}}).$$

14 By convention and consistently we set

$$(19) \quad F_0^N = \|F\|_1 = 1, E_0 := 1 \in \mathbb{L}^{\otimes 0} := \mathbf{C}.$$

15 In [26] it was shown that (18) is inverted by the following equality:

$$(20) \quad F_j^N(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(z_{i_1}) \dots F(z_{i_k}) E_{j-k}(Z_j^{\{i_1, \dots, i_k\}}), \quad j = 0, \dots, N.$$

16 i.e.  $F_j^N$  is the operator of integral kernel given by (20).

17 Theorem 2.4, Theorem 2.1 and Corollary 2.2 in [26] state the following facts, among  
18 others.



1 The kinetic errors  $E_j$ ,  $j = 1, \dots, N$ , satisfy the system of equations

$$(21) \quad \begin{aligned} \partial_t E_j &= \left( K^j + \frac{1}{N} T_j \right) E_j + D_j E_j \\ &+ D_j^1 E_{j+1} + D_j^{-1} E_{j-1} + D_j^{-2} E_{j-2}, \end{aligned}$$

2 where the operators  $D_j, D_j^1, D_j^{-1}, D_j^{-2}$ ,  $j = 0, \dots, N$ , are defined at the beginning of the  
3 Section 4, formulas (42)-(45).

4 We note that the operators  $D_j^\alpha$ ,  $\alpha = 1, -1, -2$  map functions of  $j + \alpha$  variables into  
5 functions of  $j$  variables.

6 **Theorem 2.2** (out of Theorem 2.2. and Corollary 2.3 in [26]).

7 *Let  $E_j(0)$  satisfy for some  $C_0 > 1$*

$$(22) \quad \|E_j(0)\|_1 \leq C_0^j \left( \frac{j}{\sqrt{N}} \right)^j, \quad j \geq 1.$$

8 *Then, for all  $t > 0$  and all  $j = 1, \dots, N$ , one has*

$$(23) \quad \|E_j(t)\|_1 \leq \left( C_2 e^{\frac{C_1 t \|V\|_{L^\infty}}{h}} \right)^j \left( \frac{j}{\sqrt{N}} \right)^j, \quad j \geq 1.$$

9 *for some  $C_1 > 0$ ,  $C_2 \geq 1$  explicit (see Theorem 2.2 in [26]),*

10 *Let us suppose moreover that the initial data for (1) is  $F^N(0) = F(0)^{\otimes N}$  and  $F(t)$  is*  
11 *the solution of (15) with initial data  $F(0)$ . Then*

$$(24) \quad \|F_j^N(t) - F(t)^{\otimes j}\|_1 \leq D_2 e^{\frac{D_1 t \|V\|_{L^\infty}}{h}} \frac{j^2}{N},$$

12 *where  $D_2 = \sup\{B_2, (eC_0)^2\}$ ,  $B_1 = \sup\{B_1, 2C_1\}$ ,  $B_1, B_2$  being taken in Theorem 2.2 in*  
13 *[26] at the value  $B_0 = 0$ ).*

### 14 3. ASYMPTOTIC EXPANSION AND MAIN RESULT

15

16 Two questions arise naturally:

17 (1) are the estimates (23) sharp?

18 (2) Could (24) be improved with a r.h.s. of any order we wish?

Of course, defining  $F_j^{N,n}(t)$ ,  $n = 1, \dots, j$ , by its integral kernel  $F_j^{N,n}(Z_j) = \sum_{k=j-n}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(z_{i_1}) \dots F(z_{i_k}) E_{j-k}(Z_j^{\{i_1, \dots, i_k\}})$ , we get by (20), (23) and (24) that, for any  $n \leq j$ ,  $\|F_j^N(t) - F_j^{N,n}\| = O(N^{-(n+1)/2})$ . However one cannot go further in the approximation that is, in any case useless without the knowledge of the true  $E_j$ s.

As we will see later on, one of our main results states that, not only estimates (23) are true, but  $\mathcal{E}_j(t) := N^{j/2} E_j(t)$  has a full asymptotic expansion in positive powers of  $(\frac{1}{N})^{\frac{1}{2}}$

More precisely we will show that, under the hypothesis (22) on the initial data, and for all time  $t$  and all  $j = 1, \dots, N$ , there exist sequences  $(\mathcal{E}_j^\ell(t))_{\ell \in \mathbf{N}}$  such that

$$(25) \quad \mathcal{E}_j(t) \sim \sum_{\ell=0}^{\infty} \mathcal{E}_j^\ell(t) N^{-\ell/2}$$

(in the sense that for all  $k \in \mathbf{N}$ ,  $\|\mathcal{E}_j(t) - \sum_{\ell=0}^k \mathcal{E}_j^\ell(t) N^{-\ell/2}\|_1 = o(N^{-k/2})$ ).

In fact part of our results will deal with coefficients  $\mathcal{E}_j^\ell(t)$  which will happen to have a (bounded) dependence<sup>2</sup> on  $N$ . To avoid any ambiguity with respect to this fact, we precise the meaning of  $\sim$  in (25) we consider in this paper:

$$\mathcal{E} \sim \sum_{k=0}^{\infty} \mathcal{E}_N^k N^{-\frac{k}{2}}$$

$$\Updownarrow$$

$\forall n \geq 0, \exists N_n \in \mathbf{N}, E_n, C_n > 0$  such that

$$\forall N \geq N_n, |\mathcal{E}_N^n| \leq E_n \text{ and } \|\mathcal{E} - \sum_{k=0}^n \mathcal{E}_N^k N^{-\frac{k}{2}}\|_1 \leq C_n N^{-\frac{k+1}{2}}.$$

Of course, whatever is the dependence on  $N$  of the coefficients  $\mathcal{E}^k$ , the important point is to construct an approximation of  $\mathcal{E}_j(t)$  valid up to any order in  $N^{-\frac{1}{2}}$ .

The coefficients  $\mathcal{E}_j^\ell$  can be determined as solutions of a partial differential equation which can be solved recursively. More than that,  $\mathcal{E}_j^\ell(t)$  turn out to be explicitly computed in terms of a perturbative expansion, after the knowledge of the linearization of the mean-field equation (15) around the solution of (15) with initial condition  $F(0) = (F^N(0))_1$  which will be discussed in detail later on.

<sup>2</sup>Let us remark that this situation is standard in perturbation theory, e.g. in KAM theory where the well known Arnold cut-off introduces such a dependence in the perturbation parameter.

1 The starting point of our analysis is the evolution equation for  $\mathcal{E}_j(t)$ , obtained by  
 2 the substitution  $E_j = N^{-j/2}\mathcal{E}_j$  in (21):

$$(26) \quad \partial_t \mathcal{E}_j = H_j \mathcal{E}_j + N^{-\frac{1}{2}} \Delta_j^+ \mathcal{E}_{j+1} + N^{-\frac{1}{2}} \Delta_j^- \mathcal{E}_{j-1} + \Delta_j^- \mathcal{E}_{j-2}$$

3 where

$$(27) \quad \begin{cases} H_j &= K^j + \frac{T_j}{N} + D_j(t) \\ \Delta_j^+ &= D_j^1 \\ \Delta_j^- &= ND_j^{-1} \\ \Delta_j^- &= ND_j^{-2} \end{cases}$$

4 the  $D_j$ 's being given by formulas (42)-(45) below. It follows that  $H_j, \Delta_j^+, \Delta_j^-, \Delta_j^-$  act on  
 5 functions of  $j, j+1, j-1, j-2$  particles, namely  $\mathbb{L}^{\otimes j}, \mathbb{L}^{\otimes j+1}, \mathbb{L}^{\otimes j-1}, \mathbb{L}^{\otimes j-2}$ .

6 Inserting the expansion (25) into (26) we find for  $(\mathcal{E}_j^k(t))_{j=1, \dots, N, k=0, \dots}$  the following  
 7 sequence of equations

$$(28) \quad \partial_t \mathcal{E}_j^k = H_j \mathcal{E}_j^k + \Delta_j^- \mathcal{E}_{j-2}^k + \Delta_j^+ \mathcal{E}_{j+1}^{k-1} + \Delta_j^- \mathcal{E}_{j-1}^{k-1}$$

8 with the convention,

$$(29) \quad \mathcal{E}_0^k(t) = \delta_{k,0}, \quad \mathcal{E}_{-1}^k(t) = \mathcal{E}_{-2}^k(t) = \mathcal{E}_j^{-1}(t) = 0$$

9 and the ones inherited from (46).

10 (28) can be solved recursively. Indeed we realize that

$$(30) \quad \partial_t \mathcal{E}_j^0 = H_j \mathcal{E}_j^0 + \Delta_j^- \mathcal{E}_{j-2}^0$$

11 can be solved by iteration in  $j$  (note that  $\mathcal{E}_1^0(t) = 0$ ). Thus knowing  $\mathcal{E}_j^0$ , we can also  
 12 solve

$$(31) \quad \partial_t \mathcal{E}_j^1 = H_j \mathcal{E}_j^1 + \Delta_j^- \mathcal{E}_{j-2}^1 + \Delta_j^+ \mathcal{E}_{j+1}^0 + \Delta_j^- \mathcal{E}_{j-1}^0.$$

13 by iteration in  $j$  and so on.

14 However we will see below that the computation of  $\mathcal{E}_j^k(t)$  depends actually only on  $\mathcal{E}_{j'}^{k'}$   
 15  $k' \leq k, j' \leq j+k$  through a number of operations depending only on  $j$  and  $k$  independent  
 16 of  $N$ .

17 We now introduce the two-parameter semigroup defined by

$$(32) \quad \begin{aligned} \partial_t U_j(t, s) &= H_j(t) U_j(t, s). \\ U_j(s, s) &= I. \end{aligned}$$

1 The existence of  $U_j(t)$  is guaranteed by the classical theory of perturbation of semi-  
 2 group,  $K^j$  generating an isometric semigroup and  $\frac{T_j}{N}$  and  $D_j(t)$  being bounded. More-  
 3 over, let us define  $U(t, s)$  as the linearisation of the Hartree flow around  $F(t)$ , namely

$$(33) \quad \begin{aligned} \partial_t U(t, s) &= (K_1 + \Delta_1)U(t, s), \quad \Delta_1 := Q(\cdot, F(t)) + Q(F(t), \cdot) \\ U(s, s) &= I. \end{aligned}$$

4 We will see in Section 4.3 that  $U_j(t, s)$ , when acting on symmetric states, is a per-  
 5 turbation of  $U(t, s)^{\otimes j}$ , and can be explicitly computed out of  $U(t, s)$  by a convergent,  
 6 entire, expansion in  $\frac{j^2 \|V\|}{N \hbar}$ . In particular, we'll see that expansions of  $U_j(t, s)$  up to any  
 7 power of  $\frac{1}{N}$  can be explicitly obtained under the only knowledge of the linearisation of  
 8 the Hartree flow around  $F(t)$ .

9 Using of this semigroup  $U_j(t, s)$  leads immediately to solving (28) by the family of  
 10 relations:

$$(34) \quad \left\{ \begin{array}{l} \mathcal{E}_j^k(t) = U_j(t, 0)\mathcal{E}_j^k(0) \\ \quad + \int_{s=0}^t U_j(t, s)(\Delta_j^- \mathcal{E}_{j-2}^k(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(s))ds, \\ \mathcal{E}_0^k(t) = \delta_{k,0}, \\ \Delta_1^- (\mathcal{E}_0^0) := -Q(F, F), \\ \Delta_2^- (\mathcal{E}_0^0) := T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F), \\ \mathcal{E}_{-1}^k(t) = \mathcal{E}_{-2}^k(t) = \mathcal{E}_j^{-1}(t) = 0 \text{ by convention.} \end{array} \right.$$

11 We are now in position of stating the main results of the present paper.

12 **Theorem 3.1.** *Consider for  $j = 0, \dots, N$ ,  $k = 0, \dots$ ,  $t \geq 0$  the system of recursive*  
 13 *relations (34). Then, for all  $t, s \in \mathbb{R}$ , the knowledge of  $U_j(t, s)$  (see Remark 3.6 below)*  
 14 *makes true the following*

- 15 (i)  $\mathcal{E}_j^k(t)$  is explicitly determined by  $\mathcal{E}_{j'}^{k'}(0)$ ,  $j' \leq j + k$ ,  $k' \leq k$
- 16 (ii)  $\mathcal{E}_j^k(t) = 0$  if  $\mathcal{E}_j^k(0) = 0$ , for all  $j, k$ ,  $j + k$  odd
- 17 (iii) Let  $\mathcal{E}_j(t)$  be the solution of (28) with the condition  $\|\mathcal{E}_j(0)\| \leq (Aj^2)^{j/2}$  for some  
 18  $A > 1$ . Let us take moreover  $\mathcal{E}_j^k(0) = \delta_{k,0}\mathcal{E}_j(0)$  (concerning this hypothesis, see  
 19 Remark 3.6 below). Then the following estimate holds true

$$(35) \quad \|\mathcal{E}_j(t) - \sum_{k=0}^{2n} N^{-k/2} \mathcal{E}_j^k(t)\|_1 \leq L_{2n}(t) N^{-n-\frac{1}{2}} (L'_{2n}(t) j^2)^{j/2},$$

20 where  $L_k(t), L'_k(t)$  are defined in (57) below and satisfy, as  $k, |t| \rightarrow \infty$ ,

$$(36) \quad \log L_k(t) = \frac{3k}{2} (\log k + \frac{|t| \|V\|_\infty}{\hbar}) + O(k + \frac{|t| \|V\|_\infty}{\hbar}) \text{ and } \log L'_k(t) = O(k + \frac{|t| \|V\|_\infty}{\hbar}).$$

1 The proof of the theorem is given in Sections 4.1 and 4.2.

2 Note that the estimate (36) gives that  $L_{en}(t) \sim (2n)^{3n}$  as  $n \rightarrow \infty$  so that, as expected  
 3 in perturbation theory, the bound (35) does not provide convergence of the series  
 4  $\sum_{k=0}^{\infty} N^{-k/2} \mathcal{E}_j^k(t)$ .

5 Let us set, for  $j = 1, \dots, N$ ,  $n = 0, \dots$ ,  $\mathcal{E}_j^k(0) = \delta_{k,0} \mathcal{E}_j(0)$  and

$$(37) \quad E_j^n(t) = \sum_{k=0}^{2n} N^{-\frac{j+k}{2}} \mathcal{E}_j^k(t)$$

6 and  $F_j^{N,n}(t)$  the operator of integral kernel  $F_j^{N,n}(t)(Z_j)$  defined by

$$(38) \quad F_j^{N,n}(t)(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_k}) E_{j-k}^n(Z_j^{\{i_1, \dots, i_k\}}),$$

7 (that is (20) truncated at order  $n$ ).

8 **Theorem 3.2.** *Let  $F^N(t)$  the solution of the quantum  $N$  body system (1) with initial  
 9 datum  $F^N(0) = F^{\otimes N}$ ,  $F \in \mathcal{L}(L^2(\mathbf{R}^d))$ ,  $F \geq 0$ ,  $\text{Tr}F = 1$ , and  $F(t)$  the solution of the  
 10 Hartree equation (15) with initial datum  $F$ .*

11 *Then, for all  $n \geq 0$  and  $N \geq 4(e\sqrt{L'_{2n}(t)}j)^2$ ,*

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2L_{2n}(t)e\sqrt{L'_{2n}(t)}j}{\sqrt{N}}.$$

12 *Moreover the expansion of  $F_j^{N,n}(t)$  contains only integer powers of  $\frac{1}{N}$ .*

13 **Remark 3.3.** *The condition of factorization of the initial condition  $F^N(0) = F^{\otimes N}$ ,  
 14 equivalent to  $E_j(0) = \delta_{j,0}$ , is not necessary. It can be mildly modified by taking any  
 15  $E_j(0)$  satisfying (22) and the associated sequence  $\mathcal{E}_j(0)$ . We leave to the interested  
 16 reader the elaboration of the precise corresponding statements out of Theorem 3.1.*

17 *Proof.* The proof is similar to the one of Corollary 2.2 in [26].

18 The fact that  $E_j^n(t)$ , and therefore  $F_j^{N,n}(t)$  contains only integer powers of  $\frac{1}{N}$  comes  
 19 from the fact that the factorization of  $F^N(0)$  implies that  $\mathcal{E}_j^k(0) = \delta_{k,0} \delta_{j,0}$  and therefore  
 20  $\mathcal{E}_j^k(0) = \mathcal{E}_j^k(t) = 0$  for  $j+k$  odd.

1 Moreover

$$\begin{aligned}
& \|F_j^N(t) - F_j^{N,n}(t)\|_1 \\
& \leq \sum_{k=0}^j \binom{j}{j-k} \|E_k - E_k^n\|_1 \leq N^{-n-\frac{1}{2}} \sum_{k=1}^j \binom{j}{k} L_{2n}(t) \left( \frac{L'_{2n}(t)k^2}{N} \right)^{k/2} \\
& \leq N^{-n-\frac{1}{2}} L_{2n}(t) \sum_{k=1}^j j(j-1)\dots(j-k+1) \left( \frac{\sqrt{L'_{2n}(t)}}{\sqrt{N}} \right)^k \frac{k^k}{k!} \\
& \leq N^{-n-\frac{1}{2}} L_{2n}(t) \sum_{k=1}^j \left( \frac{j e \sqrt{L'_{2n}(t)}}{\sqrt{N}} \right)^k \leq N^{-n-\frac{1}{2}} \frac{2L_{2n}(t) e \sqrt{L'_{2n}(t)} j}{\sqrt{N}}
\end{aligned}$$

2 for  $N \geq 4(e\sqrt{L'_{2n}(t)}j)^2$  (we used that  $E_0(t) = E_0^n(t) = 1$  and  $\frac{k^k}{k!} \leq \frac{e^k}{\sqrt{2\pi k}}$ ).  $\square$

3 Let us remark that, under the hypothesis of Theorem 3.2, (37) gives that  $E_j^n(t) =$   
4  $O(N^{-2})$  for  $j > 2$ ,  $E_0^n(t) = 1$ ,  $E_1^n(t) = N^{-1}\mathcal{E}_1^1(t) + O(N^{-2})$  and  $E_2^n(t) = N^{-1}\mathcal{E}_2^0(t) +$   
5  $O(N^{-2})$ .

6 Therefore, keeping in  $F_j^{N,1}(t)$ , given by (38), only the terms  $k = j-1, j-2$ , and  
7 defining  $G_1^{-1}(t) = \mathcal{E}_1^1(t)$ ,  $G_2^{-1}(t) = \mathcal{E}_2^0(t)$  and  $G_j^{-1}(t)$ ,  $j > 2$ , by its integral kernel

$$\begin{aligned}
G_j^{-1}(t)(Z_j) &= \sum_{1 \leq i_1 < \dots < i_{j-2} \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_{j-2}}) \mathcal{E}_2^0(Z_j^{\setminus \{i_1, \dots, i_{j-2}\}}) \\
&+ \sum_{1 \leq i_1 < \dots < i_{j-1} \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_{j-1}}) \mathcal{E}_1^1(Z_j^{\setminus \{i_1, \dots, i_{j-1}\}}),
\end{aligned}$$

8 we get, by Theorem 3.2, that

$$F_j^N(t) - F(t)^{\otimes j} = \frac{1}{N} G_j^{-1}(t) + O(N^{-3/2}).$$

9 For  $j = 1, 2$ ,  $G_j^{-1}(t) \neq 0$  by Lemma 4.6 below.

For  $j > 2$ , let us pick-up a neighbourhood  $\Omega \subset \mathbf{R} \times \mathbf{R}^{2d}$  such that the integral kernel  $|F(t)(z)| \geq D > 0$ . We get that,  $\forall (t, z) \in \Omega$ ,  $j > 2$ ,

$$G_j^{-1}(t)(z, \dots, z) = F(t)(z)^{j-2} \left( \binom{j}{j-2} \mathcal{E}_2^0(t)(z, z) + \binom{j}{j-1} F(t)(z) \mathcal{E}_1^1(t)(z) \right),$$

so that  $G_j^{-1}(t)(z, \dots, z) = 0$  would imply that  $F(t)(z) \mathcal{E}_1^1(t)(z) = -(j-2) \mathcal{E}_2^0(t)(z, z)$ , incompatible with (28). Therefore, for all  $j = 1, \dots$ ,

$$\|F_j^N(t) - F(t)^{\otimes j}\|_1 = \frac{1}{N} \|G_j^{-1}(t)\|_1 + O(N^{-3/2}) \geq CN^{-1}, \quad C > 0, \text{ for } N \text{ large enough,}$$

10 and we get the following by-product.

11 **Corollary 3.4.** *The rate of convergence to the mean-field limit in  $\frac{1}{N}$  is optimal.*

As we mentioned already,  $U_j(t, s)$  is given by a convergent perturbative expansion out of  $U(t, s)^{\otimes j}$  where  $U(t, s)$  is the flow generated by the linearization of the Hartree equation around its solution  $F(t)$ .

More precisely, let  $\tilde{\Delta}_j = \frac{1}{N}T_j + D_j - \Delta_j$  and, for  $n \in \mathbf{N}$ , let us define the truncated Dyson expansion of  $U_j(t, s)$  as

$$(39) \quad U_j^n(t, s) = \sum_{k=0}^{2n+1} \int_s^t dt_1 \dots \int_s^{t_{2n}} dt_{2n+1} U(t, t_1)^{\otimes j} \tilde{\Delta}_j(t_1) U(t_1, t_2)^{\otimes j} \tilde{\Delta}_j(t_2) \dots U(t_{2n}, t_{2n+1})^{\otimes j}.$$

For  $\alpha \in (0, \frac{1}{2})$ ,  $n \in \mathbf{N}$ , let us define  $F_j^{N, n, \alpha}(t)$  as the operator of integral kernel  $F_j^{N, n, \alpha}(t)(Z_j)$  given by

$$(40) \quad F_j^{N, n, \alpha}(t)(Z_j) = \sum_{k=j-\lceil \frac{n+1/2}{\alpha} \rceil}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_k}) E_{j-k}^{n, n}(Z_j^{\{i_1, \dots, i_k\}}),$$

with the convention  $E_{j-k} = 0$  for  $j - k < 0$ , and  $E_j^{n, n}(t) := N^{-\frac{j}{2}} \sum_{k=0}^{2n} N^{-\frac{k}{2}} \mathcal{E}_j^{k, n}(t)$  where  $\mathcal{E}_j^{k, n}(t)$  are the explicit solutions of the recurrence relations

$$(41) \quad \begin{cases} \mathcal{E}_j^{k, n}(t) &= U_j^n(t, 0) \mathcal{E}_j^{k, n}(0) \\ &+ \int_{s=0}^t U_j^n(t, s) (\Delta_j^- \mathcal{E}_{j-2}^{k, n}(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1, n}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1, n}(s)) ds, \\ \mathcal{E}_j^k(0) &= \delta_{k,0} \mathcal{E}_j(0) \end{cases}$$

with the same conventions as in (34) ( $U_j^n(t, s)$  is defined in (39)).

Obviously the solution of (41) satisfies the items (i) – (ii) of Theorem 3.1 and the statements of Proposition 4.1.

**Theorem 3.5.** *Let  $\alpha \in (0, \frac{1}{2})$  and  $C > 0$ . Then, under the same hypothesis as in Theorem 3.2, one has, for any  $n \in \mathbf{N}$ ,  $t \in \mathbb{R}$  and  $j \leq CN^{\frac{1}{2}-\alpha}$ ,*

$$\|F_j^N(t) - F_j^{N, n, \alpha}(t)\|_1 \leq M_{n, \alpha, C, t} N^{-n-\frac{1}{2}}$$

for all  $N > N_{n, \alpha, C, t}$  ( $M_{n, \alpha, C, t}$  and  $N_{n, \alpha, C, t}$  are given in (85)).

Note that the expansion of  $F_j^{N, n, \alpha}(t)$  contains again only integer powers of  $\frac{1}{N}$  and, by the construction of  $U_j^n$  and Proposition 4.1, its explicit computation involves a finite number of operations depending only on  $j$  and  $n$  (and not in  $N$ ) and the only knowledge of  $F(t)$  and the solution of the Hartree equation linearized around it.

The proof of the theorem is given in Section 4.3.

1 **Remark 3.6.** [Nature of the expansion in  $\frac{1}{N}$ ] In the asymptotic expansion  $E_j(t) \sim$   
2  $\sum_{k=\lfloor (j+1)/2 \rfloor}^{\infty} c_k^j(t) N^{-k}$  the coefficients  $c_k^j(t)$ , such as each coefficient  $\mathcal{E}_j^k(t)$ , depend on  $N$   
3 as well: first by the dependence in  $N$  of  $\Delta_j^+ = (1 - \frac{j}{N})C_{j+1}$  and also by the dependence  
4 in  $N$  of  $U_j(t, s)$  defined by (32). Moreover, since the condition  $\|\mathcal{E}_j(0)\|_1 \leq (Aj^2)^{j/2}$   
5 in Theorem 3.1 is a condition only on the size, all the results of this paper hold true  
6 under any dependence of  $\mathcal{E}_j(0)$ , that is of  $F^N(0)$ , on  $N$ . In particular, this allows to  
7 reincorporate in  $\mathcal{E}_j(0)$  all the terms  $\mathcal{E}_j^k(0)N^{-1/2}, k = 1 \dots$ , as done in the second item  
8 of Theorem 3.1.

#### 9 4. PROOFS OF THEOREMS 3.1 AND 3.5

10 Let us first recall from [26] the expression of the ingredients present in equation (21):  
11 For any operator  $G \in \mathbb{L}^{\otimes n}, n = 1, \dots, N$ ,  $G(Z_n)$  denotes its integral kernel and, for  
12 any function  $F(Z_n), n = 1, \dots, N$ ,  $\overline{F(Z_n)}$  is defined as the operator on  $\mathbb{L}^{\otimes n}$  of integral  
13 kernel  $F(Z_n)$ . Moreover  $J := \{1, \dots, j\}$ .

$$(42) \quad \begin{aligned} D_j &: \mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j} \\ E_j &\mapsto \frac{N-j}{N} \sum_{i \in J} C_{i,j+1} \left( \overline{F(z_i)E_j(Z_{j+1}^{\{i\}})} + \overline{F(z_{j+1})E_j(Z_j)} \right) \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l)E_j(Z_{j+1}^{\{l\}})} \end{aligned}$$

$$(43) \quad \begin{aligned} D_j^1 &: \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j} \\ E_{j+1} &\mapsto \frac{N-j}{N} C_{j+1} E_{j+1} \end{aligned}$$

$$(44) \quad \begin{aligned} D_j^{-1} &: \mathbb{L}^{\otimes(j-1)} \rightarrow \mathbb{L}^{\otimes j} \\ E_{j-1} &\mapsto \frac{1}{N} \sum_{i,r \in J} T_{i,r} \overline{F(z_i)E_{j-1}(Z_j^{\{i\}})} - \frac{j}{N} \sum_{i \in J} \overline{Q(F, F)(z_i)E_{j-1}(Z_j^{\{i\}})} \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l)F(z_{j+1})E_{j-1}(Z_j^{\{l\}})} \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l)F(z_i)E_{j-1}(Z_{j+1}^{\{i,l\}})} \end{aligned}$$



1 and

$$(45) \quad \begin{aligned} D_j^{-2} &: \mathbb{L}^{\otimes(j-2)} \rightarrow \mathbb{L}^{\otimes j} \\ E_{j-2} &\mapsto \frac{1}{N} \sum_{i,s \in J} \overline{T_{i,s} F(z_i) F(z_r) E_{j-2}(Z_j^{\{i,r\}})} \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} \overline{Q(F, F)(z_i) F(z_l) E_{j-2}(Z_l^{\{i,l\}})}, \end{aligned}$$

2 where, by convention,

$$(46) \quad \begin{cases} D_N^1 := D_1^{-2} := 0 \\ D_1^{-1}(E_0) := -\frac{1}{N} Q(F, F), \\ D_2^{-2}(E_0) := \frac{1}{N} (T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F)). \end{cases}$$

3 In (42)-(45),  $F(z)$  is meant as being the integral kernel of  $F(t)$ , solution of the Hartree  
4 equation (15).

5 **4.1. Recursive construction and proof of Theorem 3.1 (i)-(ii).** Specializing (34)

6 to  $k = 0$ , we get immediately that (we recall  $\mathcal{E}_0^0(t) = 1$ )

$$(47) \quad \mathcal{E}_j^0(t) = U_j(t, 0) \mathcal{E}_j^0(0) + U_j(t, 0) \int_0^t U_j(0, s) \Delta_j^- \mathcal{E}_{j-2}^0(s) ds, j \geq 1,$$

7 with the convention  $\mathcal{E}_l^k = 0, l < 0$ , and  $\Delta_2^- (\mathcal{E}_0^0) := T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F)$ .

8 Therefore, for  $j = 1, \dots, N, t \in \mathbf{R}$ , the knowledge of  $U_j(t, s), |s| \leq |t|$ , and  $\mathcal{E}_{j'}^0(0)$  for  
9  $j' = 1, \dots, j$  guarantees the knowledge of  $\mathcal{E}_{j'}^0(t), t \in \mathbf{R}, j' \leq j$ . We write this fact as

$$(48) \quad (\mathcal{E}_{j'}^0(0))_{j'=1, \dots, j} \rightsquigarrow (\mathcal{E}_{j'}^0(t))_{t \in \mathbf{R}, j'=1, \dots, j}$$

10 Since  $\mathcal{E}_{-1}^k(t) = 0$  by convention and  $\mathcal{E}_0^k(t) = 0$  for  $k \geq 1$  since  $E_0(t) := 1$ , we find after

11 (48) that  $\mathcal{E}_1^1(t)$  and  $\mathcal{E}_2^1(t)$  are determined by  $\mathcal{E}_1^1(0)$  and  $\mathcal{E}_2^1(0)$ . Therefore, by (34),

12  $\mathcal{E}_j^1(t), j = 1, \dots, N$  are determined by  $(\mathcal{E}_j^1(0))_{j=1, \dots, N}$ , and determine  $\mathcal{E}_1^2(t)$  and  $\mathcal{E}_2^2(t)$ .

13 These ones determine in turn all the  $\mathcal{E}_j^2(t), j = 1, \dots, N$  and so on.

14 Therefore, the knowledge of  $(\mathcal{E}_{j'}^{k'}(s))_{|s| \leq |t|, k' \leq k-1, j'=1, \dots, j+1}$  and  $\mathcal{E}_j^k(0)$  guarantees for all

15  $j, k$ , by induction, the knowledge of  $\mathcal{E}_j^k(t)$ . Thus

$$((\mathcal{E}_j^k(0), (\mathcal{E}_{j'}^{k'}(s))_{|s| \leq |t|, k' \leq k-1, j'=1, \dots, j+1}) \rightsquigarrow (\mathcal{E}_{j'}^{k'}(s))_{|s| \leq |t|, k' \leq k, j'=1, \dots, j}.$$

16 Supposing now  $(\mathcal{E}_{j'}^{k'})_{k' \leq k, j' \leq j}$  known,

$$(\mathcal{E}_{j'}^{k'}(s))_{s \leq t, k' \leq k-2, j'=1, \dots, j+2} \rightsquigarrow (\mathcal{E}_{j'}^{k'}(s))_{s \leq t, k' \leq k-1, j'=1, \dots, j+1} \rightsquigarrow \mathcal{E}_j^k(t).$$

17 and by iteration

$$(\mathcal{E}_{j'}^0(s))_{s \leq t, j'=1, \dots, j+k} \rightsquigarrow \mathcal{E}_j^k(t)$$

1 so that, by (48),

$$(\mathcal{E}_{j'}^0(0))_{j'=1,\dots,j+k} \rightsquigarrow \mathcal{E}_j^k(t).$$

2 We just proved the following result.

3 **Proposition 4.1.** *For any  $j = 1, \dots, N, t \geq 0, k = 0, \dots$ , let  $\mathcal{E}_j^k(t)$  be the solution of (34).  
4 Then  $\mathcal{E}_j^k(t)$  is determined by the values  $\mathcal{E}_{j'}^{k'}(0)$  for  $0 \leq k' \leq k, 1 \leq j' \leq j + k$ . Moreover  
5 the number of operations leading to  $\mathcal{E}_j^k(t)$  depends on  $j$  and  $k$ , but is independent of  
6  $N$ .*

7 Formula (34) will give easily the following result.

8 **Proposition 4.2.** *Let  $\mathcal{E}_{j'}^{k'}(0) = 0$  for  $j' \leq j, k' \leq k, j' + k'$  odd. Then  $\mathcal{E}_j^k(t) = 0$  for  $j + k$   
9 odd.*

10 *Proof.* Let us suppose  $\mathcal{E}_{j'}^{k'}(0) = 0$  for  $j' \leq j, k' \leq k, j' + k'$  odd. By (34) we have that  
11  $\mathcal{E}_1^0(t) = 0$  since  $\mathcal{E}_1^0(0) = 0$ . Therefore, by induction on  $j$  in (34),  $\mathcal{E}_j^0(t) = 0$  for all  $j$  odd.  
12 Since  $\mathcal{E}_0(t) := 1, \mathcal{E}_0^j(t) = 0, j > 0$ , so that  $\mathcal{E}_2^1(t) = 0$  by (34) and therefore  $\mathcal{E}_j^1(t) = 0$  for  
13 all  $j$  even, since then  $j \pm 1$  is odd, and therefore  $\mathcal{E}_{j \pm 1}^0(s) = 0$ . This gives  $\mathcal{E}_1^2(t) = 0$  by  
14 (34) and so on.  $\square$

15 Propositions 4.1 and 4.2 are precisely the contents of the two first items of Theorem  
16 3.1.

17 **4.2. Estimates and proof of Theorem 3.1 (iii).** In order to simplify the expres-  
18 sions, we will first suppose that  $\frac{\|V\|_{L^\infty}}{\hbar} = 1$ .

19 Note that one has therefore the following estimates:

$$(49) \quad \|D_j\|, \|\Delta_j^1\| \leq j \text{ and } \|\Delta_j^-\|, \|\Delta_j^{\bar{-}}\|, \|\Delta_1^-(\mathcal{E}_0)\|, \|\Delta_2^{\bar{-}}(\mathcal{E}_0)\| \leq j^2.$$

20 Let us first recall that (21) expressed on the  $\mathcal{E}_j$ s reads

$$(50) \quad \partial_t \mathcal{E}_j = H_j \mathcal{E}_j + N^{-\frac{1}{2}} \Delta_j^+ \mathcal{E}_{j+1} + N^{-\frac{1}{2}} \Delta_j^- \mathcal{E}_{j-1} + \Delta_j^{\bar{-}} \mathcal{E}_{j-2}$$

21 and that (22) and (23) can be rephrased as

$$(51) \quad \|\mathcal{E}_j(0)\| \leq (A_j^2)^{j/2} \implies \|\mathcal{E}_j(t)\| \leq (A_t j^2)^{j/2}, \quad A_t = C' A e^{Ct}$$

22 for some explicit constants  $C', C$ .

23 Furthermore for the reader's convenience we recall the equations for  $\mathcal{E}_j^k(t)$

$$(52) \quad \partial_t \mathcal{E}_j^k(t) = H_j(t) \mathcal{E}_j^k(t) + \Delta_j^{\bar{-}} \mathcal{E}_{j-2}^k(t) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(t) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(t)$$

1 . Calling  $\bar{\mathcal{E}}_j^n = \sum_{k=0}^n N^{-k/2} \mathcal{E}_j^k$ , one easily check that

$$(53) \quad \begin{aligned} \partial_t \bar{\mathcal{E}}_j^n(t) &= H_j(t) \bar{\mathcal{E}}_j^n(t) + \Delta_j^- \bar{\mathcal{E}}_{j-2}^n(t) + N^{-\frac{1}{2}} (\Delta_j^+ \bar{\mathcal{E}}_{j+1}^n(t) + \Delta_j^- \bar{\mathcal{E}}_{j-1}^n(t)) \\ &\quad - N^{-\frac{n+1}{2}} (\Delta_j^+ (\mathcal{E}_{j+1}^n(t)) + \Delta_j^- (\mathcal{E}_{j-1}^n(t))). \end{aligned}$$

2 Therefore  $R_j^n := \mathcal{E}_j - \bar{\mathcal{E}}_j^n$  satisfies the equation

$$(54) \quad \begin{aligned} \partial_t R_j^n(t) &= H_j(t) R_j^n(t) + \Delta_j^- R_{j-2}^n(t) + N^{-\frac{1}{2}} (\Delta_j^+ R_{j+1}^n(t) + \Delta_j^- R_{j-1}^n(t)) \\ &\quad + N^{-\frac{n+1}{2}} (\Delta_j^+ (\mathcal{E}_{j+1}^n(t)) + \Delta_j^- (\mathcal{E}_{j-1}^n(t))) \end{aligned}$$

Let us define the mapping

$$\mathbb{U}_j(t, s) : (\mathcal{E}_j(s))_{j=1, \dots, N} \mapsto \mathbb{U}_j(t, s) ((\mathcal{E}_j(s))_{j=1, \dots, N}) := \mathcal{E}_j(t).$$

3 In other words, the family  $(\mathbb{U}_j(t, s))_{j=1, \dots, N}$  solves the equation:

$$\begin{aligned} \partial_t \mathbb{U}_j(t, s) &= H_j(t) \mathbb{U}_j(t, s) + \Delta_j^- \mathbb{U}_{j-2}(t, s) \\ &\quad + N^{-\frac{1}{2}} (\Delta_j^+ \mathbb{U}_{j+1}(t, s) + \Delta_j^- \mathbb{U}_{j-1}(t, s)), \\ \mathbb{U}_j(s, s) &= I. \end{aligned}$$

4 Hence, the solution of (54) reads

$$(55) \quad \begin{aligned} R_j^n(t) &= \mathbb{U}_j(t, 0) ((R_j^n(0))_{j=1, \dots, N}) \\ &\quad + N^{-\frac{n+1}{2}} \int_0^t \mathbb{U}_j(t, s) ((\Delta_j^+(s) \mathcal{E}_{j+1}^n(s)) + \Delta_j^-(s) \mathcal{E}_{j-1}^n(s))_{j=1, \dots, N} ds \end{aligned}$$

5 with again the same convention on negative indices.

6 By hypothesis,  $R_j^n(0) = 0$  since  $\mathcal{E}_j^n(0) = \delta_{n,0} \mathcal{E}_j^0(0)$ .

7 Let us suppose now that

$$(56) \quad \|\Delta_j^+(\mathcal{E}_{j+1}^n(s)) + \Delta_j^-(\mathcal{E}_{j-1}^n(s))\|_1 \leq C_n(s) (C'_n(s) j^2)^{j/2}, \quad |s| \leq |t|,$$

for two increasing functions  $C_n(s), C'_n(s), C'_n(s) \geq 1$ , Then (51) implies that

$$\|\mathbb{U}_j(t, s) ((\Delta_j^+(s) \mathcal{E}_{j+1}^n(s)) + \Delta_j^-(s) \mathcal{E}_{j-1}^n(s))_{j=1, \dots, N}\|_1 \leq C_n(s) (C' C'_n(s) e^{C|t|} j^2)^{j/2},$$

8 and thus

$$\begin{aligned} \|\mathcal{E}_j(t) - \bar{\mathcal{E}}_j^n(t)\|_1 &= \|R_j^n(t)\|_1 \\ &= \left\| \int_0^t \mathbb{U}_j(t, s) ((\Delta_j^+(s) \mathcal{E}_{j+1}^n(s)) + \Delta_j^-(s) \mathcal{E}_{j-1}^n(s))_{j=1, \dots, N} ds \right\|_1 \\ &\leq N^{-\frac{n+1}{2}} L_n(t) (L'_n(t) j^2)^{j/2}, \end{aligned}$$

9 where

$$(57) \quad L_n(t) = t C_n(t) \text{ and } L'_n(t) = C' C'_n(t) e^{C|t|}.$$

1 It remains to prove an estimate like (56).

2 We will obtain such an estimate by iterating (34). We first remark that, since  $e^{K^j+T_j/N}$   
3 is unitary and  $\|D_j\| \leq j$ , the Gronwall Lemma gives that

$$(58) \quad \|U_j(t, s)\| \leq e^{j|t-s|}.$$

4 We will use

$$(59) \quad \prod_{i=0}^m e^{(j+i)(t_i-t_{i+1})} \leq e^{(j+m)|t_{m+1}-t_0|} \quad \text{for any } (t_i)_{i=0,\dots,m} \text{ (see [26]),}$$

$$(60) \quad \|\Delta^\pm\|, \|\Delta^\pm\| \leq j^2,$$

$$(61) \quad \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n = \frac{t^n}{n!}.$$

Let us remind that we have

$$\mathcal{E}_0^k(t) = \delta_{k,0}, \quad \mathcal{E}_j^k(0) = \delta_{k,0} \mathcal{E}_j(0)$$

5 together with the estimate  $\|\mathcal{E}_j(0)\| \leq (Aj^2)^{j/2}$ .

6 (52) reads:

$$(62) \quad \begin{cases} \mathcal{E}_j^0(t) &= U_j(t, 0)\mathcal{E}_j^0(0) + \int_{s=0}^t U_j(t, s)\Delta_j^\pm \mathcal{E}_{j-2}^0(s)ds, \\ \mathcal{E}_j^k(t) &= \int_0^t U_j(t, s)(\Delta_j^\pm \mathcal{E}_{j-2}^k(s) + \Delta_j^\pm \mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^\pm \mathcal{E}_{j-1}^{k-1}(s))ds, \quad k \geq 1. \end{cases}$$

7 Let us note first that (52) for  $k = 0$ , namely

$$(63) \quad \partial_t \mathcal{E}_j^0(t) = H_j(t)\mathcal{E}_j^0(t) + \Delta_j^\pm \mathcal{E}_{j-2}^0(t)$$

is verbatim (21)

$$\partial_t E_j = \left( K^j + \frac{1}{N} T_j \right) E_j + D_j E_j + D_j^1 E_{j+1} + D_j^{-1} E_{j-1} + D_j^{-2} E_{j-2},$$

8 after replacing  $E_j$  by  $N^{-\frac{j}{2}} \mathcal{E}_j^0$  and  $D_j^{\pm 1}$  by 0. On the other side, it was explained in  
9 Remark 3.2 in [26], that the proof of Theorem 2.1 in [26], Theorem 2.2 in the present  
10 paper, depends on  $D_j^\pm$  only through the bounds their norms have to satisfy. These  
11 bounds are, after (49) and (27),  $\|D_j^1\| \leq j$ ,  $\|D_j^{-1}\| \leq \frac{j^2}{N}$ , obviously satisfied by  $D_j^{\pm 1} = 0$ .

12 Therefore the conclusion of (the first part of) Theorem 2.2 holds true and we get

$$(64) \quad \|\mathcal{E}_j^0(t)\|_1 \leq (C' A e^{C|t|j^2})^{j/2}.$$

13 Thus we get, by (58), (60) and the inequality  $j^\lambda \leq e^{j\lambda/e}$ ,  $\lambda > 0$ , that

$$(65) \quad \left\| \int_0^t U_j(t, s)(\Delta_j^+ \mathcal{E}_{j+1}^0(s) + \Delta_j^- \mathcal{E}_{j-1}^0(s))ds \right\|_1 \leq 2|t|(C' A e^{4/e} e^{(C+1)|t|j^2})^{j/2},$$

Moreover  $\hat{\mathcal{E}}_j^1 := \int_0^t U_j(t, s)(\Delta_j^- \mathcal{E}_{j-2}^1(s))ds$  is again solution of (63) so that

$$\|\hat{\mathcal{E}}_j^1(t)\|_1 \leq (C' A e^{C|t|} j^2)^{j/2}$$

1 so that, by (62) and (65), we get that, for  $j$  odd,

$$(66) \quad \|\mathcal{E}_j^1(t)\|_1 \leq (1 + 2|t|)(C' A e^{4/e} e^{(C+1)|t|} j^2)^{j/2}.$$

2 For  $k > 1$  we will estimate  $\|\mathcal{E}_j^k(t)\|_1$  by iterating (62)  $M$  times, we will end up with the  
 3 sum of  $3^M$  terms involving the values  $\mathcal{E}_{j-2r+s-u}^{k-s-u}$  for any  $(r, s, u)$  such that  $M = r + s + u$   
 4 with the two constraints  $k - s - u \geq 0$ ,  $j - 2r + s - u \geq 0$ . Actually  $s, r, u$  are the numbers  
 5 of operators  $\Delta^+, \Delta^-, \Delta^-$  occurring respectively in the term under consideration.

Using the first constraint we see that

$$j - 2r + s - u \leq j - 2r + k \leq j - 2(M - k) + k = j - 2M + 3k.$$

6 So that, taking  $M = [(j + 3k)/2]$ , the second constraint reduces to  $j - 2r + s - u = 0$  and  
 7 the first one to  $s + u = k$  since  $\mathcal{E}_0^k = \delta_{k,0}$ .

8 We easily (and very roughly) estimate, using respectively  $M = [(j + 3k)/2]$ , (61),  
 9 (59) and (60),

$$\|\mathcal{E}_j^k(t)\|_1 \leq 3^{(j+3k)/2} \frac{|t|^{(j+3k)/2}}{((j+3k)/2)!} e^{3(j+k)|t|/2} ((j+k)^2)^{\frac{j+3k}{2}}$$

10 so that, using  $(1 + k/j)^j \leq e^k$ ,  $j^\lambda \leq e^{j\lambda/e}$ ,  $\lambda > 0$  and  $n! \geq n^n e^{-n}$ <sup>3</sup>, we get

$$\|\mathcal{E}_j^k(t)\|_1 \leq (2|t|e^{|t|+\frac{5}{3}}(3+k))^{3k/2} (3e^{6k/e}|t|e^{3|t|}j^2)^{j/2}, \quad k > 1$$

11 and, for all  $k \geq 0$ , using (65),

$$(67) \quad \|\mathcal{E}_j^k(t)\|_1 \leq (2|t|e^{|t|+\frac{5}{3}}(3+k))^{3k/2} ((3e^{\frac{6k}{e}}|t|e^{3|t|} + |t|C' A e^{4/e} e^{(C+1)|t|})j^2)^{j/2}.$$

12 We conclude by (60): for some constants  $C_k(s), C'_k(s)$ , we have

$$(68) \quad \|\Delta_j^+(\mathcal{E}_{j+1}^k(s)) + \Delta_j^-(\mathcal{E}_{j-1}^k(s))\|_1 \leq C_k(s)(C'_k(s)j^2)^{j/2}.$$

13 **Remark 4.3.** In the estimate of  $\|\mathcal{E}_j^k(t)\|_1$  the dangerous term is  $\Delta_j^+ \mathcal{E}_{j+1}^{k-1}$  which increases  
 14 the number of particles. However  $k$  is simultaneously decreasing so that we can stop the  
 15 iteration after a finite number of steps thus avoiding the usual short time assumption  
 16 necessary for a full iteration procedure.

<sup>3</sup>although the argument is quite standard, let us recall it:  $\log n! = \sum_{j=2}^n \log j \geq \int_1^n \log(x) dx = [x \log x - x]_1^n = n \log n - n + 1$ .

1 After restoring the dependence in  $\frac{\|V\|_{L^\infty}}{\hbar}$  by the same argument as in [26], Section 3,  
 2 namely a rescaling of the time and the kinetic part of the Hamiltonian, we find

$$(69) \quad \begin{cases} C_k(s) = 4e(2\frac{|s|\|V\|_{L^\infty}}{\hbar}e^{\frac{|s|\|V\|_{L^\infty}}{\hbar}}k)^{3k/2} \\ \quad \times (3e^{\frac{6k+4}{e}\frac{|s|\|V\|_{L^\infty}}{\hbar}}e^{3\frac{|s|\|V\|_{L^\infty}}{\hbar}} + C'Ae^{C\frac{|s|\|V\|_{L^\infty}}{\hbar}})^{1/2} \\ C'_k(s) = (3e^{\frac{6k}{e}\frac{|s|\|V\|_{L^\infty}}{\hbar}}e^{3\frac{|s|\|V\|_{L^\infty}}{\hbar}} + \frac{|s|\|V\|_{L^\infty}}{\hbar}C'Ae^{4/e}e^{(C+1)\frac{|s|\|V\|_{L^\infty}}{\hbar}})^{1/2}e^{6/e} \end{cases}$$

3 Therefore (56) is satisfied and Theorem 3.1 is proven.

4 The values of the two constants  $D_n(t), D'_n(t)$  in (57) can be expressed out of (69)  
 5 by taking, by Theorem 2.2,  $C = \sup(B_1, C_1), C' = \sup(B_2, C_2)$  where  $B_1, C_1, B_2, C_1, C_2$   
 6 are given in Theorem 2.2. in [26].

7 **Remark 4.4.** *We see that the properties (46)...(49), together with (5), are actually*  
 8 *the only ones being used in the proof of Theorem 3.1.*

9 **4.3. Computability and proof of Theorem 3.5.** The first main result of the present  
 10 paper is Theorem 3.2 which asserts the approximability of  $F_j^N(t)$ , a state of the real  
 11  $N$ -body evolution, in terms of  $F_j^{N,n}(t)$ , up to an arbitrary accuracy. Of course the  
 12 interest of the result is related to the computability of  $F_j^{N,n}(t)$ , at least in principle.  
 13 The starting point is obviously the knowledge of the solution of the Hartree equation.

14 The second ingredient is the semigroup  $U_j(t, s)$  defined by (32). We underline that  
 15 to compute  $U_j(t, s)$  we need in principle to solve a  $j$ -body problem. But we will show  
 16 now how this problem can be solved by an explicit perturbative expansion up to the  
 17 desired order of accuracy.

18 The  $N$ -independent part of the computation is the “ $j$ -kinetic linear mean-field flow”  
 19 defined by the linear kinetic mean-field equation of order  $j$ :

$$(70) \quad \frac{d}{dt}A(t) = (K^j + \Delta_j(t))A(t), \quad A(0) \in \mathbb{L}^{\otimes j},$$

20 where  $\Delta_j(t) = \lim_{N \rightarrow \infty} D_j(t)$ .

21 (70) is solved by the two parameter semigroup  $U_j^0(t, s)$  solution of

$$(71) \quad \begin{aligned} \partial_t U_j^0(t, s) &= (K^j + \Delta_j(t))U_j^0(t, s). \\ U_j^0(s, s) &= I. \end{aligned}$$

22 Note that  $U_j^0$  exists since  $K^j$  generates a unitary flow and  $\Delta_j$  is bounded.

1 The reason of the terminology comes from the fact that, as shown by (42),  $\Delta_1 =$   
 2  $Q(F, \cdot) + Q(\cdot, F)$  so that, for  $j = 1$ , (70) is the linearization of the mean-field equation  
 3 (15) around its solution  $F(t)$ :  $U(t, s) := U_1^0(t, s)$  solves (33).

4 Note moreover that, for  $G^1, G^2 \in \mathbb{L}$ ,

$$(72) \quad \Delta_2(G^1 G^2 + G^2 G^1) = (\Delta_1 G^1) G^2 + G^1 (\Delta_1 G^2) + (\Delta_1 G^2) G^1 + G^2 (\Delta_1 G^1).$$

5 and therefore

$$(73) \quad \begin{aligned} U_2^0(t, s)(G^1 G^2 + G^2 G^1) = \\ (U(t, s) G^1)(U(t, s) G^2) + (U(t, s) G^2)(U(t, s) G^1). \end{aligned}$$

6 More generally, if  $P_j : \mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j}$  is any homogeneous polynomial invariant by permu-  
 7 tations,

$$(74) \quad U_j^0(t, s) P_j(G^1, \dots, G^j) = P_j(U(t, s) G^1, \dots, U(t, s) G^j).$$

8 That is:  $U_j^0$  drives each  $G^j$  along the linearized mean-field flow “factor by factor”.  
 9 Denoting by  $\mathbb{L}_{sym}^{\otimes j}$  the subspace of symmetric (by permutations) vectors, we just proved  
 10 the following result.

**Lemma 4.5.**

$$U_j^0(t, s)|_{\mathbb{L}_{sym}^{\otimes j}} = U(t, s)^{\otimes j}.$$

11 Note also that, since  $\Delta_1 A(t)$  is a commutator, we have that  $\partial_t \text{Tr} A(t) = 0$  when  $A(t)$   
 12 solves (70). Therefore  $U_j^0(t, s)$  preserves trace on  $\mathbb{L}_{sym}^{\otimes j}$ .

13 To be more concrete, let us present the explicit computation of the first orders.

14 We have

$$\partial_t U(t, s) = \frac{1}{i\hbar} [-\hbar^2 \Delta + V_F, U(t, s)] + \frac{1}{i\hbar} [V_{U(t, s)}, F]$$

15 where, in the last term,  $V_{U(t, s)}$  acts on  $E_1(s)$  as  $V_{U(t, s)E_1(s)}$ .

16 More generally,

$$\partial_t U_j^0(t) = \frac{1}{i\hbar} [-\hbar^2 \Delta_{\mathbf{R}^{jd}} + V_F^{\otimes j}, U_j^0(t)] + P(U_j^0, F)$$

18 where

$$\begin{aligned} (P(U_j^0, F) E_j)(Z_j) = \\ \sum_i \int dx (V(x_i - x) - V(x'_i - x)) (U_j^0(t, s) E_j(Z_j^{\#i}, (x, x))) F(x_i, x'_i), \end{aligned}$$

that is, denoting by  $\star_i$  the convolution in the variable  $z_i$ ,

$$(P(U_j^0, F)E_j) = \sum_{i=1}^j [V \star_i (U_j^0(t, s)E_j), F]_i.$$

1 Finally

$$\mathcal{E}_2^0(t)(Z_2) = \int_0^t \int_{\mathbf{R}^{2d}} dsdZ'_2 U_2(t, s)(Z_2, Z'_2) V(x'_1 - x'_2) F(s)(z'_1) F(s)(z'_2) dsdZ'_2$$

2 and

$$(75) \quad \begin{aligned} \mathcal{E}_1^1(t) &= \int_0^t U_1(t, s) Q(F, F) ds \\ &+ (1 - \frac{1}{N}) \int_0^t \int_0^s U_1(t, s) \text{Tr}^2[VU_2(s, u)VF(u) \otimes F(u)] dsdu \end{aligned}$$

3 **Lemma 4.6.**  $\mathcal{E}_1^1$  and  $\mathcal{E}_2^0$  don't vanish identically.

*Proof.* By (28) and (34),  $\mathcal{E}_2^0(t) = 0$  for all  $t$  would imply that

$$T_{1,2}(F \otimes F) = Q(F, F) \otimes F - F \otimes Q(F, F) = 0,$$

4 which is wrong. Moreover  $\mathcal{E}_1^1(t) = 0$  would imply that  $\Delta_1^+ \mathcal{E}_2^0 = Q(F, F)$ , incompatible  
5 with the fact of applying  $\Delta_1^+$  to (28) for  $j = 2$ .  $\square$

6 **Proof of Theorem 3.5.** Let us first note that  $U_j^0(t, s)$  is given by a convergent Dyson  
7 expansion and that, by the isometry of the flow generated by  $K^j$  and (49), we have  
8 by Gronwall's Lemma that  $\|U_j^0(t, s)\| \leq e^{j|t-s|}$ . Since  $\|\frac{1}{N}T_j + D_j - \Delta_j\| \leq 3e^{j|t|} \frac{j^2 \|V\|_{L^\infty}}{N\hbar}$ ,  
9  $U_j(t, s)$  is itself given by a convergent Dyson expansion.

10 Let again  $\tilde{\Delta}_j = \frac{1}{N}T_j + D_j - \Delta_j$  and  $U_j^n(t, s)$  be defined by (39).

11 We get easily that, for  $3e^{2j|t|} \frac{\|V\|_{L^\infty}}{\hbar} \frac{j^2 \|V\|_{L^\infty}}{N\hbar} \leq \frac{n}{2}$ ,

$$(76) \quad \|U_j(t, s) - U_j^n(t, s)\| \leq 2 \frac{(3e^{2j|t|} \frac{\|V\|_{L^\infty}}{\hbar} \frac{j^2 \|V\|_{L^\infty}}{N\hbar})^{n+1}}{(n+1)!} := c_{n,j,t} N^{-n-1}.$$

12 Indeed,  $U(t, t')$  appearing in the right hand side of (39) is a perturbation of the unitary  
13 flow generated by  $K^1 + Q(F(t), \cdot)$ . By  $Q(\cdot, F(t))$ . Since  $\|Q(\cdot, F(t))\| \leq 2 \frac{\|V\|_{L^\infty}}{\hbar}$ ,  $U(t, t')$   
14 can be constructed by a convergent Dyson expansion and we get that  $\|U(t, t')\| \leq$   
15  $e^{2|t|} \frac{\|V\|_{L^\infty}}{\hbar}$  and therefore  $\|U(t, t')^{\otimes j}\| \leq e^{2j|t|} \frac{\|V\|_{L^\infty}}{\hbar}$ . One gets (76) by estimating the re-  
16 mainder of the Dyson expansion (39) by

$$(77) \quad \sum_{k=n+1}^{\infty} \frac{(\|U(t, 0)^{\otimes j}\|_1 \|\tilde{\Delta}_j\|_1)^k}{k!} \leq \frac{(\|U(t, 0)^{\otimes j}\|_1 \|\tilde{\Delta}_j\|_1)^{n+1}}{(n+1)!} \sum_{k=0}^{\infty} \frac{(\|U(t, 0)^{\otimes j}\|_1 \|\tilde{\Delta}_j\|_1)^k}{n^k}$$



Let us define  $\mathcal{E}_j^{k,n}(t)$  as the solution of (34) where  $U_j(t, s)$  is replaced by  $U_j^n(t, s)$  and with  $\|\mathcal{E}_j^k(0)\| \leq \delta_{k,0}(Aj^2)^{j/2}$ . One easily adapt the derivation of (67) in order to get the following result.

**Lemma 4.7.** *Let us rewrite the r.h.s. of (67) as  $d_{k,j,t}(Aj^2)^{j/2}$ . Then*

$$\|\mathcal{E}_j^k(t) - \mathcal{E}_j^{k,n}(t)\|_1 \leq (j+k-1)! c_{n,j,t}^{j+k-1} d_{k,j,t}(Aj^2)^{j/2} N^{-n-1}$$

*Proof.* Iterating  $j+k$  times the first equality of (34), we get that the difference  $\mathcal{E}_j^k(t) - \mathcal{E}_j^{k,n}(t)$  is given by the sum of  $(k+j-1)!$  expressions similar to the one for  $\mathcal{E}_j^k(t)$  with  $m$   $U_j$ s replaced by  $U_j - U_j^n$ ,  $m = 1, \dots, j+k-1$ . Since  $m \in [1, j+k-1]$ , each such expression is bounded by  $c_{n,j,\alpha,t}^{j+k-1} N^{-n-1}$  times a similar expression where the  $U_j$ s are replaced by some  $V_j$ s, equals either to  $U_j$  or to  $U_j - U_j^n$  renormalized. That is, in all cases,  $\|V_j(t, s)\| \leq e^{|t-s|}$ . Since the derivation of (67) uses only (58)-(61), the Lemma is proven.  $\square$

Defining  $\bar{\mathcal{E}}_j^{n,n} = \sum_{k=0}^n N^{-k/2} \mathcal{E}_j^{k,n}$ , Lemma 4.7 gives immediately, since obviously  $d_{k,j,t}$  is increasing in  $k$ , that

$$\|\bar{\mathcal{E}}_j^n - \bar{\mathcal{E}}_j^{n,n}\|_1 \leq (n+1)(j+n-1)! c_{n,j,t}^j (\max(1, c_{n,j,t}))^n d_{n,j,t}(Aj^2)^{j/2} N^{-n-1}.$$

Hence, defining  $E_j^{m,n} = N^{-j/2} \bar{\mathcal{E}}_j^{m,n}$ , we get, using (35) and under the hypothesis of Theorem 3.2,

$$(78) \quad \|E_j(t) - E_j^{n,n}(t)\|_1 \leq C_{n,j,t} N^{-\frac{j}{2}-n-\frac{1}{2}}$$

with

$$(79) \quad C_{n,j,t} = L_{2n}(t)(L'_{2n}(t)j^2)^{j/2} + (n+1)(j+n-1)! c_{n,j,t}^j (\max(1, c_{n,j,t}))^n d_{n,j,t}(Aj^2)^{j/2}.$$

For  $m \leq j$ ,  $n \in \mathbf{N}$  we denote by  $F_j^{N, \leq m}(t)$ ,  $F_j^{N, n, \leq m}(t)$  and  $F_j^{N, > m}(t)$  the operators of integral kernel  $F_j^{N, n, \leq m}(t)(Z_j)$  and  $F_j^{N, n, > m}(t)(Z_j)$  defined respectively by

$$(80) \quad F_j^{N, \leq m}(t)(Z_j) = \sum_{k=j-m}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_k}) E_{j-k}(Z_j^{\{i_1, \dots, i_k\}}),$$

$$(81) \quad F_j^{N, n, \leq m}(t)(Z_j) = \sum_{k=j-m}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_k}) E_{j-k}^{n,n}(Z_j^{\{i_1, \dots, i_k\}}),$$

and

$$(82) \quad F_j^{N, > m}(t)(Z_j) = \sum_{k=0}^{j-m-1} \sum_{1 \leq i_1 < \dots < i_k \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_k}) E_{j-k}(Z_j^{\{i_1, \dots, i_k\}}),$$

1 We have that

$$(83) \quad F_j^N(t) - F_j^{N,n,\leq m}(t) = F_j^{N,n,\leq m}(t) + (F_j^{N,\leq m}(t) - F_j^{N,n,\leq m}(t)).$$

2 By (23) and the same argument as in the proof of Theorem 3.2 we get that, for  
 3  $j \leq CN^{-\frac{1}{2}-\alpha}$  and  $N \geq (4e^2C_2e^{\frac{C_1t\|V\|_{L^\infty}}{h}})^{\frac{1}{2\alpha}}$ ,

$$(84) \quad \|F_j^{N,>m}(t)\|_1 \leq C_m \left(\frac{j}{\sqrt{N}}\right)^{m+1} \leq C_m C^{m+1} N^{-\alpha(m+1)} \text{ with } C_m = 2 \left(e^2 C_2 e^{\frac{C_1t\|V\|_{L^\infty}}{h}}\right)^{\frac{m+1}{2}}$$

4 Indeed

$$\begin{aligned} \|F_j^{N,>m}(t)\|_1 &\leq \sum_{k=m+1}^j \binom{j}{j-k} \|E_k(t)\|_1 \leq \sum_{k=m+1}^j \binom{j}{k} \left(\frac{C_2 e^{\frac{C_1t\|V\|_{L^\infty}}{h}}}{N}\right)^{\frac{k}{2}} \\ &\leq \sum_{k=m+1}^j \left(\frac{j}{\sqrt{N}} e \sqrt{C_2 e^{\frac{C_1t\|V\|_{L^\infty}}{h}}}\right)^k \leq C_m \left(\frac{j}{\sqrt{N}}\right)^{m+1} \end{aligned}$$

5 for  $N \geq (2e\sqrt{C_2 e^{\frac{C_1t\|V\|_{L^\infty}}{h}}})^{\frac{1}{\alpha}}$ .

Moreover, by (78),

$$\|E_j(t) - E_j^{n,n}(t)\|_1 \leq C_{n,m,t} N^{-\frac{j}{2}-n-\frac{1}{2}}, \quad j \leq m,$$

and the same argument as in the proof of Theorem 3.2 we get that,

$$\|F_j^{N,\leq m}(t) - F_j^{N,n,\leq m}(t)\| \leq C_{n,m,t} N^{-n-\frac{1}{2}} \sum_{k=0}^m \binom{j}{k} N^{-\frac{k}{2}} \leq C_{n,m,t} e^C N^{-n-\frac{1}{2}}$$

6 Taking now  $m = \lceil \frac{n+1/2}{\alpha} \rceil$  so that  $N^{-\alpha(m+1)} \leq N^{-n-\frac{1}{2}}$ , we get the result by (83) with

$$(85) \quad N_{n,\alpha,C,t} = 6e^{2m|t|} \frac{\|V\|_{L^\infty}}{h} \frac{m^2}{n} \frac{\|V\|_{L^\infty}}{h} + (4e^2C_2e^{\frac{C_1t\|V\|_{L^\infty}}{h}})^{\frac{1}{2\alpha}}, \quad M_{n,\alpha,C,t} = C_{n,m,t}e^C + C_M C^{M+1}$$

7 evaluated at  $m = \lceil \frac{n+1/2}{\alpha} \rceil$ . □

## 8 5. THE KAC AND ‘‘SOFT SPHERES’’ MODELS

9 In this section we consider the two following classes of mean-field models (see [26]  
 10 for details).

11 • *Kac model*. In this model, the  $N$ -particle system evolves according to a stochastic  
 12 process. To each particle  $i$ , we associate a velocity  $v_i \in \mathbb{R}^3$ . The vector  $\mathcal{V}_N = \{v_1, \dots, v_N\}$   
 13 changes by means of two-body collisions at random times, with random scattering

1 angle. The probability density  $F^N(\mathcal{V}_N, t)$  evolves according to the forward Kolmogorov  
2 equation

$$(86) \quad \partial_t F^N = \frac{1}{N} \sum_{i < j} \int d\omega B(\omega; v_i - v_j) \{F^N(\mathcal{V}_N^{i,j}) - F^N(\mathcal{V}_N)\},$$

3 where  $\mathcal{V}_N^{i,j} = \{v_1, \dots, v_{i-1}, v'_i, v_{i+1}, \dots, v_{j-1}, v'_j, v_{j+1}, \dots, v_N\}$  and the pair  $v'_i, v'_j$  gives the out-  
4 going velocities after a collision with scattering (unit) vector  $\omega$  and incoming velocities  
5  $v_i, v_j$ .  $\frac{B(\omega; v_i - v_j)}{|v_i - v_j|}$  is the differential cross-section of the two-body process. The resulting  
6 mean-field kinetic equation is the homogeneous Boltzmann equation

$$(87) \quad \partial_t F(v) = \int dv_1 \int d\omega B(\omega; v - v_1) \{F(v')F(v'_1) - F(v)F(v_1)\}.$$

7 • ‘Soft spheres’ model. A slightly more realistic variant, taking into account the po-  
8 sitions of particles  $X_N = \{x_1, \dots, x_N\} \in \mathbb{R}^{3N}$  and relative transport, was introduced by  
9 Cercignani [9] and further investigated in [20]. The probability density  $F^N(X_N, V_N, t)$   
10 evolves according to the equation

$$(88) \quad \begin{aligned} \partial_t F^N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} F^N &= \frac{1}{N} \sum_{i < j} h(|x_i - x_j|) B\left(\frac{x_i - x_j}{|x_i - x_j|}; v_i - v_j\right) \\ &\times \{F^N(X_N, V_N^{i,j}) - F^N(X_N, V_N)\}. \end{aligned}$$

11 Here  $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a positive function with compact support. Now a pair of particles  
12 collides at a random distance with rate modulated by  $h$ . The associated mean-field  
13 kinetic equation is the Povzner equation

$$\begin{aligned} \partial_t F(x, v) + v \cdot \nabla_x F(x, v) &= \int dv_1 \int dx_1 h(|x - x_1|) B\left(\frac{x - x_1}{|x - x_1|}; v - v_1\right) \\ &\times \{F(x, v')F(x_1, v'_1) - F(x, v)F(x_1, v_1)\}, \end{aligned}$$

14 which can be seen as an  $h$ -mollification of the inhomogeneous Boltzmann equation  
15 (formally obtained when  $h$  converges to a Dirac mass at the origin). Both classes have  
16 been treated in [26] and Theorem 2.2 applies to them, in the following sense.

The underlying space  $\mathbb{L}$  is now  $L^1(\mathbf{R}^d, dv)$  (resp.  $L^1(\mathbf{R}^{2d}, dx dv)$ ) for the Kac model  
(resp. soft spheres) both endowed with the  $L^1$  norms  $\|\cdot\|_1$ . For  $F^N \in \mathbb{L}^{\otimes N}$ ,  $F_j^N \in \mathbb{L}^{\otimes j}$  is  
defined by

$$F_j^N(Z_j) = \int_{\Omega} F^N(z_1, \dots, z_j, z_{j+1}, \dots, z_N) dz_{j+1} \dots dz_N$$

17 for  $Z_n = (z_1, \dots, z_n), n = 1, \dots, N$  with  $z_i = v_i \in \mathbf{R}^d, \Omega = \mathbf{R}^{(N-j)d}$  (resp.  $z_i = (x_i, v_i) \in$   
18  $\mathbf{R}^{2d}, \Omega = \mathbf{R}^{2(N-j)d}$ ) for the Kac (resp. soft spheres) model.

1 In both cases  $E_j(t)$  is defined by (18), inverted by (20), and it was proven in [26]  
 2 that Theorem 2.2 holds true verbatim in both cases.

3 Stating now the dynamics driven by (86) and (88) under the form (3) with  $K^N = 0$   
 4 (resp.  $K^N = - \sum_{i=1, \dots, N} v_i \partial_{x_i}$ ) for the Kac (resp. soft spheres) model and  $V^N$  given by  
 5 the right hand sides of (86),(88) respectively, one sees immediately that the proofs  
 6 contained in Sections 4.1,4.2 remain valid after an elementary redefinition of the oper-  
 7 ators  $D_j, D_j^{-1}, D_j^{-2}$  in (42)...(45) consisting in removing the “hats” in the right hand  
 8 sides of (42)...(45). The convention (46) remains verbatim the same, together with  
 9 the estimates

$$(89) \quad \|D_j\|, \|D_j^1\| \leq j \text{ and } \|D_j^{-1}\|, \|D_j^{-2}\|, \|D_1^{-1}(E_0)\|, \|D_2^{-2}(E_0)\| \leq \frac{j^2}{N}.$$

10 Therefore, by Remark 4.4, the statements contained in Theorem 3.1 and consequences  
 11 hold true, in both cases, verbatim. Moreover defining  $F_j^{N,n}$  by (38) in both cases,  
 12 Theorem 3.2 reads now as follows

13 **Theorem 5.1.** *[Kac case] Let  $F^N(t)$  the solution of the  $N$  body system (86) (resp.*  
 14 *88) with initial datum  $F^N(0) = F^{\otimes N}$ ,  $0 < F \in L^1(\mathbf{R}^d)$ ,  $\int f(v)dv = 1$  (resp,  $0 < F \in$   
 15  $L^1(\mathbf{R}^{2d})$ ,  $\int f(x,v)dx dv = 1$ ), and  $F(t)$  the solution of the homogeneous Boltzmann  
 16 equation (87) (resp. the Povzner equation(89)) with initial datum  $F$ .*

17 *Then, in both cases, for all  $n \geq 1$  and  $N \geq 4(e\sqrt{L'_{2n}(t)}j)^2$ ,*

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2tC_{2n}(t)eA_t^{2n}j}{\sqrt{N}}.$$

18 The statements of Corollary 3.4 and Theorem 3.5 (with the hypothesis of Theorem  
 19 5.1), and the Remarks 3.3 and 3.6 remain verbatim true.

## 20 APPENDIX A. THE ASBTRACT MODEL

21 **A.1. The model.** We will show in this section that the main results of [26] and of  
 22 Section 1 of the present paper remain true in the “abstract” mean-field formalism for  
 23 a dynamics of  $N$  particles that we will describe now. The present formalism contains  
 24 the abstract formalism developed in [26], without requiring a space of states endowed  
 25 with a multiplicative structure.

26 *States of the particle system and evolution equations.* Let  $\mathbb{L}$  be a vector space on the  
 27 complex numbers. We suppose the family of (algebraic) tensor products  $\{\mathbb{L}^{\otimes n}, n =$   
 28  $1, \dots, N\}$  equipped with a family of norms  $\|\cdot\|^n$  satisfying assumption **(A)** below. the

1  $N$ -body dynamics will be driven on  $\mathbb{L}^{\otimes N}$  by a one- and two- body interaction satis-  
 2 fying assumption **(B)** and the mean-field limit equation will be supposed to satisfy  
 3 assumption **(C)**.

4 Assumptions **(A)** – **(C)** below will be followed by their incarnations in the **K**(ac),  
 5 **S**(oft spheres) and **Q**(uantum) models.

6 By convention we denote  $\mathbb{L}^{\otimes 0} := \mathbf{C}$ ,  $\|z\|^0 = |z|$  and we denote by  $\mathbb{L}^{\hat{\otimes} n}$  the completion  
 7 of  $\mathbb{L}^{\otimes n}$  with respect to the norm  $\|\cdot\|^n$ .

8 For the **K**, **S** and **Q** models,  $\mathbb{L}$  is  $L^1(\mathbf{R}^d, dv)$ ,  $L^1(\mathbf{R}^{2d}, dx dv)$  and  $\mathcal{L}^1(L^2(\mathbf{R}^d))$ , the  
 9 space of trace class operators on  $L^2(\mathbf{R}^d)$ , with their associated norms.

10 **(A)** There exists a family of subsets  $\mathbb{L}_+^{\hat{\otimes} n}$  of  $\mathbb{L}^{\hat{\otimes} n}$ ,  $n = 1, \dots, N$ , of positive elements  $F$   
 11 denoted by  $F > 0$  stable by addition, multiplication by positive reals and tensor  
 12 product and there exists a linear function  $\text{Tr} : \mathbb{L} \rightarrow \mathbf{C}$ , called trace. For every  
 13  $1 \leq k, n \leq N$  and  $1 \leq i \leq j \leq n \leq N$ , let  $\text{Tr}_n^k$  and  $\sigma_{i,j}^n$  be the two mapping defined  
 14 by<sup>4</sup>

$$(90) \quad \left\{ \begin{array}{l} \text{Tr}_n^k : \mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n-1} \\ \quad \quad \quad \bigotimes_{i=1}^n v_i \mapsto \text{Tr}(v_k) \bigotimes_{\substack{i=1 \\ i \neq k}}^n v_i, \\ \sigma_{i,j}^n : \mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n} \\ \quad \quad \quad \bigotimes_{i=1}^n v_i \mapsto \bigotimes_{i=1}^n v'_i, \quad v'_k = v_k, i \neq k \neq j ; v'_i = v_j, v'_j = v_i. \end{array} \right.$$

15 We will suppose that  $\text{Tr}_N^k$  and  $\sigma_{i,j}^n$ ,  $i, j, k \leq n \leq N$ , satisfy, for any  $F \in \mathbb{L}^{\otimes n}$ ,

$$(91) \quad \left\{ \begin{array}{l} \text{Tr}_N^k(F), \sigma_{i,j}^n(F) > 0, \quad \|\text{Tr}_n^k(F)\|_{n-1} = \|F\|_n \text{ when } F > 0 \\ \|\sigma_{i,j}^n(F)\|^n = \|F\|^n \\ \|\text{Tr}_n^k(F)\|^{n-1} \leq \|F\|^n \end{array} \right.$$

16 In particular one has that  $\|F\|^n = \text{Tr}^n \dots \text{Tr}^1 F$  when  $F > 0$  and  $|\text{Tr}^n \dots \text{Tr}^1 F| \leq$   
 17  $\|F\|^n$  in general.

18 Note that (91) allows to extend  $\text{Tr}_n^k$  and  $\sigma_{i,j}^n$  to  $\mathbb{L}^{\hat{\otimes} n}$  by continuity. We will use  
 19 the same notation for these extensions.

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<sup>4</sup>The fact that the second and fourth lines of (90) define a mapping on the whole tensor space  $\mathbb{L}^{\otimes n}$  results easily from the definition of tensors products through the so-called universal property [21]. Indeed, let  $\varphi_n$  be the natural embedding  $\mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n}$ ,  $(v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n$ , and let  $h$  be any mapping  $\mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n'}$ , then the universal property of tensor products says that there is a unique map  $\tilde{h} : \mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n'}$  such that  $\tilde{h} \circ \varphi_n = \varphi_{n'} \circ h$ . Taking  $n' = n - 1$ ,  $h(v_1, \dots, v_k, \dots, v_n) = (\text{trace}(v_k)v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n)$  for  $\text{Tr}_n^k$ , and  $n' = n$ ,  $h(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = (v_1, \dots, v_j, \dots, v_i, \dots, v_n)$  for  $\sigma_{i,j}^n$  give the desired extensions.

1 For the **K**, **S** and **Q** models,  $\text{Tr}^k$  is  $\int_{\mathbf{R}^d} \cdot dv_k$ ,  $\int_{\mathbf{R}^{2d}} \cdot dx_k dv_k$  as indicated in Section  
 2 5, and the partial traces defined in Section 2. The action of  $\sigma_{i,j}^n$  consists obviously  
 3 in exchanging the variables  $v_i$  and  $v_j$ ,  $(x_i, v_i)$  and  $(x_j, v_j)$  and  $(x_i, x'_i)$  and  $(x_j, x'_j)$ ,  
 4 (in the integral kernel), respectively. Finally (91) is satisfied in the three cases.

5 From now on and when no confusion is possible, we will identify  $\mathbb{L}^{\otimes n}$  with its com-  
 6 pletion  $\mathbb{L}^{\hat{\otimes}}$  and we will denote  $\text{Tr}_N^k = \text{Tr}^k$  (note also that  $\text{Tr} = \text{Tr}_1^1 = \text{Tr}^1$ ),  $\sigma_{i,j}^N = \sigma_{i,j}$  and  
 7  $\text{Tr}(= \text{Tr}_n) = \text{Tr}_n^n \text{Tr}_n^{n-1} \dots \text{Tr}_n^1$ . Moreover, with a slight abuse of notation, we will denote

$$(92) \quad \begin{cases} \|\cdot\|_1 = \|\cdot\|^n, \quad \forall n = 1, \dots, N \\ \|\cdot\| \text{ the operator norm on any } \mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j}), \quad \forall i, j = 1, \dots, N \end{cases}$$

8 (here  $\mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j})$  is the set of bounded operators from  $\mathbb{L}^{\otimes i}$  to  $\mathbb{L}^{\otimes j}$ ).

9 We call *symmetric* any element of  $\mathbb{L}^{\otimes n}$  invariant by the action of  $\sigma_{i,j}^n$ ,  $i, j \leq n$ .

10 We call *state of the  $N$ -particle system* an element of

$$(93) \quad \mathcal{D}_N = \{F \in \mathbb{L}^{\otimes n} \mid F > 0, \quad \|F\| = 1 \quad \text{and } F \text{ is symmetric}\}.$$

11 For  $j = 0, \dots, N$ , the  $j$ -particle marginal of  $F^N$  is defined as the partial trace of order  
 12  $N - j$  of  $F^N$ , that is

$$(94) \quad F_j^N = \text{Tr}^N \text{Tr}^{N-1} \dots \text{Tr}^{j+1} F^N, \quad F_N^N := F^N.$$

Note that  $F_j^N \in \mathbb{L}^{\otimes j}$  ( $F_0^N = 1 \in \mathbb{L}^{\otimes 0} := \mathbf{C}$ ) and  $F_j^N > 0$ ,  $\|F_j^N\|^j = \|F^N\|^N$  since  $\text{Tr}$  is  
 positivity and norm preserving, and obviously  $F_j^N$  is symmetric as  $F^N$ . That is to say:

$$F_j^N \in \mathcal{D}_j.$$

13 **(B)** The evolution of a state  $F^N$  in  $\mathbb{L}^{\otimes N}$  is supposed to be given by the  $N$ -particle  
 14 dynamics associated to a two-body interaction:

$$(95) \quad \frac{d}{dt} F^N = (K^N + V^N) F^N,$$

15 where the operators on the right hand side are constructed as follows.

$$(96) \quad K^N = \sum_{i=1}^N \mathbb{I}_{\mathbb{L}^{\otimes(i-1)}} \otimes K \otimes \mathbb{I}_{\mathbb{L}^{\otimes(N-i)}}$$

16 and

$$(97) \quad V^N = \frac{1}{N} \sum_{1 \leq i < j \leq N} V_{i,j}, \quad V_{i,j} := \sigma_{1,i}^N \sigma_{2,j}^N V \otimes \mathbb{I}_{\mathbb{L}^{\otimes(N-2)}} \sigma_{1,i}^N \sigma_{2,j}^N$$

17 for a (possibly unbounded) operator  $K$  acting on  $\mathbb{L}$  and a bounded two-body  
 18 (potential) operator  $V$  acting on  $\mathbb{L}^{\otimes 2}$ .

We assume furthermore that  $K$  is the generator of a strongly continuous, isometric, positivity preserving semigroup (in  $\mathbb{L}$ )

$$(98) \quad e^{Kt}F > 0 \quad \text{if} \quad F > 0; \quad \|e^{tK}\| = 1.$$

and  $K^N + V^N$  is the generator of a strongly continuous, isometric, positivity preserving semigroup (in  $\mathbb{L}^{\otimes N}$ )

$$(99) \quad e^{(K^N+V^N)t}F^N > 0 \quad \text{if} \quad F^N > 0; \quad \|e^{t(K^N+V^N)}\| = 1.$$

Finally, for any  $F \in \mathbb{L}$ ,  $F^N \in \mathbb{L}^{\otimes N}$  and  $i, r > j$ , we assume

$$(100) \quad \text{Tr}(KF) = 0 \quad \text{and} \quad \text{Tr}^{j,N}(V_{i,r}F^N) = 0.$$

This last property is necessary to deduce the forthcoming hierarchy.

*For the  $\mathbf{K}$ ,  $\mathbf{S}$  and  $\mathbf{Q}$  models, the ingredients in (95) are given in Sections 5 and 2, where (98)-(100) are shown to be satisfied.*

Note the symmetry property of the equation (95) induced by the definition of  $V^N$ : if the initial condition  $F_0^N$  for (95) is symmetric, then  $F^N(t)$  is still symmetric.

*Hierarchies.* The family of  $j$ -marginals,  $j = 1, \dots, N$ , are solutions of the BBGKY hierarchy of equations

$$(101) \quad \partial_t F_j^N = \left( K^j + \frac{T_j}{N} \right) F_j^N + \frac{(N-j)}{N} C_{j+1} F_{j+1}^N$$

where:

$$(102) \quad K^j = \sum_{i=1}^j \mathbb{I}_{\mathbb{L}}^{\otimes(i-1)} \otimes K \otimes \mathbb{I}_{\mathbb{L}}^{\otimes(j-i)},$$

$$(103) \quad T_j = \sum_{1 \leq i < r \leq j} T_{i,r} \quad \text{with} \quad T_{i,r} = V_{ir}$$

and

$$(104) \quad C_{j+1} F_{j+1}^N = \text{Tr}^{j+1} \left( \sum_{i \leq j} V_{i,j+1} F_{j+1}^N \right) = \sum_{i=1}^j C_{i,j+1} F_{j+1}^N,$$

$$(105) \quad C_{i,j+1} : \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}, \quad C_{i,j+1} F_{j+1}^N = \text{Tr}^{j+1} (V_{i,j+1} F_{j+1}^N),$$

Indeed, thanks to (100) we get easily by applying  $\text{Tr}^{j,N}$  on (95) that

$$\frac{d}{dt} F_j^N = \left( K^j + \frac{T_j}{N} \right) F_j^N + \frac{1}{N} \text{Tr}^{j,N} \left( \sum_{1 \leq i \leq j < k \leq N} V_{i,k} F^N \right).$$

By symmetry of  $F^N$  and  $V_{i,k}$  we get  $\text{Tr}^{j,N}(V_{i,k}F^N) = \text{Tr}^{j+1}(V_{i,j}F_{j+1})$  for all  $k > j$  and

(101) follows.

1 Note that, thanks to the assumption (91) and for all  $i \leq j = 1, \dots, N$ ,

$$(106) \quad \|T_i\| \leq j^2 \|V\|, \text{ and } \|C_{i,j+1}\| \leq j \|V\|$$

2 (meant for  $(\|T_i\|_{\mathbb{L}^{\otimes i} \rightarrow \mathbb{L}^{\otimes i}}, \|C_{i,j+1}\|_{\mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}}, \|V\|_{\mathbb{L}^{\otimes 2} \rightarrow \mathbb{L}^{\otimes 2}}$  using (92)).

3 We introduce the non-linear mapping  $Q(F, F)$ ,  $Q : \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{L}$  by the formula

$$(107) \quad Q(F, F) = \text{Tr}^2(V_{1,2}(F \otimes F))$$

4 and the nonlinear mean-field equation on  $\mathbb{L}$

$$(108) \quad \partial_t F = KF + Q(F, F), \quad F(0) \geq 0, \quad \|F(0)\|_1 = 1.$$

5 Eq. (108) is the Boltzmann, Povzner or Hartree equation according to the specifications  
6 established in the table above. In full generality we will assume

7 **(C)** (108) has for all time a unique solution  $F(t) > 0$  and  $\|F(t)\| = 1$ .

8 *For the **K**, **S** and **Q** models, (C) is true by standard perturbations methods.*

9 *Correlation error.* To introduce the correlation errors, we need to extend slightly the  
10 above structure.

11 For any subset  $J \subset \{1, \dots, N\}$  we first define

$$(109) \quad \mathbb{L}_N^{\otimes J} := \bigotimes_{i=1}^N \mathbb{L}^{\otimes \chi_J(i)},$$

12 where  $\chi_J$  is the characteristic function of  $J$  and  $\mathbb{L}^{\otimes 0} = \mathbf{C}$ .

Then we introduce  $\mathbb{L}^{\otimes J}$ , the subspace of  $\mathbb{L}_N^{\otimes J}$  formed by vectors of the form  $\bigotimes_{i=1}^N v_i$   
where  $v_i = 1 \in \mathbf{C}$  for  $i \notin J$  and  $v_i \in \mathbb{L}$  for  $i \in J$ . Note that  $\mathbb{L}^{\otimes J}$  is sent to  $\mathbb{L}^{\otimes |J|}$  by the  
mapping

$$\Pi : \bigotimes_{i=1}^N v_i \in \mathbb{L}^{\otimes J} \mapsto \bigotimes_{i \in J} v_i \in \mathbb{L}^{\otimes |J|}.$$

13 We define a norm on  $\mathbb{L}^{\otimes J}$  by

$$\|\cdot\|_{\mathbb{L}^{\otimes J}} = \|\Pi(\cdot)\|_1.$$

14 For  $F \in \mathbb{L}$  and  $K \subset J \subset \{1, \dots, N\}$  we introduce the linear operator  $[F]_J^{\otimes K}$ , defined  
15 through its action on factorized elements as

$$(110) \quad [F]_J^{\otimes K} : \mathbb{L}^{\otimes J/K} \rightarrow \mathbb{L}^{\otimes J}$$

$$\bigotimes_{i=1}^N v_i \mapsto \bigotimes_{i=1}^N a_i,$$



$$1 \quad \text{where } \begin{cases} a_s = 1 \in \mathbf{C} & \text{if } s \notin J \\ a_s = F & \text{if } s \in K \\ a_s = v_s & \text{if } s \in J/K \end{cases} .$$

2 Note that, for  $K, K' \subset J$ ,  $K \cap K' = \emptyset$ , we have the composition

$$(111) \quad [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} = [F]_J^{\otimes (K \cup K')} = [F]_J^{\otimes K'} [F]_{J/K}^{\otimes K}$$

3 and more generally, for all  $F, G$ ,

$$(112) \quad [F]_J^{\otimes K} [G]_{J/K}^{\otimes K'} = [G]_J^{\otimes K'} [F]_{J/K}^{\otimes K} .$$

4 For any subset  $J \subset \{1, \dots, N\}$ , we define the *correlation error* by

$$(113) \quad E_J = \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} F_{J/K}^N$$

where  $F$  solves (108), the operator  $[F]_J^{\otimes K}$  is defined by (110) and  $F_L^N \in \mathbb{L}^{\otimes L}$  is defined through its decomposition on factorized states. Namely if

$$F^N = \sum_{\ell_1, \dots, \ell_N} c_{\ell_1, \dots, \ell_N} v_{\ell_1} \otimes \dots \otimes v_{\ell_N},$$

then

$$F_L^N = \sum_{\ell_1, \dots, \ell_N} c_{\ell_1, \dots, \ell_N} a_{\ell_1} \otimes \dots \otimes a_{\ell_N},$$

$$5 \quad \text{where } \begin{cases} a_s = \text{Tr}(v_s) \in \mathbf{C} & \text{if } s \notin L \\ a_s = v_s & \text{if } s \in L \end{cases} .$$

6 The link between the definition of  $F_L^N$  and the definition of the marginals  $F_j^N$  given in  
7 (94) is the following:

$$(114) \quad F_{\{1, \dots, \ell\}}^N = F_\ell^N \otimes (1)^{\otimes (N-\ell)} \in \mathbb{L}^{\otimes \ell} \otimes (\mathbb{L}^{\otimes 0})^{\otimes (N-\ell)} .$$

8 The formula inverse to (113) reads

$$(115) \quad F_J^N = \sum_{K \subset J} [F]_J^{\otimes K} E_{J/K} .$$

9 Note that the contribution in the right hand side of (115) corresponding to  $K = J$  and  
10  $K = \emptyset$  are  $F^{\otimes |J|}$  and  $E_J$  respectively. To prove (115), we plug (113) in the r.h.s. of

1 (115) and we use (111):

$$\begin{aligned}
\sum_{K \subset J} [F]_J^{\otimes K} E_{J/K} &= \sum_{K \subset J} [F]_J^{\otimes K} \left[ \sum_{K' \subset J/K} (-1)^{|K'|} [F]_{J/K}^{\otimes K'} F_{(J/K)/K'}^N \right] \\
&= \sum_{K \cup K' \subset J} \sum_{\substack{K \subset J \\ K' \cap K = \emptyset}} (-1)^{|K'|} [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} F_{J/(K \cup K')}^N \\
&= \sum_{L \subset J} \left( \sum_{K' \subset L} (-1)^{|K'|} \right) [F]_J^{\otimes L} F_{J/L}^N = F_J^N
\end{aligned}$$

2 since  $\sum_{K' \subset L} (-1)^{|K'|} = \sum_{k'=0}^{|L|} \binom{|L|}{k'} (-1)^{k'} = 0^{|L|} = 0$  if  $L \neq \emptyset$ , and  $= 1$  if  $L = \emptyset$  (since

3  $\sum_{K' \subset \emptyset} (-1)^{|K'|} = (-1)^0 = 1$ ).

4 One notices that since  $F_j^N$  is the marginal of some  $F^N$  which decomposes on elements  
5 of the form  $v_1 \otimes \cdots \otimes v_N$ ,  $F_j^N$  decomposes on elements of the form  $(\prod_{k=j+1}^N \text{Tr} v_k) v_1 \otimes$   
6  $\cdots \otimes v_j$ . Since one knows that  $F_j^N$  is symmetric, it is enough to choose one bijection  
7  $i_J : \{1, \dots, j\} \rightarrow J$ ,  $|J| = j$ , and consider the mapping

$$\begin{aligned}
(116) \quad \Phi_{i_J} : \mathbb{L}^{\otimes |J|} &\xrightarrow{\Phi_{i_J}} \mathbb{L}^{\otimes J} \\
\otimes_{j \in J} v_j \in \mathbb{L}^{\otimes |J|} &\mapsto \otimes_{i=1}^N a_i \in \mathbb{L}^{\otimes J}
\end{aligned}$$

$$(117) \quad F_{|J|}^N \mapsto F_J^N$$

8 where  $a_s = 1$  if  $i \notin J$  and  $a_{i_J(j)} = v_j$ .

9  $\Phi_{i_J}$  is obviously one-to-one since  $i_J$  is so, and, though (116) depends on the embedding  
10 chosen, (117) does not:  $\Phi_{i_J}$  restricted to the space  $\mathbb{L}_S^{\otimes |J|}$  of symmetric-by-permutation  
11 elements of  $\mathbb{L}^{\otimes |J|}$ , depends only on  $J$  and not on  $i_J$ . We will call  $\Phi_J$  this restriction,

$$(118) \quad \Phi_J = \Phi_{i_J}|_{\mathbb{L}_S^{\otimes |J|}}.$$

12 The same argument is also valid for  $E_J$  which enjoys the same symmetry property  
13 than  $F_J^N$  and we define

$$(119) \quad E_{|J|} = \Phi_J^{-1} E_J.$$

14  $\Phi_J$  is obviously isometric and we have that

$$(120) \quad \|E_J\|_{\mathbb{L}^{\otimes J}} = \|E_{\{1, \dots, |J|\}}\|_{\mathbb{L}^{\otimes \{1, \dots, |J|\}}} = \|E_{|J|}\|_1.$$

1 Therefore, considering the one-to-one correspondence  $\Phi_J$ , it is enough to compute/estimate  
 2 the quantities  $E_j, j = 1, \dots, N$ .  $E_j$  and  $F_j^N$  are linked by

$$(121) \quad \begin{cases} E_j &= \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \Phi_{J/K} F_{j-|K|}^N \\ F_j^N &= \sum_{K \subset J} [F]_J^{\otimes K} \Phi_{J/K} E_{j-|K|}. \end{cases}$$

3 For the **K**, **S** and **Q** models, the corresponding expressions are given in Sections 5  
 4 and 2.

5 **A.2. Main results similar to [26].** The kinetic errors  $E_j, j = 1, \dots, N$ , satisfy the  
 6 system of equations

$$(122) \quad \begin{aligned} \partial_t E_j &= \left( K^j + \frac{1}{N} T_j \right) E_j + D_j E_j \\ &+ D_j^1 E_{j+1} + D_j^{-1} E_{j-1} + D_j^{-2} E_{j-2}, \end{aligned}$$

7 where the operators  $D_j, D_j^1, D_j^{-1}, D_j^{-2}, j = 1, \dots, N$ , are defined in Appendix B below,  
 8 equations (131)-(132), together with the proof of (122). Moreover, since (133) holds  
 9 true, we know by Remark 3.2 in [26], that the proof of Theorem 2.1 (and therefore  
 10 Corollary 2.2) in [26] remain valid in our present setting.

11 We get the following result.

12 **Proposition A.1.** *The statements of Theorem 2.2 hold true in the abstract setting*  
 13 *defined in Section A.1.*

14 **A.3. Asymptotic expansion.** It is easy to see that the proofs of the main results  
 15 expressed in Section 3 are adaptable in an elementary way to the present abstract  
 16 paradigm. Indeed they use only the three properties stated in Remark 4.4, valid in the  
 17 present setting as pointed out at the very end of Appendix B, formula (133), together  
 18 with (98)-(99).

19 Therefore, the statements contained in Theorem 3.1 and Corollary 3.4 hold true,  
 20 verbatim, under the hypothesis of Theorem 2.2, and with the definition of correlation  
 21 errors given by the first line of (121) and replacing  $\frac{\|V\|_{L^\infty}}{\hbar}$  by  $\|V\|$  in (69).

Moreover defining now  $F_j^{N,n}$  by truncating the second line of (121) at order  $n$ , that is

$$F_j^{N,n} = \sum_{K \subset J} [F]_J^{\otimes K} \Phi_{J/K} E_{j-|K|}^n$$

where  $E_j^n$  is defined by (37), Theorem 3.2 reads as follows.

**Theorem A.2.** *[abstract] Let  $F^N(t)$  the solution of the  $N$  body system (95) with initial datum  $F^N(0) = F^{\otimes N}$ ,  $0 < F \in \mathbb{L}$ ,  $\|F\|_1 = 1$ , and  $F(t)$  the solution of the mean-field equation (108) with initial datum  $F$ .*

*Then, for all  $n \geq 0$  and  $N \geq 4(eA_t^{2n}j)^2$ ,*

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2tC_{2n}(t)eA_t^{2n}j}{\sqrt{N}}.$$

The statements of Corollary 3.4 and Theorem 3.5 (with the hypothesis of Theorem A.2), and the Remarks 3.3 and 3.6 remain verbatim true.

## APPENDIX B. DERIVATION OF THE CORRELATION HIERARCHY (122)

From the definition of  $E_j$  (cf. (113)) we find

$$\partial_t E_J = \sum_{K \subset J} (-1)^{|K|} \left( \partial_t ([F]_J^{\otimes K}) F_{J/K}^N + [F]_J^{\otimes K} \partial_t F_{J/K}^N \right)$$

Moreover, by (110)

$$(123) \quad \partial_t \left( [F]_J^{\otimes K} \right) = \sum_{k_0 \in K} [F]_J^{\otimes K/\{k_0\}} [\partial_t F]_{J/(K/\{k_0\})}^{\otimes \{k_0\}}.$$

Applying  $\Phi_J$  defined in (118) to the BBGKY hierarchy (101), one finds easily that  $F_J^N$  satisfies, denoting  $\alpha(j, N) := \frac{N-j}{N}$ ,

$$(124) \quad \partial_t F_J^N = K^J F_J^N + \frac{1}{N} \sum_{i < r \in J} T_{i,r} F_J^N + \alpha(j, N) \sum_{i \in J} C_{i,j+1} F_{J \cup \{j+1\}}^N$$

(for  $j+1 \notin J$ ).

By the mean-field equation (108) we deduce that

$$(125) \quad \begin{aligned} \partial_t E_J &= \sum_{K \subset J} (-1)^{|K|} \sum_{k_0 \in K} [F]_J^{\otimes K/\{k_0\}} (KF + Q(F, F))_{J/(K/\{k_0\})}^{\otimes \{k_0\}} F_{J/K}^N \\ &+ \sum_{K \subset J} (-1)^{|K|} \alpha(j - |K|, N) \sum_{i \in J/K} [F]_J^{\otimes K} C_{i,j+1} F_{(J/K) \cup \{j+1\}}^N \\ &+ \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left( \sum_{i \neq r \in J/K} T_{i,r} \right) F_{J/K}^N \\ &+ \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} (K^{J/K} F_{J/K}^N). \end{aligned}$$

We denote by  $\mathcal{T}_i$ ,  $i = 1, 2, 3, 4$ , the four terms contained in the four lines of the r.h.s. of (125), respectively. The computation of the  $\mathcal{T}_i$ s is purely algebraic and will use only the four following properties

$$\left\{ \begin{array}{l} \sum_{K \subset L} (-1)^{|K|} = \delta_{|L|, \emptyset} \\ \sum_{K \subset L} |K| (-1)^{|K|} = -\delta_{|L|, 1} \\ [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} = [F]_J^{\otimes K'} [F]_{J/K'}^{\otimes K} = [F]_J^{\otimes (K \cup K')}, \quad K, K' \subset J, K \cap K' = \emptyset \\ C_{i, j+1} [F]_{(J/K) \cup \{j+1\}}^{\otimes K} = [F]_{(J/K)}^{\otimes K} C_{i, j+1}, \quad K \subset J, j+1 \notin J. \end{array} \right.$$

- 1 In order not to make the paper too heavy, we will compute extensively two terms and
- 2 leave to the reader the straightforward (but tedious) computation of the other terms.
- 3 Using the definition (113), we get

$$\begin{aligned} \mathcal{T}_1 &:= \sum_{K \subset J} (-1)^{|K|} \sum_{k_0 \in K} [F]_J^{\otimes K/\{k_0\}} (KF + Q(F, F))_{J/(K/\{k_0\})}^{\otimes \{k_0\}} F_{J/K}^N \\ &= - \sum_{k_0 \in J} (KF + Q(F, F))_J^{\otimes \{k_0\}} \sum_{K \subset J/\{k_0\}} (-1)^{|K|} [F]_{J/\{k_0\}}^{\otimes K} F_{(J/\{k_0\})/K}^N \\ (126) \quad &= - \sum_{i \in J} (KF + Q(F, F))_J^{\otimes \{i\}} E_{J/\{i\}}. \end{aligned}$$

- 4 To compute  $\mathcal{T}_2$  we make use of the inverse definition (115):

$$\begin{aligned} \mathcal{T}_2 &:= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} C_{i, j+1} F_{(J/K) \cup \{j+1\}}^N \\ &= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\ (127) \quad &\dots C_{i, j+1} \sum_{K' \subset (J/K) \cup \{j+1\}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'}. \end{aligned}$$

- 5 Distinguishing among the belonging or not to  $K'$  of  $i$  and  $j+1$  in the r.h.s. of (127),
- 6 we decompose

$$(128) \quad \mathcal{T}_2 = \mathcal{T}_2^{i, j+1 \in K'} + \mathcal{T}_2^{i, j+1 \notin K'} + \mathcal{T}_2^{i \in K', j+1 \notin K'} + \mathcal{T}_2^{i \notin K', j+1 \in K'}$$

1 We have

$$\begin{aligned}
\mathcal{T}_2^{i,j+1 \in K'} &= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\quad \dots C_{i,j+1} \sum_{\substack{K' \subset (J/K) \cup \{j+1\} \\ i, j+1 \in K'}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'} \\
&= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\quad \dots C_{i,j+1} \sum_{K'' \subset (J/K)/\{i\}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'' \cup \{i, j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\quad \dots C_{i,j+1} \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'' \cup \{i, j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J} [F]_J^{\otimes K} \dots \\
&\quad \dots \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K)}^{\otimes K''} C_{i,j+1} [F]_{((J/K)/K'') \cup \{j+1\}}^{\otimes \{i, j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{i \in J} \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} [F]_J^{\otimes K} \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K)}^{\otimes K''} \dots \\
&\quad \dots C_{i,j+1} [F]_{((J/K)/K'') \cup \{j+1\}}^{\otimes \{i, j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{i \in J} \sum_{L \subset J/\{i\}} \left( \sum_{K \subset L} \alpha(j - |K|, N) (-1)^{|K|} \right) [F]_J^{\otimes L} \dots \\
&\quad \dots C_{i,j+1} [F]_{((J/L) \cup \{j+1\})}^{\otimes \{i, j+1\}} E_{J/(L \cup \{i\})}
\end{aligned}$$

2

$$\begin{aligned}
&= \alpha(j, N) \sum_{i \in J} C_{i,j+1} [F]_{J \cup \{j+1\}}^{\otimes \{i, j+1\}} E_{J/\{i\}} \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i,j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i, j+1\}} E_{J/(\{i, l\})} \\
&= \alpha(j, N) \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} [F]_{J \cup \{j+1\}}^{\otimes \{l\}} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i, j+1\}} E_{J/(\{i, l\})} \\
&= \alpha(j, N) \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} [F]_{J \cup \{j+1\}}^{\otimes \{i, l, j+1\}} E_{J/(\{i, l\})}
\end{aligned}$$

1 since  $\sum_{K \subset L} (-1)^{|K|} = \delta_{L, \emptyset}$ . Note that there is a crucial compensation:

$$(129) \quad \begin{aligned} \mathcal{T}_1 + \mathcal{T}_2^{i, j+1 \in K'} &= -\frac{j}{N} \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} \\ &- \frac{1}{N} \sum_{i \neq l \in J} [Q(F, F)]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{l\}} E_{J/\{i, l\}}. \end{aligned}$$

2 The computations of  $\mathcal{T}_2^{i, j+1 \notin K'}$ ,  $\mathcal{T}_2^{i \in K', j+1 \notin K'}$ ,  $\mathcal{T}_2^{i \notin K', j+1 \in K'}$  go the same way and we omit  
3 it here.

4 We consider a similar dichotomy for the term

$$\begin{aligned} \mathcal{T}_3 &:= \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left( \sum_{i \neq r \in J/K} T_{i, r} \right) F_{J/K}^N \\ &= \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left( \sum_{i \neq r \in J/K} T_{i, r} \right) \sum_{K' \subset J/K} [F]_{J/K}^{\otimes K'} E_{J/(K \cup K')}. \end{aligned}$$

5 according, this time, to the cases  $i, r \in K'$ ,  $i, r \notin K'$ ,  $i \in K', r \notin K'$  and  $i \notin K', r \in K'$ .  
6 The computation of the different terms uses the same “tricks” than for  $\mathcal{T}_2$  and we omit  
7 them.

8 Finally, we obtain easily that

$$(130) \quad \mathcal{T}_4 := \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} (K^{J/K} F_{J/K}^N) = K^J E_J.$$

9 Summing up all the contributions  $\mathcal{T}_1, 1 = 1, \dots, 4$ , we get (122) after specializing to the  
10 case  $J = \{1, \dots, j\}$ , using (119) and setting

$$\begin{aligned}
D_j &: \mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 1, \dots, N, \\
E_j &\mapsto \frac{N-j}{N} \sum_{i \in J} C_{i,j+1} \left( [F]_{J \cup \{j+1\}}^{\otimes \{i\}} \Phi_{(J \cup \{j+1\})/\{i\}} E_j + [F]_{J \cup \{j+1\}}^{\otimes \{j+1\}} E_j \right), \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} ([F]_{J \cup \{j+1\}}^{\otimes \{l\}} \Phi_{(J/\{l\}) \cup \{j+1\}} E_j) \\
D_j^1 &: \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 1, \dots, N-1, \\
E_{j+1} &\mapsto \frac{N-j}{N} C_{j+1} E_{j+1}, \\
D_j^{-1} &: \mathbb{L}^{\otimes(j-1)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 2, \dots, N, \\
E_{j-1} &\mapsto \left( -\frac{j}{N} \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} + \frac{1}{2N} \sum_{i,r \in J} T_{i,r} [F]_J^{\otimes \{i\}} \right) \Phi_{J/\{i\}} E_{j-1}, \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i,j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{j+1\}} \Phi_{J/\{l\}} E_{j-1} \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i,j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i\}} \Phi_{(J/\{i,l\}) \cup \{j+1\}} E_{j-1} \\
D_j^{-2} &: \mathbb{L}^{\otimes(j-2)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 3, \dots, N, \\
E_{j-2} &\mapsto \frac{1}{2N} \sum_{i,s \in J} T_{i,s} [F]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{s\}} \Phi_{J/\{i,s\}} E_{j-2} \\
(131) \quad &\quad - \frac{1}{N} \sum_{i \neq l \in J} [Q(F, F)]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{l\}} \Phi_{J/\{i,l\}} E_{j-2}.
\end{aligned}$$

<sup>1</sup> where, by convention,

$$(132) \quad \begin{cases} D_N^1 := D_1^{-2} := 0 \\ D_1^{-1}(E_0) := -\frac{1}{N} Q(F, F), \\ D_2^{-2}(E_0) := \frac{1}{N} (T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F)). \end{cases}$$

<sup>2</sup> Note that one has the following estimates:

$$(133) \quad \|D_j\|, \|D_j^1\| \leq j \quad \text{and} \quad \|D_j^{-1}\|, \|D_j^{-2}\|, \|D_1^{-1}(E_0)\|, \|D_2^{-2}(E_0)\| \leq \frac{j^2}{N}.$$

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