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ASYMPTOTIC EXPANSION OF THE MEAN-FIELD APPROXIMATION

THIERRY PAUL AND MARIO PULVIRENTI

ABSTRACT. We consider the N -body quantum evolution of a particle system in the mean-field approximation. We show that the j th order marginals $F_j^N(t)$, for factorized initial data $F(0)^{\otimes N}$, are explicitly expressed, modulo $N^{-\infty}$, out of the solution $F(t)$ of the corresponding non-linear mean-field equation and the solution of its linearization around $F(t)$. The result is valid for all times t , uniformly in $j = O(N^{\frac{1}{2}-\alpha})$ for any $\alpha > 0$. We establish and estimate the full asymptotic expansion in integer powers of $\frac{1}{N}$ of $F_j^N(t)$, $j = O(\sqrt{N})$, whose computation at order n involves a finite number of operations depending on j and n but not on N . Our results are also valid for more general models including Kac models. As a by-product we get that the rate of convergence to the mean-field limit in $\frac{1}{N}$ is optimal in the sense that the first correction to the mean-field limit doesn't vanish.

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1. INTRODUCTION

The mean-field limit concerns systems of interacting (classical or quantum) particles whose number diverges in a way linked with a rescaling of the interaction insuring an equilibrium between interaction and kinetic energy. In the case of an additive one-body kinetic energy part and a two-body interaction, and without taking in consideration

quantum statistics, this equilibrium is reached by putting in front of the interaction a coupling constant proportional to the inverse of the number of particles.

The system is then described by isolating the evolution of one (or j) particle(s) and averaging over all the other. This leads to a partial information on the system driven by the so-called j -marginals. The mean-field theory insures that the j -marginals tend, as the number of particles diverges, to the j -tensor power of the solution of a non-linear one-body mean-field equation (Vlasov, Hartree,...) issued from the 1-marginal on the initial N -body state. This program has been achieved in many different situations, and the literature concerning the mean-field approach is protuberant. We refer to [29] for a review and recent references.

Much less is known about the fluctuations around this limit, namely the correction to be added to the factorized limit in order to get better approximations of the true evolution of the j -marginals.

The identification of the leading order of these fluctuations with a Gaussian stochastic process has been established in the quantum context first in [16] and in the classical one in [5]. For the classical dynamics of hard spheres, the fluctuations around the Boltzmann equation have been computed at leading order in [28], generalizing to non-equilibrium states the results of [3]. More recently, for the quantum case, fluctuations near the Hartree dynamics has been derived in [22] (after [21]) and in [2] also for the grand canonical ensemble formalism (number of particles non fixed), using in both cases the methods of second quantization (Fock space) (see also [24] for a proof using the usual quantization formalism): in the case of pure states, the N -body wave function is shown to be $\frac{1}{\sqrt{N}}$ -close in L^2 norm to a sum of partially factorized states constructed out of the so-called Bogoliubov hierarchy. Note that these results rise a problem fundamentally different from the one treated in the present paper, whose goal is to compute mean-field approximation of the N -body problem with an accuracy of any order in powers of $\frac{1}{N}$.

Recently, we developed (together with S. Simonella) in [25] a method to derive mean-field limit, alternative to the ones using empirical measures or direct estimates on the “BBGKY-type” hierarchies (systems of coupled equations satisfied by the set of the j -marginals). This method rather uses the hierarchy followed by the “kinetic errors” E_{j-k} (defined below), already used (under the name “ v -functions”) to deal with kinetic limits of stochastic models [10, 7, 4, 11, 12, 6, 8, 13] and recently investigated in the

more singular low density limit of hard spheres [26] (note that error terms are also used in [22, 21, 2, 24] for the total (pure state) wave function with a quite different point of view). These quantities are, roughly speaking, the coefficient of the decomposition of the j -marginal as a linear combination of the k -th tensor powers, $k = 1, \dots, j$, of the solution of the mean-field equation issued from of the 1-marginal of the initial full state. We developed in [25] a strategy suitable in particular for Kac models (homogeneous original one [17, 18] and non-homogeneous [9]) and quantum mean-field theory. This strategy allowed us to derive the limiting factorization property of the j -marginals up to, roughly speaking, $j \lesssim \sqrt{N}$. This threshold is, on the other side, the one obtained by heuristic arguments as shown in [25] and rigorously in [15] for the Kac's model.

Here and in all this article, N denotes the number of particles of the system under consideration.

In the present paper we provide and estimate a full asymptotic expansion in powers of $\frac{1}{N}$ of the difference between the evolution of j -marginals and its factorized leading order form (Theorem 3.2), following a similar result for the kinetic errors $E_j(t)$ (Theorem 3.1). Our results are valid for $j \leq C\sqrt{N}$ for some explicit constant C and are valid for quantum, Kac's models and in the framework of the abstract formalism, slightly more general than the one developed in [25], described in Appendix A.

The non-vanishing of the first correction is established, showing therefore that the rate of the mean-field convergence is at most of order $\frac{1}{N}$ (Theorem 3.4).

Moreover, as the mean-field solution issued from the first marginal of the N body symmetrical factorized initial data determines the leading order of the the j -marginal, we show that the additional knowledge of the linearization of the mean-field flow around it gives an explicit construction of the full asymptotic expansion of the j -marginals in powers of $\frac{1}{N}$ uniformly in j, N satisfying $j \leq CN^{\frac{1}{2}-\alpha}$ for any $C, \alpha > 0$ (Theorem 3.5). Let us note the analogy with the quantum propagation of semiclassical observables, driven by the classical underlying flow at leading order in the Planck constant, and whose full asymptotic expansion is explicitly computable by the only knowledge of the linearized flow.

Let us summarized in words our main result:

The knowledge of the mean-field flow $F(t)$ and its linearization around $F(t)$ determines explicitly, modulo $N^{-\infty}$, uniformly for $j = O(N^{\frac{1}{2}-\alpha})$, $\alpha > 0$, the j -marginals of the N -body flow issued from $F(0)^{\otimes N}$.

2. QUANTUM MEAN-FIELD

Let $\mathcal{L}^1(L^2(\mathbf{R}^d))$ be the space of trace class operators on $L^2(\mathbf{R}^d)$, with their associated norms.

We consider the evolution of a system of N quantum particles interacting through a (real-valued) two-body, even potential V , described for any value of the Planck constant $\hbar > 0$ by the Schrödinger equation

$$i\hbar\partial_t\psi = H_N\psi, \quad \psi|_{t=0} = \psi_{in} \in \mathfrak{H}_N := L^2(\mathbf{R}^d)^{\otimes N},$$

where

$$H_N := -\frac{\hbar^2}{2} \sum_{k=1}^N \Delta_{x_k} + \frac{1}{2N} \sum_{\substack{1 \leq k, l \leq N \\ k \neq l}} V(x_k - x_l).$$

We will suppose in the whole present paper that V is bounded so that the N -body Hamiltonian H_N is self-adjoint on a suitable domain.

Instead of the Schrödinger equation written in terms of wave functions, we shall rather consider the quantum evolution of density matrices. An N -body density matrix is an operator F^N such that

$$0 \leq F^N = (F^N)^*, \quad \text{trace}_{\mathfrak{H}_N}(F^N) = 1.$$

The evolution of the density matrix $F^N \mapsto F^N(t)$ of a N -particle system is governed for any value of the Planck constant $\hbar > 0$ by the von Neumann equation

$$(1) \quad \partial_t F^N = \frac{1}{i\hbar} [H_N, F^N],$$

equivalent to the Schrödinger equation when $F^N(0)$ is a rank one projector, modulo a global phase.

Positivity, norm and trace are obviously preserved by (1) since H_N is self-adjoint.

For each $j = 1, \dots, N$, the j -particle marginal $F_j^N(t)$ of $F^N(t)$ is the unique trace class operator on \mathfrak{H}_j such that

$$\text{trace}_{\mathfrak{H}_N}[F^N(t)(A_1 \otimes \cdots \otimes A_j \otimes I_{\mathfrak{H}_{N-j}})] = \text{trace}_{\mathfrak{H}_j}[F_j^N(t)(A_1 \otimes \cdots \otimes A_j)].$$

for all A_1, \dots, A_j bounded operators on \mathfrak{H} . Alternatively and equivalently, the F_j^N can be defined by the partial trace of F^N on the $N-j$ last ‘‘particles’’: defining F^N through its integral kernel $F^N(x_1, x'_1; \dots; x_N, x'_N)$, the integral kernel of F_j^N is defined as (see

[1])

$$\begin{aligned}
& F_j^N(x_1, x'_1; \dots; x_j, x'_j) := (\mathrm{Tr}^{j+1} \dots \mathrm{Tr}^N F^N)(x_1, x'_1; \dots; x_j, x'_j) \\
(2) \quad & := \int_{\mathbf{R}^{d(N_j)}} F^N(x_1, x'_1; \dots; x_j, x'_j; x_{j+1}, x_{j+1}; \dots; x_N, x_N) dx_{j+1} \dots dx_N.
\end{aligned}$$

It will be convenient for the sequel to rewrite (1) in the following operator form

$$(3) \quad \partial_t F^N = (K^N + V^N) F^N$$

where K^N, V^N are operators on $\mathcal{L}^1(L^2(\mathbf{R}^{Nd}))$ defined by

$$(4) \quad K^N = \frac{1}{i\hbar} \left[-\frac{\hbar^2}{2} \Delta_{\mathbf{R}^{dN}}, \cdot \right], \quad V^N = \frac{1}{2N} \sum_{k,l} V_{k,l} \text{ with } V_{k,l} := \frac{1}{i\hbar} [V(x_k - x_l), \cdot].$$

The self-adjointness of H_N implies that

$$(5) \quad \|e^{t(K^N + V^N)}\|_{\mathcal{L}^1(L^2(\mathbf{R}^d)) \rightarrow \mathcal{L}^1(L^2(\mathbf{R}^d))} = \|e^{tK^N}\|_{\mathcal{L}^1(L^2(\mathbf{R}^d)) \rightarrow \mathcal{L}^1(L^2(\mathbf{R}^d))} = 1, \quad t \in \mathbf{R}.$$

We will denote

$$(6) \quad \mathbb{L} := \mathcal{L}^1(L^2(\mathbf{R}^d)) \text{ so that } \mathbb{L}^{\otimes n} = \mathcal{L}^1(L^2(\mathbf{R}^{nd})), \quad n = 1, \dots, N,$$

and, with a slight abuse of notation,

$$(7) \quad \begin{cases} \|\cdot\|_1 & \text{the trace norm on any } \mathbb{L}^{\otimes j}, \\ \|\cdot\| & \text{the operator norm on any } \mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j}) \end{cases}$$

for $i, j = 1, \dots, N$ (here $\mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j})$ is the set of bounded operators from $\mathbb{L}^{\otimes i}$ to $\mathbb{L}^{\otimes j}$).

A density matrix $F^n \in \mathbb{L}^{\otimes n}$ is called symmetric if its integral kernel $F^n(x_1, x'_1; \dots; x_n, x'_n)$ is invariant by any permutation

$$(x_i, x'_i) \leftrightarrow (x_j, x'_j), \quad i, j = 1, \dots, n.$$

Note that the symmetry of F^N is preserved by the equation (1) due to the particular form of the potential.

We define, for $n = 1, \dots, N$,

$$(8) \quad \mathcal{D}_n = \{F \in \mathbb{L}^{\otimes n} \mid F > 0, \quad \|F\|_1 = 1 \quad \text{and } F \text{ is symmetric}\}.$$

Note that $F_j^N \in \mathbb{L}^{\otimes j}$ ($F_0^N = 1 \in \mathbb{L}^{\otimes 0} := \mathbf{C}$) and $F_j^N > 0, \|F_j^N\|_1 = \|F^N\|_1$, and obviously F_j^N is symmetric as F^N . That is to say:

$$F_j^N \in \mathcal{D}_j.$$

The family of j -marginals, $j = 1, \dots, N$, are solutions of the BBGKY hierarchy of equations (see [27] and also [1])

$$(9) \quad \partial_t F_j^N = \left(K^j + \frac{T_j}{N} \right) F_j^N + \frac{(N-j)}{N} C_{j+1} F_{j+1}^N$$

where:

$$(10) \quad K^j = \frac{1}{i\hbar} \left[-\frac{\hbar^2}{2} \Delta_{\mathbf{R}^{jd}}, \cdot \right]$$

$$(11) \quad T_j = \sum_{1 \leq i < r \leq j} T_{i,r} \quad \text{with } T_{i,r} = V_{i,r}$$

and

$$(12) \quad C_{j+1} F_{j+1}^N = \sum_{i=1}^j C_{i,j+1} F_{j+1}^N$$

with

$$(13) \quad \begin{aligned} C_{i,j} : \quad \mathbb{L}^{\otimes(j+1)} &\rightarrow \mathbb{L}^{\otimes j} \\ C_{i,j+1} F_{j+1}^N &= \text{Tr}^{j+1} (V_{i,j+1} F_{j+1}^N), \end{aligned}$$

where Tr^{j+1} is the partial trace with respect to the $(j+1)$ th variable, as in (2).

Note that, for all $i \leq j = 1, \dots, N$,

$$(14) \quad \|T_j\| \leq j^2 \frac{\|V\|_{L^\infty}}{\hbar}, \quad \text{and } \|C_{i,j+1}\| \leq j \frac{\|V\|_{L^\infty}}{\hbar}.$$

(meant for $\|T_j\|_{\mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j}}$ and $\|C_{i,j+1}\|_{\mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}}$ in accordance with (7)).

The Hartree equation is

$$(15) \quad i\hbar \partial_t F = \left[-\frac{\hbar^2}{2} \Delta + V_F(x), F \right], \quad F(0) \in \mathcal{D}_1,$$

where $V_F(x) = \int_{\mathbf{R}^d} V(x-y) F(y, y) dy$, $F(y, y')$ being the integral kernel of F .

Note that (15) reads also

$$(16) \quad \partial_t F = K^1 F + Q(F, F),$$

with

$$(17) \quad Q(F, F) = \text{Tr}^2(V_{1,2}(F \otimes F)).$$

Since V is bounded, (15) has for all time a unique solution $F(t) > 0$ and $\|F(t)\| = 1$ (see again [27] and [1]).

In order to define the correlation error in an easy way, we need a bit more of notations concerning the variables of integral kernels.

For $i \leq j = 1, \dots, N$, we define the variables $z_i = (x_i, x'_i)$, and $Z_j = (z_1, \dots, z_j)$. For $\{i_1, \dots, i_k\} \subset \{1, \dots, j\}$, we denote by $Z_j^{\setminus \{i_1, \dots, i_k\}} \in \mathbf{R}^{2(j-k)d}$, the vector $Z_j := (z_1, \dots, z_j)$ after removing the components z_{i_1}, \dots, z_{i_k} .

Definition 2.1. For any $j = 1, \dots, N$, we define the correlation error $E_j \in \mathbb{L}^{\otimes j}$ by its integral kernel

$$(18) \quad E_j(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} (-1)^k F(z_{i_1}) \dots F(z_{i_k}) F_{j-k}^N(Z_j^{\setminus \{i_1, \dots, i_k\}}).$$

By convention and consistently we set

$$(19) \quad F_0^N = \|F\| = 1, E_0 := 1 \in \mathbb{L}^{\otimes 0} := \mathbf{C}.$$

.

In [25] it was shown that (18) is inverted by the following equality:

$$(20) \quad F_j^N(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(z_{i_1}) \dots F(z_{i_k}) E_{j-k}(Z_j^{\setminus \{i_1, \dots, i_k\}}), \quad j = 0, \dots, N.$$

i.e. F_j^N is the operator of integral kernel given by (20).

Theorem 2.4, Theorem 2.1 and Corollary 2.2 in [25] state the following facts, among others.

The kinetic errors E_j , $j = 1, \dots, N$, satisfy the system of equations

$$(21) \quad \begin{aligned} \partial_t E_j &= \left(K^j + \frac{1}{N} T_j \right) E_j + D_j E_j \\ &+ D_j^1 E_{j+1} + D_j^{-1} E_{j-1} + D_j^{-2} E_{j-2}, \end{aligned}$$

where the operators $D_j, D_j^1, D_j^{-1}, D_j^{-2}$, $j = 0, \dots, N$, are defined at the beginning of the Section 2, formulas (40)-(43).

We note that the operators D_j^α , $\alpha = 1, -1, -2$ map functions of $j + \alpha$ variables into functions of j variables.

Theorem 2.2 (out of Theorem 2.2. and Corollary 2.3 in [25]).

Let $E_j(0)$ satisfy for some $C_0 > 1$

$$(22) \quad \|E_j(0)\|_1 \leq C_0^j \left(\frac{j}{\sqrt{N}}\right)^j, \quad j \geq 1.$$

Then, for all $t > 0$ and all $j = 1, \dots, N$, one has

$$(23) \quad \|E_j(t)\|_1 \leq \left(C_2 e^{\frac{C_1 t \|V\|_{L^\infty}}{h}}\right)^j \left(\frac{j}{\sqrt{N}}\right)^j, \quad j \geq 1.$$

for some $C_1 > 0$, $C_2 \geq 1$ explicit (see Theorem 2.2 in [25]),

and

$$(24) \quad \|F_j^N(t) - F(t)^{\otimes j}\|_1 \leq D_2 e^{\frac{D_1 t \|V\|_{L^\infty}}{h}} \frac{j^2}{N},$$

where $D_2 = \sup\{B_2, (eC_0)^2\}$ ($B_1 = \sup\{B_1, 2C_1\}$, B_1, B_2 being taken in Theorem 2.2 in [25] at the value $B_0 = 0$).

3. ASYMPTOTIC EXPANSION AND MAIN RESULT

Two questions arise naturally:

- (1) are the estimates (23) sharp?
- (2) Could (24) be improved with a r.h.s. of any order we wish?

Of course, defining $F_j^{N,n}(t)$, $n = 1, \dots, j$, by its integral kernel $F_j^{N,n}(Z_j) = \sum_{k=0}^n \sum_{1 \leq i_1 < \dots < i_k \leq j} F(z_{i_1}) \dots F(z_{i_k}) E_{j-k}(Z_j^{\{i_1, \dots, i_k\}})$, we get by (20), (23) and (24) that, for any $n \leq j$, $\|F_j^N(t) - F_j^{N,n}(t)\| = O(N^{-(n+1)/2})$. However one cannot go further in the approximation that is, in any case useless without the knowledge of the true E_j s.

As we will see later on, one of our main results states that, not only estimates (23) are true, but $\mathcal{E}_j(t) := N^{j/2} E_j^N(t)$ has a full asymptotic expansion in positive powers of $(\frac{1}{N})^{\frac{1}{2}}$

More precisely we will show that, under the hypothesis (22) on the initial data, and for all time t and all $j = 1, \dots, N$, there exist sequences $(\mathcal{E}_j^\ell(t))_{\ell \in \mathbf{N}}$ such that

$$(25) \quad \mathcal{E}_j(t) \sim \sum_{\ell=0}^{\infty} \mathcal{E}_j^\ell(t) N^{-\ell/2}$$

(in the sense that for all $k \in \mathbf{N}$, $\|\mathcal{E}_j(t) - \sum_{\ell=0}^k \mathcal{E}_j^\ell(t) N^{-\ell/2}\| = o(N^{-k/2})$).

The coefficients \mathcal{E}_j^ℓ can be determined as solutions of a partial differential equations which can be solved recursively. More than that, $\mathcal{E}_j^\ell(t)$ turn out to be explicitly computed in terms of a perturbative expansion, after the knowledge of the linearization of the mean-field equation (15) around the solution of (15) with initial condition $F(0) = (F^N(0))_1$ which will be discussed in detail later on.

The starting point of our analysis is the evolution equation for $\mathcal{E}_j(t)$, obtained by the substitution $E_j = N^{-j/2} \mathcal{E}_j$ in (21):

$$(26) \quad \partial_t \mathcal{E}_j = H_j \mathcal{E}_j + N^{-\frac{1}{2}} \Delta_j^+ \mathcal{E}_{j+1} + N^{-\frac{1}{2}} \Delta_j^- \mathcal{E}_{j-1} + \Delta_j^- \mathcal{E}_{j-2}$$

where

$$(27) \quad \begin{cases} H_j &= K^j + \frac{T_j}{N} + D_j(t) \\ \Delta_j^+ &= D_j^1 \\ \Delta_j^- &= N D_j^{-1} \\ \Delta_j^- &= N D_j^{-2} \end{cases}$$

the D_j 's being given by formulas (40)-(43) below. It follows that $H_j, \Delta_j^+, \Delta_j^-, \Delta_j^-$ act on functions of $j, j+1, j-1, j-2$ particles, namely $\mathbb{L}^{\otimes j}, \mathbb{L}^{\otimes j+1}, \mathbb{L}^{\otimes j-1}, \mathbb{L}^{\otimes j-2}$.

Inserting the expansion (25) into (26) we find for $(\mathcal{E}_j^k(t))_{j=1, \dots, N, k=0, \dots}$ the following sequence of equations

$$(28) \quad \partial_t \mathcal{E}_j^k = H_j \mathcal{E}_j^k + \Delta_j^- \mathcal{E}_{j-2}^k + \Delta_j^+ \mathcal{E}_{j+1}^{k-1} + \Delta_j^- \mathcal{E}_{j-1}^{k-1}$$

with the convention,

$$(29) \quad \mathcal{E}_0^k(t) = \delta_{k,0}, \quad \mathcal{E}_{-1}^k(t) = \mathcal{E}_{-2}^k(t) = \mathcal{E}_j^{-1}(t) = 0$$

and the ones inherited from (44).

(28) can be solved recursively. Indeed we realize that

$$(30) \quad \partial_t \mathcal{E}_j^0 = H_j \mathcal{E}_j^0 + \Delta_j^- \mathcal{E}_{j-2}^0$$

can be solved by iteration in j (note that $\mathcal{E}_1^0(t) = 0$). Thus knowing \mathcal{E}_j^0 , we can also solve

$$(31) \quad \partial_t \mathcal{E}_j^1 = H_j \mathcal{E}_j^1 + \Delta_j^- \mathcal{E}_{j-2}^1 + \Delta_j^+ \mathcal{E}_{j+1}^0 + \Delta_j^- \mathcal{E}_{j-1}^0.$$

by iteration in j and so on.

However we will see below that the computation of $\mathcal{E}_j^k(t)$ depends actually only on $\mathcal{E}_{j'}^{k'}$ $k' \leq k, j' \leq j+k$ through a number of operations depending only on j and k independent of N .

We now introduce the two-parameter semigroup defined by

$$(32) \quad \begin{aligned} \partial_t U_j(t, s) &= H_j(t) U_j(t, s). \\ U_j(s, s) &= I. \end{aligned}$$

The existence of $U_j(t)$ is guaranteed by the classical theory of perturbation of semigroup, K^j generating an isometric semigroup and $\frac{T_j}{N}$ and $D_j(t)$ being bounded all the operators. Moreover, let us define $U(t, s)$ as the linearisation of the Hartree flow around $F(t)$, namely

$$(33) \quad \begin{aligned} \partial_t U(t, s) &= (K_1 + \Delta_1) U(t, s), \quad \Delta_1 := Q(\cdot, F(t)) + Q(F(t), \cdot) \\ U(s, s) &= I. \end{aligned}$$

We will see in Section 4.3 that $U_j(t, s)$, when acting on symmetric states, is a perturbation of $U(t, s)^{\otimes j}$, and can be explicitly computed out of $U(t, s)$ by a convergent, entire, expansion in $\frac{j^2 \|V\|}{N \hbar}$. In particular, we'll see that expansions of $U_j(t, s)$ up to any power of $\frac{1}{N}$ can be explicitly obtained under the only knowledge of the linearisation of the Hartree flow around $F(t)$.

Using of this semigroup $U_j(t, s)$ leads immediately to solving (28) by the family of relations:

$$(34) \quad \left\{ \begin{array}{l} \mathcal{E}_j^k(t) = U_j(t, 0) \mathcal{E}_j^k(0) \\ \quad + \int_{s=0}^t U_j(t, s) (\Delta_j^- \mathcal{E}_{j-2}^k(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(s)) ds, \\ \mathcal{E}_0^k(t) = \delta_{k,0}, \\ \Delta_1^- (\mathcal{E}_0^0) := -Q(F, F), \\ \Delta_2^- (\mathcal{E}_0^0) := T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F), \\ \mathcal{E}_{-1}^k(t) = \mathcal{E}_{-2}^k(t) = \mathcal{E}_j^{-1}(t) = 0 \text{ by convention.} \end{array} \right.$$

We are now in position of stating the main results of the present paper.

Theorem 3.1. Consider for $j = 0, \dots, N$, $k = 0, \dots$, $t \geq 0$ the system of recursive relations (34). Then, for all $t \in \mathbb{R}$, the knowledge of $U_j(t)$ (see Remark 3.6 below) makes true the following

- (i) $\mathcal{E}_j^k(t)$ is explicitly determined by $\mathcal{E}_{j'}^{k'}(0)$, $j' \leq j + k$, $k' \leq k$
- (ii) $\mathcal{E}_j^k(t) = 0$ if $\mathcal{E}_j^k(0) = 0$, both for $j + k$ odd
- (iii) Let $\mathcal{E}_j(0)$ be the solution of (28) with the condition $\|\mathcal{E}_j(0)\| \leq (Aj^2)^{j/2}$ for some $A > 1$. Let us take moreover $\mathcal{E}_j^k(0) = \delta_{k,0}\mathcal{E}_j(0)$ (concerning this hypothesis, see Remark 3.6 below). Then the following estimate holds true

$$(35) \quad \|\mathcal{E}_j(t) - \sum_{k=0}^{2n} N^{-k/2} \mathcal{E}_j^k(t)\|_1 \leq L_{2n}(t) N^{-n-\frac{1}{2}} (L'_{2n}(t) j^2)^{j/2},$$

where $L_k(t), L'_k(t)$ are defined in (55) below and satisfy, as $k, |t| \rightarrow \infty$,
 $\log L_k(t) = \frac{3k}{2}(\log k + \frac{|t|\|V\|_\infty}{\hbar}) + O(k + \frac{|t|\|V\|_\infty}{\hbar})$ and $\log L'_k(t) = O(k + \frac{|t|\|V\|_\infty}{\hbar})$.

The proof of the theorem is given in Sections 4.1 and 4.2.

Let us set, for $j = 1, \dots, N$, $n = 0, \dots$, $\mathcal{E}_j^k(0) = \delta_{k,0}\mathcal{E}_j(0)$ and

$$(36) \quad E_j^n(t) = \sum_{k=0}^{2n} N^{-\frac{j+k}{2}} \mathcal{E}_j^k(t)$$

and $F_j^{N,n}(t)$ the operator of integral kernel $F_j^{N,n}(t)(Z_j)$ defined by

$$(37) \quad F_j^{N,n}(t)(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_k}) E_{j-k}^n(Z_j^{\{i_1, \dots, i_k\}}),$$

(that is (104) truncated at order n).

Theorem 3.2. Let $F^N(t)$ the solution of the quantum N body system (1) with initial datum $F^N(0) = F^{\otimes N}$, $F \in \mathcal{L}(L^2(\mathbf{R}^d))$, $F \geq 0$, $\text{Tr}F = 1$, and $F(t)$ the solution of the Hartree equation (15) with initial datum F .

Then, for all $n \geq 0$ and $N \geq 4(e\sqrt{L'_{2n}(t)}j)^2$,

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2L_{2n}(t)e\sqrt{L'_{2n}(t)}j}{\sqrt{N}}.$$

Moreover the expansion of $F_j^{N,n}(t)$ contains only integer powers of $\frac{1}{N}$.

Remark 3.3. The condition of factorization of the initial condition $F^N(0) = F^{\otimes N}$, equivalent to $E_j(0) = \delta_{j,0}$, is not necessary. It can be mildly modified by taking any $E_j(0)$ satisfying (22) and the associated sequence $\mathcal{E}_j(0)$. We leave to the interested reader the elaboration of the precise corresponding statements out of Theorem 3.1.

Proof. The proof is similar to the one of Corollary 2.2 in [25].

The fact that $E_j^n(t)$, and therefore $F_j^{N,n}(t)$ contains only integer powers of $\frac{1}{N}$ comes from the fact that the factorization of $F^N(0)$ implies that $\mathcal{E}_j^k(0) = \delta_{k,0}\delta_{j,0}$ and therefore $\mathcal{E}_j^k(0) = \mathcal{E}_j^k(t) = 0$ for $j+k$ odd.

Moreover

$$\begin{aligned} & \|F_j^N(t) - F_j^{N,n}(t)\| \\ & \leq \sum_{k=0}^j \binom{j}{j-k} \|E_k - E_k^n\| \leq N^{-n-\frac{1}{2}} \sum_{k=1}^j \binom{j}{k} L_{2n}(t) \left(\frac{L'_{2n}(t)k^2}{N}\right)^{k/2} \\ & \leq N^{-n-\frac{1}{2}} L_{2n}(t) \sum_{k=1}^j j(j-1)\dots(j-k+1) \left(\frac{\sqrt{L'_{2n}(t)}}{\sqrt{N}}\right)^k \frac{k^k}{k!} \\ & \leq N^{-n-\frac{1}{2}} L_{2n}(t) \sum_{k=1}^j \left(\frac{je\sqrt{L'_{2n}(t)}}{\sqrt{N}}\right)^k \leq N^{-n-\frac{1}{2}} \frac{2L_{2n}(t)e\sqrt{L'_{2n}(t)}j}{\sqrt{N}} \end{aligned}$$

for $N \geq 4(e\sqrt{L'_{2n}(t)}j)^2$ (we used that $E_0(t) = E_0^n(t) = 1$ and $\frac{k^k}{k!} \leq \frac{e^k}{\sqrt{2\pi k}}$). \square

Let us remark that, under the hypothesis of Theorem 3.2, (36) gives that $E_j^n(t) = O(N^{-2})$ for $j > 2$, $E_0^n(t) = 1$, $E_1^n(t) = N^{-1}\mathcal{E}_1^1(t) + O(N^{-2})$ and $E_2^n(t) = N^{-1}\mathcal{E}_2^0(t) + O(N^{-2})$.

Therefore, keeping in $F_j^{N,1}(t)$, given by (37), only the terms $k = j-1, j-2$, and defining $G_1^{-1}(t) = \mathcal{E}_1^1(t)$, $G_2^{-1}(t) = \mathcal{E}_2^0(t)$ and $G_j^{-1}(t)$, $j > 2$, by its integral kernel

$$\begin{aligned} G_j^{-1}(t)(Z_j) &= \sum_{1 \leq i_1 < \dots < i_{j-2} \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_{j-2}}) \mathcal{E}_2^0(Z_j^{\{i_1, \dots, i_{j-2}\}}) \\ &+ \sum_{1 \leq i_1 < \dots < i_{j-1} \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_{j-1}}) \mathcal{E}_1^1(Z_j^{\{i_1, \dots, i_{j-1}\}}), \end{aligned}$$

we get, by Theorem 3.2, that

$$F_j^N(t) - F(t)^{\otimes j} = \frac{1}{N} G_j^{-1}(t) + O(N^{-3/2}).$$

Since $G_1^{-1}(t), G_2^0(t)$ don't vanish identically by Lemma 4.6 (we guess one can prove the same for all the $G_j^{-1}(t)$ s), we get the following bi-product.

Corollary 3.4. *The rate of convergence to the mean-field limit in $\frac{1}{N}$ is optimal.*

As we mentioned already, $U_j(t, s)$ is given by a convergent perturbative expansion out of $U(t, s)^{\otimes j}$ where $U(t, s)$ is the flow generated by the linearization of the Hartree equation around its solution $F(t)$.

More precisely, let $\tilde{\Delta}_j = \frac{1}{N}T_j + D_j - \Delta_j$ and, for $n \in \mathbb{N}$, let us define the truncated Dyson expansion of $U_j(t, s)$ as

$$(38) \quad U_j^n(t, s) = \sum_{k=0}^{2n+1} \int_s^t dt_1 \dots \int_s^{t_{2n}} dt_{2n+1} U(t, t_1)^{\otimes j} \tilde{\Delta}_j(t_1) U(t_1, t_2)^{\otimes j} \tilde{\Delta}_j(t_2) \dots U(t_{2n}, t_{2n+1})^{\otimes j}.$$

Let us consider $F_j^{N,n,n}(t)$ be defined as $F_j^{N,n}(t)$, but by replacing $U_j(t, s)$ by $U(t, s)^{\otimes j}$ in all underlying used expressions. Namely, the integral kernel of $F_j^{N,n,n}(t)$ is given by (37) after replacing $E_j^{N,n}(t)$ by $E_j^{N,n,n}(t) := \sum_{k=0}^{2n} \mathcal{E}_j^{n,n}(t)$ where $\mathcal{E}_j^{n,n}(t)$ are the explicit solutions of the recurrence relations

$$(39) \quad \begin{cases} \mathcal{E}_j^{k,n}(t) &= U_j^n(t, 0) \mathcal{E}_j^{k,n}(0) \\ &+ \int_{s=0}^t U_j^n(t, s) (\Delta_j^- \mathcal{E}_{j-2}^{k,n}(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1,n}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1,n}(s)) ds, \\ \mathcal{E}_j^k(0) &= \delta_{k,0} \mathcal{E}_j(0) \end{cases}$$

with the same conventions as in (34).

Obviously the solution of (39) satisfies the items (i) – (ii) of Theorem 3.1 and the statements of Proposition 4.1.

Theorem 3.5. *Let $\alpha(0, \frac{1}{2})$ and $C > 0$. Then, under the same hypothesis than in Theorem 3.2, one has, for any $n \in \mathbb{N}$, $t \in \mathbb{R}$ and $j \leq CN^{\frac{1}{2}-\alpha}$,*

$$\|F_j^N(t) - F_j^{N,n,n}(t)\| \leq M_{n,\alpha,t} N^{-n-\frac{1}{2}}$$

for all $N > M'_{\alpha,t}$ ($M_{n,\alpha,t}$ and $M'_{\alpha,t}$ are given in (74)).

Note that the expansion of $F_j^{N,n,n}(t)$ contains again only integer powers of $\frac{1}{N}$ and, by the construction of U_j^n and Proposition 4.1, its explicit computation involves a finite number of operations depending only on j and n (and not in N) and the only knowledge of $F(t)$ and the solution of the Hartree equation linearized around it.

The proof of the theorem is given in Section 4.3.

Remark 3.6. [Nature of the expansion in $\frac{1}{N}$] *In the asymptotic expansion $E_j(t) \sim \sum_{k=[(j+1)/2]}^{\infty} c_k^j(t) N^{-k}$ the coefficients $c_k^j(t)$, such as Let us remark finally that each coefficient $\mathcal{E}_j^k(t)$ depend on N as well: first by the dependence in N of $\Delta_j^+ = (1 - \frac{j}{N})C_{j+1}$ and also $U_j(t, s)$ defined by (32). Moreover, since the condition $\|\mathcal{E}_j(0)\| \leq (Aj^2)^{j/2}$ in Theorem 3.1 is a condition only on the size, all the result of this paper hold true under any dependence of $\mathcal{E}_j(0)$ in N . In particular, this allows to reincorporate in $\mathcal{E}_j(0)$ all the terms $\mathcal{E}_j^k(0) N^{-1/2}$, $k = 1 \dots$, as done in the second item of Theorem 3.1.*

4. PROOFS OF THEOREMS 3.1 AND 3.5

Let us first recall from [25] the expression of the ingredients present in equation (21): For any operator $G \in \mathbb{L}^{\otimes n}$, $n = 1, \dots, N$, $G(Z_n)$ denotes its integral kernel and, for any function $F(Z_n)$, $n = 1, \dots, N$, $\overline{F(Z_n)}$ is defined as the operator on $\mathbb{L}^{\otimes n}$ of integral kernel $F(Z_n)$. Moreover $J := \{1, \dots, j\}$.

$$(40) \quad \begin{aligned} D_j &: \mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j} \\ E_j &\mapsto \frac{N-j}{N} \sum_{i \in J} C_{i,j+1} \left(\overline{F(z_i) E_j(Z_{j+1}^{\{i\}})} + \overline{F(z_{j+1}) E_j(Z_j)} \right) \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l) E_j(Z_{j+1}^{\{l\}})} \end{aligned}$$

$$(41) \quad \begin{aligned} D_j^1 &: \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j} \\ E_{j+1} &\mapsto \frac{N-j}{N} C_{j+1} E_{j+1} \end{aligned}$$

$$(42) \quad \begin{aligned} D_j^{-1} &: \mathbb{L}^{\otimes(j-1)} \rightarrow \mathbb{L}^{\otimes j} \\ E_{j-1} &\mapsto \frac{1}{N} \sum_{i,r \in J} T_{i,r} \overline{F(z_i) E_{j-1}(Z_j^{\{i\}})} - \frac{j}{N} \sum_{i \in J} \overline{Q(F, F)(z_i) E_{j-1}(Z_j^{\{i\}})} \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l) F(z_{j+1}) E_{j-1}(Z_j^{\{l\}})} \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l) F(z_i) E_{j-1}(Z_{j+1}^{\{i,l\}})} \end{aligned}$$

and

$$(43) \quad \begin{aligned} D_j^{-2} &: \mathbb{L}^{\otimes(j-2)} \rightarrow \mathbb{L}^{\otimes j} \\ E_{j-2} &\mapsto \frac{1}{N} \sum_{i,s \in J} T_{i,s} \overline{F(z_i) F(z_r) E_{j-2}(Z_j^{\{i,r\}})} \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} \overline{Q(F, F)(z_i) F(z_l) E_{j-2}(Z_l^{\{i,l\}})}. \end{aligned}$$

where, by convention,

$$(44) \quad \begin{cases} D_N^1 := D_1^{-2} := 0 \\ D_1^{-1}(E_0) := -\frac{1}{N} Q(F, F), \\ D_2^{-2}(E_0) := \frac{1}{N} (T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F)). \end{cases}$$

In (40)-(43), $F(z)$ is meant as being the integral kernel of $F(t)$ solution of the Hartree equation 15.

4.1. Recursive construction and proof of Theorem 3.1 (i)-(ii). Specializing (34) to $k = 0$, we get immediately that (we recall $\mathcal{E}_0^0(t) = 1$)

$$(45) \quad \mathcal{E}_j^0(t) = U_j(t, 0)\mathcal{E}_j^0(0) + U_j(t, 0) \int_0^t U_j(0, s)\Delta_j^{\bar{}}\mathcal{E}_{j-2}^0(s)ds, j \geq 1,$$

with the convention $\mathcal{E}_l^k = 0, l < 0$, and $\Delta_2^{\bar{}}(\mathcal{E}_0^0) := (T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F))$.

Therefore, for $j = 1, \dots, N, t \in \mathbf{R}$, the knowledge of $U_j(t, s), |s| \leq |t|$, and $\mathcal{E}_{j'}^0(0), j' = 1, \dots, j$ guarantees the knowledge of $\mathcal{E}_{j'}^0(t), t \in \mathbf{R}, j' \leq j$. We write this fact as

$$(46) \quad (\mathcal{E}_{j'}^0(0))_{j'=1, \dots, j} \rightsquigarrow (\mathcal{E}_{j'}^0(t))_{t \in \mathbf{R}, j'=1, \dots, j}$$

Since $\mathcal{E}_{-1}^k(t) = 0$ by convention and $\mathcal{E}_0^k(t) = 0$ for $k \geq 1$ since $E_0(t) := 1$, we find after (46) that $\mathcal{E}_1^1(t)$ and $\mathcal{E}_2^1(t)$ are determined by $\mathcal{E}_1^1(0)$ and $\mathcal{E}_2^1(0)$. Therefore, by (34), $\mathcal{E}_j^1(t), j = 1, \dots, N$ are determined by $(\mathcal{E}_j^1(0))_{j=1, \dots, N}$, and determine $\mathcal{E}_1^2(t)$ and $\mathcal{E}_2^2(t)$. These ones determine in turn all the $\mathcal{E}_j^2(t), j = 1, \dots, N$ and so on.

Therefore, the knowledge of $(\mathcal{E}_{j'}^{k'}(s))_{|s| \leq |t|, k' \leq k-1, j'=1, \dots, j+1}$ and $\mathcal{E}_j^k(0)$ guarantees for all j, k , by induction, the knowledge of $\mathcal{E}_j^k(t)$. Thus

$$((\mathcal{E}_j^k(0), (\mathcal{E}_{j'}^{k'}(s))_{|s| \leq |t|, k' \leq k-1, j'=1, \dots, j+1}) \rightsquigarrow (\mathcal{E}_{j'}^{k'}(s))_{|s| \leq |t|, k' \leq k, j'=1, \dots, j}.$$

Supposing now $(\mathcal{E}_{j'}^{k'})_{k' \leq k, j' \leq j}$ known,

$$(\mathcal{E}_{j'}^{k'}(s))_{s \leq t, k' \leq k-2, j'=1, \dots, j+2} \rightsquigarrow (\mathcal{E}_{j'}^{k'}(s))_{s \leq t, k' \leq k-1, j'=1, \dots, j+1} \rightsquigarrow \mathcal{E}_j^k(t).$$

and by iteration

$$(\mathcal{E}_{j'}^0(s))_{s \leq t, j'=1, \dots, j+k} \rightsquigarrow \mathcal{E}_j^k(t)$$

so that, by (46),

$$(\mathcal{E}_{j'}^0(0))_{j'=1, \dots, j+k} \rightsquigarrow \mathcal{E}_j^k(t).$$

We just proved the following result.

Proposition 4.1. *For any $j = 1, \dots, N, t \geq 0, k = 0, \dots$, let $\mathcal{E}_j^k(t)$ be the solution of (34). Then $\mathcal{E}_j^k(t)$ is determined by the values $\mathcal{E}_{j'}^{k'}(0)$ for $0 \leq k' \leq k, 1 \leq j' \leq j + k$. Moreover the number of operations leading to $\mathcal{E}_j^k(t)$ depends on j and k , but is independent of N .*

Formula (34) will give easily the following result.

Proposition 4.2. *Let $\mathcal{E}_{j'}^{k'}(0) = 0$ for $j' \leq j, k' \leq k, j' + k'$ odd. Then $\mathcal{E}_j^k(t) = 0$ for $j + k$ odd.*

Proof. Let us suppose $\mathcal{E}_{j'}^{k'}(0) = 0$ for $j' \leq j, k' \leq k, j' + k'$ odd. By (34) we have that $\mathcal{E}_1^0(t) = 0$ since $\mathcal{E}_1^0(0) = 0$. Therefore, by induction on j in (34), $\mathcal{E}_j^0(t) = 0$ for all j odd. Since $\mathcal{E}_0(t) := 1$, $\mathcal{E}_0^j(t) = 0, j > 0$, so that $\mathcal{E}_2^1(t) = 0$ by (34) and therefore $\mathcal{E}_j^1(t) = 0$ for all j even, since then $j \pm 1$ is odd, and therefore $\mathcal{E}_{j \pm 1}^0(s) = 0$. This gives $\mathcal{E}_1^2(t) = 0$ by (34) and so on. \square

Propositions 4.1 and 4.2 are precisely the contents of the two first items of Theorem 3.1.

4.2. Estimates and proof of Theorem 3.1 (iii). In order to simplify the expressions, we will first suppose that $\frac{\|V\|_{L^\infty}}{h} = 1$.

Note that one has therefore the following estimates:

$$(47) \quad \|D_j\|, \|\Delta_j^1\| \leq j \text{ and } \|\Delta_j^-\|, \|\Delta_j^{\bar{-}}\|, \|\Delta_1^-(\mathcal{E}_0)\|, \|\Delta_2^{\bar{-}}(\mathcal{E}_0)\| \leq j^2.$$

Let us first recall that (21) expressed on the \mathcal{E}_j s reads

$$(48) \quad \partial_t \mathcal{E}_j = H_j \mathcal{E}_j + N^{-\frac{1}{2}} \Delta_j^+ \mathcal{E}_{j+1} + N^{-\frac{1}{2}} \Delta_j^- \mathcal{E}_{j-1} + \Delta_j^{\bar{-}} \mathcal{E}_{j-2}$$

and that (22) and (23) can be rephrased as

$$(49) \quad \|\mathcal{E}_j(0)\| \leq (A_j^2)^{j/2} \implies \|\mathcal{E}_j(t)\| \leq (A_t j^2)^{j/2}, \quad A_t = C' A e^{Ct}$$

for some explicit constants A', C .

Furthermore for the reader's convenience we recall the equations for $\mathcal{E}_j^k(t)$

$$(50) \quad \partial_t \mathcal{E}_j^k(t) = H_j(t) \mathcal{E}_j^k(t) + \Delta_j^{\bar{-}} \mathcal{E}_{j-2}^k(t) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(t) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(t)$$

Calling $\bar{\mathcal{E}}_j^n = \sum_{k=0}^n N^{-k/2} \mathcal{E}_j^k$, one easily check that

$$(51) \quad \begin{aligned} \partial_t \bar{\mathcal{E}}_j^n(t) &= H_j(t) \bar{\mathcal{E}}_j^n(t) + \Delta_j^{\bar{-}} \bar{\mathcal{E}}_{j-2}^n(t) + N^{-\frac{1}{2}} (\Delta_j^+ \bar{\mathcal{E}}_{j+1}^n(t) + \Delta_j^- \bar{\mathcal{E}}_{j-1}^n(t)) \\ &\quad - N^{-\frac{n+1}{2}} (\Delta_j^+ (\mathcal{E}_{j+1}^n(t)) + \Delta_j^- (\mathcal{E}_{j-1}^n(t))). \end{aligned}$$

Therefore $R_j^n := \mathcal{E}_j - \bar{\mathcal{E}}_j^n$ satisfies the equation

$$(52) \quad \begin{aligned} \partial_t R_j^n(t) &= H_j(t) R_j^n(t) + \Delta_j^{\bar{-}} R_{j-2}^n(t) + N^{-\frac{1}{2}} (\Delta_j^+ R_{j+1}^n(t) + \Delta_j^- R_{j-1}^n(t)) \\ &\quad + N^{-\frac{n+1}{2}} (\Delta_j^+ (\mathcal{E}_{j+1}^n(t)) + \Delta_j^- (\mathcal{E}_{j-1}^n(t))) \end{aligned}$$

Let us define the mapping

$$\mathbb{U}_j(t, s) : (\mathcal{E}_j(s))_{j=1, \dots, N} \mapsto \mathbb{U}_j(t, s) \left((\mathcal{E}_j(s))_{j=1, \dots, N} \right) := \mathcal{E}_j(t).$$

In other words, the family $(\mathbb{U}_j(t, s))_{j=1, \dots, N}$ solves the equation:

$$\begin{aligned} \partial_t \mathbb{U}_j(t, s) &= H_j(t) \mathbb{U}_j(t, s) + \Delta_j^- \mathbb{U}_{j-2}(t, s) \\ &\quad + N^{-\frac{1}{2}} (\Delta_j^+ \mathbb{U}_{j+1}(t, s) + \Delta_j^- \mathbb{U}_{j-1}(t, s)), \\ \mathbb{U}_j(s, s) &= I. \end{aligned}$$

Hence, the solution of (52) reads

$$(53) \quad \begin{aligned} R_j^n(t) &= \mathbb{U}_j(t, 0) ((R_j^n(0))_{j=1, \dots, N}) \\ &\quad + N^{-\frac{n+1}{2}} \int_0^t \mathbb{U}_j(t, s) ((\Delta_j^+(s) \mathcal{E}_{j+1}^n(s)) + \Delta_j^-(s) \mathcal{E}_{j-1}^n(s))_{j=1, \dots, N} ds \end{aligned}$$

with again the same convention on negative indices.

By hypothese, $R_j^n(0) = 0$ since $\mathcal{E}_j^n(0) = \delta_{n,0} \mathcal{E}_j^0(0)$.

Let us suppose now that

$$(54) \quad \|\Delta_j^+(\mathcal{E}_{j+1}^n(s)) + \Delta_j^-(\mathcal{E}_{j-1}^n(s))\| \leq C_n(s) (C_n'(s) j^2)^{j/2}, \quad |s| \leq |t|,$$

for two increasing functions $C_n(s), C_n'(s), C_n''(s) \geq 1$, Then (49) implies that

$$\|\mathbb{U}_j(t, s) ((\Delta_j^+(s) \mathcal{E}_{j+1}^n(s)) + \Delta_j^-(s) \mathcal{E}_{j-1}^n(s))_{j=1, \dots, N}\| \leq C_n(s) (C_n' C_n''(s) e^{C|t|} j^2)^{j/2},$$

and thus

$$\begin{aligned} \|\mathcal{E}_j(t) - \bar{\mathcal{E}}_j^n(t)\| &= \|R_j^n(t)\| \\ &= \left\| \int_0^t \mathbb{U}_j(t, s) ((\Delta_j^+(s) \mathcal{E}_{j+1}^n(s)) + \Delta_j^-(s) \mathcal{E}_{j-1}^n(s))_{j=1, \dots, N} ds \right\| \\ &\leq N^{-\frac{n+1}{2}} L_n(t) (L_n'(t) j^2)^{j/2}, \end{aligned}$$

where

$$(55) \quad L_n(t) = t C_n(t) \text{ and } L_n'(t) = C_n' C_n''(t) e^{C|t|}.$$

It remains to prove an estimate like (54).

We will obtain such an estimate by iterating (34). We first remark that, since $e^{K^j + T_j/N}$ is unitary and $\|D_j\| \leq j$, the Gronwall Lemma gives that

$$(56) \quad \|\mathbb{U}_j(t, s)\| \leq e^{j|t-s|}.$$

We will use

$$(57) \quad \prod_{i=0}^m e^{(j+i)(t_i - t_{i+1})} \leq e^{(j+m)|t_{m+1} - t_0|} \text{ for any } (t_i)_{i=0, \dots, m} \text{ (see [25]),}$$

$$(58) \quad \|\Delta^\pm\|, \|\Delta^\pm\| \leq j^2,$$

$$(59) \quad \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n = \frac{t^n}{n!}.$$

Let us remind that we have

$$\mathcal{E}_0^k(t) = \delta_{k,0}, \quad \mathcal{E}_j^k(0) = \delta_{k,0}\mathcal{E}_j(0)$$

together with the estimate $\|\mathcal{E}_j(0)\| \leq (Aj^2)^{j/2}$.

(50) reads:

$$\begin{cases} \mathcal{E}_j^0(t) &= U_j(t,0)\mathcal{E}_j^0(0) + \int_{s=0}^t U_j(t,s)\Delta_j^-\mathcal{E}_{j-2}^0(s)ds, \\ \mathcal{E}_j^k(t) &= \int_0^t U_j(t,s)(\Delta_j^-\mathcal{E}_{j-2}^k(s) + \Delta_j^+\mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^-\mathcal{E}_{j-1}^{k-1}(s))ds, \quad k \geq 1. \end{cases}$$

Let us note first that (50) for $k = 0$ is verbatim (21) after replacing E_j by \mathcal{E}_j^0 and D_j^\pm by 0. On the other side, we know by Remark 3.2 in [25], that the proof of Theorem 2.1 in [25], Theorem 3.1 in the present paper, depends on D_j^\pm only through its norm $\|D_j^\pm\|$ required to satisfy (47). Therefore we get immediately,

$$(60) \quad \|\mathcal{E}_j^0(t)\| \leq (C'Ae^{C|t|j^2})^{j/2},$$

and thus, by (56), (58) and using $j^\lambda \leq e^{j\lambda/e}$, $\lambda > 0$,

$$(61) \quad \left\| \int_0^t U_j(t,s)(\Delta_j^+\mathcal{E}_{j+1}^0(s) + \Delta_j^-\mathcal{E}_{j-1}^0(s))ds \right\|_1 \leq 2|t|(C'Ae^{4/e}e^{(C+1)|t|j^2})^{j/2},$$

and the same argument as the one which leads to (60), we get, for j odd,

$$(62) \quad \|\mathcal{E}_j^1(t)\| \leq (1 + 2|t|)(C'Ae^{4/e}e^{(C+1)|t|j^2})^{j/2}.$$

For $k > 1$ we will estimate $\|\mathcal{E}_j^k(t)\|$ by iterating the third line M times, we will end up with the sum of 3^M terms involving the values $\mathcal{E}_{j-2r+s-u}^{k-s-u}$ for any (r,s,u) such that $M = r + s + u$ with the two constraints $k - s - u \geq 0$, $j - 2r + s - u \geq 0$. Actually s, r, u are the numbers of operators $\Delta^+, \Delta^-, \Delta^+$ occurring respectively in the term under consideration.

Using the first constraint we see that

$$j - 2r + s - u \leq j - 2r + k \leq j - 2(M - k) + k = j - 2M + 3k.$$

So that, taking $M = [(j + 3k)/2]$, the second constraint reduces to $j - 2r + s - u = 0$ and the first one to $s + u = k$ since $\mathcal{E}_0^k = \delta_{k,0}$.

We easily (and very roughly) estimate, using respectively $M = [(j + 3k)/2]$, (59), (57) and (58),

$$\|\mathcal{E}_j^k(t)\| \leq 3^{(j+3k)/2} \frac{|t|^{(j+3k)/2}}{((j+3k)/2)!} e^{3(j+k)|t|/2} ((j+k)^2)^{\frac{j+3k}{2}}$$

so that, using $(1 + k/j)^j \leq e^k$, $j^\lambda \leq e^{j\lambda/e}$, $\lambda > 0$ and $n! \geq n^n e^{-n-1}$, we get

$$\|\mathcal{E}_j^k(t)\| \leq (2|t|e^{|t|+\frac{5}{3}}(3+k))^{3k/2}(3e^{6k/e}|t|e^{3|t|}j^2)^{j/2}, \quad k > 1$$

and, for all $k \geq 0$, using (61),

$$(63) \quad \|\mathcal{E}_j^k(t)\| \leq (2|t|e^{|t|+\frac{5}{3}}(3+k))^{3k/2}((3e^{\frac{6k}{e}}|t|e^{3|t|} + |t|C'Ae^{4/e}e^{(C+1)|t|})j^2)^{j/2}.$$

We conclude by (58): for some constants $C_k(s), C'_k(s)$, we have

$$(64) \quad \|\Delta_j^+(\mathcal{E}_{j+1}^k(s)) + \Delta_j^-(\mathcal{E}_{j-1}^k(s))\| \leq C_k(s)(C'_k(s)j^2)^{j/2}.$$

Remark 4.3. *In the estimate of $\|\mathcal{E}_j^k(t)\|$ the dangerous term is $\Delta_j^+\mathcal{E}_{j+1}^{k-1}$ which increases the number of particles. However k is simultaneously decreasing so that we can stop the iteration after a finite number of steps thus avoiding the usual short time assumption necessary for a full iteration procedure.*

After restoring the dependence in $\frac{\|V\|_{L^\infty}}{\hbar}$ by the same argument as in [25], Section 3, namely a rescaling of the time and the kinetic part of the Hamiltonian, we find

$$(65) \quad \begin{cases} C_k(s) = 4e(2\frac{|s|\|V\|_{L^\infty}}{\hbar}e^{\frac{|s|\|V\|_{L^\infty}}{\hbar}}k)^{3k/2} \\ \quad \times (3e^{\frac{6k+4}{e}}\frac{|s|\|V\|_{L^\infty}}{\hbar}e^{3\frac{|s|\|V\|_{L^\infty}}{\hbar}} + C'Ae^{C\frac{|s|\|V\|_{L^\infty}}{\hbar}})^{1/2} \\ C'_k(s) = (3e^{\frac{6k}{e}}\frac{|s|\|V\|_{L^\infty}}{\hbar}e^{3\frac{|s|\|V\|_{L^\infty}}{\hbar}} + \frac{|s|\|V\|_{L^\infty}}{\hbar}C'Ae^{4/e}e^{(C+1)\frac{|s|\|V\|_{L^\infty}}{\hbar}})^{1/2}e^{6/e} \end{cases}$$

Therefore (54) is satisfied and Theorem 3.1 is proven.

The values of the two constants $D_n(t), D'_n(t)$ in (55) can be expressed out of (65) by taking, by Theorem 2.2, $C = \sup(B_1, C_1), C' = \sup(B_2, C_2)$ where B_1, C_1, B_2, C_1, C_2 are given in Theorem 2.2. in [25].

Remark 4.4. *We see that the properties (44)-(47), together with (5), are actually the only ones being used in the proof of Theorem 3.1.*

4.3. Computability and proof of Theorem 3.5. The main result of the present paper is Theorem 1.4 which asserts the approximability of $F_j^N(t)$, a state of the real N -body evolution, in terms of $F_j^{N,n}(t)$, up to an arbitrary accuracy. Of course the interest of the result is related to the computability of $F_j^{N,n}(t)$, at least in principle. The starting point is obviously the knowledge of the solution of the Hartree equation.

The second ingredient is the semigroup $U_j(t, s)$ defined by (32). We underline that to compute $U_j(t, s)$ we need in principal to solve a j -body problem. But we will show

¹although the argument is quite standard, let us recall it: $\log n! = \sum_{j=2}^n \log j \geq \int_1^n \log(x)dx = [x \log x - x]_1^n = n \log n - n + 1$.

now how, up to the desired order of accuracy, this problem can be solved by a explicit perturbative expansion.

The N -independent part of the computation is the “ j -kinteic linear mean-field flow” defined by the linear kinetic mean-field equation of order j :

$$(66) \quad \frac{d}{dt}A(t) = (K^j + \Delta_j(t))A(t), \quad A(0) \in \mathbb{L}^{\otimes j},$$

where $\Delta_j(t) = \lim_{N \rightarrow \infty} D_j(t)$.

(66) is solved by the two parameter semigroup $U_j^0(t, s)$ solving

$$(67) \quad \begin{aligned} \partial_t U_j^0(t, s) &= (K^j + \Delta_j(t))U_j^0(t, s). \\ U_j^0(s, s) &= I. \end{aligned}$$

Note that U_j^0 exists since K^j generates a unitary flow and Δ_j is bounded.

The reason of the terminology comes form the fact that, as shown by (40), $\Delta_1 = Q(F, \cdot) + Q(\cdot, F)$ so that, for $j = 1$, (66) is the linearization of the mean-field equation (15) around its solution $F(t)$: $U(t, s) := U_1^0(t, s)$ solves (33).

Note moreover that, for $G^1, G^2 \in \mathbb{L}$,

$$(68) \quad \Delta_2(G^1 G^2 + G^2 G^1) = (\Delta_1 G^1)G^2 + G^1(\Delta_1 G^2) + (\Delta_1 G^2)G^1 + G^2(\Delta_1 G^1).$$

and therefore

$$(69) \quad \begin{aligned} U_2^0(t, s)(G^1 G^2 + G^2 G^1) &= \\ (U(t, s)G^1)(U(t, s)G^2) &+ (U(t, s)G^2)(U(t, s)G^1). \end{aligned}$$

More generally, if $P_j : \mathbb{L}^j \rightarrow \mathbb{L}^{\otimes j}$ is any homogeneous polynomial invariant by permutations,

$$(70) \quad U_j^0(t, s)P_j(G^1, \dots, G^j) = P_j(U(t, s)G^1, \dots, U(t, s)G^j).$$

That is: U_j^0 drives each G^j along the linearized mean-field flow “factor by factor”. Denoting by $\mathbb{L}_{sym}^{\otimes j}$ the subspace of symmetric (by permutations) vectors, we just prove the following result.

Lemma 4.5.

$$U_j^0(t, s)|_{\mathbb{L}_{sym}^{\otimes j}} = U(t, s)^{\otimes j}.$$

Note also that, since $\Delta_1 A(t)$ is a commutator, we have that $\partial_t \text{Tr} A(t) = 0$ when $A(t)$ solves (66). Therefore $U_j^0(t, s)$ preserves trace on $\mathbb{L}_{sym}^{\otimes j}$.

To be more concrete, let us present the explicit computation of the first orders.

We have

$$\partial_t U(t, s) = \frac{1}{i\hbar} [-\hbar^2 \Delta + V_F, U(t, s)] + \frac{1}{i\hbar} [V_{U(t,s)}, F]$$

where, in the last term, $V_{U(t,s)}$ acts on $E_1(s)$ as $V_{U(t,s)E_1(s)}$.

More generally,

$$\partial_t U_j^0(t) = \frac{1}{i\hbar} [-\hbar^2 \Delta_{\mathbf{R}^{jd}} + V_F^{\otimes j}, U_j^0(t)] + P(U_j^o, F)$$

where

$$(P(U_j^o, F)E_j)(Z_j) = \sum_i \int dx (V(x_i - x) - V(x'_i - x)) (U_j^0(t, s) E_j(Z_j^{\neq i}, (x, x)) F(x_i, x'_i)),$$

that is

$$(P(U_j^o, F)E_j) = \sum_{i=1}^j [V \star_i (U_j^0(t, s) E_j), F]_i.$$

Finally

$$\mathcal{E}_2^0(t)(Z_2) = \int_0^t \int_{\mathbf{R}^{2d}} ds dZ'_2 U_2(t, s)(Z_2, Z'_2) V(x'_1 - x'_2) F(s)(z'_1) F(s)(z'_2) ds dZ'_2$$

and

$$(71) \quad \begin{aligned} \mathcal{E}_1^1(t) &= \int_0^t U_1(t, s) Q(F, F) ds \\ &+ (1 - \frac{1}{N}) \int_0^t \int_0^s U_1(t, s) \text{Tr}^2 [V U_2(s, u) V F(u) \otimes F(u)] ds du \end{aligned}$$

Lemma 4.6. \mathcal{E}_1^1 and \mathcal{E}_2^0 don't vanish identically.

Proof. By (28) and (34), $\mathcal{E}_2^0(t) = 0$ for all t would imply that $T_{1,2}(F \otimes F) = Q(F, F) \otimes F - F \otimes Q(F, F) = 0$, which is wrong, and $\mathcal{E}_1^1(t)$ would imply that $\Delta_1^+ \mathcal{E}_2^0 = Q(F, F)$, incompatible with applying Δ_1^+ to (28) taken at $j = 2$. \square

Proof of Theorem 3.5. Let us first note that $U_j^0(t, s)$ is given by a convergent Dyson expansion and that, by the isometry of the flow generated by K^j and (47), we have by Gronwall Lemma that $\|U_j^o(t, s)\| \leq e^{j|t-s|}$. Since $\|\frac{1}{N} T_j + D_j - \Delta_j\| \leq 3e^{j|t|} \frac{j^2 \|V\|_{L^\infty}}{N \hbar}$, $U_j(t, s)$ is itself given by a convergent Dyson expansion.

Let again $\tilde{\Delta}_j = \frac{1}{N} T_j + D_j - \Delta_j$. and $U_j^n(t, s)$ be defined by (38).

We get easily that, for $j \leq CN^{\frac{1}{2}-\alpha}$,

$$\|U_j(t, s) - U_j^n(t, s)\| + 1 \leq \frac{(3e^{j|t|} \frac{j^2 \|V\|_{L^\infty}}{N h})^{n+1}}{(n+1)!} e^{3e^{j|t|} C^2 \frac{\|V\|_{L^\infty}}{h}} := c_{n,j,\alpha,t} N^{-n-1}.$$

Let us define $\mathcal{E}_j^{k,n}(t)$ the solution of (34) where $U_j(t, s)$ is replaced by $U_j^n(t, s)$ and $\|\mathcal{E}_j^k(0)\| \leq \delta_{k,j} (Aj^2)^{j/2}$. One easily adapt the derivation of (63) in order to get the following result.

Lemma 4.7. *Let us rewrite the r.h.s. of (63) as $d_{k,j,t} (Aj^2)^{j/2}$. Then*

$$\|\mathcal{E}_j^k(t) - \mathcal{E}_j^{k,n}(t)\| \leq (j+k-1)! c_{n,j,\alpha,t}^{j+k-1} d_{k,j,t} (Aj^2)^{j/2} N^{-n-1}$$

Proof. Iterating $j+k$ times the first equality of (34), we get that the difference $\mathcal{E}_j^k(t) - \mathcal{E}_j^{k,n}(t)$ is given by the sum of $(k+j-1)!$ expressions similar to the one for $\mathcal{E}_j^k(t)$ with m U_j s replaced by $U_j - U_j^n$, $m = 1, \dots, j+k-1$. Since $m \in [1, j+k-1]$, each such expression is bounded by $c_{n,j,\alpha,t}^{j+k-1} N^{-n-1}$ times a similar expression where the U_j s are replaced by some V_j s, equals either to U_j or to $U_j - U_j^n$ renormalized. That is, in all cases, $\|V_j(t, s)\| \leq e^{j|t-s|}$. Since the derivation of (63) uses only (56)-(59), the Lemma is proven. \square

Defining $\bar{\mathcal{E}}_j^{n,n} = \sum_{k=0}^n N^{-k/2} \mathcal{E}_j^{k,n}$, Lemma 4.7 gives immediately that

$$\|\bar{\mathcal{E}}_j^n - \bar{\mathcal{E}}_j^{n,n}\| \leq (n+1)(j+k-1)! c_{n,j,\alpha,t}^{j+k} d_{k,j,t} (Aj^2)^{j/2} N^{-n-1}.$$

Hence, defining $E_j^{n,n} = N^{-j/2} \bar{\mathcal{E}}_j^{n,n}$, we get, using (35) and under the hypothesis of Theorem 3.2,

$$(72) \quad \|E_j(t) - E_j^{n,n}(t)\| \leq C_{n,j,t} N^{-n-\frac{1}{2}}$$

with

$$(73) \quad C_{n,j,t} = L_{2n}(t) N^{-n-\frac{1}{2}} (L'_{2n}(t) j^2)^{j/2} + (n+1)(j+k-1)! c_{n,j,\alpha,t}^{j+k} d_{k,j,t} (Aj^2)^{j/2}.$$

Let us fix n and let us define $j_\alpha = [(n + \frac{1}{2})/\alpha] + 1$ (so that $N^{-\alpha j_\alpha} \leq N^{-n-\frac{1}{2}}$). If $j \leq j_\alpha$, (72) gives the result with $M_{n,\alpha,t} = C_{n,j_\alpha,t}$. When $j > n$, let us decompose $F_j^N(t) = F_{j, \leq j_\alpha}^N(t) + F_{j, > j_\alpha}^N(t)$ where the integral kernel of $F_{j, \leq j_\alpha}^N(t)$ is given by the r.h.s. of (37) where the sum is restricted to $k \in \{0, \dots, j - j_\alpha\}$.

By (23), $\|F_{j, \leq j_\alpha}^N(t)\| \leq 2(C_2 e^{\frac{CC_1 t \|V\|_{L^\infty}}{h}})^{j_\alpha} N^{-n-\frac{1}{2}}$ for $N > (2C_2 e^{\frac{CC_1 t \|V\|_{L^\infty}}{h}})^{1/\alpha}$ (by the same argument as in the proof of Theorem 3.2) and, by (72), $\|F_{j, > j_\alpha}^N(t)\| \leq (\sum_{k=0}^{j_\alpha} \binom{j_\alpha}{k} C_{n,k,t}) N^{-n-\frac{1}{2}}$.

Theorem 3.5 is proven by setting

$$(74) \quad M'_{\alpha,t} = (2C_2 e^{\frac{CC_1 t \|V\|_{L^\infty}}{h}})^{1/\alpha}, \quad M_{n,\alpha,t} = C_{n,n,t} + 2 \left(C_2 e^{\frac{CC_1 t \|V\|_{L^\infty}}{h}} \right)^{j_\alpha} + \sum_{k=0}^{j_\alpha} \binom{j_\alpha}{k} C_{n,k,t}.$$

□

5. THE KAC AND “SOFT SPHERES” MODELS

In this section we consider the two following classes of mean-field models (see [25] for details).

- *Kac model.* In this model, the N -particle system evolves according to a stochastic process. To each particle i , we associate a velocity $v_i \in \mathbb{R}^3$. The vector $\mathcal{V}_N = \{v_1, \dots, v_N\}$ changes by means of two-body collisions at random times, with random scattering angle. The probability density $F^N(\mathcal{V}_N, t)$ evolves according to the forward Kolmogorov equation

$$(75) \quad \partial_t F^N = \frac{1}{N} \sum_{i < j} \int d\omega B(\omega; v_i - v_j) \{F^N(\mathcal{V}_N^{i,j}) - F^N(\mathcal{V}_N)\},$$

where $\mathcal{V}_N^{i,j} = \{v_1, \dots, v_{i-1}, v'_i, v_{i+1}, \dots, v_{j-1}, v'_j, v_{j+1}, \dots, v_N\}$ and the pair v'_i, v'_j gives the outgoing velocities after a collision with scattering (unit) vector ω and incoming velocities v_i, v_j . $\frac{B(\omega; v_i - v_j)}{|v_i - v_j|}$ is the differential cross-section of the two-body process. The resulting mean-field kinetic equation is the homogeneous Boltzmann equation

$$(76) \quad \partial_t F(v) = \int dv_1 \int d\omega B(\omega; v - v_1) \{F(v')F(v'_1) - F(v)F(v_1)\}.$$

- *‘Soft spheres’ model.* A slightly more realistic variant, taking into account the positions of particles $X_N = \{x_1, \dots, x_N\} \in \mathbb{R}^{3N}$ and relative transport, was introduced by Cercignani [9] and further investigated in [19]. The probability density $F^N(X_N, V_N, t)$ evolves according to the equation

$$(77) \quad \begin{aligned} \partial_t F^N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} F^N &= \frac{1}{N} \sum_{i < j} h(|x_i - x_j|) B\left(\frac{x_i - x_j}{|x_i - x_j|}; v_i - v_j\right) \\ &\times \{F^N(X_N, V_N^{i,j}) - F^N(X_N, V_N)\}. \end{aligned}$$

Here $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a positive function with compact support. Now a pair of particles collides at a random distance with rate modulated by h . The associated mean-field

kinetic equation is the Povzner equation

$$\begin{aligned} \partial_t F(x, v) + v \cdot \nabla_x F(x, v) &= \int dv_1 \int dx_1 h(|x - x_1|) B\left(\frac{x - x_1}{|x - x_1|}; v - v_1\right) \\ &\quad \times \{F(x, v')F(x_1, v'_1) - F(x, v)F(x_1, v_1)\}, \end{aligned}$$

which can be seen as an h -mollification of the inhomogeneous Boltzmann equation (formally obtained when h converges to a Dirac mass at the origin). Both classes have been treated in [25] and Theorem 2.2 apply to them, in the following sense.

The underlying space \mathbb{L} is now $L^1(\mathbf{R}^d, dv)$ (resp. $L^1(\mathbf{R}^{2d}, dx dv)$) for the Kac model (resp. soft spheres) both endowed with the L^1 norms $\|\cdot\|_1$. For $F^N \in \mathbb{L}^{\otimes N}$, $F_j^N \in \mathbb{L}^{\otimes j}$ is defined by

$$F_j^N(Z_j) = \int_{\Omega} F^N(z_1, \dots, z_j, z_{j+1}, \dots, z_N) dz_{j+1} \dots dz_N$$

for $Z_n = (z_1, \dots, z_n)$, $n = 1, \dots, N$ with $z_i = v_i \in \mathbf{R}^d$, $\Omega = \mathbf{R}^{(N-j)d}$ (resp. $z_i = (x_i, v_i) \in \mathbf{R}^{2d}$, $\Omega = \mathbf{R}^{2(N-j)d}$) for the Kac (resp. soft spheres) model.

In both cases $E_j(t)$ is defined by (18), inverted by (20), and it was proven in [25] that Theorem 2.2 holds true verbatim in both cases.

Stating now the dynamics driven by (75) and (77) under the form (3) with $K^N = 0$ (resp. $K^N = -\sum_{i=1, \dots, N} v_i \partial_{x_i}$) for the Kac (resp. soft spheres) model and V^N given by the right hand-sides of (75), (77) respectively, one sees immediately that the proofs contained in Sections 4.1, 4 remain valid after an elementary redefinition of the operators D_j, D_j^{-1}, D_j^{-2} in (40)-(43) consisting in removing the bottom and overhead straight lines in the right hand sides and, by a slight abuse of notation, identifying functions with their evaluations. The convention (44) remains verbatim the same, together with the estimates

$$(78) \quad \|D_j\|, \|D_j^1\| \leq j \text{ and } \|D_j^{-1}\|, \|D_j^{-2}\|, \|D_1^{-1}(E_0)\|, \|D_2^{-2}(E_0)\| \leq \frac{j^2}{N}.$$

Therefore, by Remark 4.4, the statements contained in Theorem 3.1 and consequences hold true, in both cases, verbatim. Moreover defining $F_j^{N,n}$ by (37) in both cases, Theorem 3.2 reads now as follows

Theorem 5.1. *[Kac case] Let $F^N(t)$ the solution of the N body system (75) (resp. 77) with initial datum $F^N(0) = F^{\otimes N}$, $0 < F \in L^1(\mathbf{R}^d)$, $\int_{\mathbf{R}^d} f(v) dv = 1$ (resp. $0 < F \in L^1(\mathbf{R}^{2d})$, $\int_{\mathbf{R}^{2d}} f(x, v) dx dv = 1$), and $F(t)$ the solution of the homogeneous Boltzmann equation (76) (resp. the Povzner equation (78)) with initial datum F .*

Then, in both cases, for all $n \geq 1$ and $N \geq 4(eA_t^{2n}j)^2$,

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2tC_{2n}(t)eA_t^{2n}j}{\sqrt{N}}.$$

The statements of Corollary 3.4 and Theorem 3.5 (with the hypothesis of Theorem 5.1), and the Remarks 3.3 and 3.6 remain verbatim true.

APPENDIX A. THE ASBTRACT MODEL

A.1. The model. We will show in this section that the main results of [25] and of Section 1 of the present paper remain true in the “abstract” mean-field formalism for a dynamics of N particles that we will describe now. The present formalism contains the abstract formalism developed in [25], without requiring a space of states endowed with a multiplicative structure.

States of the particle system and evolution equations. Let \mathbb{L} be a vector space on the complex numbers. We suppose the family of (algebraic) tensor products $\{\mathbb{L}^{\otimes n}, n = 1, \dots, N\}$ equipped with a family of norms $\|\cdot\|^n$ satisfying assumption **(A)** below. the N -body dynamics will be driven on $\mathbb{L}^{\otimes N}$ by a one- and two- body interaction satisfying assumption **(B)** and the mean-field limit equation will be supposed to satisfy assumption **(C)**.

Assumptions **(A)** – **(C)** below will be followed by their incarnations in the **K**(ac), **S**(oft spheres) and **Q**(uantum) models.

By convention we denote $\mathbb{L}^{\otimes 0} := \mathbf{C}$, $\|z\|^0 = |z|$ and we denote by $\mathbb{L}^{\hat{\otimes} n}$ the completion of $\mathbb{L}^{\otimes n}$ with respect to the norm $\|\cdot\|^n$.

For the **K**, **S** and **Q** models, $\mathbb{L}^{\otimes n}$ is $L^1(\mathbf{R}^d, dv)$, $L^1(\mathbf{R}^{2d}, dx dv)$ and $\mathcal{L}^1(L^2(\mathbf{R}^d))$, the space of trace class operators on $L^2(\mathbf{R}^d)$, with their associated norms.

- (A)** There exists a family of subsets $\mathbb{L}_+^{\hat{\otimes} n}$ of $\mathbb{L}^{\hat{\otimes} n}$, $n = 1, \dots, N$, of positive elements F denoted by $F > 0$ stable by addition, multiplication by positive reals and tensor product and there exists a linear function $\text{Tr} : \mathbb{L} \rightarrow \mathbf{C}$, called trace. For every $1 \leq k, n \leq N$ and $1 \leq i \leq j \leq n \leq N$, let Tr_n^k and $\sigma_{i,j}^n$ be the two mapping defined

by²

$$(79) \quad \left\{ \begin{array}{l} \text{Tr}_n^k : \mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n-1} \\ \quad \quad \quad \bigotimes_{i=1}^n v_i \mapsto \text{Tr}(v_k) \bigotimes_{\substack{i=1 \\ i \neq k}}^n v_i, \\ \\ \sigma_{i,j}^n : \mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n} \\ \quad \quad \quad \bigotimes_{i=1}^n v_i \mapsto \bigotimes_{i=1}^n v'_i, \quad v'_k = v_k, i \neq k \neq j ; v'_i = v_j, v'_j = v_i. \end{array} \right.$$

We will suppose that Tr_N^k and $\sigma_{i,j}^n$, $i, j, k \leq n \leq N$, satisfy, for any $F \in \mathbb{L}^{\otimes n}$,

$$(80) \quad \left\{ \begin{array}{l} \text{Tr}_N^k(F), \sigma_{i,j}^n(F) > 0, \quad \|\text{Tr}_n^k(F)\|_{n-1} = \|F\|_n \text{ when } F > 0 \\ \|\sigma_{i,j}^n(F)\|^n = \|F\|^n \\ \|\text{Tr}_n^k(F)\|^{n-1} \leq \|F\|^n \end{array} \right.$$

In particular one has that $\|F\|^n = \text{Tr}^n \dots \text{Tr}^1 F$ when $F > 0$ and $|\text{Tr}^n \dots \text{Tr}^1 F| \leq \|F\|^n$ in general.

Note that (80) allows to extend Tr_n^k and $\sigma_{i,j}^n$ to $\mathbb{L}^{\hat{\otimes} n}$ by continuity. We will use the same notation for these extensions.

For the **K**, **S** and **Q** models, Tr^k is $\int_{\mathbf{R}^d} \cdot dv_k$, $\int_{\mathbf{R}^{2d}} \cdot dx_k dv_k$ as indicated in Section 5, and the partial traces defined in Section 2. The action of $\sigma_{i,j}^n$ consists obviously in exchanging the variables v_i and v_j , (x_i, v_i) and (x_j, v_j) and (x_i, x'_i) and (x_j, x'_j) , (in the integral kernel), respectively. Finally (80) is satisfied in the three cases.

From now on and when no confusion is possible, we will identify $\mathbb{L}^{\otimes n}$ with its completion $\mathbb{L}^{\hat{\otimes}}$ and we will denote $\text{Tr}_N^k = \text{Tr}^k$ (note also that $\text{Tr} = \text{Tr}_1^1 = \text{Tr}^1$), $\sigma_{i,j}^N = \sigma_{i,j}$ and $\text{Tr}(= \text{Tr}_n) = \text{Tr}_n^n \text{Tr}_n^{n-1} \dots \text{Tr}_n^1$. Moreover, with a slight abuse of notation, we will denote

$$(81) \quad \left\{ \begin{array}{l} \|\cdot\|_1 = \|\cdot\|^n, \quad \forall n = 1, \dots, N \\ \|\cdot\| \text{ the operator norm on any } \mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j}), \quad \forall i, j = 1, \dots, N \end{array} \right.$$

(here $\mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j})$ is the set of bounded operators from $\mathbb{L}^{\otimes i}$ to $\mathbb{L}^{\otimes j}$).

We call *symmetric* any element of $\mathbb{L}^{\otimes n}$ invariant by the action of $\sigma_{i,j}^n$, $i, j \leq n$.

We call *state of the N-particle system* an element of

$$(82) \quad \mathcal{D}_N = \{F \in \mathbb{L}^{\otimes n} \mid F > 0, \quad \|F\| = 1 \quad \text{and } F \text{ is symmetric}\}.$$

²The fact that the second and fourth lines of (79) define a mapping on the whole tensor space $\mathbb{L}^{\otimes n}$ results easily from the definition of tensors products through the so-called universal property [20]. Indeed, let φ_n be the natural embedding $\mathbb{L}^{\times n} \rightarrow \mathbb{L}^{\otimes n}$, $(v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n$, and let h be any mapping $\mathbb{L}^{\times n} \rightarrow \mathbb{L}^{\times n'}$, then the universal property of tensor products says that there is a unique map $\tilde{h} : \mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n'}$ such that $\tilde{h} \circ \varphi_n = \varphi_{n'} \circ h$. Taking $n' = n-1$, $h(v_1, \dots, v_k, \dots, v_n) = (\text{trace}(v_k)v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n)$ for Tr_n^k , and $n' = n$, $h(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = (v_1, \dots, v_j, \dots, v_i, \dots, v_n)$ for $\sigma_{i,j}^n$ give the desired extensions.

For $j = 0, \dots, N$, the j -particle marginal of $F^N \in (\mathbb{L}^{\otimes N})_1^+$ is defined as the the partial trace of order $N - j$ of F^N , that is

$$(83) \quad F_j^N = \text{Tr}^N \text{Tr}^{N-1} \dots \text{Tr}^{j+1} F^N, \quad F_N^N := F^N.$$

Note that $F_j^N \in \mathbb{L}^{\otimes j}$ ($F_0^N = 1 \in \mathbb{L}^{\otimes 0} := \mathbf{C}$) and $F_j^N > 0$, $\|F_j^N\|^j = \|F^N\|^N$ since Tr is positivity and norm preserving, and obviously F_j^N is symmetric as F^N . That is to say:

$$F_j^N \in \mathcal{D}_j.$$

(B) The evolution of a state F^N in $\mathbb{L}^{\otimes N}$ is supposed to be given by the N -particle dynamics associated to a two-body interaction:

$$(84) \quad \frac{d}{dt} F^N = (K^N + V^N) F^N,$$

where the operators on the right hand side are constructed as follows.

$$(85) \quad K^N = \sum_{i=1}^N \mathbb{I}_{\mathbb{L}}^{\otimes(i-1)} \otimes K \otimes \mathbb{I}_{\mathbb{L}}^{\otimes(N-i)}$$

and

$$(86) \quad V^N = \frac{1}{N} \sum_{1 \leq i < j \leq N} V_{i,j}, \quad V_{i,j} := \sigma_{1,i}^N \sigma_{2,j}^N V \otimes \mathbb{I}_{\mathbb{L}^{\otimes(N-2)}} \sigma_{1,i}^N \sigma_{2,j}^N$$

for a (possibly unbounded) operator K acting on \mathbb{L} and a bounded two-body (potential) operator V acting on $\mathbb{L}^{\otimes 2}$.

We assume furthermore that K is the generator of a strongly continuous, isometric, positivity preserving semigroup (in \mathbb{L})

$$(87) \quad e^{Kt} F > 0 \quad \text{if} \quad F > 0; \quad \|e^{tK}\| = 1.$$

and $K^N + V^N$ is the generator of a strongly continuous, isometric, positivity preserving semigroup (in $\mathbb{L}^{\otimes N}$)

$$(88) \quad e^{(K^N + V^N)t} F^N > 0 \quad \text{if} \quad F^N > 0; \quad \|e^{t(K^N + V^N)}\| = 1.$$

Finally, for any $F \in \mathbb{L}$, $F^N \in \mathbb{L}^{\otimes N}$ and $i, r > j$, we assume

$$(89) \quad \text{Tr}(KF) = 0 \quad \text{and} \quad \text{Tr}^{j,N}(V_{i,r} F^N) = 0.$$

This last property is necessary to deduce the forthcoming hierarchy.

*For the **K**, **S** and **Q** models, the ingredients in (84) are given in Sections 5 and 2, where (87)-(89) are shown to be satisfied.*

Note the symmetry property of the equation (84) induced by the definition of V^N : if the initial condition F_0^N for (84) is symmetric, then $F^N(t)$ is still symmetric.

Hierarchies. The family of j -marginals, $j = 1, \dots, N$, are solutions of the BBGKY hierarchy of equations

$$(90) \quad \partial_t F_j^N = \left(K^j + \frac{T_j}{N} \right) F_j^N + \frac{(N-j)}{N} C_{j+1} F_{j+1}^N$$

where:

$$(91) \quad K^j = \sum_{i=1}^j \mathbb{I}_{\mathbb{L}}^{\otimes(i-1)} \otimes K \otimes \mathbb{I}_{\mathbb{L}}^{\otimes(j-i)},$$

$$(92) \quad T_j = \sum_{1 \leq i < r \leq j} T_{i,r} \quad \text{with } T_{i,r} = V_{ir}$$

and

$$(93) \quad C_{j+1} F_{j+1}^N = \text{Tr}^{j+1} \left(\sum_{i \leq j} V_{i,j+1} F_{j+1}^N \right) = \sum_{i=1}^j C_{i,j+1} F_{j+1}^N,$$

$$(94) \quad C_{i,j+1} : \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}, \quad C_{i,j+1} F_{j+1}^N = \text{Tr}^{j+1} (V_{i,j+1} F_{j+1}^N),$$

Indeed, thanks to (89) we get easily by applying $\text{Tr}^{j,N}$ on (84) that

$$\frac{d}{dt} F_j^N = \left(K^j + \frac{T_j}{N} \right) F_j^N + \frac{1}{N} \text{Tr}^{j,N} \left(\sum_{1 \leq i \leq j < k \leq N} V_{i,k} F^N \right)$$

By symmetry of F^N and $V_{i,k}$ we get $\text{Tr}^{j,N} (V_{i,k} F^N) = \text{Tr}^{j+1} (V_{i,j+1} F_{j+1}^N)$ for all $k > j$ and (90) follows.

Note that, thanks to the assumption (80) and for all $i \leq j = 1, \dots, N$,

$$(95) \quad \|T_i\| \leq j^2 \|V\|, \quad \text{and } \|C_{i,j+1}\| \leq j \|V\|$$

(meant for $(\|T_i\|_{\mathbb{L}^{\otimes i} \rightarrow \mathbb{L}^{\otimes i}}, \|C_{i,j+1}\|_{\mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}}, \|V\|_{\mathbb{L}^{\otimes 2} \rightarrow \mathbb{L}^{\otimes 2}}$ using (81)).

We introduce the non-linear mapping $Q(F, F)$, $Q : \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{L}$ by the formula

$$(96) \quad Q(F, F) = \text{Tr}^2 (V_{1,2} (F \otimes F))$$

and the nonlinear mean-field equation on \mathbb{L}

$$(97) \quad \partial_t F = KF + Q(F, F), \quad F(0) \geq 0, \quad \|F(0)\|_1 = 1.$$

Eq. (97) is the Boltzmann, Povzner or Hartree equation according to the specifications established in the table above. In full generality we will assume

(C) (97) has for all time a unique solution $F(t) > 0$ and $\|F(t)\| = 1$.

For the **K**, **S** and **Q** models, (C) is true by standard perturbations methods.

Correlation error. To introduce the correlation errors, we need to extend slightly the above structure.

For any subset $J \subset \{1, \dots, N\}$ we first define

$$(98) \quad \mathbb{L}_N^{\otimes J} := \bigotimes_{i=1}^N \mathbb{L}^{\otimes \chi_J(i)},$$

where χ_J is the characteristic function of J and $\mathbb{L}^{\otimes 0} = \mathbf{C}$.

Then we introduce $\mathbb{L}^{\otimes J}$, the subspace of $\mathbb{L}_N^{\otimes J}$ formed by vectors of the form $\bigotimes_{i=1}^N v_i$ where $v_i = 1 \in \mathbf{C}$ for $i \notin J$ and $v_i \in \mathbb{L}$ for $i \in J$. Note that $\mathbb{L}^{\otimes J}$ is sent to $\mathbb{L}^{\otimes |J|}$ by the mapping

$$\Pi : \bigotimes_{i=1}^N v_i \in \mathbb{L}^{\otimes J} \mapsto \bigotimes_{i \in J} v_i \in \mathbb{L}^{\otimes |J|}.$$

We define a norm on $\mathbb{L}^{\otimes J}$ by

$$\|\cdot\|_{\mathbb{L}^{\otimes J}} = \|\Pi(\cdot)\|_1.$$

For $F \in \mathbb{L}$ and $K \subset J \subset \{1, \dots, N\}$ we introduce the linear operator $[F]_J^{\otimes K}$, defined through its action on factorized elements as

$$(99) \quad [F]_J^{\otimes K} : \mathbb{L}^{\otimes J/K} \rightarrow \mathbb{L}^{\otimes J}$$

$$\bigotimes_{i=1}^N v_i \mapsto \bigotimes_{i=1}^N a_i,$$

$$\text{where } \begin{cases} a_s = 1 \in \mathbf{C} & \text{if } s \notin J \\ a_s = F & \text{if } s \in K \\ a_s = v_s & \text{if } s \in J/K \end{cases}.$$

Note that, for $K, K' \subset J$, $K \cap K' = \emptyset$, we have the composition

$$(100) \quad [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} = [F]_J^{\otimes (K \cup K')} = [F]_J^{\otimes K'} [F]_{J/K}^{\otimes K},$$

and more generally, for all F, G ,

$$(101) \quad [F]_J^{\otimes K} [G]_{J/K}^{\otimes K'} = [G]_J^{\otimes K'} [F]_{J/K}^{\otimes K}.$$

For any subset $J \subset \{1, \dots, N\}$, we define the *correlation error* by

$$(102) \quad E_J = \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} F_{J/K}^N$$

where F solves (97), the operator $[F]_J^{\otimes K}$ is defined by (99) and $F_L^N \in \mathbb{L}^{\otimes L}$ is defined through its decomposition on factorized states. Namely if

$$F^N = \sum_{\ell_1, \dots, \ell_N} c_{\ell_1, \dots, \ell_N} v_{\ell_1} \otimes \cdots \otimes v_{\ell_N},$$

then

$$F_L^N = \sum_{\ell_1, \dots, \ell_N} c_{\ell_1, \dots, \ell_N} a_{\ell_1} \otimes \cdots \otimes a_{\ell_N},$$

$$\text{where } \begin{cases} a_s = \text{Tr}(v_s) \in \mathbf{C} & \text{if } s \notin L \\ a_s = v_s & \text{if } s \in L \end{cases}.$$

The link between the definition of F_L^N and the definition of the marginals F_j^N given in (83) is the following:

$$(103) \quad F_{\{1, \dots, \ell\}}^N = F_\ell^N \otimes (1)^{\otimes (N-\ell)} \in \mathbb{L}^{\otimes \ell} \otimes (\mathbb{L}^{\otimes 0})^{\otimes (N-\ell)}.$$

The formula inverse to (102) reads

$$(104) \quad F_J^N = \sum_{K \subset J} [F]_J^{\otimes K} E_{J/K}.$$

Note that the contribution in the right hand side of (104) corresponding to $K = J$ and $K = \emptyset$ are $F^{\otimes |J|}$ and E_J respectively. To prove (104), we plug (102) in the r.h.s. of (104) and we use (100):

$$\begin{aligned} \sum_{K \subset J} [F]_J^{\otimes K} E_{J/K} &= \sum_{K \subset J} [F]_J^{\otimes K} \left[\sum_{K' \subset J/K} (-1)^{|K'|} [F]_{J/K}^{\otimes K'} F_{(J/K)/K'}^N \right] \\ &= \sum_{K \cup K' \subset J} \sum_{\substack{K \subset J \\ K' \cap K = \emptyset}} (-1)^{|K'|} [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} F_{J/(K \cup K')}^N \\ &= \sum_{L \subset J} \left(\sum_{K' \subset L} (-1)^{|K'|} \right) [F]_J^{\otimes L} F_{J/L}^N = F_J^N \end{aligned}$$

since $\sum_{K' \subset L} (-1)^{|K'|} = \sum_{k'=0}^{|L|} \binom{|L|}{k'} (-1)^{k'} = 0^{|L|} = 0$ if $L \neq \emptyset$, and $= 1$ if $L = \emptyset$ (since $\sum_{K' \subset \emptyset} (-1)^{|K'|} = (-1)^0 = 1$).

One notices that since F_j^N is the marginal of some F^N which decomposes on elements of the form $v_1 \otimes \cdots \otimes v_N$, F_j^N decomposes on elements of the form $(\prod_{k=j+1}^N \text{Tr} v_k) v_1 \otimes \cdots \otimes v_j$. Since one knows that F_j^N is symmetric, it is enough to choose one bijection

$i_J : \{1, \dots, j\} \rightarrow J$, $|J| = j$, and consider the mapping

$$(105) \quad \begin{aligned} \Phi_{i_J} : \mathbb{L}^{\otimes |J|} &\xrightarrow{\Phi_{i_J}} \mathbb{L}^{\otimes J} \\ \otimes_{j \in J} v_j \in \mathbb{L}^{\otimes |J|} &\mapsto \otimes_{i=1}^N a_i \in \mathbb{L}^{\otimes J} \end{aligned}$$

$$(106) \quad F_{|J|}^N \mapsto F_J^N$$

where $a_s = 1$ if $i \notin J$ and $a_{i_J(j)} = v_j$.

Φ_{i_J} is obviously one-to-one since i_J is so, and, though (105) depends on the embedding chosen, (106) does not: Φ_{i_J} restricted to the space $\mathbb{L}_S^{\otimes |J|}$ of symmetric-by-permutation elements of $\mathbb{L}^{\otimes |J|}$, depends only on J and not on i_J . We will call Φ_J this restriction,

$$(107) \quad \Phi_J = \Phi_{i_J}|_{\mathbb{L}_S^{\otimes |J|}}.$$

The same argument is also valid for E_J which enjoys the same symmetry property than F_J^N and we define

$$(108) \quad E_{|J|} = \Phi_J^{-1} E_J.$$

Φ_J is obviously isometric and we have that

$$(109) \quad \|E_J\|_{\mathbb{L}^{\otimes J}} = \|E_{\{1, \dots, |J|\}}\|_{\mathbb{L}^{\otimes \{1, \dots, |J|\}}} = \|E_{|J|}\|_1.$$

Therefore, considering the one-to-one correspondence Φ_J , it is enough to compute/estimate the quantities $E_j, j = 1, \dots, N$. E_j and F_j^N are linked by

$$(110) \quad \begin{cases} E_j &= \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \Phi_{J/K} F_{j-|K|}^N \\ F_j^N &= \sum_{K \subset J} [F]_J^{\otimes K} \Phi_{J/K} E_{j-|K|}. \end{cases}$$

For the \mathbf{K} , \mathbf{S} and \mathbf{Q} models, the corresponding expressions are given in Sections 5 and 2.

A.2. Main results similar to [25]. The kinetic errors $E_j, j = 1, \dots, N$, satisfy the system of equations

$$(111) \quad \begin{aligned} \partial_t E_j &= \left(K^j + \frac{1}{N} T_j \right) E_j + D_j E_j \\ &+ D_j^1 E_{j+1} + D_j^{-1} E_{j-1} + D_j^{-2} E_{j-2}, \end{aligned}$$

where the operators $D_j, D_j^1, D_j^{-1}, D_j^{-2}, j = 1, \dots, N$, are defined in Appendix B below, equations (120)-(121), together with the proof of (111). Moreover, since (122) holds true, we know by Remark 3.2 in [25], that the proof of Theorem 2.1 (and therefore Corollary 2.2) in [25] remain valid in our present setting.

We get the following result.

Proposition A.1. *The statements of Theorem 2.2 hold true in the abstract setting defined in Section A.1.*

A.3. Asymptotic expansion. It is easy to see that the proofs of the main results expressed in Section 3 are adaptable in an elementary way to the present abstract paradigm. Indeed they use only the three properties stated in Remark 4.4, valid in the present setting as pointed out at the very end of Appendix B, formula (122), together with (87)-(88).

Therefore, the statements contained in Theorem 3.1 and Corollary 3.4 hold true, verbatim, under the hypothesis of Theorem 2.2, and with the definition of corrections errors given by the first line of (110) and replacing $\frac{\|V\|_{L^\infty}}{\hbar}$ by $\|V\|$ in (65).

Moreover defining now $F_j^{N,n}$ by truncating the second line of (110) at order n , that is

$$F_j^{N,n} = \sum_{K \subset J} [F]_J^{\otimes K} \Phi_{J/K} E_{j-|K|}^n$$

where E_j^n is defined by (36), Theorem 3.2 reads as follows.

Theorem A.2. *[abstract] Let $F^N(t)$ the solution of the N body system (84) with initial datum $F^N(0) = F^{\otimes N}$, $0 < F \in \mathbb{L}$, $\|F\|_1 = 1$, and $F(t)$ the solution of the mean-field equation (97) with initial datum F .*

Then, for all $n \geq 0$ and $N \geq 4(eA_t^{2n}j)^2$,

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2tC_{2n}(t)eA_t^{2n}j}{\sqrt{N}}.$$

The statements of Corollary 3.4 and Theorem 3.5 (with the hypothesis of Theorem A.2), and the Remarks 3.3 and 3.6 remain verbatim true.

APPENDIX B. DERIVATION OF THE CORRELATION HIERARCHY (111)

From the definition of E_j (cf. (102)) we find

$$\partial_t E_J = \sum_{K \subset J} (-1)^{|K|} \left(\partial_t ([F]_J^{\otimes K}) F_{J/K}^N + [F]_J^{\otimes K} \partial_t F_{J/K}^N \right)$$

Moreover, by (99)

$$(112) \quad \partial_t \left([F]_J^{\otimes K} \right) = \sum_{k_0 \in K} [F]_J^{\otimes K/\{k_0\}} [\partial_t F]_{J/(K/\{k_0\})}^{\otimes \{k_0\}}.$$

Applying Φ_J defined in (108) to the BBGKY hierarchy (90), one finds easily that F_J^N satisfies, denoting $\alpha(j, N) := \frac{N-j}{N}$,

$$(113) \quad \partial_t F_J^N = K^J F_J^N + \frac{1}{N} \sum_{i < r \in J} T_{i,r} F_J^N + \alpha(j, N) \sum_{i \in J} C_{i,j+1} F_{J \cup \{j+1\}}^N$$

(for $j+1 \notin J$).

By the mean-field equation (97) we deduce that

$$(114) \quad \begin{aligned} \partial_t E_J &= \sum_{K \subset J} (-1)^{|K|} \sum_{k_0 \in K} [F]_J^{\otimes K / \{k_0\}} (KF + Q(F, F))_{J/(K/\{k_0\})}^{\otimes \{k_0\}} F_{J/K}^N \\ &+ \sum_{K \subset J} (-1)^{|K|} \alpha(j - |K|, N) \sum_{i \in J/K} [F]_J^{\otimes K} C_{i,j+1} F_{(J/K) \cup \{j+1\}}^N \\ &+ \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left(\sum_{i \neq r \in J/K} T_{i,r} \right) F_{J/K}^N \\ &+ \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} (K^{J/K} F_{J/K}^N). \end{aligned}$$

We denote by \mathcal{T}_i , $i = 1, 2, 3, 4$, the four terms contained in the four lines of the r.h.s. of (114), respectively. The computation of the \mathcal{T}_i s is purely algebraic and will use only the four following properties

$$\left\{ \begin{array}{l} \sum_{K \subset L} (-1)^{|K|} = \delta_{|L|, \emptyset} \\ \sum_{K \subset L} |K| (-1)^{|K|} = -\delta_{|L|, 1} \\ [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} = [F]_J^{\otimes K'} [F]_{J/K}^{\otimes K} = [F]_J^{\otimes (K \cup K')}, \quad K, K' \subset J, \quad K \cap K' = \emptyset \\ C_{i,j+1} [F]_{(J/K) \cup \{j+1\}}^{\otimes K} = [F]_{(J/K)}^{\otimes K} C_{i,j+1}, \quad K \subset J, \quad j+1 \notin J. \end{array} \right.$$

In order not to make the paper too heavy, we will compute extensively two terms and leave to the reader the straightforward (but tedious) computation of the other terms.

Using the definition (102), we get

$$(115) \quad \begin{aligned} \mathcal{T}_1 &:= \sum_{K \subset J} (-1)^{|K|} \sum_{k_0 \in K} [F]_J^{\otimes K / \{k_0\}} (KF + Q(F, F))_{J/(K/\{k_0\})}^{\otimes \{k_0\}} F_{J/K}^N \\ &= - \sum_{k_0 \in J} (KF + Q(F, F))_J^{\otimes \{k_0\}} \sum_{K \subset J / \{k_0\}} (-1)^{|K|} [F]_{J/\{k_0\}}^{\otimes K} F_{(J/\{k_0\})/K}^N \\ &= - \sum_{i \in J} (KF + Q(F, F))_J^{\otimes \{i\}} E_{J/\{i\}}. \end{aligned}$$

To compute \mathcal{T}_2 we make use of the inverse definition (104):

$$\begin{aligned}
\mathcal{T}_2 &:= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} C_{i,j+1} F_{(J/K) \cup \{j+1\}}^N \\
&= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
(116) \quad &\dots C_{i,j+1} \sum_{K' \subset (J/K) \cup \{j+1\}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'}.
\end{aligned}$$

Distinguishing among the belonging or not to K' of i and $j+1$ in the r.h.s. of (116), we decompose

$$(117) \quad \mathcal{T}_2 = \mathcal{T}_2^{i,j+1 \in K'} + \mathcal{T}_2^{i,j+1 \notin K'} + \mathcal{T}_2^{i \in K', j+1 \notin K'} + \mathcal{T}_2^{i \notin K', j+1 \in K'}$$

We have

$$\begin{aligned}
\mathcal{T}_2^{i,j+1 \in K'} &= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\dots C_{i,j+1} \sum_{\substack{K' \subset (J/K) \cup \{j+1\} \\ i,j+1 \in K'}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'} \\
&= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\dots C_{i,j+1} \sum_{K'' \subset (J/K)/\{i\}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'' \cup \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\dots C_{i,j+1} \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'' \cup \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J} [F]_J^{\otimes K} \dots \\
&\dots \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K)}^{\otimes K''} C_{i,j+1} [F]_{((J/K)/K'') \cup \{j+1\}}^{\otimes \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{i \in J} \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} [F]_J^{\otimes K} \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K)}^{\otimes K''} \dots \\
&\dots C_{i,j+1} [F]_{((J/K)/K'') \cup \{j+1\}}^{\otimes \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{i \in J} \sum_{L \subset J/\{i\}} \left(\sum_{K \subset L} \alpha(j - |K|, N) (-1)^{|K|} \right) [F]_J^{\otimes L} \dots \\
&\dots C_{i,j+1} [F]_{((J/L) \cup \{j+1\})}^{\otimes \{i,j+1\}} E_{J/(L \cup \{i\})}
\end{aligned}$$

$$\begin{aligned}
&= \alpha(j, N) \sum_{i \in J} C_{i, j+1} [F]_{J \cup \{j+1\}}^{\otimes \{i, j+1\}} E_{J/\{i\}} \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i, j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i, j+1\}} E_{J/(\{i, l\})} \\
&= \alpha(j, N) \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i, j+1} [F]_{J \cup \{j+1\}}^{\otimes \{l\}} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i, j+1\}} E_{J/(\{i, l\})} \\
&= \alpha(j, N) \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} - \frac{1}{N} \sum_{i \neq l \in J} C_{i, j+1} [F]_{J \cup \{j+1\}}^{\otimes \{i, l, j+1\}} E_{J/(\{i, l\})}
\end{aligned}$$

since $\sum_{K \subset L} (-1)^{|K|} = \delta_{L, \emptyset}$. Note that there is a crucial compensation:

$$\begin{aligned}
(118) \quad \mathcal{T}_1 + \mathcal{T}_2^{i, j+1 \in K'} &= -\frac{j}{N} \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} [Q(F, F)]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{l\}} E_{J/\{i, l\}}.
\end{aligned}$$

The computations of $\mathcal{T}_2^{i, j+1 \notin K'}$, $\mathcal{T}_2^{i \in K', j+1 \notin K'}$, $\mathcal{T}_2^{i \notin K', j+1 \in K'}$ go the same way and we omit it here.

We consider a similar dichotomy for the term

$$\begin{aligned}
\mathcal{T}_3 &:= \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left(\sum_{i \neq r \in J/K} T_{i, r} \right) F_{J/K}^N \\
&= \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left(\sum_{i \neq r \in J/K} T_{i, r} \right) \sum_{K' \subset J/K} [F]_{J/K}^{\otimes K'} E_{J/(K \cup K')}.
\end{aligned}$$

according, this time, to the cases $i, r \in K'$, $i, r \notin K'$, $i \in K', r \notin K'$ and $i \notin K', r \in K'$. The computation of the different terms uses the same “tricks” than for \mathcal{T}_2 and we omit them.

Finally, we obtain easily that

$$(119) \quad \mathcal{T}_4 := \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} (K^{J/K} F_{J/K}^N) = K^J E_J.$$

Summing up all the contributions $\mathcal{T}_1, 1 = 1, \dots, 4$, we get (111) after specializing to the case $J = \{1, \dots, j\}$, using (108) and setting

$$\begin{aligned}
D_j &: \mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 1, \dots, N, \\
E_j &\mapsto \frac{N-j}{N} \sum_{i \in J} C_{i,j+1} \left([F]_{J \cup \{j+1\}}^{\otimes \{i\}} \Phi_{(J \cup \{j+1\})/\{i\}} E_j + [F]_{J \cup \{j+1\}}^{\otimes \{j+1\}} E_j \right), \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} ([F]_{J \cup \{j+1\}}^{\otimes \{l\}} \Phi_{(J/\{l\}) \cup \{j+1\}} E_j) \\
D_j^1 &: \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 1, \dots, N-1, \\
E_{j+1} &\mapsto \frac{N-j}{N} C_{j+1} E_{j+1}, \\
D_j^{-1} &: \mathbb{L}^{\otimes(j-1)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 2, \dots, N, \\
E_{j-1} &\mapsto \left(-\frac{j}{N} \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} + \frac{1}{2N} \sum_{i,r \in J} T_{i,r} [F]_J^{\otimes \{i\}} \right) \Phi_{J/\{i\}} E_{j-1}, \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i,j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{j+1\}} \Phi_{J/\{l\}} E_{j-1} \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i,j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i\}} \Phi_{(J/\{i,l\}) \cup \{j+1\}} E_{j-1} \\
D_j^{-2} &: \mathbb{L}^{\otimes(j-2)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 3, \dots, N, \\
E_{j-2} &\mapsto \frac{1}{2N} \sum_{i,s \in J} T_{i,s} [F]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{s\}} \Phi_{J/\{i,s\}} E_{j-2} \\
(120) \quad &\quad - \frac{1}{N} \sum_{i \neq l \in J} [Q(F, F)]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{l\}} \Phi_{J/\{i,l\}} E_{j-2}.
\end{aligned}$$

where, by convention,

$$(121) \quad \begin{cases} D_N^1 := D_1^{-2} := 0 \\ D_1^{-1}(E_0) := -\frac{1}{N} Q(F, F), \\ D_2^{-2}(E_0) := \frac{1}{N} (T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F)). \end{cases}$$

Note that one has the following estimates:

$$(122) \quad \|D_j\|, \|D_j^1\| \leq j \text{ and } \|D_j^{-1}\|, \|D_j^{-2}\|, \|D_1^{-1}(E_0)\|, \|D_2^{-2}(E_0)\| \leq \frac{j^2}{N}.$$

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