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► **To cite this version:**

Thierry Paul, Mario Pulvirenti. Asymptotic expansion of the mean-field approximation. 2017. hal-01579763v3

HAL Id: hal-01579763

<https://hal.science/hal-01579763v3>

Preprint submitted on 10 Oct 2017 (v3), last revised 12 Oct 2018 (v5)

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ASYMPTOTIC EXPANSION OF THE MEAN-FIELD APPROXIMATION

THIERRY PAUL AND MARIO PULVIRENTI

ABSTRACT. We established and estimate the full asymptotic expansion in integer powers of $\frac{1}{N}$ of the $[\sqrt{N}]$ first marginals of N -body systems evolution lying in a general paradigm containing Kac models and non-relativistic quantum evolution. We prove that the coefficients of the expansion are, at any time, explicitly computable given the knowledge of the linearization on the one-body meanfield kinetic limit equation. Instead of working directly with the corresponding BBGKY-type hierarchy, we follow a method developed in [24] for the meanfield limit, dealing with error terms analogue to the v -functions used in previous works. As a by-product we get that the rate of convergence to the meanfield limit in $\frac{1}{N}$ is optimal.

1. INTRODUCTION: MOTIVATION AND MAIN RESULTS

Mean field limit concerns systems of interacting (classical or quantum) particles whose number diverges in a way linked with a rescaling of the interaction insuring an equilibrium between interaction and residual kinetic energies. In the case of an additive one-body kinetic energy part and a two-body interaction, and without taking in consideration quantum statistics, this equilibrium is reached by putting in front of the interaction a coupling constant proportional to the inverse of the number of particles.

The system is then described by isolating the evolution of one (or j) particle(s) and averaging over all the other. This leads to a partial information on the system driven by the so-called j -marginals. The mean field theory insures that the j -marginals tend, as the number of particles diverges, to the j -tensor power of the solution of a non-linear one-body meanfield equation (Vlasov, Hartree,...) issued from the 1-marginal on the initial N -body state. This program has been achieved in many different situations, and the literature concerning the mean field approach is protuberant. We refer to the review article [27] for a reasonable bibliography.

Much less is known about the fluctuations around this limit, namely the correction to be added to the factorized limit in order to get better approximations of the true evolution of the j -marginals.

The identification of the leading order of these fluctuations with a Gaussian stochastic process has been established in the quantum context first in [15] and in the classical one in [5]. For the classical dynamics of hard

spheres, the fluctuations around the Boltzmann equation have been computed at leading order in [26], generalizing to non-equilibrium states the results of [3]. More recently, for the quantum case, fluctuations near the Hartree dynamics has been derived in [21] (after [20]) and in [2] also for the grand canonical ensemble formalism (number of particles non fixed), using in both cases the methods of second quantization (Fock space).

Recently, we developed (together with S. Simonella) in [24] a method to derive mean field limit, alternative to the ones using empirical measures or direct estimates on the “BBGKY-type” hierarchies (systems of coupled equations satisfied by the set of the j -marginals). This method rather uses the hierarchy followed by the “kinetic errors” E_{j-k} (defined below), already used (under the name “ v -functions”) to deal with kinetic limits of stochastic models [10, 7, 4, 11, 12, 6, 8, 13] and recently investigated in the more singular low density limit of hard spheres [25]. These quantities are, roughly speaking, the coefficient of the decomposition of the j -marginal as a linear combination of the k th tensor powers, $k = 1, \dots, j$, of the solution of the mean-field equation issued from of the 1-marginal of the initial full state. We developed in [24] a strategy suitable in particular for Kac models (homogeneous original one [16, 17] and non-homogeneous [9]) and quantum mean field theory. This strategy allowed us to derive the limiting factorization property of the j -marginals up to, roughly speaking, $j \lesssim \sqrt{N}$. This threshold is, on the other side, the one obtained by heuristic arguments as shown in [24].

Here and in all this article, N denotes the number of particles of the system under consideration.

In the present note we provide and estimate a full asymptotic expansion in powers of $\frac{1}{N}$ of the difference between the evolution of j -marginals and its factorized leading order form (Corollary 1.5), following a similar result for the kinetic errors $E_j(t)$ (Theorem 1.4). Our results are valid for $j \lesssim \sqrt{N}$ in an abstract paradigm, generalization of the abstract formalism developed in [24] and described in Appendix A, and applies of course to the different Kac models and quantum mean field theory treated in [24]. Moreover, we show that the additional knowledge of the linearization of the mean field

flow, around the meanfield solution issued from the 1-marginal of the initial data, gives an explicit construction of the full asymptotic expansion of the j -marginals in powers of $\frac{1}{N}$.

We will state in this section the quantum results and postpone in Section 5 and in the Appendix A the Kac's type and the abstract results, respectively. Sections 2 and 3 contain the algebraic and analytical proofs of our results in the quantum case, immediately transposable to the Kac and abstract situations as shown in Section 5 and Appendix A. In Section 4 one compute explicitly the first terms of the asymptotic expansions obtained in the quantum case and rely them to previous works.

1.1. Quantum mean-field. Let $\mathcal{L}^1(L^2(\mathbf{R}^d))$ be the space of trace class operators on $L^2(\mathbf{R}^d)$, with their associated norms.

We consider the evolution of a system of N quantum particles interacting through a (real-valued) two-body, even potential V , described for any value of the Planck constant $\hbar > 0$ by the Schrödinger equation

$$i\hbar\partial_t\psi = H_N\psi, \quad \psi|_{t=0} = \psi_{in} \in \mathfrak{H}_N := L^2(\mathbf{R}^d)^{\otimes N},$$

where

$$H_N := -\frac{1}{2}\hbar^2 \sum_{k=1}^N \Delta_{x_k} + \frac{1}{2N} \sum_{1 \leq k, l \leq N} V(x_k - x_l).$$

We will suppose in the whole present paper that the N -body Hamiltonian H_N is essentially self-adjoint.

Instead of the Schrödinger equation written in terms of wave functions, we shall rather consider the quantum evolution of density matrices. An N -body density matrix is an operator F^N such that

$$0 \leq F^N = (F^N)^*, \quad \text{trace}_{\mathfrak{H}_N}(F^N) = 1.$$

The evolution of the density matrix $F^N \mapsto F^N(t)$ of a N -particle system is governed for any value of the Planck constant $\hbar > 0$ by the von Neumann equation

$$(1) \quad \partial_t F^N = \frac{1}{i\hbar} [H_N, F^N],$$

equivalent to the Schrödinger equation when $F^N(0)$ is a rank one projector, modulo a global phase.

Positivity, norm and trace are obviously preserved by (1) since H_N is taken essentially self-adjoint.

For each $j = 1, \dots, N$, the j -particle marginal $F_j^N(t)$ of $F^N(t)$ is the unique operator on \mathfrak{H}_j such that

$$\text{trace}_{\mathfrak{H}_N}[F^N(t)(A_1 \otimes \dots \otimes A_j \otimes I_{\mathfrak{H}_{N-j}})] = \text{trace}_{\mathfrak{H}_j}[F_j^N(t)(A_1 \otimes \dots \otimes A_j)].$$

for all A_1, \dots, A_j bounded operators on \mathfrak{H} . Alternatively and equivalently, the F_j^N can be defined by the partial trace of F^N on the $N - j$ last ‘‘particles’’: defining F^N through its integral kernel $F^N(x_1, x'_1; \dots; x_N, x'_N)$, the integral kernel of F_j^N is defined as (see [1])

$$(2) \quad \begin{aligned} & F_j^N(x_1, x'_1; \dots; x_j, x'_j) := (\text{Tr}^{j+1} \dots \text{Tr}^N F^N)(x_1, x'_1; \dots; x_j, x'_j) \\ & := \int_{\mathbf{R}^{d(N-j)}} F^N(x_1, x'_1; \dots; x_j, x'_j; x_{j+1}, x'_{j+1}; \dots; x_N, x'_N) dx_{j+1} dx_N. \end{aligned}$$

It will be convenient for the sequel to rewrite (1) in the following operator form

$$(3) \quad \partial_t F^N = (K^N + V^N) F^N$$

where K^N, V^N are operators on $\mathcal{L}^1(L^2(\mathbf{R}^{Nd}))$ defined by

$$(4) \quad K^N = \frac{1}{i\hbar} \left[-\frac{\hbar^2}{2} \Delta_{\mathbf{R}^{dN}}, \cdot \right], \quad V^N = \frac{1}{2N} \sum_{k,l} V_{k,l} \text{ with } V_{k,l} := \frac{1}{i\hbar} [V(x_k - x_l), \cdot].$$

As mentioned before the (essential) self-adjointness of H_N implies that

$$(5) \quad \|e^{t(K^N + V^N)}\|_{\mathcal{L}^1(L^2(\mathbf{R}^d)) \rightarrow \mathcal{L}^1(L^2(\mathbf{R}^d))} = \|e^{tK^N}\|_{\mathcal{L}^1(L^2(\mathbf{R}^d)) \rightarrow \mathcal{L}^1(L^2(\mathbf{R}^d))} = 1, \quad t \in \mathbf{R}.$$

We will denote

$$(6) \quad \mathbb{L} := \mathcal{L}^1(L^2(\mathbf{R}^d)) \text{ so that } \mathbb{L}^{\otimes n} = \mathcal{L}^1(L^2(\mathbf{R}^{nd})), \quad n = 1, \dots, N,$$

and, with a slight abuse of notation,

$$(7) \quad \begin{cases} \|\cdot\|_1 & \text{the trace norm on any } \mathbb{L}^{\otimes j}, \\ \|\cdot\| & \text{the operator norm on any } \mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j}) \end{cases}$$

for $i, j = 1, \dots, N$ (here $\mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j})$ is the set of bounded operators from $\mathbb{L}^{\otimes i}$ to $\mathbb{L}^{\otimes j}$).

A density matrix $F^n \in \mathbb{L}^{\otimes n}$ is called symmetric if its integral kernel $F^n(x_1, x'_1; \dots; x_n, x'_n)$ is invariant by any permutation

$$(x_i, x'_i) \leftrightarrow (x_j, x'_j), \quad i, j = 1, \dots, n.$$

Note that the symmetry of F^N is preserved by the equation (1) due to the particular form of the potential.

We define, for $n = 1, \dots, N$,

$$(8) \quad \mathcal{D}_n = \{F \in \mathbb{L}^{\otimes n} \mid F > 0, \quad \|F\|_1 = 1 \quad \text{and } F \text{ is symmetric}\}.$$

Note that $F_j^N \in \mathbb{L}^{\otimes j}$ ($F_0^N = 1 \in \mathbb{L}^{\otimes 0} := \mathbf{C}$) and $F_j^N > 0$, $\|F_j^N\|_1 = \|F^N\|_1$, and obviously F_j^N is symmetric as F^N . That is to say:

$$F_j^N \in \mathcal{D}_j.$$

The family of j -marginals, $j = 1, \dots, N$, are solutions of the BBGKY hierarchy of equations (see [1])

$$(9) \quad \partial_t F_j^N = \left(K^j + \frac{T_j}{N} \right) F_j^N + \frac{(N-j)}{N} C_{j+1} F_{j+1}^N$$

where:

$$(10) \quad K^j = \frac{1}{i\hbar} \left[-\frac{\hbar^2}{2} \Delta_{\mathbf{R}^{jd}}, \cdot \right]$$

$$(11) \quad T_j = \sum_{1 \leq i < r \leq j} T_{i,r} \quad \text{with } T_{i,r} = V_{ir}$$

and

$$(12) \quad C_{j+1} F_{j+1}^N = \sum_{i=1}^j C_{i,j+1} F_{j+1}^N$$

with

$$(13) \quad \begin{aligned} C_{i,j} : \quad \mathbb{L}^{\otimes(j+1)} &\rightarrow \mathbb{L}^{\otimes j} \\ C_{i,j+1} F_{j+1}^N &= \text{Tr}^{j+1} (V_{i,j+1} F_{j+1}^N), \end{aligned}$$

where Tr^{j+1} is the partial trace with respect to the $(j+1)$ th variable, as in (2).

Note that, for all $i \leq j = 1, \dots, N$,

$$(14) \quad \|T_j\| \leq j^2 \frac{\|V\|_{L^\infty}}{\hbar}, \quad \text{and } \|C_{i,j+1}\| \leq j \frac{\|V\|_{L^\infty}}{\hbar}.$$

(meant for $\|T_j\|_{\mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j}}$ and $\|C_{i,j+1}\|_{\mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}}$ in accordance with (7)).

The Hartree equation is

$$(15) \quad i\hbar \partial_t F = \left[-\frac{\hbar^2}{2} \Delta + V_F(x), F \right], \quad F(0) \in \mathcal{D}_1,$$

where $V_F(x) = \int_{\mathbf{R}^d} V(x-y) F(y, y) dy$, $F(y, y')$ being the integral kernel of F .

Note that (15) reads also

$$(16) \quad \partial_t F = K^1 F + Q(F, F),$$

with

$$(17) \quad Q(F, F) = \text{Tr}^2(V_{1,2}(F \otimes F)).$$

Since V is bounded, (15) has for all time a unique solution $F(t) > 0$ and $\|F(t)\| = 1$ (see again [1]).

In order to define the correlation error in an easy way, we need a bit more of notations concerning the variables of integral kernels.

For $i \leq j = 1, \dots, N$, we define the variables $z_i = (x_i, x'_i)$, and $Z_j = (z_1, \dots, z_j)$. For $\{i_1, \dots, i_k\} \subset \{1, \dots, j\}$, we denote by $Z_j^{\setminus \{i_1, \dots, i_k\}} \in \mathbf{R}^{2(j-k)d}$, the vector $Z_j := (z_1, \dots, z_j)$ after removing the components z_{i_1}, \dots, z_{i_k} .

Definition 1.1. For any $j = 1, \dots, N$, we define the correlation error $E_j \in \mathbb{L}^{\otimes j}$ by its integral kernel

$$(18) \quad E_j(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} (-1)^k F(z_{i_1}) \dots F(z_{i_k}) F_{j-k}^N(Z_j^{\setminus \{i_1, \dots, i_k\}}).$$

By convention and consistently with $F_0^N = \|F\| = 1$, we define

$$(19) \quad E_0 := 1 \in \mathbb{L}^{\otimes 0} := \mathbf{C}.$$

.

In [24] was shown that (18) is inverted by the following equality.

$$(20) \quad F_j^N(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(z_{i_1}) \dots F(z_{i_k}) E_{j-k}(Z_j^{\setminus \{i_1, \dots, i_k\}}), \quad j = 0, \dots, N..$$

i.e. F_j^N is the operator of integral kernel given by (20).

1.2. Main results of [24]. Theorem 2.4, theorem 2.1 and Corollary 2.2 in [24] state the following facts.

The kinetic errors E_j , $j = 1, \dots, N$, satisfy the system of equations

$$(21) \quad \begin{aligned} \partial_t E_j &= \left(K^j + \frac{1}{N} T_j \right) E_j + D_j E_j \\ &+ D_j^1 E_{j+1} + D_j^{-1} E_{j-1} + D_j^{-2} E_{j-2}, \end{aligned}$$

where the operators $D_J, D_j^1, D_j^{-1}, D_j^{-2}$, $j = 0, \dots, N$, are defined at the beginning of the Section 2, formulas (37)-(40).

Theorem 1.2 (Theorem 2.2. and Corollary 2.3 in [24]).

Let $E_j(0)$ satisfy for some $C_0 > 1, B > 0$

$$(22) \quad \begin{cases} \|E_1(0)\|_1 \leq \frac{B}{N} \\ \|E_j(0)\|_1 \leq \left(\frac{j^2}{N}\right)^{j/2} C_0^j, \quad j \geq 2. \end{cases}$$

Then, for all $t > 0$ and all $j = 1, \dots, N$, one has

$$(23) \quad \begin{cases} \|E_1(t)\|_1 \leq \frac{1}{N} \left(B_2 e^{\frac{B_1 t \|V\|_{L^\infty}}{h}} \right) \\ \|E_j(t)\|_1 \leq \left(C_2 e^{\frac{C_1 t \|V\|_{L^\infty}}{h}} \right)^j \left(\frac{j}{\sqrt{N}} \right)^j, \quad j \geq 2. \end{cases}$$

for some $B_1, C_1 > 0$, $B_2, C_2 \geq 1$ explicit (see Theorem 2.2 in [24]), and

$$(24) \quad \|F_j^N(t) - F(t)^{\otimes j}\|_1 \leq D_2 e^{\frac{D_1 t \|V\|_{L^\infty}}{h}} \frac{j^2}{N},$$

where $D_2 = \sup\{B_2, (eC_0)^2\}$, $B_1 = \sup\{B_1, 2C_1\}$.

1.3. Asymptotic expansion and main result of the present article.

Two questions arise naturally:

- (1) are the estimates (23) sharp?
- (2) could (24) be improved with a r.h.s. of any order we wish?¹

We will see below that, indeed, not only the estimates (23) are true, but $N^{j/2} E_j^N(t)$ has a full asymptotic expansion in positive powers of $(\frac{1}{N})^{\frac{1}{2}}$ (actually we will show that this expansion contains only powers $(\frac{1}{N})^{\frac{k}{2}}$ with $k + j$ even) if $N^{j/2} E_j^N(0)$ do possess such an expansion in half powers of $1/N$.

¹Of course (23) and (24) imply that

$$(25) \quad \|F_j^N(t) - F^{\otimes j} - \sum_{\substack{J=K \subset J \\ |K| \leq k \leq |J|}} F^{\otimes |K|} E_{[J]-|K|}^N\| = O(N^{-(k+1)/2}),$$

but first one cannot go further in the approximation, and second (25) is meaningless without the knowledge of the E_j^N 's.

More precisely we will show that, under the hypothesis (22) on the initial data and for all time t and all $j = 1, \dots, N$,

$$(26) \quad E_j(t) = N^{-j/2} \mathcal{E}_j, \quad \mathcal{E}_j(t) \sim \sum_{\ell=0}^{\infty} \mathcal{E}_j^\ell(t) N^{-\ell/2} \text{ with } \mathcal{E}_j^k(t) = 0 \text{ for } j+k \text{ odd}$$

when the same is true at $t = 0$.

Moreover we will see that all the \mathcal{E}_j^ℓ can be explicitly recursively computed after the knowledge of the linearization of the mean field equation (15) around the solution of (15) with initial condition $F(0) = (F^N(0))_1$. Indeed the proof will involve the “ j -kinteic linear mean field flow” defined by the linear kinetic mean field equation of order j :

$$(27) \quad \frac{d}{dt} A(t) = (K^j + \Delta_j(t)) A(t), \quad A(0) \in \mathbb{L}^{\otimes j},$$

where $\Delta_j(t) = \lim_{N \rightarrow \infty} D_j(t)$.

(27) is solved by the two parameter semigroup $U_j^0(t, s)$ solving

$$(28) \quad \begin{aligned} \partial_t U_j^0(t, s) &= (K^j + \Delta_j(t)) U_j^0(t, s). \\ U_j^0(s, s) &= I. \end{aligned}$$

Note that U_j^0 exists since K^j generates a unitary flow and Δ_j is bounded.

The reason of the terminology comes from the fact that, as shown by (37), $\Delta_1 = Q(F, \cdot) + Q(\cdot, F)$ so that, for $j = 1$, (27) is the linearization of the mean field equation (15) around its solution $F(t)$. Note moreover that, for $G^1, G^2 \in \mathbb{L}$,

$$(29) \quad \begin{aligned} \Delta_2(G^1 G^2 + G^2 G^1) &= \\ (\Delta_1 G^1) G^2 + G^1 (\Delta_1 G^2) + (\Delta_1 G^2) G^1 + G^2 (\Delta_1 G^1). \end{aligned}$$

and therefore

$$(30) \quad U_2^0(t, s)(G^1 G^2 + G^2 G^1) = (U_1^0(t, s) G^1)(U_1^0(t, s) G^2).$$

More generally, if $P_j: \mathbb{L}^j \rightarrow \mathbb{L}^{\otimes j}$ is any homogeneous polynomial invariant by permutations,

$$(31) \quad U_j^0(t, s) P_j(G^1, \dots, G^j) = P_j(U_1^0(t, s) G^1, \dots, U_1^0(t, s) G^j).$$

That is: U_j^0 drives each G^j along the linearized mean field flow “factor by factor”. Denoting by $\mathbb{L}_{sym}^{\otimes j}$ the subspace of symmetric (by permutations) vectors, we just prove the following result.

Lemma 1.3.

$$U_j^0(t, s)|_{\mathbb{L}_{sym}^{\otimes j}} = U_1^0(t, s)^{\otimes j}.$$

Note also that $U_j^0(t, s)$ is given by a convergent Dyson expansion and that, by the isometry of the flow generated by K^j and the bound (54) below, we have by Gronwall Lemma that $\|U_j^o(t, s)\| \leq e^{j|t-s|}$.

We will also need in the sequel the semigroup defined by

$$(32) \quad \begin{aligned} \partial_t U_j(t, s) &= (K^j + \frac{T_j}{N} + \frac{N-j}{N} D_j(t)) U_j(t, s). \\ U_j(s, s) &= I. \end{aligned}$$

$U_j(t)$ exists by the same argument as for $U_j^0(t)$. Moreover, in the regime j^2/N small, U_j can be also computed out of U_j^0 by a convergent perturbation expansion (in j^2/N). Indeed (37), (92) and (54) show clearly that $\|\frac{T_j}{N} + \frac{N-j}{N} D_j - \Delta_j\| \leq \frac{j^2}{N}$. Therefore, for any t , $U_j(t)$ can be approximated, up to any power of j^2/N , by a finite Dyson expansion.

Finally, we will extend (24) as we will show that F_j^N has an asymptotic expansion in positive powers of $1/N$ whose partial sums up to any order $n \geq 0$ is $O(jN^{-n-1})$ -close to F_j^N .

The main results of the present note are the following.

Theorem 1.4. *Consider for $j = 0, \dots, N$, $k = 0, \dots$, $t \geq 0$ the system of recursive relations*

$$(33) \quad \begin{cases} \mathcal{E}_j^k(t) = U_j(t, 0) \mathcal{E}_j^k(0) \\ \quad + \int_{s=0}^t U_j(t, s) (\Delta_j^- \mathcal{E}_{j-2}^k(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(s)) ds \\ \mathcal{E}_0^k(t) = \delta_{k,0}, \\ \mathcal{E}_{-1}^k(t) = \mathcal{E}_{-2}^k(t) = \mathcal{E}_j^{-1}(t) = 0 \text{ by convention.} \end{cases}$$

where $U_j(t, s)$ is the two times flow defined by (32) and $\Delta_j^+ D_j^1$, $\Delta_j^- = N D_j^{-1}$, $\Delta_j^- = N D_j^{-2}$, the D_j s being given by (37)-(40).

Then, for any $j = 1, \dots, N$, $k = 0, \dots$, $t \geq 0$, the knowledge of $\mathcal{E}_{j'}^{k'}(0)$ for $j' = 1, \dots, j+k$, $k' = 0, \dots, k$, determine in a unique way $\mathcal{E}_j^k(t)$, and

$$\mathcal{E}_j^k(t) = 0 \text{ when } j+k \text{ is odd}$$

if $\mathcal{E}_j^k(0)$ satisfies the same property.

Moreover, for all $t \in \mathbf{R}$, the solution $E_j(t)$ of (21) with initial data $E_j(0)$ satisfying (22). has an asymptotic expansion $E_j(t) \sim \sum_{k=0}^{\infty} N^{-\frac{j+k}{2}} \mathcal{E}_j^k(0)$, where $\mathcal{E}_j^k(t)$ is the solution of (33) with initial condition $\mathcal{E}_j^k(0) = \delta_{k,0} N^{\frac{j}{2}} E_j(0)$ (resp. $\delta_{k,1} N^{\frac{j}{2}} E_j(0)$) if j is even (resp. j is odd) and one has

$$\|E_j(t) - \sum_{k=0}^{2n} N^{-k/2} E_j^k(t)\|_1 \leq D_{2n}(t) N^{-n-\frac{1}{2}} (D'_{2n}(t) \frac{j^2}{N})^{j/2},$$

where $D_k(t), D'_k(t)$ are defined in (61) below and satisfy, as $k, |t| \rightarrow \infty$,

$$\log D_k(t) = \frac{3k}{2} (\log k + \frac{|t| \|V\|_{\infty}}{h}) + O(k + \frac{|t| \|V\|_{\infty}}{h}) \quad \text{and} \quad \log D'_k(t) = O(k + \frac{|t| \|V\|_{\infty}}{h}).$$

Let us define, for $j = 1, \dots, N$, $n = 0, \dots$

$$(34) \quad E_j^n(t) = \sum_{k=0}^{2n} N^{-\frac{j+k}{2}} \mathcal{E}_j^k(t)$$

(note that $E_j^n(t)$ contains only integer powers of N^{-1} , since $\mathcal{E}_j^k = 0$ when $j+k$ is odd, that is $E_j^n = \sum_{k=[(j+1)/2]}^n c_k^j N^{-k}$), and $F_j^{N,n}(t)$ the operator of integral kernel $F_j^{N,n}(t)(Z_j)$ defined by

$$(35) \quad F_j^{N,n}(t)(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_k}) E_{j-k}^n(Z_j^{\{i_1, \dots, i_k\}}),$$

(that is (101) truncated at order n , same slight abuse of notation). $F_j^{N,n}$ is therefore a polynomial of order n in $\frac{1}{N}$.

Corollary 1.5. *Let $F^N(t)$ the solution of the quantum N body system (1) with initial datum $F^N(0) = F^{\otimes N}$, $F \in \mathcal{L}(L^2(\mathbf{R}^d))$, $F \geq 0$, $\text{Tr} F = 1$, and $F(t)$ the solution of the Hartree equation (15) with initial datum F .*

Then, for all $n \geq 1$ and $N \geq 4(eA_t^{2n} j)^2$,

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2D_{2n}(t)eA_t^{2n}D'_{2n}(t)j}{\sqrt{N}}.$$

Remark 1.6. *If one is interested only to the expansion up to order $n < j$, we can change the sum in the l.h.s. of the inequality in Corollary 1.5 by a sum up to $\ell = n$.*

Proof. The proof is similar to the one of Corollary 2.2 in [24].

$$\begin{aligned}
& \|F_j^N(t) - \sum_{K \subset \{1, \dots, j\}} \prod_K^j (F^{\otimes |K|} \otimes E_{j-|K|}^n)\| \\
& \leq \sum_{k=1}^j \binom{j-k}{k} \|E_k - E_k^n\| \\
& \leq N^{-n-\frac{1}{2}} \sum_{k=1}^j \binom{j}{k} C_{2n-k}(t) \left(\frac{A_t^{2n-k} k^2}{N}\right)^{k/2} \\
& \leq N^{-n-\frac{1}{2}} C_{2n}(t) \sum_{k=1}^j j(j-1) \dots (j-k+1) \left(\frac{A_t^{2n}}{\sqrt{N}}\right)^k \frac{k^k}{k!} \\
& \leq N^{-n-\frac{1}{2}} C_{2n}(t) \sum_{k=1}^j \left(\frac{j e A_t^{2n}}{\sqrt{N}}\right)^k \leq N^{-n-\frac{1}{2}} \frac{2C_{2n}(t) e A_t^{2n} j}{\sqrt{N}}
\end{aligned}$$

for $N \geq 4(eA_t^{2n} j)^2$ (we used $\frac{k^k}{k!} \leq \frac{e^k}{\sqrt{2\pi k}}$). \square

Corollary 1.7. *The rate of convergence to the meanfield limit in $\frac{1}{N}$ is optimal.*

Remark 1.8. *In the asymptotic expansion $E_j(t) \sim \sum_{k=[(j+1)/2]}^{\infty} c_k^j(t) N^{-k}$ the coefficients $c_k^j(t)$, such as $\mathcal{E}_j^k(t)$, depend on N as well: first by the dependence of $\Delta_j^+ = (1 - \frac{j}{N})C_{j+1}$ and also by $U_j(t, s)$ defined by (42). As mentioned already the latter can be expressed as a (convergent) series in $\frac{1}{N}$ out of the linearization of the meanfield equation so that obtaining a full asymptotic expansion of $E_j(t)$ with the only knowledge of the linearization of the meanfield equation is (tedious but) elementary. Let us note also that $E_j(0)$ is allowed to depend on N , without any restriction as soon as it satisfies (22).*

2. THE RECURSIVE CONSTRUCTION

Let us recall from [24] that the hierarchy of error terms satisfy the following equation:

$$\begin{aligned}
(36) \quad \partial_t E_j &= \left(K^j + \frac{1}{N} T_j\right) E_j + D_j E_j \\
&+ D_j^1 E_{j+1} + D_j^{-1} E_{j-1} + D_j^{-2} E_{j-2}.
\end{aligned}$$

Here the four operator $D_j, D_j^1, D_j^{-1}, D_j^{-2}$, $j = 0, \dots, N$, are defined as follows (here again, $J = \{1, \dots, j\}$): for any operator $G \in \mathbb{L}^{\otimes n}$, $n = 1, \dots, N$, we denote by $G(Z_n)$ its integral kernel, and for any function $F(Z_n)$, $n = 1, \dots, N$, we define $\overline{F(Z_n)}$ as being the operator on $\mathbb{L}^{\otimes n}$ of integral kernel $F(Z_n)$ then

$$(37) \quad \begin{aligned} D_j &: \mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j} \\ E_j &\mapsto \frac{N-j}{N} \sum_{i \in J} C_{i,j+1} \left(\overline{F(z_i) E_j(Z_{j+1}^{\{i\}})} + \overline{F(z_{j+1}) E_j(Z_j)} \right) \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l) E_j(Z_{j+1}^{\{l\}})} \end{aligned}$$

$$(38) \quad \begin{aligned} D_j^1 &: \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j} \\ E_{j+1} &\mapsto \frac{N-j}{N} C_{j+1} E_{j+1} \end{aligned}$$

$$(39) \quad \begin{aligned} D_j^{-1} &: \mathbb{L}^{\otimes(j-1)} \rightarrow \mathbb{L}^{\otimes j} \\ E_{j-1} &\mapsto \frac{1}{N} \sum_{i,r \in J} T_{i,r} \overline{F(z_i) E_{j-1}(Z_j^{\{i\}})} - \frac{j}{N} \sum_{i \in J} \overline{Q(F, F)(z_i) E_{j-1}(Z_j^{\{i\}})} \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l) F(z_{j+1}) E_{j-1}(Z_j^{\{l\}})} \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l) F(z_i) E_{j-1}(Z_{j+1}^{\{i,l\}})} \end{aligned}$$

$$(40) \quad \begin{aligned} D_j^{-2} &: \mathbb{L}^{\otimes(j-2)} \rightarrow \mathbb{L}^{\otimes j} \\ E_{j-2} &\mapsto \frac{1}{N} \sum_{i,s \in J} T_{i,s} \overline{F(z_i) F(z_r) E_{j-2}(Z_j^{\{i,r\}})} \\ &\quad - \frac{1}{N} \sum_{i \neq l \in J} \overline{Q(F, F)(z_i) F(z_l) E_{j-2}(Z_l^{\{i,l\}})}. \end{aligned}$$

where, by convention,

$$(41) \quad \begin{cases} D_N^1 := D_1^{-2} := 0 \\ D_1^{-1}(E_0) := -\frac{1}{N} Q(F, F), \\ D_2^{-2}(E_0) := \frac{1}{N} (T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F)). \end{cases}$$

In (37)-(40), $F(z)$ is meant as being the integral kernel of $F(t)$ solution of the Hartree equation 15.

We define $H_j(t) = K^j + T_j/N + D_j(t)$ and recall the definition of the (two parameters) semigroup $U_j(t, s)$ satisfying, for all $s, t \in \mathbf{R}$,

$$(42) \quad \begin{aligned} \partial_t U_j(t, s) &= H_j(t)U_j(t, s), \quad j = 1, \dots, N \\ U_j(s, s) &= I =: U_0(t, s) \end{aligned}$$

Let us perform the following rescaling

$$(43) \quad \begin{cases} E_j &= N^{-j/2} \mathcal{E}_j \\ \Delta_j^+ &= D_j^1 \\ \Delta_j^- &= ND_j^{-1} \\ \Delta_j^- &= ND_j^{-2} \end{cases}$$

We find easily that, again with the convention (41),

$$(44) \quad \begin{aligned} \partial_t \mathcal{E}_j &:= H_j \mathcal{E}_j + N^{-\frac{1}{2}} D_j^1 \mathcal{E}_{j+1} + N^{\frac{1}{2}} D_j^{-1} \mathcal{E}_{j-1} + ND_j^{-2} \mathcal{E}_{j-2} \\ &= H_j \mathcal{E}_j + N^{-\frac{1}{2}} \Delta_j^+ \mathcal{E}_{j+1} + N^{-\frac{1}{2}} \Delta_j^- \mathcal{E}_{j-1} + \Delta_j^- \mathcal{E}_{j-2} \\ &= H_j \mathcal{E}_j + \Delta_j^- \mathcal{E}_{j-2} + O(N^{-\frac{1}{2}}). \end{aligned}$$

We define $\mathcal{E}_j^0(t)$ as the solution of

$$(45) \quad \partial_t \mathcal{E}_1^0 = H_1 \mathcal{E}_1^0$$

$$(46) \quad \partial_t \mathcal{E}_2^0 = H_2 \mathcal{E}_2^0 + T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F)$$

$$(47) \quad \partial_t \mathcal{E}_j^0 = H_j \mathcal{E}_j^0 + \Delta_j^- \mathcal{E}_{j-2}^0, \quad j \geq 3.$$

(45) and (46) are two closed equations whose solutions are given by

$$(48) \quad \mathcal{E}_1^0(t) = U_1(t, 0) \mathcal{E}_1^0(0) = 0$$

since we supposed $E_1(0) = O(N^{-1})$, and

$$(49) \quad \begin{aligned} \mathcal{E}_2^0(t) &= U_1(t, 0) \mathcal{E}_2^0(0) \\ &+ U_2(t, 0) \int_0^t U_2(0, s) (T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F)) ds. \end{aligned}$$

Iterating till j , we get explicitly the solution of (45)-(47) given by

$$(50) \quad \mathcal{E}_j^0(t) = U_j(t, 0) \mathcal{E}_j^0(0) + U_j(t, 0) \int_0^t U_j(0, s) \Delta_j^- \mathcal{E}_{j-2}^0(s) ds, \quad j \geq 1,$$

with the convention $\mathcal{E}_l^k = 0, l < 0$.

Therefore, for $j = 1, \dots, N, t \in \mathbf{R}$, the knowledge of $U_j(t, s)$, $|s| \leq |t|$, and $\mathcal{E}_{j'}^0(0)$, $j' = 1, \dots, j$ guarantees the knowledge of $\mathcal{E}_{j'}^0(t)$, $t \in \mathbf{R}, j' \leq j$. We write this fact as

$$(51) \quad (\mathcal{E}_{j'}^0(0))_{j'=1, \dots, j} \rightsquigarrow (\mathcal{E}_{j'}^0(t))_{t \in \mathbf{R}, j'=1, \dots, j}$$

Making now the ansatz $\mathcal{E}_j(t) \sim \sum_{k=0}^{\infty} \mathcal{E}_j^k(t) N^{-k/2}$ we find that the family $(\mathcal{E}_j^k(t))_{j=1, \dots, N, k=0, \dots}$ must satisfy

$$(52) \quad \partial_t \mathcal{E}_j^k = H_j \mathcal{E}_j^k + D_j^{-2} \mathcal{E}_{j-2}^k + \Delta_j^+ \mathcal{E}_{j+1}^{k-1} + \Delta_j^- \mathcal{E}_{j-1}^{k-1},$$

with again the conventions (41) and $\mathcal{E}_j^l = 0$ for $l < 0$, solved by

$$(53) \quad \begin{aligned} \mathcal{E}_j^k(t) &= U_j(t, 0) \mathcal{E}_j^k(0) \\ &+ \int_{s=0}^t U_j(t, s) (\Delta_j^- \mathcal{E}_{j-2}^{k-1}(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(s)) ds \end{aligned}$$

Since $\mathcal{E}_{-1}^k(t) = 0$ by convention and $\mathcal{E}_0^k(t) = 0$ for $k \geq 1$ since $E_0(t) := 1$, we find after (51) that $\mathcal{E}_1^1(t)$ and $\mathcal{E}_2^1(t)$ are determined by $\mathcal{E}_1^1(0)$ and $\mathcal{E}_2^1(0)$. Therefore, by (53), $\mathcal{E}_j^1(t), j = 1, \dots, N$ are determined by $(\mathcal{E}_j^1(0))_{j=1, \dots, N}$, and determine $\mathcal{E}_1^2(t)$ and $\mathcal{E}_2^2(t)$. These ones determine in turn all the $\mathcal{E}_j^2(t), j = 1, \dots, N$ and so on.

Therefore, the knowledge of $(\mathcal{E}_{j'}^{k'}(s))_{|s| \leq |t|, k' \leq k-1, j'=1, \dots, j+1}$ and $\mathcal{E}_j^k(0)$ guarantees for all j, k , by induction, the knowledge of $\mathcal{E}_j^k(t)$. Thus

$$((\mathcal{E}_j^k(0), (\mathcal{E}_{j'}^{k'}(s))_{|s| \leq |t|, k' \leq k-1, j'=1, \dots, j+1}) \rightsquigarrow (\mathcal{E}_{j'}^{k'}(s))_{|s| \leq |t|, k' \leq k, j'=1, \dots, j}.$$

Therefore, supposing known $(\mathcal{E}_{j'}^{k'})_{k' \leq k, j' \leq j}$,

$$(\mathcal{E}_{j'}^{k'}(s))_{s \leq t, k' \leq k-2, j'=1, \dots, j+2} \rightsquigarrow (\mathcal{E}_{j'}^{k'}(s))_{s \leq t, k' \leq k-1, j'=1, \dots, j+1} \rightsquigarrow \mathcal{E}_j^k(t).$$

and by iteration

$$(\mathcal{E}_{j'}^0(s))_{s \leq t, j'=1, \dots, j+k} \rightsquigarrow \mathcal{E}_j^k(t)$$

so that, by (51),

$$(\mathcal{E}_{j'}^0(0))_{j'=1, \dots, j+k} \rightsquigarrow \mathcal{E}_j^k(t).$$

We just proved the following result.

Proposition 2.1. *For any $j = 1, \dots, N, t \geq 0, k = 0, \dots$, let $\mathcal{E}_j^k(t)$ be the solution of (53). Then $\mathcal{E}_j^k(t)$ is determined by the values $\mathcal{E}_{j'}^{k'}(0)$ for $0 \leq k' \leq k, 1 \leq j' \leq j+k$.*

Formula (53) will give easily the following result.

Proposition 2.2. *Let $\mathcal{E}_{j'}^{k'}(0) = 0$ for $j' \leq j, k' \leq k, j'+k'$ odd. Then $\mathcal{E}_j^k(t) = 0$ for $j+k$ odd.*

Proof. Let us suppose $\mathcal{E}_{j'}^{k'}(0) = 0$ for $j' \leq j, k' \leq k, j' + k'$ odd. By (45) we have that $\mathcal{E}_1^0(t) = 0$ since $\mathcal{E}_1^0(0) = 0$. Therefore, by induction on j in (53), $\mathcal{E}_j^0(t) = 0$ for all j odd.

Since $\mathcal{E}_0(t) := 1$, $\mathcal{E}_0^j(t) = 0, j > 0$, so that $\mathcal{E}_2^1(t) = 0$ by (53) and therefore $\mathcal{E}_j^1(t) = 0$ for all j even, since then $j \pm 1$ is odd, and therefore $\mathcal{E}_{j \pm 1}^0(s) = 0$. This gives $\mathcal{E}_1^2(t) = 0$ by (53) and so on. \square

Corollary 2.3. $E_j(t)$ has an asymptotic expansion in powers of N^{-1} with leading order $N^{-\lfloor \frac{j+1}{2} \rfloor}$ when $E_j(0)$ is so.

3. ESTIMATES AND PROOF OF THEOREM 1.4

In order to simplify the expressions, we will first suppose that $\frac{\|V\|_{L^\infty}}{h} = 1$. Note that one has therefore the following estimates:

$$(54) \quad \|D_j\|, \|D_j^1\| \leq j \text{ and } \|D_j^{-1}\|, \|D_j^{-2}\|, \|D_1^{-1}(E_0)\|, \|D_2^{-2}(E_0)\| \leq \frac{j^2}{N}.$$

Let us first notice that (21) expressed on the \mathcal{E}_j s reads

$$(55) \quad \partial_t \mathcal{E}_j = H_j \mathcal{E}_j + N^{-\frac{1}{2}} \Delta_j^+ \mathcal{E}_{j+1} + N^{-\frac{1}{2}} \Delta_j^- \mathcal{E}_{j-1} + \Delta_j^- \mathcal{E}_{j-2}$$

and that (22) and (23) can be rephrased as

$$(56) \quad \|\mathcal{E}_j(0)\| \leq (A_j^2)^{j/2} \implies \|\mathcal{E}_j(t)\| \leq (A_t j^2)^{j/2}, \quad A_t = C' A e^{Ct}$$

for some explicit constants A', C .

We get

$$\begin{aligned} \partial_t \mathcal{E}_j^k(t) &= H_j(t) \mathcal{E}_j^k(t) + \Delta_j^- \mathcal{E}_{j-2}^k(t) + N^{-\frac{1}{2}} (\Delta_j^+ \mathcal{E}_{j+1}^{k-1}(t) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(t)) \\ &= H_j(t) \mathcal{E}_j^k(t) + \Delta_j^- \mathcal{E}_{j-2}^k(t) + \Delta_j^+ \mathcal{E}_{j+1}^k(t) + \Delta_j^- \mathcal{E}_{j-1}^k(t) \\ &\quad + N^{-\frac{1}{2}} ((\Delta_j^+ (\mathcal{E}_{j+1}^{k-1}(t) - \mathcal{E}_{j+1}^k(t)) + (\Delta_j^- (\mathcal{E}_{j-1}^{k-1}(t) - \mathcal{E}_{j-1}^k(t)))), \end{aligned}$$

and, calling $\bar{\mathcal{E}}_j^n = \sum_{k=0}^n N^{-k/2} \mathcal{E}_j^k$, one easily check that

$$(57) \quad \begin{aligned} \partial_t \bar{\mathcal{E}}_j^n(t) &= H_j(t) \bar{\mathcal{E}}_j^n(t) + \Delta_j^- \bar{\mathcal{E}}_{j-2}^n(t) + N^{-\frac{1}{2}} (\Delta_j^+ \bar{\mathcal{E}}_{j+1}^n(t) + \Delta_j^- \bar{\mathcal{E}}_{j-1}^n(t)) \\ &\quad - N^{-\frac{n+1}{2}} (\Delta_j^+ (\mathcal{E}_{j+1}^n(t)) + \Delta_j^- (\mathcal{E}_{j-1}^n(t))). \end{aligned}$$

Therefore $R_j^n := \mathcal{E}_j - \bar{\mathcal{E}}_j^n$ satisfies the equation

$$(58) \quad \begin{aligned} \partial_t R_j^n(t) &= H_j(t) R_j^n(t) + \Delta_j^- R_{j-2}^n(t) + N^{-\frac{1}{2}} (\Delta_j^+ R_{j+1}^n(t) + \Delta_j^- R_{j-1}^n(t)) \\ &\quad + N^{-\frac{n+1}{2}} (\Delta_j^+ (\mathcal{E}_{j+1}^n(t)) + \Delta_j^- (\mathcal{E}_{j-1}^n(t))) \end{aligned}$$

Let us define the mapping

$$U_j^N(t, s) : (\mathcal{E}_j(s))_{j=1, \dots, N} \mapsto U_j^N(t, s) ((\mathcal{E}_j(s))_{j=1, \dots, N}) := \mathcal{E}_j(t).$$

In other words, the family $(U_j^N(t, s))_{j=1, \dots, N}$ solves the equation:

$$\begin{aligned} \partial_t U_j^N(t, s) &= H_j(t)U_j^N(t, s) + \Delta_j^- U_{j-2}^N(t, s) \\ &\quad + N^{-\frac{1}{2}}(\Delta_j^+ U_{j+1}^N(t, s) + \Delta_j^- U_{j-1}^N(t, s)), \\ U_j^N(s, s) &= I. \end{aligned}$$

Hence, the solution of (58) reads

$$(59) \quad \begin{aligned} R_j^n(t) &= U_j^N(t, 0)((R_j^n(0))_{j=1, \dots, N}) \\ &+ N^{-\frac{n+1}{2}} \int_0^t U_j^N(t, s)((\Delta_j^+(s)\mathcal{E}_{j+1}^n(s)) + \Delta_j^-(s)\mathcal{E}_{j-1}^n(s))_{j=1, \dots, N} ds \end{aligned}$$

with again the same convention on negative indices.

By hypothese, $R_j^n(0) = 0$ since $\mathcal{E}_j^n(0) = \delta_{n,0}\mathcal{E}_j^0(0)$.

Let us suppose now that

$$(60) \quad \|\Delta_j^+(\mathcal{E}_{j+1}^n(s)) + \Delta_j^-(\mathcal{E}_{j-1}^n(s))\| \leq C_n(s)(C'_n(s)j^2)^{j/2}, \quad |s| \leq |t|,$$

for two increasing functions $C_n(s), C'_n(s), C'_n(s) \geq 1$, Then (56) implies that

$$\|U_j^N(t, s)((\Delta_j^+(s)\mathcal{E}_{j+1}^n(s)) + \Delta_j^-(s)\mathcal{E}_{j-1}^n(s))_{j=1, \dots, N}\| \leq C_n(s)(C'_n C'_n(s)e^{C|t|}j^2)^{j/2},$$

and thus

$$\begin{aligned} \|\mathcal{E}_j(t) - \bar{\mathcal{E}}_j^n(t)\| &= \|R_j^n(t)\| \\ &= \left\| \int_0^t U_j^N(t, s)((\Delta_j^+(s)\mathcal{E}_{j+1}^n(s)) + \Delta_j^-(s)\mathcal{E}_{j-1}^n(s))_{j=1, \dots, N} ds \right\| \\ &\leq N^{-\frac{n+1}{2}} D_n(t)(D'_n(t)j^2)^{j/2}, \end{aligned}$$

where

$$(61) \quad D_n(t) = tC_n(t) \text{ and } D'_n(t) = C'_n C'_n(t)e^{C|t|}.$$

It remains to prove an estimate like (60).

We will obtain such an estimate by iterating (53). We first remark that, since $e^{K^j+T_j/N}$ is unitary and $\|D_j\| \leq j$, the Gronwall Lemma gives that

$$(62) \quad \|U_j(t, s)\| \leq e^{j|t-s|}.$$

We will use

$$(63) \quad \prod_{i=0}^m e^{(j+i)(t_i-t_{i+1})} \leq e^{(j+m)|t_{m+1}-t_0|} \text{ for any } (t_i)_{i=0, \dots, m} \text{ (see [24])},$$

$$(64) \quad \|\Delta^\pm\|, \|\Delta^\pm\| \leq j^2,$$

$$(65) \quad \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n = \frac{t^n}{n!}.$$

Let us remind that we have $\mathcal{E}_0^k(t) = \delta_{k,0}$ for all t $\mathcal{E}_j^k(0) = \delta_{k,0}N^{\frac{j}{2}}E_j(0)$ (resp. $\delta_{k,1}N^{\frac{j}{2}}E_j(0)$) if j is even (resp. j is odd) with $\|E_j(0)\|_1 \leq (A\frac{j^2}{N})^{j/2}$. Therefore (53) reads:

$$\begin{cases} \mathcal{E}_j^0(t) = U_j(t,0)\mathcal{E}_j^0(0) + \int_{s=0}^t U_j(t,s)\Delta_j^- \mathcal{E}_{j-2}^0(s)ds, & \mathcal{E}_j^1(t) = 0, \quad j \text{ even} \\ \mathcal{E}_j^1(t) = U_j(t,0)\mathcal{E}_j^1(0) + \int_{s=0}^t U_j(t,s)\Delta_j^- \mathcal{E}_{j-2}^1(s)ds \\ \quad + \int_0^t U_j(t,s)(\Delta_j^+ \mathcal{E}_{j+1}^0(s) + \Delta_j^- \mathcal{E}_{j-1}^0(s))ds, & \mathcal{E}_j^0(t) = 0, \quad j \text{ odd}, \\ \mathcal{E}_j^k(t) = \int_0^t U_j(t,s)(\Delta_j^- \mathcal{E}_{j-2}^k(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(s))ds, & k > 1. \end{cases}$$

Let us note first that (52) for $k = 0$ is verbatim (21) after replacing E_j by \mathcal{E}_j^0 and D_j^\pm by 0. On the other side, we know by Remark 3.2 in [24], that the proof of Theorem 2.1 in [24], Theorem 1.4 in the present paper, depends on D_j^\pm only through its norm $\|D_j^\pm\|$ required to be bounded by j^2 . Therefore we get immediately, for j even,

$$(66) \quad \|\mathcal{E}_j^0(t)\| \leq (C' Ae^{C|t|j^2})^{j/2},$$

and thus, by (62), (64) and using $j^\lambda \leq e^{j\lambda/e}$, $\lambda > 0$,

$$(67) \quad \left\| \int_0^t U_j(t,s)(\Delta_j^+ \mathcal{E}_{j+1}^0(s) + \Delta_j^- \mathcal{E}_{j-1}^0(s))ds \right\|_1 \leq 2t(C' Ae^{4/e} e^{(C+1)|t|j^2})^{j/2},$$

and the same argument as the one which leads to (66), we get, for j odd,

$$(68) \quad \|\mathcal{E}_j^1(t)\| \leq (1 + 2|t|)(C' Ae^{4/e} e^{(C+1)|t|j^2})^{j/2}.$$

For $k > 1$ we will estimate $\|cE_j^k(t)\|_1$ by iterating the second line M times, we will end up with the sum of 3^M terms involving the values $\mathcal{E}_{j-2r+s-t}^{k-s-t}$ for any (r,s,t) such that $M = r + s + t$ with the two constraints $k - s - t \geq 0$, $j - 2r + s - t \geq 0$. Using the first constraint we see that

$$j - 2r + s - t \leq j - 2r + k \leq j - 2(M - k) + k = j - 2M + 3k.$$

So that, taking $M = [(j + 3k)/2]$, the second constraint reduces to $j - 2r + s - t = 0$ and the first one to $s + t = k$ since $\mathcal{E}_0^k = \delta_{k,0}$.

We easily (and very roughly) estimate, using respectively $M = [(j + 3k)/2]$, (65), (63) and (64),

$$\|\mathcal{E}_j^k(t)\| \leq 3^{(j+3k)/2} \frac{|t|^{(j+3k)/2}}{((j+3k)/2)!} e^{3(j+k)|t|/2} ((j+k)^2)^{\frac{j+3k}{2}}$$

so that, using $(1 + k/j)^j \leq e^k$, $j^\lambda \leq e^{j\lambda/e}$, $\lambda > 0$ and $n! \geq n^n e^{-n}$ ², we get

$$\|\mathcal{E}_j^k(t)\| \leq (2|t|e^{|t|+\frac{5}{3}}(3+k))^{3k/2} (3e^{6k/e}|t|e^{3|t|j^2})^{j/2}, \quad k > 1$$

²since $\log n! = \sum_{j=2}^n \log j \geq \int_1^n \log(x)dx = [x \log x - x]_1^n = n \log n - n + 1$.

and,

$$\|\mathcal{E}_j^k(t)\| \leq (2(1+|t|)e^{|t|+\frac{5}{3}}(3+k))^{3k/2}((3e^{\frac{6k}{e}}|t|e^{3|t|}+(1+2|t|)C'Ae^{4/e}e^{(C+1)|t|})j^2)^{j/2}, \quad k \geq 0.$$

We conclude by (64):

$$(69) \quad \|\Delta_j^+(\mathcal{E}_{j+1}^k(s)) + \Delta_j^-(\mathcal{E}_{j-1}^k(s))\| \leq C_k(s)(C'_k(s)j^2)^{j/2}$$

with, after restoring the dependence in $\frac{\|V\|_{L^\infty}}{\hbar}$ by the same argument as in [24], Section 3, namely a rescaling of the time and the kinetic part of the Hamiltonian,

$$(70) \quad \begin{cases} C_k(s) &= 4e(2(1 + \frac{\|s\|V\|_{L^\infty}}{\hbar})e^{\frac{\|s\|V\|_{L^\infty}}{\hbar}}k)^{3k/2} \\ &\times (3e^{\frac{6k+4}{e}}\frac{\|s\|V\|_{L^\infty}}{\hbar}e^{3\frac{\|s\|V\|_{L^\infty}}{\hbar}} + C'Ae^{C\frac{\|s\|V\|_{L^\infty}}{\hbar}})^{1/2} \\ C'_k(s) &= (3e^{\frac{6k}{e}}\frac{\|s\|V\|_{L^\infty}}{\hbar}e^{3\frac{\|s\|V\|_{L^\infty}}{\hbar}} + (1 + 2\frac{\|s\|V\|_{L^\infty}}{\hbar})C'Ae^{4/e}e^{(C+1)\frac{\|s\|V\|_{L^\infty}}{\hbar}})^{1/2}e^{6/e} \end{cases}$$

Therefore (60) is satisfied and Theorem 1.4 is proven.

The values of the two constants $D_n(t), D'_n(t)$ in (61) can be expressed out of (70) by taking, by Theorem 1.2, $C = \sup(B_1, C_1), C' = \sup(B_2, C_2)$ where B_1, C_1, B_2, C_2 are given in Theorem 2.2. in [24].

Remark 3.1. *We see that the properties (41)-(54), together with (5), are actually the only ones being used in the proof of Theorem 1.4.*

4. EXPLICIT COMPUTATIONS OF FIRST ORDERS

We have

$$\partial_t U_1^0(t, s) = \frac{1}{i\hbar}[-\hbar^2 \Delta + V_F, U_1^0(t, s)] + \frac{1}{i\hbar}[V_{U_1^0(t, s)}, F]$$

where, in the last term, $V_{U_1^0(t, s)}$ acts on $E_1(s)$ as $V_{U_1^0(t, s)E_1(s)}$.

More generally,

$$\partial_t U_j^0(t) = \frac{1}{i\hbar}[-\hbar^2 \Delta_{\mathbf{R}^{jd}} + V_F^{\otimes j}, U_j^0(t)] + P(U_j^0, F)$$

where

$$(P(U_j^0, F)E_j)(Z_j) = \sum_i \int dx (V(x_i - x) - V(x'_i - x))(U_j^0(t, s)E_j(Z_j^{\neq i}, (x, x))F(x_i, x'_i)),$$

that is

$$(P(U_j^0, F)E_j) = \sum_{i=1}^j [V \star_i (U_j^0(t, s)E_j), F]_i.$$

Finally

$$\mathcal{E}_2^0(t)(Z_2) = \int_0^t \int_{\mathbb{R}^{2d}} ds dZ'_2 U_2(t, s)(Z_2, Z'_2) V(x'_1 - x'_2) F(s)(z'_1) F(s)(z'_2) ds dZ'_2$$

and

$$\begin{aligned} \mathcal{E}_1^1(t) &= \int_0^t U_1(t, s) Q(F, F) ds \\ &\quad + \left(1 - \frac{1}{N}\right) \int_0^t \int_0^s U_1(t, s) \text{Tr}^2[VU_2(s, u)VF(u) \otimes F(u)] ds du \end{aligned} \quad (71)$$

5. THE KAC AND “SOFT SPHERES” MODELS

In this section we consider the two following classes of mean field models (see [24] for details).

- *Kac model.* In this model, the N -particle system evolves according to a stochastic process. To each particle i , we associate a velocity $v_i \in \mathbb{R}^3$. The vector $\mathcal{V}_N = \{v_1, \dots, v_N\}$ changes by means of two-body collisions at random times, with random scattering angle. The probability density $F^N(\mathcal{V}_N, t)$ evolves according to the forward Kolmogorov equation

$$(72) \quad \partial_t F^N = \frac{1}{N} \sum_{i < j} \int d\omega B(\omega; v_i - v_j) \{F^N(\mathcal{V}_N^{i,j}) - F^N(\mathcal{V}_N)\},$$

where $\mathcal{V}_N^{i,j} = \{v_1, \dots, v_{i-1}, v'_i, v_{i+1}, \dots, v_{j-1}, v'_j, v_{j+1}, \dots, v_N\}$ and the pair v'_i, v'_j gives the outgoing velocities after a collision with scattering (unit) vector ω and incoming velocities v_i, v_j . $\frac{B(\omega; v_i - v_j)}{|v_i - v_j|}$ is the differential cross-section of the two-body process. The resulting mean-field kinetic equation is the homogeneous Boltzmann equation

$$(73) \quad \partial_t F(v) = \int dv_1 \int d\omega B(\omega; v - v_1) \{F(v')F(v'_1) - F(v)F(v_1)\}.$$

- *‘Soft spheres’ model.* A slightly more realistic variant, taking into account the positions of particles $X_N = \{x_1, \dots, x_N\} \in \mathbb{R}^{3N}$ and relative transport, was introduced by Cercignani [9] and further investigated in [18]. The probability density $F^N(X_N, V_N, t)$ evolves according to the equation

$$(74) \quad \begin{aligned} \partial_t F^N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} F^N &= \frac{1}{N} \sum_{i < j} h(|x_i - x_j|) B\left(\frac{x_i - x_j}{|x_i - x_j|}; v_i - v_j\right) \\ &\quad \times \{F^N(X_N, V_N^{i,j}) - F^N(X_N, V_N)\}. \end{aligned}$$

Here $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a positive function with compact support. Now a pair of particles collides at a random distance with rate modulated by h . The

associated mean-field kinetic equation is the Povzner equation

$$\begin{aligned} \partial_t F(x, v) + v \cdot \nabla_x F(x, v) &= \int dv_1 \int dx_1 h(|x - x_1|) B\left(\frac{x - x_1}{|x - x_1|}; v - v_1\right) \\ &\quad \times \{F(x, v') F(x_1, v'_1) - F(x, v) F(x_1, v_1)\}, \end{aligned}$$

which can be seen as an h -mollification of the inhomogeneous Boltzmann equation (formally obtained when h converges to a Dirac mass at the origin). Both classes have been treated in [24] and Theorem 1.2 apply to them, in the following sense.

The underlying space \mathbb{L} is now $L^1(\mathbf{R}^d, dv)$ (resp. $L^1(\mathbf{R}^{2d}, dx dv)$) for the Kac model (resp. soft spheres) both endowed with the L^1 norms $\|\cdot\|_1$. For $F^N \in \mathbb{L}^{\otimes N}$, $F_j^N \in \mathbb{L}^{\otimes j}$ is defined by

$$F_j^N(Z_j) = \int_{\Omega} F^N(z_1, \dots, z_j, z_{j+1}, \dots, z_N) dz_{j+1} \dots dz_N$$

for $Z_n = (z_1, \dots, z_n)$, $n = 1, \dots, N$ with $z_i = v_i \in \mathbf{R}^d$, $\Omega = \mathbf{R}^{(N-j)d}$ (resp. $z_i = (x_i, v_i) \in \mathbf{R}^{2d}$, $\Omega = \mathbf{R}^{2(N-j)d}$) for the Kac (resp. soft spheres) model.

In both cases $E_j(t)$ is defined by (18), inverted by (20), and it was proven in [24] that Theorem 1.2 holds true verbatim in both cases.

Stating now the dynamics driven by (72) and (74) under the form (3) with $K^N = 0$ (resp. $K^N = -\sum_{i=1, \dots, N} v_i \partial_{x_i}$) for the Kac (resp. soft spheres) model and V^N given by the right hand-sides of (72), (74) respectively, one sees immediately that the proofs contained in Sections 2,3 remain valid after an elementary redefinition of the operators D_j, D_j^{-1}, D_j^{-2} in (37)-(40) consisting in removing the bottom and overhead straight lines in the right hand sides and, by a slight abuse of notation, identifying functions with their evaluations. The convention (41) remains verbatim the same, together with the estimates

$$(75) \quad \|D_j\|, \|D_j^1\| \leq j \text{ and } \|D_j^{-1}\|, \|D_j^{-2}\|, \|D_1^{-1}(E_0)\|, \|D_2^{-2}(E_0)\| \leq \frac{j^2}{N}.$$

Therefore, by Remark 3.1, the statements contained in Theorem 1.4 and Corollary 1.7 hold true, in both cases, verbatim. Moreover defining $F_j^{N,n}$ by (35) in both cases, Corollary 1.5 reads now as follows

Corollary 5.1. *[Kac case] Let $F^N(t)$ the solution of the N body system (72) (resp. (74)) with initial datum $F^N(0) = F^{\otimes N}$, $0 < F \in L^1(\mathbf{R}^d)$, $\int_{\mathbf{R}^d} f(v) dv = 1$ (resp. $0 < F \in L^1(\mathbf{R}^{2d})$, $\int_{\mathbf{R}^{2d}} f(x, v) dx dv = 1$), and $F(t)$ the solution of the*

homogeneous Boltzmann equation (73) (resp. the Povzner equation(75)) with initial datum F .

Then, in both cases, for all $n \geq 1$ and $N \geq 4(eA_t^{2n}j)^2$,

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2tC_{2n}(t)eA_t^{2n}j}{\sqrt{N}}.$$

APPENDIX A. THE ASBTRACT MODEL

A.1. The model. We will show in this section that the main results of [24] and of Section 1 of the present paper remain true in the “abstract” mean field formalism for a dynamics of N particles that we will describe now. The present formalism contains the abstract formalism developed in [24], without requiring a space of states endowed with a multiplicative structure.

States of the particle system and evolution equations. Let \mathbb{L} be a vector space on the complex numbers. We suppose the family of (algebraic) tensor products $\{\mathbb{L}^{\otimes n}, n = 1, \dots, N\}$ equipped with a family of norms $\|\cdot\|^n$ satisfying assumption **(A)** below. the N -body dynamics will be driven on $\mathbb{L}^{\otimes N}$ by a one- and two- body interaction satisfying assumption **(B)** and the mean field limit equation will be supposed to satisfy assumption **(C)**.

Assumptions **(A)** – **(C)** below will be followed by their incarnations in the **K**(ac), **S**(oft spheres) and **Q**(uantum) models.

By convention we denote $\mathbb{L}^{\otimes 0} := \mathbf{C}$, $\|z\|^0 = |z|$ and we denote by $\mathbb{L}^{\hat{\otimes} n}$ the completion of $\mathbb{L}^{\otimes n}$ with respect to the norm $\|\cdot\|^n$.

For the **K**, **S** and **Q** models, $\mathbb{L}^{\otimes n}$ is $L^1(\mathbf{R}^d, dv)$, $L^1(\mathbf{R}^{2d}, dx dv)$ and $\mathcal{L}^1(L^2(\mathbf{R}^d))$, the space of trace class operators on $L^2(\mathbf{R}^d)$, with their associated norms.

- (A)** There exists a family of subsets $\mathbb{L}_+^{\hat{\otimes} n}$ of $\mathbb{L}^{\hat{\otimes} n}$, $n = 1, \dots, N$, of positive elements F denoted by $F > 0$ stable by addition, multiplication by positive reals and tensor product and there exists a linear function $\text{Tr} : \mathbb{L} \rightarrow \mathbf{C}$, called trace. For every $1 \leq k, n \leq N$ and $1 \leq i \leq j \leq n \leq N$, let Tr_n^k and $\sigma_{i,j}^n$ be the two mapping defined by³

³The fact that the second and fourth lines of (76) define a mapping on the whole tensor space $\mathbb{L}^{\otimes n}$ results easily from the definition of tensors products through the so-called universal property [19]. Indeed, let φ_n be the natural embedding $\mathbb{L}^{\times n} \rightarrow \mathbb{L}^{\otimes n}$, $(v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n$, and let h be any mapping $\mathbb{L}^{\times n} \rightarrow \mathbb{L}^{\times n'}$, then the universal property of tensor products says that there is a unique map $\tilde{h} : \mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n'}$ such that $\tilde{h} \circ \varphi_n = \varphi_{n'} \circ h$. Taking $n' = n-1$, $h(v_1, \dots, v_k, \dots, v_n) = (\text{trace}(v_k)v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n)$ for Tr_n^k , and $n' = n$, $h(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = (v_1, \dots, v_j, \dots, v_i, \dots, v_n)$ for $\sigma_{i,j}^n$ give the desired extensions.

$$(76) \quad \left\{ \begin{array}{l} \text{Tr}_n^k : \mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n-1} \\ \quad \quad \quad \bigotimes_{i=1}^n v_i \mapsto \text{Tr}(v_k) \bigotimes_{\substack{i=1 \\ i \neq k}}^n v_i, \\ \sigma_{i,j}^n : \mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n} \\ \quad \quad \quad \bigotimes_{i=1}^n v_i \mapsto \bigotimes_{i=1}^n v'_i, \quad v'_k = v_k, i \neq k \neq j ; v'_i = v_j, v'_j = v_i. \end{array} \right.$$

We will suppose that Tr_N^k and $\sigma_{i,j}^n$, $i, j, k \leq n \leq N$, satisfy, for any $F \in \mathbb{L}^{\otimes n}$,

$$(77) \quad \left\{ \begin{array}{l} \text{Tr}_N^k(F), \sigma_{i,j}^n(F) > 0, \quad \|\text{Tr}_n^k(F)\|_{n-1} = \|F\|_n \text{ when } F > 0 \\ \|\sigma_{i,j}^n(F)\|^n = \|F\|^n \\ \|\text{Tr}_n^k(F)\|^{n-1} \leq \|F\|^n \end{array} \right.$$

In particular one has that $\|F\|^n = \text{Tr}^n \dots \text{Tr}^1 F$ when $F > 0$ and $|\text{Tr}^n \dots \text{Tr}^1 F| \leq \|F\|^n$ in general.

Note that (77) allows to extend Tr_n^k and $\sigma_{i,j}^n$ to $\mathbb{L}^{\hat{\otimes} n}$ by continuity.

We will use the same notation for these extensions.

For the \mathbf{K} , \mathbf{S} and \mathbf{Q} models, Tr^k is $\int_{\mathbf{R}^d} \cdot dv_k$, $\int_{\mathbf{R}^{2d}} \cdot dx_k dv_k$ as indicated in Section 5, and the partial traces defined in Section 1.1. The action of $\sigma_{i,j}^n$ consists obviously in exchanging the variables v_i and v_j , (x_i, v_i) and (x_j, v_j) and (x_i, x'_i) and (x_j, x'_j) , (in the integral kernel), respectively. Finally (77) is satisfied in the three cases.

From now on and when no confusion is possible, we will identify $\mathbb{L}^{\otimes n}$ with its completion $\mathbb{L}^{\hat{\otimes}}$ and we will denote $\text{Tr}_N^k = \text{Tr}^k$ (note also that $\text{Tr} = \text{Tr}_1^1 = \text{Tr}^1$), $\sigma_{i,j}^N = \sigma_{i,j}$ and $\text{Tr}(= \text{Tr}_n) = \text{Tr}_n^n \text{Tr}_n^{n-1} \dots \text{Tr}_n^1$. Moreover, with a slight abuse of notation, we will denote

$$(78) \quad \left\{ \begin{array}{l} \|\cdot\|_1 = \|\cdot\|^n, \quad \forall n = 1, \dots, N \\ \|\cdot\| \text{ the operator norm on any } \mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j}), \quad \forall i, j = 1, \dots, N \end{array} \right.$$

(here $\mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j})$ is the set of bounded operators from $\mathbb{L}^{\otimes i}$ to $\mathbb{L}^{\otimes j}$).

We call *symmetric* any element of $\mathbb{L}^{\otimes n}$ invariant by the action of $\sigma_{i,j}^n$, $i, j \leq n$.

We call *state of the N -particle system* an element of

$$(79) \quad \mathcal{D}_N = \{F \in \mathbb{L}^{\otimes n} \mid F > 0, \quad \|F\| = 1 \quad \text{and } F \text{ is symmetric}\}.$$

For $j = 0, \dots, N$, the j -particle marginal of $F^N \in (\mathbb{L}^{\otimes N})_1^\dagger$ is defined as the partial trace of order $N - j$ of F^N , that is

$$(80) \quad F_j^N = \text{Tr}^N \text{Tr}^{N-1} \dots \text{Tr}^{j+1} F^N, \quad F_N^N := F^N.$$

Note that $F_j^N \in \mathbb{L}^{\otimes j}$ ($F_0^N = 1 \in \mathbb{L}^{\otimes 0} := \mathbf{C}$) and $F_j^N > 0$, $\|F_j^N\|^j = \|F^N\|^N$ since Tr is positivity and norm preserving, and obviously F_j^N is symmetric as F^N . That is to say:

$$F_j^N \in \mathcal{D}_j.$$

(B) The evolution of a state F^N in $\mathbb{L}^{\otimes N}$ is supposed to be given by the N -particle dynamics associated to a two-body interaction:

$$(81) \quad \frac{d}{dt} F^N = (K^N + V^N) F^N,$$

where the operators on the right hand side are constructed as follows.

$$(82) \quad K^N = \sum_{i=1}^N \mathbb{I}_{\mathbb{L}}^{\otimes(i-1)} \otimes K \otimes \mathbb{I}_{\mathbb{L}}^{\otimes(N-i)}$$

and

$$(83) \quad V^N = \frac{1}{N} \sum_{1 \leq i < j \leq N} V_{i,j}, \quad V_{i,j} := \sigma_{1,i}^N \sigma_{2,j}^N V \otimes \mathbb{I}_{\mathbb{L}^{\otimes(N-2)}} \sigma_{1,i}^N \sigma_{2,j}^N$$

for a (possibly unbounded) operator K acting on \mathbb{L} and a bounded two-body (potential) operator V acting on $\mathbb{L}^{\otimes 2}$.

We assume furthermore that K is the generator of a strongly continuous, isometric, positivity preserving semigroup (in \mathbb{L})

$$(84) \quad e^{Kt} F > 0 \quad \text{if} \quad F > 0; \quad \|e^{tK}\| = 1.$$

and $K^N + V^N$ is the generator of a strongly continuous, isometric, positivity preserving semigroup (in $\mathbb{L}^{\otimes N}$)

$$(85) \quad e^{(K^N + V^N)t} F^N > 0 \quad \text{if} \quad F^N > 0; \quad \|e^{t(K^N + V^N)}\| = 1.$$

Finally, for any $F \in \mathbb{L}$, $F^N \in \mathbb{L}^{\otimes N}$ and $i, r > j$, we assume

$$(86) \quad \text{Tr}(KF) = 0 \quad \text{and} \quad \text{Tr}^{j,N}(V_{i,r} F^N) = 0.$$

This last property is necessary to deduce the forthcoming hierarchy.

*For the **K**, **S** and **Q** models, the ingredients in (81) are given in Sections 5 and 1.1, where (84)-(86) are shown to be satisfied.*

Note the symmetry property of the equation (81) induced by the definition of V^N : if the initial condition F_0^N for (81) is symmetric, then $F^N(t)$ is still symmetric.

Hierarchies. The family of j -marginals, $j = 1, \dots, N$, are solutions of the BBGKY hierarchy of equations

$$(87) \quad \partial_t F_j^N = \left(K^j + \frac{T_j}{N} \right) F_j^N + \frac{(N-j)}{N} C_{j+1} F_{j+1}^N$$

where:

$$(88) \quad K^j = \sum_{i=1}^j \mathbb{I}_{\mathbb{L}}^{\otimes(i-1)} \otimes K \otimes \mathbb{I}_{\mathbb{L}}^{\otimes(j-i)},$$

$$(89) \quad T_j = \sum_{1 \leq i < r \leq j} T_{i,r} \quad \text{with} \quad T_{i,r} = V_{ir}$$

and

$$(90) \quad C_{j+1} F_{j+1}^N = \text{Tr}^{j+1} \left(\sum_{i \leq j} V_{i,j+1} F_{j+1}^N \right) = \sum_{i=1}^j C_{i,j+1} F_{j+1}^N,$$

$$(91) \quad C_{i,j+1} : \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}, \quad C_{i,j+1} F_{j+1}^N = \text{Tr}^{j+1} (V_{i,j+1} F_{j+1}^N),$$

Indeed, thanks to (86) we get easily by applying $\text{Tr}^{j,N}$ on (81) that

$$\frac{d}{dt} F_j^N = \left(K^j + \frac{T_j}{N} \right) F_j^N + \frac{1}{N} \text{Tr}^{j,N} \left(\sum_{1 \leq i \leq j < k \leq N} V_{i,k} F^N \right)$$

By symmetry of F^N and $V_{i,k}$ we get $\text{Tr}^{j,N}(V_{i,k} F^N) = \text{Tr}^{j+1}(V_{i,j+1} F_{j+1}^N)$ for all $k > j$ and (87) follows.

Note that, thanks to the assumption (77) and for all $i \leq j = 1, \dots, N$,

$$(92) \quad \|T_i\| \leq j^2 \|V\|, \quad \text{and} \quad \|C_{i,j+1}\| \leq j \|V\|$$

(meant for $(\|T_i\|_{\mathbb{L}^{\otimes i} \rightarrow \mathbb{L}^{\otimes i}}, \|C_{i,j+1}\|_{\mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}}, \|V\|_{\mathbb{L}^{\otimes 2} \rightarrow \mathbb{L}^{\otimes 2}}$ using (78)).

We introduce the non-linear mapping $Q(F, F)$, $Q : \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{L}$ by the formula

$$(93) \quad Q(F, F) = \text{Tr}^2(V_{1,2}(F \otimes F))$$

and the nonlinear mean field equation on \mathbb{L}

$$(94) \quad \partial_t F = KF + Q(F, F), \quad F(0) \geq 0, \quad \|F(0)\|_1 = 1.$$

Eq. (94) is the Boltzmann, Povzner or Hartree equation according to the specifications established in the table above. In full generality we will assume

(C) (94) has for all time a unique solution $F(t) > 0$ and $\|F(t)\| = 1$.

For the **K**, **S** and **Q** models, (C) is true by standard perturbations methods.

Correlation error. To introduce the correlation errors, we need to extend slightly the above structure.

For any subset $J \subset \{1, \dots, N\}$ we first define

$$(95) \quad \mathbb{L}_N^{\otimes J} := \bigotimes_{i=1}^N \mathbb{L}^{\otimes \chi_J(i)},$$

where χ_J is the characteristic function of J and $\mathbb{L}^{\otimes 0} = \mathbf{C}$.

Then we introduce $\mathbb{L}^{\otimes J}$, the subspace of $\mathbb{L}_N^{\otimes J}$ formed by vectors of the form $\bigotimes_{i=1}^N v_i$ where $v_i = 1 \in \mathbf{C}$ for $i \notin J$ and $v_i \in \mathbb{L}$ for $i \in J$. Note that $\mathbb{L}^{\otimes J}$ is sent to $\mathbb{L}^{\otimes |J|}$ by the mapping

$$\Pi : \bigotimes_{i=1}^N v_i \in \mathbb{L}^{\otimes J} \mapsto \bigotimes_{i \in J} v_i \in \mathbb{L}^{\otimes |J|}.$$

We define a norm on $\mathbb{L}^{\otimes J}$ by

$$\|\cdot\|_{\mathbb{L}^{\otimes J}} = \|\Pi(\cdot)\|_1.$$

For $F \in \mathbb{L}$ and $K \subset J \subset \{1, \dots, N\}$ we introduce the linear operator $[F]_J^{\otimes K}$, defined through its action on factorized elements as

$$(96) \quad \begin{aligned} [F]_J^{\otimes K} : \mathbb{L}^{\otimes J/K} &\rightarrow \mathbb{L}^{\otimes J} \\ \bigotimes_{i=1}^N v_i &\mapsto \bigotimes_{i=1}^N a_i, \end{aligned}$$

$$\text{where } \begin{cases} a_s = 1 \in \mathbf{C} & \text{if } s \notin J \\ a_s = F & \text{if } s \in K \\ a_s = v_s & \text{if } s \in J/K \end{cases}.$$

Note that, for $K, K' \subset J$, $K \cap K' = \emptyset$, we have the composition

$$(97) \quad [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} = [F]_J^{\otimes (K \cup K')} = [F]_J^{\otimes K'} [F]_{J/K'}^{\otimes K}$$

and more generally, for all F, G ,

$$(98) \quad [F]_J^{\otimes K} [G]_{J/K}^{\otimes K'} = [G]_J^{\otimes K'} [F]_{J/K'}^{\otimes K}.$$

For any subset $J \subset \{1, \dots, N\}$, we define the *correlation error* by

$$(99) \quad E_J = \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} F_{J/K}^N$$

where F solves (94), the operator $[F]_J^{\otimes K}$ is defined by (96) and $F_L^N \in \mathbb{L}^{\otimes L}$ is defined through its decomposition on factorized states. Namely if

$$F^N = \sum_{\ell_1, \dots, \ell_N} c_{\ell_1, \dots, \ell_N} v_{\ell_1} \otimes \cdots \otimes v_{\ell_N},$$

then

$$F_L^N = \sum_{\ell_1, \dots, \ell_N} c_{\ell_1, \dots, \ell_N} a_{\ell_1} \otimes \cdots \otimes a_{\ell_N},$$

$$\text{where } \begin{cases} a_s = \text{Tr}(v_s) \in \mathbf{C} & \text{if } s \notin L \\ a_s = v_s & \text{if } s \in L \end{cases}.$$

The link between the definition of F_L^N and the definition of the marginals F_j^N given in (80) is the following:

$$(100) \quad F_{\{1, \dots, \ell\}}^N = F_\ell^N \otimes (1)^{\otimes (N-\ell)} \in \mathbb{L}^{\otimes \ell} \otimes (\mathbb{L}^{\otimes 0})^{\otimes (N-\ell)}.$$

The formula inverse to (99) reads

$$(101) \quad F_J^N = \sum_{K \subset J} [F]_J^{\otimes K} E_{J/K}.$$

Note that the contribution in the right hand side of (101) corresponding to $K = J$ and $K = \emptyset$ are $F^{\otimes |J|}$ and E_J respectively. To prove (101), we plug (99) in the r.h.s. of (101) and we use (97):

$$\begin{aligned} \sum_{K \subset J} [F]_J^{\otimes K} E_{J/K} &= \sum_{K \subset J} [F]_J^{\otimes K} \left[\sum_{K' \subset J/K} (-1)^{|K'|} [F]_{J/K}^{\otimes K'} F_{(J/K)/K'}^N \right] \\ &= \sum_{K \cup K' \subset J} \sum_{\substack{K \subset J \\ K' \cap K = \emptyset}} (-1)^{|K'|} [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} F_{J/(K \cup K')}^N \\ &= \sum_{L \subset J} \left(\sum_{K' \subset L} (-1)^{|K'|} \right) [F]_J^{\otimes L} F_{J/L}^N = F_J^N \end{aligned}$$

since $\sum_{K' \subset L} (-1)^{|K'|} = \sum_{k'=0}^{|L|} \binom{|L|}{k'} (-1)^{k'} = 0^{|L|} = 0$ if $L \neq \emptyset$, and $= 1$ if $L = \emptyset$ (since $\sum_{K' \subset \emptyset} (-1)^{|K'|} = (-1)^0 = 1$).

One notices that since F_j^N is the marginal of some F^N which decomposes on elements of the form $v_1 \otimes \cdots \otimes v_N$, F_j^N decomposes on elements of the form $(\prod_{k=j+1}^N \text{Tr} v_k) v_1 \otimes \cdots \otimes v_j$. Since one knows that F_j^N is symmetric, it is enough to choose one bijection $i_J : \{1, \dots, j\} \rightarrow J$, $|J| = j$, and consider the

mapping

$$(102) \quad \begin{aligned} \Phi_{i_J} : \mathbb{L}^{\otimes |J|} &\xrightarrow{\Phi_{i_J}} \mathbb{L}^{\otimes J} \\ \otimes_{j \in J} v_j \in \mathbb{L}^{\otimes |J|} &\mapsto \otimes_{i=1}^N a_i \in \mathbb{L}^{\otimes J} \end{aligned}$$

$$(103) \quad F_{|J|}^N \mapsto F_J^N$$

where $a_s = 1$ if $i \notin J$ and $a_{i_J(j)} = v_j$.

Φ_{i_J} is obviously one-to-one since i_J is so, and, though (102) depends on the embedding chosen, (103) does not: Φ_{i_J} restricted to the space $\mathbb{L}_S^{\otimes |J|}$ of symmetric-by-permutation elements of $\mathbb{L}^{\otimes |J|}$, depends only on J and not on i_J . We will call Φ_J this restriction,

$$(104) \quad \Phi_J = \Phi_{i_J}|_{\mathbb{L}_S^{\otimes |J|}}.$$

The same argument is also valid for E_J which enjoys the same symmetry property than F_J^N and we define

$$(105) \quad E_{|J|} = \Phi_J^{-1} E_J.$$

Φ_J is obviously isometric and we have that

$$(106) \quad \|E_J\|_{\mathbb{L}^{\otimes J}} = \|E_{\{1, \dots, |J|\}}\|_{\mathbb{L}^{\otimes \{1, \dots, |J|\}}} = \|E_{|J|}\|_1.$$

Therefore, considering the one-to-one correspondence Φ_J , it is enough to compute/estimate the quantities $E_j, j = 1, \dots, N$. E_j and F_j^N are linked by

$$(107) \quad \begin{cases} E_j &= \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \Phi_{J/K} F_{j-|K|}^N \\ F_j^N &= \sum_{K \subset J} [F]_J^{\otimes K} \Phi_{J/K} E_{j-|K|}. \end{cases}$$

For the \mathbf{K} , \mathbf{S} and \mathbf{Q} models, the corresponding expressions are given in Sections 5 and 1.1.

A.2. Main results similar to [24]. The kinetic errors $E_j, j = 1, \dots, N$, satisfy the system of equations

$$(108) \quad \begin{aligned} \partial_t E_j &= \left(K^j + \frac{1}{N} T_j \right) E_j + D_j E_j \\ &+ D_j^1 E_{j+1} + D_j^{-1} E_{j-1} + D_j^{-2} E_{j-2}, \end{aligned}$$

where the operators $D_j, D_j^1, D_j^{-1}, D_j^{-2}, j = 1, \dots, N$, are defined in Appendix B below, equations (117)-(118), together with the proof of (108). Moreover, since (119) holds true, we know by Remark 3.2 in [24], that the proof

of Theorem 2.1 (and therefore Corollary 2.2) in [24] remain valid in our present setting.

We get the following result.

Proposition A.1. *The statements of Theorem 1.2 hold true in the abstract setting defined in Section A.1.*

A.3. Asymptotic expansion. It is easy to see that the proofs of the main results expressed in Section 1.3 are adaptable in an elementary way to the present abstract paradigm. Indeed they use only the three properties stated in Remark 3.1, valid in the present setting as pointed out at the very end of Appendix B, formula (119), together with (84)-(85).

Therefore, the statements contained in Theorem 1.4 and Corollary 1.7 hold true, verbatim, under the hypothesis of Theorem 1.2, and with the definition of corrections errors given by the first line of (107) and replacing $\frac{\|V\|_{L^\infty}}{h}$ by $\|V\|$ in (70).

Moreover defining now $F_j^{N,n}$ by truncating the second line of (107) at order n , that is

$$F_j^{N,n} = \sum_{K \subset J} [F]_J^{\otimes K} \Phi_{J/K} E_{j-|K|}^n$$

where E_j^n is defined by (34), Corollary 1.5 reads as follows.

Corollary A.2. *[abstract] Let $F^N(t)$ the solution of the N body system (81) with initial datum $F^N(0) = F^{\otimes N}$, $0 < F \in \mathbb{L}$, $\|F\|_1 = 1$, and $F(t)$ the solution of the mean-field equation (94) with initial datum F .*

Then, for all $n \geq 0$ and $N \geq 4(eA_t^{2n}j)^2$,

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2tC_{2n}(t)eA_t^{2n}j}{\sqrt{N}}.$$

APPENDIX B. DERIVATION OF THE CORRELATION HIERARCHY (108)

From the definition of E_j (cf. (99)) we find

$$\partial_t E_J = \sum_{K \subset J} (-1)^{|K|} \left(\partial_t ([F]_J^{\otimes K}) F_{J/K}^N + [F]_J^{\otimes K} \partial_t F_{J/K}^N \right)$$

Moreover, by (96)

$$(109) \quad \partial_t \left([F]_J^{\otimes K} \right) = \sum_{k_0 \in K} [F]_J^{\otimes K/\{k_0\}} [\partial_t F]_{J/(K/\{k_0\})}^{\otimes \{k_0\}}.$$

Applying Φ_J defined in (105) to the BBGKY hierarchy (87), one finds easily that F_J^N satisfies, denoting $\alpha(j, N) := \frac{N-j}{N}$,

$$(110) \quad \partial_t F_J^N = K^J F_J^N + \frac{1}{N} \sum_{i < r \in J} T_{i,r} F_J^N + \alpha(j, N) \sum_{i \in J} C_{i,j+1} F_{J \cup \{j+1\}}^N$$

(for $j+1 \notin J$).

By the mean-field equation (94) we deduce that

$$(111) \quad \begin{aligned} \partial_t E_J &= \sum_{K \subset J} (-1)^{|K|} \sum_{k_0 \in K} [F]_J^{\otimes K / \{k_0\}} (KF + Q(F, F))_{J/(K/\{k_0\})}^{\otimes \{k_0\}} F_{J/K}^N \\ &+ \sum_{K \subset J} (-1)^{|K|} \alpha(j - |K|, N) \sum_{i \in J/K} [F]_J^{\otimes K} C_{i,j+1} F_{(J/K) \cup \{j+1\}}^N \\ &+ \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left(\sum_{i \neq r \in J/K} T_{i,r} \right) F_{J/K}^N \\ &+ \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} (K^{J/K} F_{J/K}^N). \end{aligned}$$

We denote by \mathcal{T}_i , $i = 1, 2, 3, 4$, the four terms contained in the four lines of the r.h.s. of (111), respectively. The computation of the \mathcal{T}_i s is purely algebraic and will use only the four following properties

$$\left\{ \begin{array}{l} \sum_{K \subset L} (-1)^{|K|} = \delta_{|L|, \emptyset} \\ \sum_{K \subset L} |K| (-1)^{|K|} = -\delta_{|L|, 1} \\ [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} = [F]_J^{\otimes K'} [F]_{J/K'}^{\otimes K} = [F]_J^{\otimes (K \cup K')}, \quad K, K' \subset J, \quad K \cap K' = \emptyset \\ C_{i,j+1} [F]_{(J/K) \cup \{j+1\}}^{\otimes K} = [F]_{(J/K)}^{\otimes K} C_{i,j+1}, \quad K \subset J, \quad j+1 \notin J. \end{array} \right.$$

In order not to make the paper too heavy, we will compute extensively two terms and refer to an earlier version, [23], for an exhaustive calculus.

Using the definition (99), we get

$$(112) \quad \begin{aligned} \mathcal{T}_1 &:= \sum_{K \subset J} (-1)^{|K|} \sum_{k_0 \in K} [F]_J^{\otimes K / \{k_0\}} (KF + Q(F, F))_{J/(K/\{k_0\})}^{\otimes \{k_0\}} F_{J/K}^N \\ &= - \sum_{k_0 \in J} (KF + Q(F, F))_J^{\otimes \{k_0\}} \sum_{K \subset J / \{k_0\}} (-1)^{|K|} [F]_{J/\{k_0\}}^{\otimes K} F_{(J/\{k_0\})/K}^N \\ &= - \sum_{i \in J} (KF + Q(F, F))_J^{\otimes \{i\}} E_{J/\{i\}}. \end{aligned}$$

To compute \mathcal{T}_2 we make use of the inverse definition (101):

$$\begin{aligned}
\mathcal{T}_2 &:= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} C_{i,j+1} F_{(J/K) \cup \{j+1\}}^N \\
&= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
(113) \quad &\dots C_{i,j+1} \sum_{K' \subset (J/K) \cup \{j+1\}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'} \cdot
\end{aligned}$$

Distinguishing among the belonging or not to K' of i and $j+1$ in the r.h.s. of (113), we decompose

$$(114) \quad \mathcal{T}_2 = \mathcal{T}_2^{i,j+1 \in K'} + \mathcal{T}_2^{i,j+1 \notin K'} + \mathcal{T}_2^{i \in K', j+1 \notin K'} + \mathcal{T}_2^{i \notin K', j+1 \in K'}$$

We have

$$\begin{aligned}
\mathcal{T}_2^{i,j+1 \in K'} &= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\dots C_{i,j+1} \sum_{\substack{K' \subset (J/K) \cup \{j+1\} \\ i,j+1 \in K'}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'} \\
&= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\dots C_{i,j+1} \sum_{K'' \subset (J/K)/\{i\}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'' \cup \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\dots C_{i,j+1} \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'' \cup \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J} [F]_J^{\otimes K} \dots \\
&\dots \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K)}^{\otimes K''} C_{i,j+1} [F]_{((J/K)/K'') \cup \{j+1\}}^{\otimes \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{i \in J} \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} [F]_J^{\otimes K} \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K)}^{\otimes K''} \dots \\
&\dots C_{i,j+1} [F]_{((J/K)/K'') \cup \{j+1\}}^{\otimes \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{i \in J} \sum_{L \subset J/\{i\}} \left(\sum_{K \subset L} \alpha(j - |K|, N) (-1)^{|K|} \right) [F]_J^{\otimes L} \dots \\
&\dots C_{i,j+1} [F]_{((J/L) \cup \{j+1\})}^{\otimes \{i,j+1\}} E_{J/(L \cup \{i\})}
\end{aligned}$$

$$\begin{aligned}
&= \alpha(j, N) \sum_{i \in J} C_{i, j+1} [F]_{J \cup \{j+1\}}^{\otimes \{i, j+1\}} E_{J/\{i\}} \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i, j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i, j+1\}} E_{J/(\{i, l\})} \\
&= \alpha(j, N) \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i, j+1} [F]_{J \cup \{j+1\}}^{\otimes \{l\}} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i, j+1\}} E_{J/(\{i, l\})} \\
&= \alpha(j, N) \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} - \frac{1}{N} \sum_{i \neq l \in J} C_{i, j+1} [F]_{J \cup \{j+1\}}^{\otimes \{i, l, j+1\}} E_{J/(\{i, l\})}
\end{aligned}$$

since $\sum_{K \subset L} (-1)^{|K|} = \delta_{L, \emptyset}$. Note that there is a crucial compensation:

$$\begin{aligned}
(115) \quad \mathcal{T}_1 + \mathcal{T}_2^{i, j+1 \in K'} &= -\frac{j}{N} \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} [Q(F, F)]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{l\}} E_{J/\{i, l\}}.
\end{aligned}$$

The computations of $\mathcal{T}_2^{i, j+1 \notin K'}$, $\mathcal{T}_2^{i \in K', j+1 \notin K'}$, $\mathcal{T}_2^{i \notin K', j+1 \in K'}$ go the same way and we omit it here (see [23] for a complete derivation).

We consider a similar dichotomy for the term

$$\begin{aligned}
\mathcal{T}_3 &:= \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left(\sum_{i \neq r \in J/K} T_{i, r} \right) F_{J/K}^N \\
&= \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left(\sum_{i \neq r \in J/K} T_{i, r} \right) \sum_{K' \subset J/K} [F]_{J/K}^{\otimes K'} E_{J/(K \cup K')}.
\end{aligned}$$

according, this time, to the cases $i, r \in K'$, $i, r \notin K'$, $i \in K', r \notin K'$ and $i \notin K', r \in K'$. The computation of the different terms uses the same “tricks” than for \mathcal{T}_2 and we omit them (see again [23]).

Finally, we obtain easily (see [23]) that

$$(116) \quad \mathcal{T}_4 := \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} (K^{J/K} F_{J/K}^N) = K^J E_J.$$

Summing up all the contributions $\mathcal{T}_1, 1 = 1, \dots, 4$ (see [23]), we get (108) after specializing to the case $J = \{1, \dots, j\}$, using (105) and setting

$$\begin{aligned}
D_j &: \mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 1, \dots, N, \\
E_j &\mapsto \frac{N-j}{N} \sum_{i \in J} C_{i,j+1} \left([F]_{J \cup \{j+1\}}^{\otimes \{i\}} \Phi_{(J \cup \{j+1\})/\{i\}} E_j + [F]_{J \cup \{j+1\}}^{\otimes \{j+1\}} E_j \right), \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} ([F]_{J \cup \{j+1\}}^{\otimes \{l\}} \Phi_{(J/\{l\}) \cup \{j+1\}} E_j) \\
D_j^1 &: \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 1, \dots, N-1, \\
E_{j+1} &\mapsto \frac{N-j}{N} C_{j+1} E_{j+1}, \\
D_j^{-1} &: \mathbb{L}^{\otimes(j-1)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 2, \dots, N, \\
E_{j-1} &\mapsto \left(-\frac{j}{N} \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} + \frac{1}{2N} \sum_{i,r \in J} T_{i,r} [F]_J^{\otimes \{i\}} \right) \Phi_{J/\{i\}} E_{j-1}, \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i,j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{j+1\}} \Phi_{J/\{l\}} E_{j-1} \\
&\quad - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i,j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i\}} \Phi_{(J/\{i,l\}) \cup \{j+1\}} E_{j-1} \\
D_j^{-2} &: \mathbb{L}^{\otimes(j-2)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 3, \dots, N, \\
E_{j-2} &\mapsto \frac{1}{2N} \sum_{i,s \in J} T_{i,s} [F]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{s\}} \Phi_{J/\{i,s\}} E_{j-2} \\
(117) \quad &\quad - \frac{1}{N} \sum_{i \neq l \in J} [Q(F, F)]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{l\}} \Phi_{J/\{i,l\}} E_{j-2}.
\end{aligned}$$

where, by convention,

$$(118) \quad \begin{cases} D_N^1 := D_1^{-2} := 0 \\ D_1^{-1}(E_0) := -\frac{1}{N} Q(F, F), \\ D_2^{-2}(E_0) := \frac{1}{N} (T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F)). \end{cases}$$

Note that one has the following estimates:

$$(119) \quad \|D_j\|, \|D_j^1\| \leq j \quad \text{and} \quad \|D_j^{-1}\|, \|D_j^{-2}\|, \|D_1^{-1}(E_0)\|, \|D_2^{-2}(E_0)\| \leq \frac{j^2}{N}.$$

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