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# ASYMPTOTIC EXPANSION OF THE MEAN-FIELD APPROXIMATION

THIERRY PAUL AND MARIO PULVIRENTI

ABSTRACT. We established and estimate the full asymptotic expansion in integer powers of  $\frac{1}{N}$  of the  $[\sqrt{N}]$  first marginals of  $N$ -body evolutions lying in a general paradigm containing Kac models and non-relativistic quantum evolution. We prove that the coefficients of the expansion are, at any time, explicitly computable given the knowledge of the linearization on the one-body meanfield kinetic limit equation. Instead of working directly with the corresponding BBGKY-type hierarchy, we follow a method developed in [22] for the meanfield limit, dealing with error terms analogue to the  $v$ -functions used in previous works. As a by-product we get that the rate of convergence to the meanfield limit in  $\frac{1}{N}$  is optimal.

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## 1. INTRODUCTION: MOTIVATION AND MAIN RESULTS

Mean field limit concerns systems of interacting (classical or quantum) particles whose number diverges in a way linked with a rescaling of the interaction insuring an equilibrium between interaction and residual kinetic energies. In the case of an additive one-body kinetic energy part and a two-body interaction, and without taking in consideration quantum statistics,

this equilibrium is reached by putting in front of the interaction a coupling constant proportional to the inverse of the number of particles.

The system is then described by isolating the evolution of one (or  $j$ ) particle(s) and averaging over all the other. This leads to a partial information on the system driven by the so-called  $j$ -marginals. The mean field theory insures then that the  $j$ -marginals tends, as the number of particles diverges, to the  $j$ -tensor power of the solution of a non-linear one-body meanfield equation (Vlasov, Hartree,...) issued from the 1-marginal on the initial  $N$ -body state. This program has been achieved in many different situations, and the literature concerning the mean field approach is protuberant. We refer to the review article [25] for a reasonable bibliography.

Much less is known about the fluctuations around this limit, namely the correction to be added in order to the factorized limit to get better approximations of the true evolution of the  $j$ -marginals.

The identification of the leading order of these fluctuations with a Gaussian stochastic process has been established in the quantum context first in [14] and in the classical one in [5]. For the classical dynamics of hard spheres, the fluctuations around the Boltzmann equation have been computed at leading order in [24], generalizing to non-equilibrium states the results of [3]. More recently, for the quantum case, fluctuations near the Hartree dynamics has been derived in [2] and in [20] (after [19]) and, in the grand canonical ensemble (number of particles non fixed) in [2], using methods of second quantization (Fock space).

Recently, we developed in [22] (together with S. Simonella) a method to derive mean field limit, alternative to the ones using empirical measures or direct estimates on the “BBGKY-type” hierarchies (systems of coupled equations satisfied by the set of  $j$ -marginals). This method rather uses the hierarchy followed by the “kinetic errors”  $E_{j-k}$  (defined below), already used (under the name “ $v$ -functions”) to deal with kinetic limits of stochastic models [10, 7, 4, 11, 12, 6, 8, 13] and recently investigated in the more singular low density limit of hard spheres [23]. These quantities are, roughly speaking, the coefficient of the decomposition of the  $j$ -marginal as a linear combination of the  $k$ -tensor powers,  $k = 1, \dots, j$ , of the solution

of the mean-field equation issued from of the 1-marginal of the initial full state. We developed in [22] a strategy suitable in particular for Kac models (homogeneous original one [15, 16] and non-homogeneous [9]) and quantum mean field theory. This strategy allowed us to derive the limiting factorization property of the  $j$ -marginals up to, roughly speaking,  $j \lesssim \sqrt{N}$ . This threshold is, on the other side, the one obtained by heuristic arguments as shown in [22].

Here and in all this article,  $N$  is the number of particles of the system under consideration.

In the present note we provide and estimate a full asymptotic expansion in powers of  $\frac{1}{N}$  of the difference between the evolution of  $j$ -marginals and its factorized leading order form (Corollary 1.5), following a similar result for the kinetic errors  $E_j(t)$  (Theorem 1.4). Our results are valid for  $j \lesssim \sqrt{N}$  in an abstract paradigm, slight generalization of the abstract formalism developed in [22], which is described in the next paragraph and applies of course to the different Kac models and quantum mean field theory treated in [22]. Moreover, we show that the additional knowledge of the linearization of the mean field flow, around the meanfield solution issued from the 1-marginal of the initial data, gives an explicit construction of the full asymptotic expansion of the  $j$ -marginals in powers of  $\frac{1}{N}$ .

In the present article, we will establish the results just described in the following situations:

- the quantum mean-field model
- the homogeneous and non-homogeneous Kac models
- a general “abstract” model containing the two first situations

We will state in this section the quantum results and postpone in Section 5 and in the Appendix A the Kac’s type and the abstract results, respectively. Sections 2 and 3 contain the algebraic and analytical proofs of our results in the quantum case, immediately transposable to the Kac and abstract situations as shown in Section 5 and Appendix A. In Section 4 one compute explicitly the first terms of the asymptotic expansions obtained in the quantum case and rely them to previous works.

1.1. **Quantum mean-field.**  $\mathcal{L}^1(L^2(\mathbf{R}^d))$ , the space of trace class operators on  $L^2(\mathbf{R}^d)$ , with their associated norms.

We consider the evolution of a system of  $N$  quantum particles interacting through a (real-valued) two-body, even potential  $V$ , described for any value of the Planck constant  $\hbar > 0$  by the Schrödinger equation

$$i\hbar\partial_t\psi = H_N\psi, \quad \psi|_{t=0} = \psi_{in} \in \mathfrak{H}_N := L^2(\mathbf{R}^d)^{\otimes N},$$

where

$$H_N := -\frac{1}{2}\hbar^2 \sum_{k=1}^N \Delta_{x_k} + \frac{1}{2N} \sum_{1 \leq k, l \leq N} V(x_k - x_l).$$

Instead of the Schrödinger equation written in terms of wave functions, we shall rather consider the quantum evolution of density matrices. An  $N$ -body density matrix is an operator  $F^N$  such that

$$0 \leq F^N = (F^N)^*, \quad \text{trace}_{\mathfrak{H}_N}(F^N) = 1.$$

The evolution of the density matrix  $F^N \mapsto F^N(t)$  of a  $N$ -particle system is governed for any value of the Planck constant  $\hbar > 0$  by the von Neumann equation

$$(1) \quad \partial_t F^N = \frac{1}{i\hbar} [H_N, F^N].$$

Positivity, norm and trace are obviously preserved by (1).

For each  $j = 1, \dots, N$ , the  $j$ -particle marginal  $F_j^N(t)$  of  $F^N(t)$  is the unique operator on  $\mathfrak{H}_j$  such that

$$\text{trace}_{\mathfrak{H}_N}[F^N(t)(A_1 \otimes \dots \otimes A_j \otimes I_{\mathfrak{H}_{N-j}})] = \text{trace}_{\mathfrak{H}_j}[F_j^N(t)(A_1 \otimes \dots \otimes A_j)].$$

for all  $A_1, \dots, A_j$  bounded operators on  $\mathfrak{H}$ . Alternatively the  $F_j^N$  can be defined by the partial trace of  $F^N$  on the  $N - j$  last variables (see [1]. Namely, defining  $F^N$  through its integral kernel  $F^N(x_1, \dots, x_n; x'_1, \dots, x'_n)$ , the integral kernel of  $F_j^N$  is defined as

$$(2) \quad \begin{aligned} F_j^N(x_1, \dots, x_j; x'_1, \dots, x'_j) &:= (\text{Tr}^{j+1} \dots \text{Tr}^N F^N)(x_1, \dots, x_j; x'_1, \dots, x'_j) \\ &:= \int_{\mathbf{R}^{d(N-j)}} F^N(x_1, \dots, x_N; x'_1, \dots, x'_j, x_{j+1}, \dots, x_N) dx_{j+1} dx_N. \end{aligned}$$

It will be convenient to rewrite (1) in the following operator form

$$(3) \quad \partial_t F^N = (K^N + V^N)F^N$$

where  $K^N, V^N$  are operators on  $\mathcal{L}^1(L^2(\mathbf{R}^{Nd}))$  defined by

$$(4) \quad K^N = \frac{1}{i\hbar} \left[ -\frac{\hbar^2}{2} \Delta_{\mathbf{R}^{dN}}, \cdot \right], \quad V^N = \frac{1}{2N} \sum_{k,l} V_{k,l} \quad \text{with} \quad V_{k,l} := \frac{1}{i\hbar} [V(x_k - x_l), \cdot].$$

Note that the (essantial) self-adjointness of  $H_N$  implies that

$$(5) \quad \|e^{t(K^N + V^N)}\|_{\mathcal{L}^1(L^2(\mathbf{R}^d)) \rightarrow \mathcal{L}^1(L^2(\mathbf{R}^d))} = \|e^{tK^N}\|_{\mathcal{L}^1(L^2(\mathbf{R}^d)) \rightarrow \mathcal{L}^1(L^2(\mathbf{R}^d))} = 1, \quad t \in \mathbf{R}.$$

We will also denote by

$$(6) \quad \mathbb{L} = \mathcal{L}^1(L^2(\mathbf{R}^d)) \quad \text{so that} \quad \mathbb{L}^{\otimes n} = \mathcal{L}^1(L^2(\mathbf{R}^{nd})), \quad n = 1, \dots, N,$$

and, with a slight abuse of notation, by

$$(7) \quad \begin{cases} \|\cdot\|_1 & \text{the trace norm on any } \mathbb{L}^{\otimes j}, \\ \|\cdot\| & \text{the operator norm on any } \mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j}) \end{cases}$$

for  $i, j = 1, \dots, N$  (here  $\mathcal{L}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j})$  is the set of bounded operators from  $\mathbb{L}^{\otimes i}$  to  $\mathbb{L}^{\otimes j}$ ).

A density matrix  $F^n \in \mathbb{L}^{\otimes n}$  is called symmetric if its integral kernel  $F^n(x_1, \dots, x_n; x'_1, \dots, x'_n)$  is invariant by any permutation

$$(x_i, x'_i) \leftrightarrow (x_j, x'_j), \quad i, j = 1, \dots, n.$$

Note that the symmetry of  $F^N$  is preserved by the equation (1) due to the particular form of the potential.

We define, for  $n = 1, \dots, N$ ,

$$(8) \quad \mathcal{D}_n = \{F \in \mathbb{L}^{\otimes n} \mid F > 0, \quad \|F\|_1 = 1 \quad \text{and} \quad F \text{ is symmetric}\}.$$

Note that  $F_j^N \in \mathbb{L}^{\otimes j}$  ( $F_0^N = 1 \in \mathbb{L}^{\otimes 0} := \mathbf{C}$ ) and  $F_j^N > 0, \|F_j^N\|_1 = \|F^N\|_1$ , and obviously  $F_j^N$  is symmetric as  $F^N$ . That is to say:

$$F_j^N \in \mathcal{D}_j.$$

The family of  $j$ -marginals,  $j = 1, \dots, N$ , are solutions of the BBGKY hierarchy of equations (see [1])

$$(9) \quad \partial_t F_j^N = \left( K^j + \frac{T_j}{N} \right) F_j^N + \frac{(N-j)}{N} C_{j+1} F_{j+1}^N$$

where:

$$(10) \quad K^j = \frac{1}{i\hbar} \left[ -\frac{\hbar^2}{2} \Delta_{\mathbf{R}^{jd}}, \cdot \right]$$

$$(11) \quad T_j = \sum_{1 \leq i < r \leq j} T_{i,r} \quad \text{with} \quad T_{i,r} = V_{ir}$$

and

$$(12) \quad C_{j+1}F_{j+1}^N = \sum_{i=1}^j C_{i,j+1}F_{j+1}^N$$

with

$$(13) \quad \begin{aligned} C_{i,j} &: \mathbb{L}^{\otimes(j+1)} &\rightarrow &\mathbb{L}^{\otimes j} \\ C_{i,j+1}F_{j+1}^N &= \text{Tr}^{j+1}(V_{i,j+1}F_{j+1}^N), \end{aligned}$$

where  $\text{Tr}^{j+1}$  is the partial trace with respect to the  $(j+1)$ th variable, as in (2).

Note that, for all  $i \leq j = 1, \dots, N$ ,

$$(14) \quad \|T_j\| \leq j^2 \frac{\|V\|_{L^\infty}}{\hbar}, \quad \text{and} \quad \|C_{i,j+1}\| \leq j \frac{\|V\|_{L^\infty}}{\hbar}.$$

(meant for  $\|T_j\|_{\mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j}}$  and  $\|C_{i,j+1}\|_{\mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}}$  in accordance with (7)).

The Hartree equation reads

$$(15) \quad i\hbar \partial_t F = \left[ -\frac{\hbar^2}{2} \Delta + V_F(x), F \right], \quad F(0) \in \mathcal{D}_1,$$

where  $V_F(x) = \int_{\mathbf{R}^d} V(x-y)F(y,y)dy$ ,  $F(y,y')$  being the integral kernel of  $F$ .

Note that (15) reads also

$$(16) \quad \partial_t F = K^1 F + Q(F, F),$$

with

$$(17) \quad Q(F, F) = \text{Tr}^2(V_{1,2}(F \otimes F)).$$

Since  $V$  is bounded, (15) has for all time a unique solution  $F(t) > 0$  and  $\|F(t)\| = 1$  (see again [1]).

In order to define the correlation error in an easy way, we need a bit more of notations concerning the variables of integral kernels.

We define the variables  $z_i, i = 1, \dots, N$ , equal to  $(x_i, x'_i)$ , and  $Z_j = (z_1, \dots, z_j)$ . We will also denote by  $Z_j^{\setminus \{i_1, \dots, i_k\}} \in \mathbf{R}^{2(j-k)d}$ , where  $\{i_1, \dots, i_k\} \subset \{1, \dots, j\}$ , the vector  $Z_j := (z_1, \dots, z_j)$  after removing the components  $z_{i_1}, \dots, z_{i_k}$ .

**Definition 1.1.** For any  $j = 1, \dots, N$ , we define the correlation error  $E_j \in \mathbb{L}^{\otimes j}$  by its integral kernel

$$(18) \quad E_j(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} (-1)^k F(z_{i_1}) \dots F(z_{i_k}) F_{j-k}^N(Z_j^{\{i_1, \dots, i_k\}}).$$

By convention and consistently with  $F_0^N = \|F\| = 1$ , we define  $E_0 := 1$ .

In [22] was shown that (18) is inverted by the following equality.

$$(19) \quad F_j^N(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(z_{i_1}) \dots F(z_{i_k}) E_{j-k}(Z_j^{\{i_1, \dots, i_k\}}), \quad j = 0, \dots, N..$$

i.e.  $F_j^N$  is the operator of integral kernel given by (19).

Note that, in order to avoid too heavy notations, we made, in (18) and (19), a slight abuse of notation as writing integral kernels as functions of  $Z_j = ((x_1, x'_1), \dots, (x_j, x'_j))$  and not functions of  $(x_1, \dots, x_j; x'_1, \dots, x'_j)$ .

In the sequel, we will take  $\hbar = 1$ , the correct value being easy to restore in the different expressions.

**1.2. Main results of [22].** Theorem 2.4, theorem 2.1 and Corollary 2.2 in [22] state the following facts.

The kinetic errors  $E_j$ ,  $j = 1, \dots, N$ , satisfy the system of equations

$$(20) \quad \begin{aligned} \partial_t E_j &= \left( K^j + \frac{1}{N} T_j \right) E_j + D_j E_j \\ &+ D_j^1 E_{j+1} + D_j^{-1} E_{j-1} + D_j^{-2} E_{j-2}, \end{aligned}$$

where the operators  $D_j, D_j^1, D_j^{-1}, D_j^{-2}$ ,  $j = 0, \dots, N$ , are defined at the beginning of the Section 2, formulas (35)-(38).

**Theorem 1.2** (Theorem 2.2. and Corollary 2.3 in [22]).

Let  $E_j(0)$  satisfy for some  $C_0 > 1, B > 0$

$$(21) \quad \begin{cases} \|E_1(0)\|_1 \leq \frac{B}{N} \\ \|E_j(0)\|_1 \leq \left(\frac{j^2}{N}\right)^{j/2} C_0^j, \quad j \geq 2. \end{cases}$$

Then, for all  $t > 0$  and all  $j = 1, \dots, N$ , one has



$$(22) \quad \begin{cases} \|E_1(t)\|_1 \leq \frac{1}{N} (B_2 e^{\frac{B_1 t \|V\|_{L^\infty}}{h}}) \\ \|E_j(t)\|_1 \leq \left( C_2 e^{\frac{C_1 t \|V\|_{L^\infty}}{h}} \right)^j \left( \frac{j}{\sqrt{N}} \right)^j \end{cases}$$

for some  $B_1, C_1 > 0$ ,  $B_2, C_2 \geq 1$  explicit (see formulas (78), (81) in [22]), and

$$(23) \quad \|F_j^N(t) - F(t)^{\otimes j}\|_1 \leq D_2 e^{\frac{D_1 t \|V\|_{L^\infty}}{h}} \frac{j^2}{N},$$

where  $D_2 = \sup\{B_2, (eC_0)^2\}$ ,  $B_1 = \sup\{B_1, 2C_1\}$ .

### 1.3. Asymptotic expansion and main result of the present article.

Two questions arise naturally:

- (1) are the estimates (22) sharp?
- (2) could (23) be improved with a r.h.s. of any order we wish?<sup>1</sup>

We will see below that, indeed, not only the estimates (22) are true, but  $N^{j/2} E_j^N(t)$  has a full asymptotic expansion in  $(\frac{1}{N})^{\frac{1}{2}}$  (actually we will show that this expansion contains only powers  $(\frac{1}{N})^{\frac{k}{2}}$  with  $k + j$  even) if  $N^{j/2} E_j^N(0)$  do posses such an expansion in half powers of  $1/N$ .

More precisely we will show that, under the hypothesis (21) on the initial data,

$$(25) \quad E_j(t) = N^{-j/2} \mathcal{E}_j, \quad \mathcal{E}_j(t) \sim \sum_{\ell=0}^{\infty} \mathcal{E}_j^\ell(t) N^{-\ell/2} \text{ with } \mathcal{E}_j^k = 0 \text{ for } j + k \text{ odd.}$$

for all time  $t$  and all  $j = 1, \dots, N$ .

Moreover we will see that all the  $\mathcal{E}_j^\ell$  can be explicitly recursively computed after the knowledge of the linearization of the mean field equation (15) around the solution of (15) with initial condition  $F(0) = (F^N(0))_1$ . Indeed the proof will involve the “ $j$ -kinteic linear mean field flow” defined by the

<sup>1</sup>Of course (22) and (23) imply that

$$(24) \quad \|F_j^N(t) - F^{\otimes j} - \sum_{\substack{J=K \subset J \\ |K| \leq k \leq |J|}} F^{\otimes |K|} E_{|J|-|K|}^N\| = O(N^{-(k+1)/2}),$$

but first one cannot go further in the approximation, and second (24) is meaningless without the knowledge of the  $E_j^N$ s.

linear kinetic mean field equation of order  $j$ :

$$(26) \quad \frac{d}{dt}A(t) = (K^j + \Delta_j(t))A(t), \quad A(0) \in \mathbb{L}^{\otimes j},$$

solved by the two parameter semigroup  $U_j^0(t, s)$  solving

$$(27) \quad \begin{aligned} \partial_t U_j^0(t, s) &= (K^j + \Delta_j(t))U_j^0(t, s). \\ U_j^0(s, s) &= I. \end{aligned}$$

Here  $\Delta_j(t) = \lim_{N \rightarrow \infty} D_j(t)$  and  $U_j^0$  exists since  $K^j$  generates a unitary flow and  $\Delta_j$  is bounded.

The reason of the terminology comes from the fact that, as shown by (35),  $\Delta_1 = Q(F, \cdot) + Q(\cdot, F)$  so that, for  $j = 1$ , (26) is the linearization of the mean field equation (15) around its solution  $F(t)$ . Note moreover that, for  $G^1, G^2 \in \mathbb{L}$ ,

$$(28) \quad \begin{aligned} \Delta_2(G^1 G^2 + G^2 G^1) &= \\ (\Delta_1 G^1)G^2 + G^1(\Delta_1 G^2) &+ (\Delta_1 G^2)G^1 + G^2(\Delta_1 G^1). \end{aligned}$$

and therefore

$$(29) \quad U_2^0(t, s)(G^1 G^2 + G^2 G^1) = (U_1^0(t, s)G^1)(U_1^0(t, s)G^2).$$

More generally, if  $P_j : \mathbb{L}^j \rightarrow \mathbb{L}^{\otimes j}$  is any homogeneous polynomial invariant by permutations,

$$(30) \quad U_j^0(t, s)P_j(G^1, \dots, G^j) = P_j(U_1^0(t, s)G^1, \dots, U_1^0(t, s)G^j).$$

That is:  $U_j^0$  drives each  $G^j$  along the linearized mean field flow “factor by factor”. Denoting by  $\mathbb{L}_{sym}^{\otimes j}$  the subspace of symmetric (by permutations) vectors, we just prove the following result.

**Lemma 1.3.**

$$U_j^0(t, s)|_{\mathbb{L}_{sym}^{\otimes j}} = U_1^0(t, s)^{\otimes j}.$$

Note also that  $U_j^0(t, s)$  is given by a convergent Dyson expansion and that, by Gronwall Lemma and the isometry of the flow generated by  $K^j$ , we have that  $\|U_j^0(t, s)\| \leq e^{j|t-s|}$ . Moreover, we will also need in the sequel the semigroup defined by

$$(31) \quad \begin{aligned} \partial_t U_j(t, s) &= (K^j + \frac{T_j}{N} + \frac{N-j}{N}D_j(t))U_j(t, s). \\ U_j(s, s) &= I. \end{aligned}$$

$U_j(t)$  exists by the same argument as for  $U_j^0(t)$ . Moreover, since we are in the regime  $j^2/N$  small,  $U_j$  can be also computed out of  $U_j^0$  by convergent perturbation expansion (in  $j^2/N$ ). Indeed (35), (100) and (40) show clearly that  $\|\frac{T_j}{N} + \frac{N-j}{N}D_j - \Delta_j\| \leq \frac{j^2}{N}$ . Therefore  $U_j(t)$  can be approximated for any  $t$  by a finite Dyson expansion up to any power of  $j^2/N$ .

Finally, we will extend (23) as we will show that  $F_j^N$  has an asymptotic expansion in positive powers of  $1/N$  whose partial sums up to any order  $n \geq 0$  is  $O(jN^{-n-1})$ -close to  $F_j^N$ .

The main results of the present note are the following.

**Theorem 1.4.** *Consider for  $j = 0, \dots, N$ ,  $k = 0, \dots$ ,  $t \geq 0$  the system of recursive relations*

$$\begin{cases} \mathcal{E}_j^k(t) = U_j(t, 0)\mathcal{E}_j^k(0) \\ \quad + \int_{s=0}^t U_j(t, s)(\Delta_j^- \mathcal{E}_{j-2}^k(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(s))ds \\ \mathcal{E}_0^k(t) = \delta_{k,0}, \\ \mathcal{E}_{-1}^k(t) = \mathcal{E}_{-2}^k(t) = \mathcal{E}_j^{-1}(t) = 0 \text{ by convention.} \end{cases}$$

where  $U_j(t, s)$  is the two times flow defined by (31) and  $\Delta_j^+ D_j^1$ ,  $\Delta_j^- = ND_j^{-1}$ ,  $\Delta_j^- = ND_j^{-2}$ , the  $D_j$ s being given by (35)-(38).

Then, for any  $j = 1, \dots, N$ ,  $k = 0, \dots$ ,  $t \geq 0$ , the knowledge of  $\mathcal{E}_{j'}^{k'}(0)$  for  $j' = 1, \dots, j+k$ ,  $k' = 0, \dots, k$ , determine in a unique way  $\mathcal{E}_j^k(t)$ , and

$$\mathcal{E}_j^k(t) = 0 \text{ when } j+k \text{ is odd}$$

if  $\mathcal{E}_j^k(0)$  satisfies the same property.

Moreover, if  $E_j(t) := N^{-j/2}\mathcal{E}_j(t)$  solves the equation (117) and satisfies (21), one has

$$\|\mathcal{E}_j(t) - \sum_{k=1}^{2n} N^{-k/2}\mathcal{E}_j^k(t)\|_1 \leq tC_{2n}(t)N^{-n-\frac{1}{2}}(A_t^{2n}j^2)^{j/2},$$

where  $C_n(t), A_t^n$  are defined in (77) below.

Let us define, for  $j = 1, \dots, N$ ,  $n = 0, \dots$

$$(32) \quad E_j^n(t) = \sum_{k=0}^{2n} N^{-\frac{j+k}{2}} \mathcal{E}_j^k(t)$$

(note that  $E_j^n(t)$  contains only integer powers of  $N^{-1}$ , since  $\mathcal{E}_j^k = 0$  when  $j+k$  is odd, that is  $E_j^n = \sum_{k=[(j+1)/2]}^n c_k^j N^{-k}$ ), and  $F_j^{N,n}(t)$  the operator of integral kernel  $F_j^{N,n}(t)(Z_j)$  defined by

$$(33) \quad F_j^{N,n}(t)(Z_j) = \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(t)(z_{i_1}) \dots F(t)(z_{i_k}) E_{j-k}^n(Z_j^{\{i_1, \dots, i_k\}}),$$

(that is (109) truncated at order  $n$ , same slight abuse of notation).  $F_j^{N,n}$  is therefore a polynomial of order  $n$  in  $\frac{1}{N}$ .

**Corollary 1.5.** *Let  $F^N(t)$  the solution of the quantum  $N$  body system (1) with initial datum  $F^N(0) = F^{\otimes N}$ ,  $F \in \mathcal{L}(L^2(\mathbf{R}^d))$ ,  $F \geq 0$ ,  $\text{Tr}F = 1$ , and  $F(t)$  the solution of the Hartree equation (15) with initial datum  $F$ .*

*Then, for all  $n \geq 1$ ,*

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2tC_{2n}(t)eA_t^{2n}j}{\sqrt{N}}.$$

for  $N \geq 4(eA_t^{2n}j)^2$ .

**Remark 1.6.** *If one is interested only to the expansion up to order  $n < j$ , we can change the sum in the l.h.s. of the inequality in Corollary 1.5 by a sum up to  $\ell = n$ .*

*Proof.* The proof is similar to the one of Corollary 2.2 in [22].

$$\begin{aligned} & \|F_j^N(t) - \sum_{K \subset \{1, \dots, j\}} \prod_K^j (F^{\otimes |K|} \otimes E_{j-|K|}^n)\| \\ & \leq \sum_{k=1}^j \binom{j-k}{k} \|E_k - E_k^n\| \\ & \leq N^{-n-\frac{1}{2}} \sum_{k=1}^j \binom{j}{k} C_{2n-k}(t) \left(\frac{A_t^{2n-k} k^2}{N}\right)^{k/2} \\ & \leq N^{-n-\frac{1}{2}} C_{2n}(t) \sum_{k=1}^j j(j-1) \dots (j-k+1) \left(\frac{A_t^{2n}}{\sqrt{N}}\right)^k \frac{k^k}{k!} \\ & \leq N^{-n-\frac{1}{2}} C_{2n}(t) \sum_{k=1}^j \left(\frac{j e A_t^{2n}}{\sqrt{N}}\right)^k \leq N^{-n-\frac{1}{2}} \frac{2C_{2n}(t)eA_t^{2n}j}{\sqrt{N}} \end{aligned}$$

for  $N \geq 4(eA_t^{2n}j)^2$  (we used  $\frac{k^k}{k!} \leq \frac{e^k}{\sqrt{2\pi k}}$ ). □

**Corollary 1.7.** *The rate of convergence to the meanfield limit in  $\frac{1}{N}$  is optimal.*

**Remark 1.8.** *In the asymptotic expansion  $E_j(t) \sim \sum_{k=[(j+1)/2]}^{\infty} c_k^j(t) N^{-k}$  the coefficients  $c_k^j(t)$ , such as  $\mathcal{E}_j^k(t)$ , depend on  $N$  as well: first by the dependence of  $\Delta_j^+ = (1 - \frac{j}{N})C_{j+1}$  and also by  $U_j(t, s)$  defined by (41). As mentioned already the latter can be expressed as a (convergent) series in  $\frac{1}{N}$  out of the linearization of the meanfield equation so that obtaining a full asymptotic expansion of  $E_j(t)$  with the only knowledge of the linearization of the meanfield equation is (tedious but) elementary.*

## 2. THE RECURSIVE CONSTRUCTION

Let us recall from [22] that the hierarchy of error terms satisfy the following equation:

$$(34) \quad \begin{aligned} \partial_t E_j &= \left( K^j + \frac{1}{N} T_j \right) E_j + D_j E_j \\ &+ D_j^1 E_{j+1} + D_j^{-1} E_{j-1} + D_j^{-2} E_{j-2}. \end{aligned}$$

Here the four operator  $D_j, D_j^1, D_j^{-1}, D_j^{-2}$ ,  $j = 0, \dots, N$ , are defined as follows (here again,  $J = \{1, \dots, j\}$ ): for any operator  $G \in \mathbb{L}^{\otimes n}$ ,  $n = 1, \dots, N$ , we denote by  $G(Z_n)$  its integral kernel, and for any function  $F(Z_n)$ ,  $n = 1, \dots, N$ , we define  $\overline{F(Z_n)}$  as being the operator on  $\mathbb{L}^{\otimes n}$  of integral kernel  $F(Z_n)$  then

$$\begin{aligned}
D_j &: \mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j} \\
E_j &\mapsto \frac{N-j}{N} \sum_{i \in J} C_{i,j+1} \left( \overline{F(z_i)E_j(Z_{j+1}^{\{i\}})} + \overline{F(z_{j+1})E_j(Z_j)} \right) \\
(35) \quad & - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l)E_j(Z_{j+1}^{\{l\}})}
\end{aligned}$$

$$\begin{aligned}
D_j^1 &: \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j} \\
E_{j+1} &\mapsto \frac{N-j}{N} C_{j+1} E_{j+1} \\
(36) \quad &
\end{aligned}$$

$$\begin{aligned}
D_j^{-1} &: \mathbb{L}^{\otimes(j-1)} \rightarrow \mathbb{L}^{\otimes j} \\
E_{j-1} &\mapsto \frac{1}{N} \sum_{i,r \in J} T_{i,r} \overline{F(z_i)E_{j-1}(Z_j^{\{i\}})} - \frac{j}{N} \sum_{i \in J} \overline{Q(F,F)(z_i)E_{j-1}(Z_j^{\{i\}})} \\
& - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l)F(z_{j+1})E_{j-1}(Z_j^{\{l\}})} \\
(37) \quad & - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \overline{F(z_l)F(z_i)E_{j-1}(Z_{j+1}^{\{i,l\}})}
\end{aligned}$$

$$\begin{aligned}
D_j^{-2} &: \mathbb{L}^{\otimes(j-2)} \rightarrow \mathbb{L}^{\otimes j} \\
E_{j-2} &\mapsto \frac{1}{N} \sum_{i,s \in J} T_{i,s} \overline{F(z_i)F(z_r)E_{j-2}(Z_j^{\{i,r\}})} \\
(38) \quad & - \frac{1}{N} \sum_{i \neq l \in J} \overline{Q(F,F)(z_i)F(z_l)E_{j-2}(Z_l^{\{i,l\}})}.
\end{aligned}$$

where, by convention,

$$(39) \quad \begin{cases} D_N^1 := D_1^{-2} := 0 \\ D_1^{-1}(E_0) := -\frac{1}{N}Q(F,F), \\ D_2^{-2}(E_0) := \frac{1}{N}(T_{1,2}(F \otimes F) - Q(F,F) \otimes F - F \otimes Q(F,F)). \end{cases}$$

In (35)-(38),  $F(z)$  is meant as being the integral kernel of  $F(t)$  solution of the Hartree equation 15.

Note that one has the following estimates:

$$(40) \quad \|D_j\|, \|D_j^1\| \leq j \text{ and } \|D_j^{-1}\|, \|D_j^{-2}\|, \|D_1^{-1}(E_0)\|, \|D_2^{-2}(E_0)\| \leq \frac{j^2}{N}.$$

**Remark 2.1.** *We shall see that the properties (39)-(40), together with (5), are actually the only ones being used in the proof of Theorem 1.4.*

We define the (two parameters) semigroup  $U_j(t, s)$  for  $s \leq t$ , satisfying

$$(41) \quad \begin{aligned} \partial_t U_j(t, s) &= (K^J + \frac{T_j}{N} + D_j(t))U_j(t, s), \quad j = 1, \dots, N \\ U_j(s, s) &= I =: U_0(t, s) \end{aligned}$$

Obviously  $U_j(t, s)U_j(s, t) = I$  so that

$$(42) \quad U_j(t, s)^{-1} = U_j(s, t).$$

Having in mind the result of Theorem 2.1. of [22] and calling  $E_j = N^{-j/2}\mathcal{E}_j$ ,  $H_j = K^j + T_j/N + D_j$ ,  $\Delta_j^+ = D_j^1$ ,  $\Delta_j^- = ND_j^{-1}$ ,  $\Delta_j^- := ND_j^{-2}$  we find easily that

$$(43) \quad \begin{aligned} \partial_t \mathcal{E}_1 &= H_1 \mathcal{E}_1 + N^{-\frac{1}{2}} \Delta_1^+ \mathcal{E}_2 + N^{-\frac{1}{2}} \Delta_1^- \mathcal{E}_0 \\ &= H_1 \mathcal{E}_1 + N^{-\frac{1}{2}} \Delta_1^+ \mathcal{E}_2 - N^{-\frac{1}{2}} Q(F, F) \\ \partial_t \mathcal{E}_j &= H_j \mathcal{E}_j + N^{-\frac{1}{2}} D_j^1 \mathcal{E}_{j+1} + N^{\frac{1}{2}} D_j^{-1} \mathcal{E}_{j-1} + ND_j^{-2} \mathcal{E}_{j-2} \\ &= H_j \mathcal{E}_j + N^{-\frac{1}{2}} \Delta_j^+ \mathcal{E}_{j+1} + N^{-\frac{1}{2}} \Delta_j^- \mathcal{E}_{j-1} + \Delta_j^- \mathcal{E}_{j-2}, \quad j \geq 2 \end{aligned}$$

Therefore

$$\begin{aligned} \partial_t \mathcal{E}_1 &= H_1 \mathcal{E}_1 + O(N^{-\frac{1}{2}}) \\ \partial_t \mathcal{E}_2 &= H_2 \mathcal{E}_2 + \Delta_2^- \mathcal{E}_0 + O(N^{-\frac{1}{2}}) \\ &= H_2 \mathcal{E}_2 + V^2 F \otimes F + O(N^{-\frac{1}{2}}) \\ &\quad \cdot \\ \partial_t \mathcal{E}_j &= H_j \mathcal{E}_j + \Delta_j^- \mathcal{E}_{j-2} + O(N^{-\frac{1}{2}}), \quad j \geq 3. \end{aligned}$$

We define  $\mathcal{E}_j^0(t)$  as the solution of

$$(44) \quad \partial_t \mathcal{E}_1^0 = H_1 \mathcal{E}_1^0$$

$$(45) \quad \partial_t \mathcal{E}_2^0 = H_2 \mathcal{E}_2^0 + V^2 F^{\otimes 2}$$

$$(46) \quad \partial_t \mathcal{E}_j^0 = H_j \mathcal{E}_j^0 + \Delta_j^- \mathcal{E}_{j-2}^0, \quad j \geq 3.$$

(67) and (68) are closed equations whose solutions are given by

$$(47) \quad \mathcal{E}_1^0(t) = U_1(t, 0)\mathcal{E}_1^0(0) = 0$$

since we supposed  $E_1(0) = O(N^{-1})$ , and

$$(48) \quad \mathcal{E}_2^0(t) = U_1(t, 0)\mathcal{E}_2^0(0) + U_2(t, 0) \int_0^t U_2(0, s) V F(s)^{\otimes 2} ds.$$

Iterating till  $j$ , we get

$$(49) \quad \mathcal{E}_j^0(t) = U_j(t, 0)\mathcal{E}_j^0(0) + U_j(t, 0) \int_0^t U_j(0, s)\Delta_j^- \mathcal{E}_{j-2}^0(s)ds, j \geq 1,$$

with the convention  $\mathcal{E}_l^k = 0, l < 0$ .

Therefore, for  $j = 1, \dots, N, t \geq 0$ , the knowledge of  $U_j(t, s), s \leq t$ , and  $\mathcal{E}_{j_0}^0(0), j_0 = 1, \dots, j$  guarantees the knowledge of  $\mathcal{E}_j^0(t)$ . We write this fact as

$$(50) \quad (\mathcal{E}_{j_0}^0(0))_{j_0=1, \dots, j} \rightsquigarrow (\mathcal{E}_{j_0}^0(t))_{t \geq 0, j_0=1, \dots, j}$$

To go one step further, one find

$$(51) \quad \begin{aligned} \partial_t \mathcal{E}_1^1 &= H_1 \mathcal{E}_1^1 + \Delta_1^+ \mathcal{E}_2^0 + \Delta_1^- \mathcal{E}_0^0 \\ &\quad + \Delta_1^+ \mathcal{E}_2^0 - Q(F, F) \end{aligned}$$

$$(52) \quad \partial_t \mathcal{E}_2^1 = H_2 \mathcal{E}_2^1 + \Delta_2^+ \mathcal{E}_3^0 + \Delta_2^- \mathcal{E}_1^0$$

.

$$(53) \quad \partial_t \mathcal{E}_j^1 = H_1 \mathcal{E}_j^1 + \Delta_j^- \mathcal{E}_{j-2}^1 + \Delta_j^+ \mathcal{E}_{j+1}^0 + \Delta_j^- \mathcal{E}_{j-1}^0, j > 2$$

Recursively we find:

$$(54) \quad \partial_t \mathcal{E}_1^k = H_1 \mathcal{E}_1^k + \Delta_1^+ \mathcal{E}_2^{k-1} + \Delta_1^- \mathcal{E}_0^{k-1}$$

$$(55) \quad \partial_t \mathcal{E}_2^k = H_2 \mathcal{E}_2^k + \Delta_2^+ \mathcal{E}_3^{k-1} + \Delta_2^- \mathcal{E}_1^{k-1}$$

.

$$(56) \quad \partial_t \mathcal{E}_j^k = H_j \mathcal{E}_j^k + D_j^{-2} \mathcal{E}_{j-2}^k + \Delta_j^+ \mathcal{E}_{j+1}^{k-1} + \Delta_j^- \mathcal{E}_{j-1}^{k-1},$$

solved respectively by

$$(57) \quad \mathcal{E}_1^k(t) = U_1(t, 0)\mathcal{E}_1^k(0) + U_1(t, 0) \int_0^t U_1(0, s)(\Delta_1^+ \mathcal{E}_2^{k-1} + \Delta_1^- \mathcal{E}_0^{k-1})ds$$

$$(58) \quad \mathcal{E}_2^k(t) = U_2(t, 0)\mathcal{E}_2^k(0) + U_2(t, 0) \int_0^t U_2(0, s)(\Delta_2^+ \mathcal{E}_3^{k-1}(s) + \Delta_2^- \mathcal{E}_1^{k-1}(s))ds$$

$$(59) \quad \begin{aligned} \mathcal{E}_j^k(t) &= U_j(t, 0)\mathcal{E}_j^k(0) \\ &\quad + \int_{s=0}^t U_j(t, s)(\Delta_j^- \mathcal{E}_{j-2}^k(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(s))ds \end{aligned}$$

Note that (59) reduces to (57),(58) for  $j = 1, 2$ , by convention (39)



Therefore, the knowledge of  $(\mathcal{E}_{j'}^{k'}(s))_{s \leq t, k' \leq k-1, j'=1, \dots, j+1}$  guarantees, by (57),(58), the knowledge of  $\mathcal{E}_1^k(t), \mathcal{E}_2^k(t)$  and therefore, by (59), the knowledge of  $\mathcal{E}_3^k(t), \mathcal{E}_4^k(t)$  and so forth till  $\mathcal{E}_j^k(t)$ :

$$(\mathcal{E}_{j'}^{k'}(s))_{s \leq t, k' \leq k-1, j'=1, \dots, j+1} \rightsquigarrow \mathcal{E}_j^k(t).$$

Therefore

$$(\mathcal{E}_{j'}^{k'}(s))_{s \leq t, k' \leq k-2, j'=1, \dots, j+2} \rightsquigarrow (\mathcal{E}_{j'}^{k'}(s))_{s \leq t, k' \leq k-1, j'=1, \dots, j+1} \rightsquigarrow \mathcal{E}_j^k(t).$$

and by iteration

$$(\mathcal{E}_{j'}^0(s))_{s \leq t, j'=1, \dots, j+k} \rightsquigarrow \mathcal{E}_j^k(t)$$

so that, by (50),

$$(\mathcal{E}_{j'}^0(0))_{j'=1, \dots, j+k} \rightsquigarrow \mathcal{E}_j^k(t).$$

We just proved the following result.

**Proposition 2.2.** *For any  $j = 1, \dots, N, t \geq 0, k = 0, \dots$ , let  $\mathcal{E}_j^k(t)$  be the solution of (59). Then  $\mathcal{E}_j^k(t)$  is determined by the values  $\mathcal{E}_{j'}^{k'}(0)$  for  $0 \leq k' \leq k, 1 \leq j' \leq j+k$ .*

Formula (59) will give easily the following result.

**Proposition 2.3.**  $\mathcal{E}_j^k(t) = 0$  if  $j$  and  $k$  have inverse parity<sup>2</sup>.

*Proof.* Let us write (59) again

$$(60) \quad \mathcal{E}_j^k(t) = \int_0^t U_j(t, s) (\Delta_j^- \mathcal{E}_{j-2}^k(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(s)) ds$$

By (67) we have that  $\mathcal{E}_1^0(t) = 0$  since  $\mathcal{E}_1^0(0) = 0$ . Since (60) gives  $\mathcal{E}_j^k(t) = \int_{s=0}^t U_j(t, s) \Delta_j^- \mathcal{E}_{j-2}^0(s) ds$ , we get  $\mathcal{E}_j^0(t) = 0$  for all  $j$  odd.

Since  $\mathcal{E}_0(t) = 1, \mathcal{E}_0^j(t) = 0, j > 0$ , and we have that  $\mathcal{E}_1^1(t) = 0$  and therefore  $\mathcal{E}_j^1(t) = 0$  for all  $j$  even, since then  $j \pm 1$  is odd, and therefore  $\mathcal{E}_{j \pm 1}^0 = 0$ .

This gives  $\mathcal{E}_0^2 = 0$  and so on.  $\square$

**Corollary 2.4.**  $E_j(t)$  has an asymptotic expansion in powers of  $N^{-1}$  with leading order  $N^{-[\frac{j+1}{2}]}$ .

---

<sup>2</sup> $_j$  even,  $k$  odd or  $j$  odd,  $k$  even

### 3. ESTIMATES AND PROOF OF THEOREM 1.4

In order to simplify the expressions, we will first suppose that  $\frac{\|V\|_{L^\infty}}{h} = 1$ .

Let us first notice that (117) expressed on the  $\mathcal{E}_j$ s reads

$$(61) \quad \partial_t \mathcal{E}_j = H_j \mathcal{E}_j + N^{-\frac{1}{2}} \Delta_j^+ \mathcal{E}_{j+1} + N^{-\frac{1}{2}} \Delta_j^- \mathcal{E}_{j-1} + \Delta_j^- \mathcal{E}_{j-2}$$

and that (21) and (22) can be rephrased as

$$(62) \quad \|\mathcal{E}_j(0)\| \leq (A_j^2)^{j/2} \implies \|\mathcal{E}_j(t)\| \leq (A_t j^2)^{j/2}, \quad A_t = A' e^{Ct}$$

for some explicit constants  $A', C$ .

Let us consider the solution at order  $k$  of (61) given by (59):

$$\mathcal{E}_j^k(t) = \int_{s=0}^t U_j(t, s) (\Delta_j^- \mathcal{E}_{j-2}^k(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(s)) ds + U_j(t, 0) \mathcal{E}_j^k.$$

We get

$$\begin{aligned} \partial_t \mathcal{E}_j^k(t) &= H_j(t) \mathcal{E}_j^k(t) + \Delta_j^- \mathcal{E}_{j-2}^k(t) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(t) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(t) \\ &= H_j(t) \mathcal{E}_j^k(t) + \Delta_j^- \mathcal{E}_{j-2}^k(t) + \Delta_j^+ \mathcal{E}_{j+1}^k(t) + \Delta_j^- \mathcal{E}_{j-1}^k(t) \\ &\quad + \Delta_j^+ (\mathcal{E}_{j+1}^{k-1}(t) - \mathcal{E}_{j+1}^k(t)) + \Delta_j^- (\mathcal{E}_{j-1}^{k-1}(t) - \mathcal{E}_{j-1}^k(t)), \end{aligned}$$

and, calling  $\bar{\mathcal{E}}_j^n = \sum_{k=0}^n N^{-k/2} \mathcal{E}_j^k$ , one easily check that

$$(63) \quad \begin{aligned} \partial_t \bar{\mathcal{E}}_j^n(t) &= H_j(t) \bar{\mathcal{E}}_j^n(t) + \Delta_j^- \bar{\mathcal{E}}_{j-2}^n(t) + \frac{1}{\sqrt{N}} (\Delta_j^+ \bar{\mathcal{E}}_{j+1}^n(t) + \Delta_j^- \bar{\mathcal{E}}_{j-1}^n(t)) \\ &\quad + \left(\frac{1}{\sqrt{N}}\right)^{n+1} (\Delta_j^+ (\mathcal{E}_{j+1}^n(t)) + \Delta_j^- (\mathcal{E}_{j-1}^n(t))). \end{aligned}$$

Therefore  $A_j^n := \mathcal{E}_j - \bar{\mathcal{E}}_j^n$  satisfies the equation

$$(64) \quad \begin{aligned} \partial_t A_j^n(t) &= H_j(t) A_j^n(t) + \Delta_j^- A_{j-2}^n(t) + \frac{1}{\sqrt{N}} (\Delta_j^+ A_{j+1}^n(t) + \Delta_j^- A_{j-1}^n(t)) \\ &\quad + \left(\frac{1}{\sqrt{N}}\right)^{n+1} (\Delta_j^+ (\mathcal{E}_{j+1}^n(t)) + \Delta_j^- (\mathcal{E}_{j-1}^n(t))) \end{aligned}$$

Let us define

$$U_j^N(t, s) : (\mathcal{E}_j(s))_{j=1, \dots, N} \mapsto U_j^N(t, s) ((\mathcal{E}_j(s))_{j=1, \dots, N}) := \mathcal{E}_j(t).$$

In other words, the family  $(U_j^N(t, s))_{j=1, \dots, N}$  solves the equation:

$$\begin{aligned} \partial_t U_j^N(t, s) &= H_j(t) U_j^N(t, s) + \Delta_j^- U_{j-2}^N(t, s) \\ &\quad + \frac{1}{\sqrt{N}} (\Delta_j^+ U_{j+1}^N(t, s) + \Delta_j^- U_{j-1}^N(t, s)), \\ U_j^N(s, s) &= I. \end{aligned}$$

Hence, the solution of (64) reads

(65)

$$A_j^n(t) = N^{-\frac{n+1}{2}} \int_0^t U_j^N(t, s) (\Delta_j^+(s) \mathcal{E}_{j+1}^n(s) + \Delta_j^-(s) \mathcal{E}_{j-1}^n(s)) ds + U_j^N(t, 0) (A_j^n(0))$$

where  $\Delta_j^+(s) \mathcal{E}_{j+1}^n(s) + \Delta_j^-(s) \mathcal{E}_{j-1}^n(s)$  (resp.  $A_j^n(0)$ ) is meant for the sequence  $((\Delta_j^+(s) \mathcal{E}_{j+1}^n(s) + \Delta_j^-(s) \mathcal{E}_{j-1}^n(s)) \delta_{k,j})_{k=1, \dots, N}$  (resp.  $(A_k^n(0) \delta_{k,j})_{k=1, \dots, N}$ ).

Let us suppose that that  $\mathcal{E}_j(0)$  has an asymptotic expansion of the form  $\sum_0^\infty N^{-k/2} \mathcal{E}_j^k(0)$  so that  $\|A_j^n(0)\| \leq N^{-\frac{n+1}{2}} D_n (A_j^2)^{j/2}$  for some positive constants  $D_n, A$ . Then (62) implies that

$$(66) \quad \|U_j^N(t, 0) (A_j^n(0))\| \leq N^{-\frac{n+1}{2}} D_n (A' e^{Ct} j^2)^{j/2}.$$

By the same idea we get that, if

$$(67) \quad \|\Delta_j^+(\mathcal{E}_{j+1}^n(s)) + \Delta_j^-(\mathcal{E}_{j-1}^n(s))\| \leq C_n(s) (A_s^n j^2)^{j/2}, \quad \forall s \leq t,$$

for two increasing functions  $C(s, A_s^n)$  of  $s$  (say), then

$$(68) \quad \|\mathcal{E}_j(t) - \bar{\mathcal{E}}_j^n(t)\| \leq N^{-\frac{n+1}{2}} t C_n(t) ((A_t^n)' e^{C|t|} j^2)^{j/2},$$

where  $(A_t^n)'$  depends only on  $A_t^n$ , and on the same way that  $A'$  depends on  $A$  in (62).

It remains to prove an estimate like (67).

We will obtain such an estimate by iterating (59). We first remark that, since  $e^{K^j + T_j/N}$  is unitary and  $\|D_j\| \leq j$ , the Gronwall Lemma gives that

$$(69) \quad \|U_j(t, s)\| \leq e^{j|t-s|}.$$

We will use

$$(70) \quad \prod_{i=0}^m e^{(j+i)(t_i - t_{i+1})} \leq e^{(j+m)|t_{m+1} - t_0|} \quad \text{for any } (t_i)_{i=0, \dots, m} \text{ (see [22]),}$$

$$(71) \quad \|\Delta^\pm\| \leq j^2,$$

$$\|\mathcal{E}_{j'}(t)\| \leq (A_t(j')^2)^{(j')/2} \leq (A_t(j+n)^2)^{(j+n)/2}, \quad j' \leq j+n,$$

and

$$(72) \quad \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n = \frac{t^n}{n!}.$$

Let us remind that we have  $\mathcal{E}_0^k(t) = \delta_{k,0}$  for all  $t$ . Without loss of generality, we might take  $\mathcal{E}_j^k(0) = \delta_{k,0} \mathcal{E}_j^0(0)$  (possibly depending on  $N$ ) such that  $\|\mathcal{E}_j^0(0)\| \leq (A[j+1]^2)^{[j+1]/2}$ ,  $A \geq 1$ . Under this hypothesis, (59) reads:

$$\begin{cases} \mathcal{E}_j^0(t) &= U_j(t, 0) \mathcal{E}_j^0(0) + \int_{s=0}^t U_j(t, s) \Delta_j^- \mathcal{E}_{j-2}^0(s) ds \\ \mathcal{E}_j^k(t) &= \int_{s=0}^t U_j(t, s) (\Delta_j^- \mathcal{E}_{j-2}^k(s) + \Delta_j^+ \mathcal{E}_{j+1}^{k-1}(s) + \Delta_j^- \mathcal{E}_{j-1}^{k-1}(s)) ds, \quad k > 0. \end{cases}$$

Therefore we get easily, using (70),(71),(72),

$$(73) \quad \|\mathcal{E}_j^0(t)\| \leq e^{2j|t|} j (A_j^2)^{j/2}.$$

Moreover iterating the second line  $M$  times, we will end up with the sum of less than  $M^3$  terms involving the values  $\mathcal{E}_{j-2r+s-t}^{n-s-t}$  for any  $(r,s,t)$  such that  $M = r + s + t$  with the two constraints  $n - s - t \geq 0$ ,  $j - 2r + s - t \geq 0$ . Using the first constraint we see that

$$j - 2r + s - t \leq j - 2r + n \leq j - 2(M - n) + n = j - 2M + 3n.$$

So that, taking  $M = [(j + 3n)/2] := M_{j,n}$ , the second constraint reduces to  $j - 2r + s - t = 0$  and the first one to  $s + t = n$  since  $\mathcal{E}_0^k = \delta_{k,0}$ .

We easily (and very roughly) estimate

$$\begin{aligned} \|\mathcal{E}_j^k(t)\| &\leq ((j + 3n)/2)^3 e^{(j+3n)|t|} ((j + n)^2)^{\frac{j+3n}{2}} \\ &\leq (1 + 3n)^{3(n+1)} e^{3n|t|} j^{3(n+1)} (1 + n/j)^j (e^{2|t|} j^2)^{j/2} \end{aligned}$$

so that, using  $(1 + n/j)^j \leq e^n$  and  $j^\lambda \leq e^{j\lambda/e}$ ,  $\lambda > 0$ , we get

$$(74) \quad \|\mathcal{E}_j^k(t)\| \leq (1 + 3n)^{3(n+1)} e^{(3n+1)|t|} (e^{2|t|+3(n+1)/2} j^2)^{j/2}, \quad k > 0$$

and, since  $A \geq 1$ ,

$$(75) \quad \|\mathcal{E}_j^k(t)\| \leq (1 + 3n)^{3(n+1)} e^{(3n+1)|t|} (Ae^{2|t|+3(n+1)/2} j^2)^{j/2}, \quad k \geq 0.$$

We conclude by

$$(76) \quad \|\Delta_j^+(\mathcal{E}_{j+1}^n(s)) + \Delta_j^-(\mathcal{E}_{j-1}^n(s))\| \leq C_n(s) (A_s^n j^2)^{j/2}$$

with, after restoring the dependence in  $\frac{\|V\|_{L^\infty}}{h}$  by the same argument as in [22], Section 3,

$$(77) \quad C_n(s) = j(j+1)(1+3n)^{3(n+1)} e^{(3n+1)|t|} \frac{\|V\|_{L^\infty}}{h} \quad \text{and} \quad A_s^n = A e^{2|t|} \frac{\|V\|_{L^\infty}}{h} + 3(n+1)/2.$$

Therefore (68) is satisfied and Theorem 1.4 is proven.

#### 4. EXPLICIT COMPUTATIONS OF FIRST ORDERS

We have

$$\partial_t U_1^0(t, s) = \frac{1}{i\hbar} [-\hbar^2 \Delta + V_F, U_1^0(t, s)] + \frac{1}{i\hbar} [V_{U_1^0(t, s)}, F]$$

where, in the last term,  $V_{U_1^0(t, s)}$  acts on  $E_1(s)$  as  $V_{U_1^0(t, s)E_1(s)}$ .

More generally,

$$\partial_t U_j^0(t) = \frac{1}{i\hbar} [-\hbar^2 \Delta_{\mathbf{R}^{jd}} + V_F^{\otimes j}, U_j^0(t)] + P(U_j^0, F)$$

where

$$(P(U_j^0, F)E_j)(Z_j) = \sum_i \int dx (V(x_i - x) - V(x'_i - x))(U_j^0(t, s)E_j(Z_j^{\neq i}, (x, x))F(x_i, x'_i),$$

that is

$$(P(U_j^0, F)E_j) = \sum_{i=1}^j [V \star_i (U_j^0(t, s)E_j), F]_i.$$

Finally

$$\mathcal{E}_2^0(t)(Z_2) = \int_0^t \int_{\mathbf{R}^{2d}} ds dZ'_2 U_2(t, s)(Z_2, Z'_2) V(x'_1 - x'_2) F(s)(z'_1) F(s)(z'_2) ds dZ'_2$$

and

$$\begin{aligned} \mathcal{E}_1^1(t) &= \int_0^t U_1(t, s) Q(F, F) ds \\ &+ (1 - \frac{1}{N}) \int_0^t \int_0^s U_1(t, s) \text{Tr}^2 [V U_2(s, u) V F(u) \otimes F(u)] ds du \end{aligned}$$

(78)

#### 5. THE KAC AND “SOFT SPHERES” MODELS

In this section we consider the two following classes of mean field models (see [22] for details).

- *Kac model.* In this model, the  $N$ -particle system evolves according to a stochastic process. To each particle  $i$ , we associate a velocity  $v_i \in \mathbb{R}^3$ . The vector  $\mathcal{V}_N = \{v_1, \dots, v_N\}$  changes by means of two-body collisions at random times, with random scattering angle. The probability density  $F^N(\mathcal{V}_N, t)$  evolves according to the forward Kolmogorov equation

$$(79) \quad \partial_t F^N = \frac{1}{N} \sum_{i < j} \int d\omega B(\omega; v_i - v_j) \{F^N(\mathcal{V}_N^{i, j}) - F^N(\mathcal{V}_N)\},$$

where  $\mathcal{V}_N^{i, j} = \{v_1, \dots, v_{i-1}, v'_i, v_{i+1}, \dots, v_{j-1}, v'_j, v_{j+1}, \dots, v_N\}$  and the pair  $v'_i, v'_j$  gives the outgoing velocities after a collision with scattering (unit) vector

$\omega$  and incoming velocities  $v_i, v_j$ .  $\frac{B(\omega; v_i - v_j)}{|v_i - v_j|}$  is the differential cross-section of the two-body process. The resulting mean-field kinetic equation is the homogeneous Boltzmann equation

$$(80) \quad \partial_t F(v) = \int dv_1 \int d\omega B(\omega; v - v_1) \{F(v')F(v'_1) - F(v)F(v_1)\}.$$

• *‘Soft spheres’ model.* A slightly more realistic variant, taking into account the positions of particles  $X_N = \{x_1, \dots, x_N\} \in \mathbb{R}^{3N}$  and relative transport, was introduced by Cercignani [9] and further investigated in [17]. The probability density  $F^N(X_N, V_N, t)$  evolves according to the equation

$$(81) \quad \begin{aligned} \partial_t F^N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} F^N &= \frac{1}{N} \sum_{i < j} h(|x_i - x_j|) B\left(\frac{x_i - x_j}{|x_i - x_j|}; v_i - v_j\right) \\ &\times \{F^N(X_N, V_N^{i,j}) - F^N(X_N, V_N)\}. \end{aligned}$$

Here  $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a positive function with compact support. Now a pair of particles collides at a random distance with rate modulated by  $h$ . The associated mean-field kinetic equation is the Povzner equation

$$\begin{aligned} \partial_t F(x, v) + v \cdot \nabla_x F(x, v) &= \int dv_1 \int dx_1 h(|x - x_1|) B\left(\frac{x - x_1}{|x - x_1|}; v - v_1\right) \\ &\times \{F(x, v')F(x_1, v'_1) - F(x, v)F(x_1, v_1)\}, \end{aligned}$$

which can be seen as an  $h$ -mollification of the inhomogeneous Boltzmann equation (formally obtained when  $h$  converges to a Dirac mass at the origin). Both classes have been treated in [22] and Theorem 1.2 apply to them, in the following sense.

The underlying space  $\mathbb{L}$  is now  $L^1(\mathbf{R}^d, dv)$  (resp.  $L^1(\mathbf{R}^{2d}, dx dv)$ ) for the Kac model (resp. soft spheres) both endowed with the  $L^1$  norms  $\|\cdot\|_1$ . For  $F^N \in \mathbb{L}^{\otimes N}$ ,  $F_j^N \in \mathbb{L}^{\otimes j}$  is defined by

$$F_j^N(Z_j) = \int_{\Omega} F^N(z_1, \dots, z_j, z_{j+1}, \dots, z_N) dz_{j+1} \dots dz_N$$

for  $Z_n = (z_1, \dots, z_n)$ ,  $n = 1, \dots, N$  with  $z_i = v_i \in \mathbf{R}^d$ ,  $\Omega = \mathbf{R}^{(N-j)d}$  (resp.  $z_i = (x_i, v_i) \in \mathbf{R}^{2d}$ ,  $\Omega = \mathbf{R}^{2(N-j)d}$ ) for the Kac (resp. soft spheres) model.

In both cases  $E_j(t)$  is defined by (18), inverted by (19), and it was proven in [22] that Theorem 1.2 holds true verbatim in both cases.

Stating now the dynamics driven by (79) and (81) under the form (3) with  $K^N = 0$  (resp.  $K^N = - \sum_{i=1, \dots, N} v_i \partial_{x_i}$ ) for the Kac (resp. soft spheres) model and  $V^N$  given by the right hand-sides of (79), (81) respectively, one sees

immediately that the proofs contained in Sections 2,3 remain valid after an elementary redefinition of the operators  $D_j, D_j^{-1}, D_j^{-2}$  in (35)-(38) as (in the left hand sides below, we use a slight abuse of notation by identifying functions with evaluations)

$$\left\{ \begin{array}{l} (D_j E_j)(Z_j) = \frac{N-j}{N} \sum_{i \in J} C_{i,j+1} \left( F(z_i) E_j(Z_{j+1}^{\{i\}}) + F(z_{j+1}) E_j(Z_j) \right) \\ \quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} (F(z_l) E_j(Z_{j+1}^{\{l\}})) \\ (D_j^{-1} E_{j-1})(Z_j) = \frac{1}{N} \sum_{i,r \in J} T_{i,r} (F(z_i) E_{j-1}(Z_j^{\{i\}})) \\ \quad - \frac{j}{N} \sum_{i \in J} Q(F, F)(z_i) E_{j-1}(Z_j^{\{i\}}) \\ \quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} (F(z_l) F(z_{j+1}) E_{j-1}(Z_j^{\{l\}})) \\ \quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} (F(z_l) F(z_i) E_{j-1}(Z_{j+1}^{\{i,l\}})) \\ (D_j^{-2} E_{j-2})(Z_j) = \frac{1}{N} \sum_{i,s \in J} T_{i,s} (F(z_i) F(z_r) E_{j-2}(Z_j^{\{i,r\}})) \\ \quad - \frac{1}{N} \sum_{i \neq l \in J} Q(F, F)(z_i) F(z_l) E_{j-2}(Z_l^{\{i,l\}}). \end{array} \right.$$

with

$$(82) \quad \left\{ \begin{array}{l} D_N^1 := D_1^{-2} := 0 \\ D_1^{-1}(E_0)(z_1) := -\frac{1}{N} Q(F, F)(z_1), \\ D_2^{-2}(E_0)(z_1, z_2) := \frac{1}{N} (T_{1,2}(F \otimes F)(z_1, z_2) - Q(F, F)(z_1) F(z_2) \\ \quad - F(z_1) Q(F, F)(z_2)). \end{array} \right.$$

Note that one has the following estimates:

$$(83) \quad \|D_j\|, \|D_j^1\| \leq j \text{ and } \|D_j^{-1}\|, \|D_j^{-2}\|, \|D_1^{-1}(E_0)\|, \|D_2^{-2}(E_0)\| \leq \frac{j^2}{N}.$$

Therefore, by Remark 2.1, the statements contained in Theorem 1.4 and Corollary 1.7 hold true, in both cases, verbatim. Moreover defining  $F_j^{N,n}$  by (33) in both cases, Corollary 1.5 reads now as follows

**Corollary 5.1.** *[Kac case] Let  $F^N(t)$  the solution of the  $N$  body system (79) (resp. 81) with initial datum  $F^N(0) = F^{\otimes N}$ ,  $0 < F \in L^1(\mathbf{R}^d)$ ,  $\int_{\mathbf{R}^d} f(v) dv = 1$  (resp,  $0 < F \in L^1(\mathbf{R}^{2d})$ ,  $\int_{\mathbf{R}^{2d}} f(x, v) dx dv = 1$ ), and  $F(t)$  the solution of the homogeneous Boltzmann equation (80) (resp. the Povzner equation(82)) with initial datum  $F$ .*

Then, in both cases, for all  $n \geq 1$ ,

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2tC_{2n}(t)\epsilon A_t^{2n}j}{\sqrt{N}}.$$

for  $N \geq 4(eA_t^{2n}j)^2$ .

## APPENDIX A. THE ASBTRACT MODEL

**A.1. The model.** We will show in this section that the main results of [22] and of Section 1 of the present paper remain true in the “abstract” mean field formalism for a dynamics of  $N$  particles that we will describe now. The present formalism contains the abstract formalism developed in [22], without requiring a space of states endowed with a multiplicative structure.

*States of the particle system and evolution equations.* Let  $\mathbb{L}$  be a vector space on the complex numbers. We suppose the family of (algebraic) tensor products  $\{\mathbb{L}^{\otimes n}, n = 1, \dots, N\}$  equipped with norms  $\|\cdot\|_n$  satisfying assumption **(A)** below. the  $N$ -body dynamics will be driven on  $\mathbb{L}^{\otimes N}$  by a one- and two- body interaction satisfying assumption **(B)** and the mean field limit equation will be supposed to satisfy assumption **(C)**.

Assumptions **(A)** – **(C)** below will be followed by their incarnations in the **K**(ac), **S**(oft spheres) and **Q**(uantum) models.

By convention we denote  $\mathbb{L}^{\otimes 0} := \mathbf{C}$ ,  $\|z\|_0 = |z|$  and we denote by  $\mathbb{L}^{\hat{\otimes} n}$  the completion of  $\mathbb{L}^{\otimes n}$  with respect to the norm  $\|\cdot\|_n$ .

*For the **K**, **S** and **Q** models,  $\mathbb{L}^{\otimes n}$  is  $L^1(\mathbf{R}^d, dv)$ ,  $L^1(\mathbf{R}^{2d}, dx dv)$  and  $\mathcal{L}^1(L^2(\mathbf{R}^d))$ , the space of trace class operators on  $L^2(\mathbf{R}^d)$ , with their associated norms.*

- (A)** There exists a family of subsets  $\mathbb{L}_+^{\hat{\otimes} n}$  of  $\mathbb{L}^{\hat{\otimes} n}$ ,  $n = 1, \dots, N$  of positive elements  $F$  denoted by  $F > 0$  stable by addition, multiplication by positive reals and tensor product and there exists a linear function  $\text{Tr} : \mathbb{L} \rightarrow \mathbf{C}$ , called trace.



For every  $1 \leq k, n \leq N$  and  $1 \leq i \leq j \leq n \leq N$ , let  $\text{Tr}_n^k$  and  $\sigma_{i,j}^n$  be the two mapping defined by<sup>3</sup>

$$(84) \quad \begin{aligned} \text{Tr}_n^k : \mathbb{L}^{\otimes n} &\rightarrow \mathbb{L}^{\otimes n-1} \\ \bigotimes_{i=1}^n v_i &\mapsto \text{Tr}(v_k) \bigotimes_{\substack{i=1 \\ i \neq k}}^n v_i, \\ \sigma_{i,j}^n : \mathbb{L}^{\otimes n} &\rightarrow \mathbb{L}^{\otimes n} \\ \bigotimes_{i=1}^n v_i &\mapsto \bigotimes_{i=1}^n v'_i, \quad v'_k = v_k, i \neq k \neq j ; v'_i = v_j, v'_j = v_i. \end{aligned}$$

We will suppose that  $\text{Tr}_N^k$  and  $\sigma_{i,j}^n$ ,  $i, j, k \leq n \leq N$ , satisfy, for any  $F \in \mathbb{L}^{\otimes n}$ ,

$$(85) \quad \begin{cases} \text{Tr}_N^k(F), \sigma_{i,j}^n(F) > 0, \quad \|\text{Tr}_n^k(F)\|_{n-1} = \|F\|_n \text{ when } F > 0 \\ \|\sigma_{i,j}^n(F)\|_n = \|F\|_n \\ \|\text{Tr}_n^k(F)\|_{n-1} \leq \|F\|_n \end{cases}$$

In particular one has that  $\|F\|_n = \text{Tr}^n \dots \text{Tr}^1 F$  when  $F > 0$  and  $|\text{Tr}^n \dots \text{Tr}^1 F| \leq \|F\|_n$  in general.

Note that (85) allows to extend  $\text{Tr}_n^k$  and  $\sigma_{i,j}^n$  to  $\mathbb{L}^{\hat{\otimes} n}$  by continuity.

We will use the same notation for these extensions.

For the **K**, **S** and **Q** models,  $\text{Tr}^k$  is  $\int_{\mathbf{R}^d} \cdot dv_k$ ,  $\int_{\mathbf{R}^{2d}} \cdot dx_k dv_k$  and the  $k$ -partial trace, namely, defining  $A$  through its integral kernel  $a(x_1, \dots, x_n, x'_1, \dots, x'_n)$ , the integral kernel of  $\text{Tr}^k A$  is defined as  $\int_{\mathbf{R}^{2d}} a(x_1, \dots, x_n, x'_1, \dots, x'_n) dx_k$ , respectively. Moreover, the action of  $\sigma_{i,j}^n$  consists obviously in exchanging the variables  $v_i$  and  $v_j$ ,  $(x_i, v_i)$  and  $(x_j, v_j)$  and  $(x_i, y_i)$  and  $(x_j, y_j)$  (in the integral kernel), respectively. Finally (85) is satisfied in the three cases.

From now on and when no confusion is possible, we will identify  $\mathbb{L}^{\otimes n}$  with its completion  $\mathbb{L}^{\hat{\otimes}}$  and we will denote  $\text{Tr}_N^k = \text{Tr}^k$  (note also that  $\text{Tr} = \text{Tr}_1^1 = \text{Tr}^1$ ),  $\sigma_{i,j}^N = \sigma_{i,j}$  and  $\text{Tr}(= \text{Tr}_n) = \text{Tr}_n^n \text{Tr}_n^{n-1} \dots \text{Tr}_n^1$ .

We call *symmetric* any element of  $\mathbb{L}^{\otimes n}$  invariant by the action of  $\sigma_{i,j}^n$ ,  $\forall i, j \leq n$ .

We call *state of the  $N$ -particle system* an element of

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<sup>3</sup>The fact that the second and fourth lines of (84) define a mapping on the whole tensor space  $\mathbb{L}^{\otimes n}$  results easily from the definition of tensors products through the so-called universal property [18]. Indeed, let  $\varphi_n$  be the natural embedding  $\mathbb{L}^{\times n} \rightarrow \mathbb{L}^{\otimes n}$ ,  $(v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n$ , and let  $h$  be any mapping  $\mathbb{L}^{\times n} \rightarrow \mathbb{L}^{\times n'}$ , then the universal property of tensor products says that there is a unique map  $\tilde{h} : \mathbb{L}^{\otimes n} \rightarrow \mathbb{L}^{\otimes n'}$  such that  $\tilde{h} \circ \varphi_n = \varphi_{n'} \circ h$ . Taking  $n' = n-1$ ,  $h(v_1, \dots, v_k, \dots, v_n) = (\text{trace}(v_k)v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n)$  for  $\text{Tr}_n^k$ , and  $n' = n$ ,  $h(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = (v_1, \dots, v_j, \dots, v_i, \dots, v_n)$  for  $\sigma_{i,j}^n$  give the desired extensions.

$$(86) \quad \mathcal{D}_N = \{F \in \mathbb{L}^{\otimes n} \mid F > 0, \quad \|F\| = 1 \quad \text{and } F \text{ is symmetric}\}.$$

For  $j = 0, \dots, N$ , the  $j$ -particle marginal of  $F^N \in (\mathbb{L}^{\otimes N})_1^+$  is defined as the partial trace of order  $N - j$  of  $F^N$ , that is

$$(87) \quad F_j^N = \text{Tr}^N \text{Tr}^{N-1} \dots \text{Tr}^{j+1} F^N, \quad F_N^N := F^N.$$

Note that  $F_j^N \in \mathbb{L}^{\otimes j}$  ( $F_0^N = 1 \in \mathbb{L}^{\otimes 0} := \mathbf{C}$ ) and  $F_j^N > 0$ ,  $\|F_j^N\|_j = \|F^N\|_N$  since  $\text{Tr}$  is positivity and norm preserving, and obviously  $F_j^N$  is symmetric as  $F^N$ . That is to say:

$$F_j^N \in \mathcal{D}_j.$$

**(B)** The evolution of a state  $F^N$  in  $\mathbb{L}^{\otimes N}$  is supposed to be given by the  $N$ -particle dynamics associated to a two-body interaction:

$$(88) \quad \frac{d}{dt} F^N = (K^N + V^N) F^N,$$

where the operators on the right hand side are constructed as follows.

$$(89) \quad K^N = \sum_{i=1}^N \mathbb{I}_{\mathbb{L}}^{\otimes(i-1)} \otimes K \otimes \mathbb{I}_{\mathbb{L}}^{\otimes(N-i)}$$

and

$$(90) \quad V^N = \frac{1}{N} \sum_{1 \leq i < j \leq N} V_{i,j}, \quad V_{i,j} := \sigma_{1,i}^N \sigma_{2,j}^N V \otimes \mathbb{I}_{\mathbb{L}^{\otimes(N-2)}} \sigma_{1,i}^N \sigma_{2,j}^N$$

for a (possibly unbounded) operator  $K$  acting on  $\mathbb{L}$  and a bounded two-body (potential) operator  $V$  acting on  $\mathbb{L}^{\otimes 2}$ .

We assume furthermore that  $K$  is the generator of a strongly continuous, isometric, positivity preserving semigroup (in  $\mathbb{L}$ )

$$(91) \quad e^{Kt} F > 0 \quad \text{if } F > 0; \quad \|e^{tK}\| = 1.$$

and  $K^N + V^N$  is the generator of a strongly continuous, isometric, positivity preserving semigroup (in  $\mathbb{L}^{\otimes N}$ )

$$(92) \quad e^{(K^N + V^N)t} F^N > 0 \quad \text{if } F^N > 0; \quad \|e^{t(K^N + V^N)}\| = 1.$$

Finally, for any  $F \in \mathbb{L}$ ,  $F^N \in \mathbb{L}^{\otimes N}$  and  $i, r > j$ , we assume

$$(93) \quad \text{Tr}(KF) = 0 \quad \text{and} \quad \text{Tr}^{j,N}(V_{i,r} F^N) = 0.$$

This last property is necessary to deduce the forthcoming hierarchy.

For the **K**, **S** and **Q** models, the ingredients in (88) are given in the table bellow and (91)-(93) are easily shown to be satisfied.

Note the symmetry property of the equation (88) induced by the definition of  $V^N$ : if the initial condition  $F_0^N$  for (88) is symmetric, then  $F^N(t)$  is still symmetric.

The following table specifies the applications of the above abstract model to the three models presented in the introduction.

	Kac	Soft Spheres	Quantum Mean field
$\mathbb{L}$	$L^1(v)$	$L^1(x, v)$	$\mathcal{L}^1(L^2(\mathbf{R}^d))$
$ \cdot $	$f \rightarrow  f  = \sqrt{\bar{f}f}$	$f \rightarrow  f  = \sqrt{\bar{f}f}$	$A \rightarrow  A  = \sqrt{A^*A}$
Tr	$f \rightarrow \int dv f$	$f \rightarrow \int \int f dx dv$	$A \rightarrow Tr A$
$K^n$	0	$-v \cdot \nabla_x$	$\frac{1}{i\hbar}[-\Delta, \cdot]$
$V_{i,j}$	$\int d\omega$ $B(\omega; v_i - v_j)$ $\{f^N(V_N^{i,j})$ $-f^N(V_N)\}$	$h( x_i - x_j )$ $B\left(\frac{x_i - x_j}{ x_i - x_j }; v_i - v_j\right)$ $\{f^N(X_N, V_N^{i,j})$ $-f^N(X_N, V_N)\}$	$\frac{1}{i\hbar}[V(x_i - x_j), \cdot]$

*Hierarchies.* The family of  $j$ -marginals,  $j = 1, \dots, N$ , are solutions of the BBGKY hierarchy of equations

$$(94) \quad \partial_t F_j^N = \left(K^j + \frac{T_j}{N}\right) F_j^N + \frac{(N-j)}{N} C_{j+1} F_{j+1}^N$$

where:

$$(95) \quad K^j = \sum_{i=1}^j \mathbb{I}_{\mathbb{L}}^{\otimes(i-1)} \otimes K \otimes \mathbb{I}_{\mathbb{L}}^{\otimes(j-i)},$$

$$(96) \quad T_j = \sum_{1 \leq i < r \leq j} T_{i,r} \quad \text{with } T_{i,r} = V_{ir}$$

and

$$(97) \quad C_{j+1} F_{j+1}^N = \text{Tr}^{j+1} \left( \sum_{i \leq j} V_{i,j+1} F_{j+1}^N \right) = \sum_{i=1}^j C_{i,j+1} F_{j+1}^N$$

with

$$(98) \quad C_{i,j+1} F_{j+1}^N = \text{Tr}^{j+1} (V_{i,j+1} F_{j+1}^N),$$

$$(99) \quad C_{i,j} : \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}.$$

Indeed, thanks to (93) we get easily by applying  $\text{Tr}^{j,N}$  on (88) that

$$\frac{d}{dt}F_j^N = (K^j + \frac{T_j}{N})F_j^N + \frac{1}{N}\text{Tr}^{j,N}(\sum_{1 \leq i \leq j < k \leq N} V_{i,k}F^N)$$

By symmetry of  $F^N$  and  $V_{i,k}$  we get  $\text{Tr}^{j,N}(V_{i,k}F^N) = \text{Tr}^{j+1}(V_{i,j=1}F_{j+1}^N)$  for all  $k > j$  and (94) follows.

Note that, thanks to the assumption (85) and for all  $i \leq j = 1, \dots, N$ ,

$$(100) \quad \|T_i\| \leq j^2\|V\|, \text{ and } \|C_{i,j+1}\| \leq j\|V\|$$

(meant for  $(\|T_i\|_{\mathbb{L}^{\otimes i} \rightarrow \mathbb{L}^{\otimes i}}, \|C_{i,j+1}\|_{\mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}}, \|V\|_{\mathbb{L}^{\otimes 2} \rightarrow \mathbb{L}^{\otimes 2}})$ .

We introduce the non-linear mapping  $Q(F, F)$ ,  $Q : \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{L}$  by the formula

$$(101) \quad Q(F, F) = \text{Tr}^2(V_{1,2}(F \otimes F))$$

and the nonlinear mean field equation on  $\mathbb{L}$

$$(102) \quad \partial_t F = KF + Q(F, F), \quad F(0) \geq 0, \quad \|F(0)\|_1 = 1.$$

Eq. (102) is the Boltzmann, Povzner or Hartree equation according to the specifications established in the table above. In full generality we will assume

**(C)** (102) has for all time a unique solution  $F(t) > 0$  and  $\|F(t)\| = 1$ .

*For the **K**, **S** and **Q** models, (C) is true by standard perturbations methods.*

*Correlation error.* To introduce the correlation errors, we need to extend slightly the above structure.

For any subset  $J \subset \{1, \dots, N\}$  we first define

$$(103) \quad \mathbb{L}_N^{\otimes J} := \bigotimes_{i=1}^N \mathbb{L}^{\otimes \chi_J(i)},$$

where  $\chi_J$  is the characteristic function of  $J$  and  $\mathbb{L}^{\otimes 0} = \mathbf{C}$ .

Then we introduce  $\mathbb{L}^{\otimes J}$ , the subspace of  $\mathbb{L}_N^{\otimes J}$  formed by vectors of the form  $\bigotimes_{i=1}^N v_i$  where  $v_i = 1 \in \mathbf{C}$  for  $i \notin J$  and  $v_i \in \mathbb{L}$  for  $i \in J$ . Note that  $\mathbb{L}^{\otimes J}$  is sent to  $\mathbb{L}^{\otimes |J|}$  by the mapping

$$\Pi : \bigotimes_{i=1}^N v_i \in \mathbb{L}^{\otimes J} \mapsto \bigotimes_{i \in J} v_i \in \mathbb{L}^{\otimes |J|}.$$

We define

$$\|\cdot\|_{\mathbb{L}^{\otimes J}} = \|\Pi(\cdot)\|_{\mathbb{L}^{\otimes |J|}}.$$

For  $F \in \mathbb{L}$  and  $K \subset J \subset \{1, \dots, N\}$  we introduce the linear operator  $[F]_J^{\otimes K}$ , defined through its action on factorized elements as

$$(104) \quad \begin{aligned} [F]_J^{\otimes K} : \mathbb{L}^{\otimes J/K} &\rightarrow \mathbb{L}^{\otimes J} \\ \bigotimes_{i=1}^N v_i &\mapsto \bigotimes_{i=1}^N a_i, \end{aligned}$$

$$\text{where } \begin{cases} a_s = 1 \in \mathbf{C} & \text{if } s \notin J \\ a_s = F & \text{if } s \in K \\ a_s = v_s & \text{if } s \in J/K \end{cases}.$$

Note that, for  $K, K' \subset J$ ,  $K \cap K' = \emptyset$ , we have the composition

$$(105) \quad [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} = [F]_J^{\otimes (K \cup K')} = [F]_J^{\otimes K'} [F]_{J/K'}^{\otimes K}$$

and more generally, for all  $F, G$ ,

$$(106) \quad [F]_J^{\otimes K} [G]_{J/K}^{\otimes K'} = [G]_J^{\otimes K'} [F]_{J/K'}^{\otimes K}.$$

For any subset  $J \subset \{1, \dots, N\}$ , we define the *correlation error* by

$$(107) \quad E_J = \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} F_{J/K}^N$$

where  $F$  solves (102), the operator  $[F]_J^{\otimes K}$  is defined by (104) and  $F_L^N \in \mathbb{L}^{\otimes L}$  is defined through its decomposition on factorized states. Namely if

$$F^N = \sum_{\ell_1, \dots, \ell_N} c_{\ell_1, \dots, \ell_N} v_{\ell_1} \otimes \dots \otimes v_{\ell_N},$$

then

$$F_L^N = \sum_{\ell_1, \dots, \ell_N} c_{\ell_1, \dots, \ell_N} a_{\ell_1} \otimes \dots \otimes a_{\ell_N},$$

$$\text{where } \begin{cases} a_s = \text{Tr}(v_s) \in \mathbf{C} & \text{if } s \notin L \\ a_s = v_s & \text{if } s \in L \end{cases}.$$

The link between the definition of  $F_L^N$  and the definition of the marginals  $F_j^N$  given in (87) is the following:

$$(108) \quad F_{\{1, \dots, \ell\}}^N = F_\ell^N \otimes (1)^{\otimes (N-\ell)} \in \mathbb{L}^{\otimes \ell} \otimes (\mathbb{L}^{\otimes 0})^{\otimes (N-\ell)}.$$

The formula inverse to (107) reads

$$(109) \quad F_J^N = \sum_{K \subset J} [F]_J^{\otimes K} E_{J/K}.$$

Note that the contribution in the right hand side of (109) corresponding to  $K = J$  and  $K = \emptyset$  are  $F^{\otimes|J|}$  and  $E_J$  respectively. To prove (109), we plug (107) in the r.h.s. of (109) and we use (105):

$$\begin{aligned} \sum_{K \subset J} [F]_J^{\otimes K} E_{J/K} &= \sum_{K \subset J} [F]_J^{\otimes K} \left[ \sum_{K' \subset J/K} (-1)^{|K'|} [F]_{J/K}^{\otimes K'} F_{(J/K)/K'}^N \right] \\ &= \sum_{K \cup K' \subset J} \sum_{\substack{K \subset J \\ K' \cap K = \emptyset}} (-1)^{|K'|} [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} F_{J/(K \cup K')}^N \\ &= \sum_{L \subset J} \left( \sum_{K' \subset L} (-1)^{|K'|} \right) [F]_J^{\otimes L} F_{J/L}^N = F_J^N \end{aligned}$$

since  $\sum_{K' \subset L} (-1)^{|K'|} = \sum_{k'=0}^{|L|} \binom{|L|}{k'} (-1)^{k'} = 0^{|L|} = 0$  if  $L \neq \emptyset$ , and  $= 1$  if  $L = \emptyset$  (since  $\sum_{K' \subset \emptyset} (-1)^{|K'|} = (-1)^0 = 1$ ).

We will use the notation

$$(110) \quad \delta_{K, K'} = 1 \text{ if } K = K' \text{ and } 0 \text{ if } K \neq K'.$$

One notices that since  $F_j^N$  is the marginal of some  $F^N$  which decomposes on elements of the form  $v_1 \otimes \cdots \otimes v_N$ ,  $F_j^N$  decomposes on elements of the form  $(\prod_{k=j+1}^N \text{Tr} v_k) v_1 \otimes \cdots \otimes v_j$ . Since one knows that  $F_j^N$  is symmetric, it is enough to choose one bijection  $i_J : \{1, \dots, j\} \rightarrow J$ ,  $|J| = j$ , and consider the mapping

$$(111) \quad \begin{aligned} \Phi_{i_J} : \mathbb{L}^{\otimes |J|} &\xrightarrow{\Phi_{i_J}} \mathbb{L}^{\otimes J} \\ \otimes_{j \in J} v_j \in \mathbb{L}^{\otimes |J|} &\mapsto \otimes_{i=1}^N a_i \in \mathbb{L}^{\otimes J} \end{aligned}$$

$$(112) \quad F_{|J|}^N \mapsto F_J^N$$

where  $a_s = 1$  if  $i \notin J$  and  $a_{i_J(j)} = v_j$ .

$\Phi_{i_J}$  is obviously one-to-one since  $i_J$  is so, and, though (111) depends on the embedding chosen, (112) does not:  $\Phi_{i_J}$  restricted to the space  $\mathbb{L}_S^{\otimes |J|}$  of symmetric-by-permutation elements of  $\mathbb{L}^{\otimes |J|}$ , depends only on  $J$  and not on  $i_J$ . We will call  $\Phi_J$  this restriction,

$$(113) \quad \Phi_J = \Phi_{i_J}|_{\mathbb{L}_S^{\otimes |J|}}.$$

The same argument is also valid for  $E_J$  which enjoys the same symmetry property than  $F_J^N$  and we define

$$(114) \quad E_{|J|} = \Phi_J^{-1} E_J.$$

$\Phi_J$  is obviously isometric and we have that

$$(115) \quad \|E_J\|_{\mathbb{L}^{\otimes J}} = \|E_{\{1, \dots, |J|\}}\|_{\mathbb{L}^{\otimes \{1, \dots, |J|\}}} = \|E_{|J|}\|_{\mathbb{L}^{\otimes |J|}}.$$

Therefore, considering the one-to-one correspondence  $\Phi_J$ , it is enough to compute/estimate the quantities  $E_j, j = 1, \dots, N$ .  $E_j$  and  $F_j^N$  are linked by

$$(116) \quad \begin{cases} E_j &= \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \Phi_{J/K} F_{j-|K|}^N \\ F_j^N &= \sum_{K \subset J} [F]_J^{\otimes K} \Phi_{J/K} E_{j-|K|}. \end{cases}$$

For the **K**, **S** and **Q** models, let us define the variables  $z_i, i = 1, \dots, N$ , equal to  $v_i, (x_i, v_i)$  and  $(x_i, x'_i)$  respectively. In the three situations, elements of  $\mathbb{L}^{\otimes j}, j = 1, \dots, N$ , are identified as functions of the variables  $z_i, i = 1, \dots, j$  (through, as before, the integral kernel of the density matrix in the quantum case), that is functions  $F(Z_j)$  where we denote  $Z_j = (z_1, \dots, z_j)$ . . Let us also denote by  $Z_j^{\setminus \{i_1, \dots, i_k\}}$ , where  $\{i_1, \dots, i_k\} \subset \{1, \dots, j\}$ , the vector  $Z_j := (z_1, \dots, z_j)$  after removing the components  $z_{i_1}, \dots, z_{i_k}$ . The formulas defining the error and its inverse read

$$\begin{aligned} E_j(Z_j) &= \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} (-1)^k F(z_{i_1}) \dots F(z_{i_k}) F_{j-k}^N(Z_j^{\setminus \{i_1, \dots, i_k\}}), \\ F_j^N(Z_j) &= \sum_{k=0}^j \sum_{1 \leq i_1 < \dots < i_k \leq j} F(z_{i_1}) \dots F(z_{i_k}) E_{j-k}(Z_j^{\setminus \{i_1, \dots, i_k\}}). \end{aligned}$$

**A.2. Main results similar to [22].** The kinetic errors  $E_j, j = 1, \dots, N$ , satisfy the system of equations

$$(117) \quad \begin{aligned} \partial_t E_j &= \left( K^j + \frac{1}{N} T_j \right) E_j + D_j E_j \\ &+ D_j^1 E_{j+1} + D_j^{-1} E_{j-1} + D_j^{-2} E_{j-2}, \end{aligned}$$

where the operators  $D_j, D_j^1, D_j^{-1}, D_j^{-2}, j = 1, \dots, N$ , are defined in Appendix B below, equations (138)-(139), together with the proof of (117). Moreover, since (140) holds true, we know by Remark 3.2 in [22], that the proof of Theorem 2.1 (and therefore Corollary 2.2) in [22] remain valid in our present setting. We get the following result.

**Theorem A.1** (Analog to Theorem 1.2 (Theorem 2.2. and Corollary 2.3 in [22])).

Let  $E_j(0)$  satisfy for some  $C_0 > 1, B > 0$

$$(118) \quad \begin{cases} \|E_1(0)\|_1 \leq \frac{B}{N} \\ \|E_j(0)\|_1 \leq \left(\frac{j^2}{N}\right)^{j/2} C_0^j, \quad j \geq 2. \end{cases}$$

Then, for all  $t > 0$  and all  $j = 1, \dots, N$ , one has

$$(119) \quad \begin{cases} \|E_1(t)\|_1 \leq \frac{1}{N} (B_2 e^{B_1 t \|V\|}) \\ \|E_j(t)\|_1 \leq (C_2 e^{C_1 t \|V\|})^j \left(\frac{j}{\sqrt{N}}\right)^j \end{cases}$$

for some  $B_1, C_1 > 0$ ,  $B_2, C_2 \geq 1$  explicit (see formulas (78), (81) in [22]), and

$$(120) \quad \|F_j^N(t) - F(t)^{\otimes j}\|_1 \leq D_2 e^{D_1 t \|V\|} \frac{j^2}{N},$$

where  $D_2 = \sup\{B_2, (eC_0)^2\}$ ,  $D_1 = \sup\{B_1, 2C_1\}$ .

**A.3. Asymptotic expansion.** It is easy to see that the proofs of the main results expressed in Section 1.3 are adaptable in an elementary way to the present abstract paradigm. Indeed they use only the three properties stated in Remark 2.1, valid in the present setting as pointed out at the very end of Appendix B, formula (140), together with (91)-(92).

Therefore, the statements contained in Theorem 1.4 and Corollary 1.7 hold true, verbatim, under the hypothesis of Theorem A.1, and with the definition of corrections errors given by the first line of (116) and replacing  $\frac{\|V\|_{L^\infty}}{h}$  by  $\|V\|$  in (77).

Moreover defining now  $F_j^{N,n}$  by truncating the second line of (116) at order  $n$ , that is

$$F_j^{N,n} = \sum_{K < j} [F]_J^{\otimes K} \Phi_{J/K} E_{j-|K|}^n$$

where  $E_j^n$  is defined by (32), Corollary 1.5 reads as follows.

**Corollary A.2.** [abstract] Let  $F^N(t)$  the solution of the  $N$  body system (88) with initial datum  $F^N(0) = F^{\otimes N}$ ,  $0 < F \in \mathbb{L}$ ,  $\|F\|_1 = 1$ , and  $F(t)$  the solution of the mean-field equation (15) with initial datum  $F$ .



Then, for all  $n \geq 0$ ,

$$\|F_j^N(t) - F_j^{N,n}(t)\|_1 \leq N^{-n-\frac{1}{2}} \frac{2tC_{2n}(t)eA_t^{2n}j}{\sqrt{N}}.$$

for  $N \geq 4(eA_t^{2n}j)^2$ .

## APPENDIX B. DERIVATION OF THE CORRELATION HIERARCHY (117)

From the definition of  $E_j$  (cf. (107)) we find

$$\partial_t E_J = \sum_{K \subset J} (-1)^{|K|} \left( \partial_t ([F]_J^{\otimes K}) F_{J/K}^N + [F]_J^{\otimes K} \partial_t F_{J/K}^N \right)$$

Moreover, by (104)

$$(121) \quad \partial_t \left( [F]_J^{\otimes K} \right) = \sum_{k_0 \in K} [F]_J^{\otimes K/\{k_0\}} [\partial_t F]_{J/(K/\{k_0\})}^{\otimes \{k_0\}}.$$

Applying  $\Phi_J$  defined in (114) to the BBGKY hierarchy (94), one finds easily that  $F_J^N$  satisfies, denoting  $\alpha(j, N) := \frac{N-j}{N}$ ,

$$(122) \quad \partial_t F_J^N = K^J F_J^N + \frac{1}{N} \sum_{i < r \in J} T_{i,r} F_J^N + \alpha(j, N) \sum_{i \in J} C_{i,j+1} F_{J \cup \{j+1\}}^N$$

(for  $j+1 \notin J$ ). Finally, we recall the mean-field equation

$$\partial_t F = KF + Q(F, F).$$

From this we deduce that

$$(123) \quad \begin{aligned} \partial_t E_J &= \sum_{K \subset J} (-1)^{|K|} \sum_{k_0 \in K} [F]_J^{\otimes K/\{k_0\}} (KF + Q(F, F))_{J/(K/\{k_0\})}^{\otimes \{k_0\}} F_{J/K}^N \\ &+ \sum_{K \subset J} (-1)^{|K|} \alpha(j - |K|, N) \sum_{i \in J/K} [F]_J^{\otimes K} C_{i,j+1} F_{(J/K) \cup \{j+1\}}^N \\ &+ \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left( \sum_{i \neq r \in J/K} T_{i,r} \right) F_{J/K}^N \\ &+ \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} (K^{J/K} F_{J/K}^N). \end{aligned}$$

We compute now the four terms in the r.h.s. of (123), denoted by  $\mathcal{T}_i$ ,  $i = 1, 2, 3, 4$  respectively. As we shall see, this computation is purely algebraic and uses only the decompositions  $\sum_{K \subset L} (-1)^{|K|} = \delta_{L, \emptyset}$ ,  $\sum_{K \subset L} |K| (-1)^{|K|} = -\delta_{|L|, 1}$  (cf. (110)) and the two following properties

$$\begin{cases} [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} = [F]_J^{\otimes K'} [F]_{J/K'}^{\otimes K} = [F]_J^{\otimes (K \cup K')}, & K, K' \subset J, K \cap K' = \emptyset \\ C_{i,j+1} [F]_{(J/K) \cup \{j+1\}}^{\otimes K} = [F]_{(J/K)}^{\otimes K} C_{i,j+1}, & K \subset J, j+1 \notin J. \end{cases}$$

Using again the definition (107),

$$\begin{aligned}
\mathcal{T}_1 &:= \sum_{K \subset J} (-1)^{|K|} \sum_{k_0 \in K} [F]_J^{\otimes K/\{k_0\}} (KF + Q(F, F))_{J/(K/\{k_0\})}^{\otimes \{k_0\}} F_{J/K}^N \\
&= - \sum_{k_0 \in J} (KF + Q(F, F))_J^{\otimes \{k_0\}} \sum_{K \subset J/\{k_0\}} (-1)^{|K|} [F]_{J/\{k_0\}}^{\otimes K} F_{(J/\{k_0\})/K}^N \\
&= - \sum_{i \in J} (KF + Q(F, F))_J^{\otimes \{i\}} E_{J/\{i\}}.
\end{aligned}
\tag{124}$$

To compute  $\mathcal{T}_2$  we make use of the inverse definition (109):

$$\begin{aligned}
\mathcal{T}_2 &:= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} C_{i, j+1} F_{(J/K) \cup \{j+1\}}^N \\
&= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\quad \dots C_{i, j+1} \sum_{K' \subset (J/K) \cup \{j+1\}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'}.
\end{aligned}$$

Now we have to distinguish among several cases so we decompose

$$\mathcal{T}_2 = \sum_{s=1}^4 \mathcal{T}_2^s
\tag{125}$$

where each  $\mathcal{T}_2^s$  concerns the part of the sum corresponding to the following constraints:

- $s = 1$ :  $i, j + 1 \in K'$ ,
- $s = 2$ :  $i, j + 1 \notin K'$ ,
- $s = 3$ :  $i \in K', j + 1 \notin K'$ ,
- $s = 4$ :  $i \notin K', j + 1 \in K'$ .

For  $s = 1$  we have

$$\begin{aligned}
\mathcal{T}_2^1 &= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\quad \dots C_{i,j+1} \sum_{\substack{K' \subset (J/K) \cup \{j+1\} \\ i, j+1 \in K'}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'} \\
&= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\quad \dots C_{i,j+1} \sum_{K'' \subset (J/K)/\{i\}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'' \cup \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\quad \dots C_{i,j+1} \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'' \cup \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J} [F]_J^{\otimes K} \dots \\
&\quad \dots \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K)}^{\otimes K''} C_{i,j+1} [F]_{((J/K)/K'') \cup \{j+1\}}^{\otimes \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{i \in J} \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} [F]_J^{\otimes K} \sum_{K'' \subset (J/\{i\})/K} [F]_{(J/K)}^{\otimes K''} \dots \\
&\quad \dots C_{i,j+1} [F]_{((J/K)/K'') \cup \{j+1\}}^{\otimes \{i,j+1\}} E_{(J/K)/(K'' \cup \{i\})} \\
&= \sum_{i \in J} \sum_{L \subset J/\{i\}} \left( \sum_{K \subset L} \alpha(j - |K|, N) (-1)^{|K|} \right) [F]_J^{\otimes L} \dots \\
&\quad \dots C_{i,j+1} [F]_{((J/L) \cup \{j+1\})}^{\otimes \{i,j+1\}} E_{J/(L \cup \{i\})} \\
&= \alpha(j, N) \sum_{i \in J} C_{i,j+1} [F]_{J \cup \{j+1\}}^{\otimes \{i,j+1\}} E_{J/\{i\}} - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i,j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i,j+1\}} E_{J/(\{i,l\})} \\
&= \alpha(j, N) \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} [F]_{J \cup \{j+1\}}^{\otimes \{l\}} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i,j+1\}} E_{J/(\{i,l\})} \\
&= \alpha(j, N) \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} [F]_{J \cup \{j+1\}}^{\otimes \{i,l,j+1\}} E_{J/(\{i,l\})}
\end{aligned}$$

since  $\sum_{K \subset L} (-1)^{|K|} = \delta_{L, \emptyset}$ . We get

$$\begin{aligned}
\mathcal{T}_2^1 &= \alpha(j, N) \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} E_{J/\{i\}} \\
(126) \quad &- \frac{1}{N} \sum_{i \neq l \in J} [Q(F, F)]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{l\}} E_{J/\{i,l\}}
\end{aligned}$$

and therefore there is a crucial compensation:

$$(127) \quad \begin{aligned} \mathcal{T}_1 + \mathcal{T}_2^1 &= -\frac{j}{N} \sum_{i \in J} [Q(F, F)]_J^{\otimes i} E_{J/\{i\}} \\ &- \frac{1}{N} \sum_{i \neq l \in J} [Q(F, F)]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{l\}} E_{J/\{i, l\}}. \end{aligned}$$

Furthermore,

$$(128) \quad \begin{aligned} \mathcal{T}_2^2 &= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\ &\dots C_{i, j+1} \sum_{\substack{K' \subset (J/K) \cup \{j+1\} \\ i \notin K', j+1 \notin K'}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'} \\ &= \sum_{i \in J} \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} [F]_J^{\otimes K} \dots \\ &\dots C_{i, j+1} \sum_{K' \in ((J/\{i\})/K)} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'} \\ &= \sum_{i \in J} \sum_{L \subset J/\{i\}} \left( \sum_{K \subset L} \alpha(j - |K|, N) (-1)^{|K|} \right) [F]_J^{\otimes L} C_{i, j+1} E_{((J/L) \cup \{j+1\})} \\ &= \alpha(j, N) \sum_{i \in J} C_{i, j+1} E_{(J \cup \{j+1\})} - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i, j+1} E_{((J/\{l\}) \cup \{j+1\})} \end{aligned}$$

having used  $C_{i, j+1} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} = [F]_{(J/K)}^{\otimes K'} C_{i, j+1}$  when  $j+1 \notin K'$ .

By the same argument,

$$(129) \quad \begin{aligned} \mathcal{T}_2^3 &= \sum_{i \in J} \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} [F]_J^{\otimes K} \dots \\ &\dots C_{i, j+1} \sum_{\substack{K' \subset (J/K) \cup \{j+1\} \\ i \notin K', j+1 \notin K'}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'} \\ &= \sum_{i \in J} \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} [F]_J^{\otimes K} \sum_{K'' \subset (J/\{i\})/K} [F]_{((J/\{i\})/K)}^{\otimes K''} \dots \\ &\dots C_{i, j+1} [F]_{J \cup \{j+1\}/(K \cup K'')}^{\otimes i} E_{((J/(\{i\} \cup K \cup K'')) \cup \{j+1\})} \\ &= \alpha(j, N) \sum_{i \in J} C_{i, j+1} [F]_{J \cup \{j+1\}}^{\otimes i} E_{((J/\{i\}) \cup \{j+1\})} \\ &- \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i, j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i\}} E_{((J/\{i, l\}) \cup \{j+1\})}. \end{aligned}$$

Finally,

$$\begin{aligned}
\mathcal{T}_2^4 &= \sum_{K \subset J} \alpha(j - |K|, N) (-1)^{|K|} \sum_{i \in J/K} [F]_J^{\otimes K} \dots \\
&\quad \dots C_{i,j+1} \sum_{\substack{K' \subset (J/K) \cup \{j+1\} \\ i \notin K', j+1 \in K'}} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/K'} \\
&= \sum_{i \in J} \sum_{K \subset J/\{i\}} \alpha(j - |K|, N) (-1)^{|K|} C_{i,j+1} [F]_{J \cup \{j+1\}}^{\otimes K} \dots \\
&\quad \dots \sum_{K' \subset (J/\{i\})/K} [F]_{(J/K) \cup \{j+1\}}^{\otimes K'} E_{((J/K) \cup \{j+1\})/(K' \cup \{j+1\})} \\
&= \sum_{i \in J} \sum_{L \subset J/\{i\}} \left( \sum_{K \subset L} \alpha(j - |K|, N) (-1)^{|K|} \right) C_{i,j+1} [F]_{J \cup \{j+1\}}^{\otimes L \cup \{j+1\}} E_{J/L} \\
&= \alpha(j, N) \sum_{i \in J} C_{i,j+1} [F]_{J \cup \{j+1\}}^{\otimes \{j+1\}} E_J - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} [F]_{J \cup \{j+1\}}^{\otimes \{l, j+1\}} E_{J/\{l\}} \\
&= \alpha(j, N) \sum_{i \in J} C_{i,j+1} [F]_{J \cup \{j+1\}}^{\otimes \{j+1\}} E_J - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i,j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{j+1\}} E_{J/\{l\}}.
\end{aligned} \tag{130}$$

Next we compute the term

$$\begin{aligned}
\mathcal{T}_3 &:= \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left( \sum_{i \neq r \in J/K} T_{i,r} \right) F_{J/K}^N \\
&= \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left( \sum_{i \neq r \in J/K} T_{i,r} \right) \sum_{K' \subset J/K} [F]_{J/K}^{\otimes K'} E_{J/(K \cup K')}.
\end{aligned}$$

We set

$$(131) \quad \mathcal{T}_3 = \sum_{s=1}^4 \mathcal{T}_3^s$$

where each term  $\mathcal{T}_3^i$  corresponds to the constraints

- $s = 1$   $i, r \in K'$ ,
- $s = 2$   $i, r \notin K'$ ,
- $s = 3$   $i \in K', r \notin K'$ ,
- $s = 4$   $i \notin K', r \in K'$ . We obtain

$$\begin{aligned}
\mathcal{T}_3^1 &= \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left( \sum_{i \neq r \in J/K} T_{i,r} \right) \sum_{\substack{K' \subset J/K \\ i,r \in K'}} [F]_{J/K}^{\otimes K'} E_{J/(K \cup K')} \\
&= \frac{1}{2N} \sum_{i \neq r \in J} \sum_{K \subset J/\{i,r\}} (-1)^{|K|} [F]_J^{\otimes K} T_{i,r} \sum_{\substack{K' \subset J/K \\ i,r \in K'}} [F]_{J/K}^{\otimes K'} E_{J/(K \cup K')} \\
&= \frac{1}{2N} \sum_{i \neq r \in J} \sum_{K \subset J/\{i,r\}} (-1)^{|K|} [F]_J^{\otimes K} T_{i,r} \\
&\quad \sum_{K'' \subset (J/K)/\{i,r\}} [F]_{J/K}^{\otimes K'' \cup \{i,r\}} E_{J/(K \cup K')} \\
&= \frac{1}{2N} \sum_{i \neq r \in J} \sum_{K \subset J/\{i,r\}} (-1)^{|K|} [F]_J^{\otimes K} \\
&\quad \sum_{K'' \subset (J/K)/\{i,r\}} T_{i,r} [F]_{J/K}^{\otimes K''} [F]_{J/(K \cup K'')}^{\otimes \{i,r\}} E_{J/(K \cup K'' \cup \{i,r\})} \\
&= \frac{1}{2N} \sum_{i \neq r \in J} \sum_{K \subset J/\{i,r\}} (-1)^{|K|} [F]_J^{\otimes K} \\
&\quad \sum_{K'' \subset (J/K)/\{i,r\}} [F]_{J/K}^{\otimes K''} T_{i,r} [F]_{J/(K \cup K'')}^{\otimes \{i,r\}} E_{J/(K \cup K'' \cup \{i,r\})} \\
&= \frac{1}{2N} \sum_{i \neq r \in J} \sum_{L \subset J/\{i,r\}} \left( \sum_{K \subset L} (-1)^{|K|} \right) [F]_J^{\otimes L} T_{i,r} [F]_{J/(L)}^{\otimes \{i,r\}} E_{J/(L \cup \{i,r\})} \\
(132) \quad &= \frac{1}{2N} \left( \sum_{i \neq r \in J} T_{i,r} [F]_J^{\otimes \{i,r\}} \right) E_{J/\{i,r\}},
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_3^2 &= \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left( \sum_{i \neq r \in J/K} T_{i,r} \right) \sum_{\substack{K' \subset J/K \\ i,r \notin K'}} [F]_{J/K}^{\otimes K'} E_{J/(K \cup K')} \\
&= \frac{1}{2N} \sum_{i \neq r \in J} \sum_{K \subset J/\{i,r\}} (-1)^{|K|} [F]_J^{\otimes K} \\
&\quad \sum_{K' \subset (J/\{i,r\})/K} [F]_{J/K}^{\otimes K'} T_{i,r} E_{J/(K \cup K')} \\
&= \frac{1}{2N} \sum_{i \neq r \in J} \sum_{L \subset J/\{i,r\}} \left( \sum_{K \subset L} (-1)^{|K|} \right) [F]_J^{\otimes L} T_{i,r} E_{J/L} \\
(133) \quad &= \frac{1}{2N} \left( \sum_{i \neq r \in J} T_{i,r} \right) E_J = \frac{1}{2N} T_J E_J
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}_3^3 &= \frac{1}{2N} \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} \left( \sum_{i \neq r \in J/K} T_{i,r} \right) \sum_{\substack{K' \subset J/K \\ i \in K', r \notin K'}} [F]_{J/K}^{\otimes K'} E_{J/(K \cup K')} \\
&= \frac{1}{2N} \sum_{i \neq r \in J} \sum_{K \subset J/\{i,r\}} (-1)^{|K|} [F]_J^{\otimes K} \\
&\quad \sum_{K'' \subset (J/\{i,r\})/K} T_{i,r} [F]_{J/K}^{\otimes K'' \cup \{i\}} E_{J/(K \cup K'' \cup \{i\})} \\
&= \frac{1}{2N} \sum_{i \neq r \in J} \sum_{K \subset J/\{i,r\}} (-1)^{|K|} [F]_J^{\otimes K} \\
&\quad \sum_{K'' \subset (J/\{i,r\})/K} [F]_{J/K}^{\otimes K''} T_{i,r} [F]_{(J/K)/K''}^{\otimes \{i\}} E_{J/(K \cup K'' \cup \{i\})} \\
&= \frac{1}{2N} \sum_{i \neq r \in J} \sum_{L \subset J/\{i,r\}} \left( \sum_{K \subset L} (-1)^{|K|} \right) [F]_{J/\{i\}}^{\otimes L} T_{i,r} [F]_{J/L}^{\otimes \{i\}} E_{J/(L \cup \{i\})} \\
&= \frac{1}{2N} \sum_{i \neq r \in J} T_{i,r} [F]_J^{\otimes \{i\}} E_{J/\{i\}}.
\end{aligned}$$

Note also that

$$(134) \quad \mathcal{T}_3^4 = \mathcal{T}_3^3|_{i \leftrightarrow r},$$

therefore

$$(135) \quad \mathcal{T}_3^4 + \mathcal{T}_3^3 = \frac{1}{N} \sum_{i,r \in J} T_{i,r} [F]_J^{\otimes \{i\}} E_{J/\{i\}}.$$

Finally,

$$\begin{aligned}
\mathcal{T}_4 &= \sum_{K \subset J} (-1)^{|K|} [F]_J^{\otimes K} (K^{J/K} F_{J/K}^N) \\
&= \sum_{K \subset J} \sum_{K' \subset J/K} (-1)^{|K|} [F]_J^{\otimes K} K^{J/K} [F]_{J/K}^{\otimes K'} E_{(J/K)/K'} \\
&= \sum_{K \subset J} \sum_{K' \subset J/K} (-1)^{|K|} [F]_J^{\otimes K} \left( \sum_{j \in J/K} K^j [F]_{J/K}^{\otimes K'} E_{(J/K)/K'} \right) \\
&= \sum_{K \subset J} \sum_{j \in J/K} K^j \sum_{K' \subset J/K} (-1)^{|K|} [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} E_{(J/K)/K'} \\
&= \sum_{j \in J} K^j \sum_{K \subset J/\{j\}} \sum_{K' \subset J/K} (-1)^{|K|} [F]_J^{\otimes K} [F]_{J/K}^{\otimes K'} E_{(J/K)/K'} \\
&= \sum_{j \in J} K^j \sum_{L \subset J/\{j\}} \left( \sum_{K \subset L} (-1)^{|K|} \right) [F]_J^{\otimes L} E_{(J/L)} \\
&= \sum_{j \in J} K^j E_{(J)} \\
(136) \quad &= K^J E_J.
\end{aligned}$$

**Remark B.1.** *Note that, in the Kac model case,  $K = 0$  so  $\mathcal{T}_4 = 0$ .*

Summing (127), (128), (129), (130), (132), (133), (135) and (136), we get

$$\begin{aligned}
& \partial_t E_J = \\
& (K^J + \frac{1}{N}T_J) E_J + \frac{N-j}{N} \sum_{i \in J} C_{i,j+1} \left( [F]_{J \cup \{j+1\}}^{\otimes i} E_{(J \cup \{j+1\})/\{i\}} + [F]_{J \cup \{j+1\}}^{\otimes j+1} E_J \right) \\
& - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \left( [F]_{J \cup \{j+1\}}^{\otimes \{l\}} E_{((J/\{l\}) \cup \{j+1\})} \right) \\
& + \frac{N-j}{N} C_{j+1} E_{J \cup \{j+1\}} \\
(137) \quad & + \left( \frac{1}{N} \sum_{i,r \in J} T_{i,r} [F]_J^{\otimes i} - \frac{j}{N} \sum_{i \in J} [Q(F, F)]_J^{\otimes i} \right) E_{J/\{i\}} \\
& - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \left( [F]_{J \cup \{j+1\}}^{\otimes \{l\}} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{j+1\}} E_{J/\{l\}} \right) \\
& - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \left( [F]_{J \cup \{j+1\}}^{\otimes \{l\}} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i\}} E_{((J/\{i,l\}) \cup \{j+1\})} \right) \\
& + \frac{1}{2N} \left( \sum_{i \neq s \in J} T_{i,s} [F]_J^{\otimes i} [F]_{J/\{i\}}^{\otimes s} \right) E_{J/\{i,s\}} - \frac{1}{N} \sum_{i \neq l \in J} [Q(F, F)]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{l\}} E_{J/\{i,l\}}.
\end{aligned}$$

We organized the terms in the above equation, according to the following rule. The two first lines contains operators which do not change the particle number. The third line increases the number of particles by one. The three next lines decrease it by one and the last one by two. We get (117) after specializing (137) to  $J = \{1, \dots, j\}$ , using (114) and setting



$$\begin{aligned}
D_j & : \mathbb{L}^{\otimes j} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 1, \dots, N, \\
E_j & \mapsto \frac{N-j}{N} \sum_{i \in J} C_{i,j+1} \left( [F]_{J \cup \{j+1\}}^{\otimes \{i\}} \Phi_{(J \cup \{j+1\})/\{i\}} E_j + [F]_{J \cup \{j+1\}}^{\otimes \{j+1\}} E_j \right), \\
& \quad - \frac{1}{N} \sum_{i \neq l \in J} C_{i,j+1} \left( [F]_{J \cup \{j+1\}}^{\otimes \{l\}} \Phi_{(J/\{l\}) \cup \{j+1\}} E_j \right) \\
D_j^1 & : \mathbb{L}^{\otimes(j+1)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 1, \dots, N-1, \\
E_{j+1} & \mapsto \frac{N-j}{N} C_{j+1} E_{j+1}, \\
D_j^{-1} & : \mathbb{L}^{\otimes(j-1)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 2, \dots, N, \\
E_{j-1} & \mapsto \left( -\frac{j}{N} \sum_{i \in J} [Q(F, F)]_J^{\otimes \{i\}} + \frac{1}{2N} \sum_{i,r \in J} T_{i,r} [F]_J^{\otimes \{i\}} \right) \Phi_{J/\{i\}} E_{j-1}, \\
& \quad - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i,j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{j+1\}} \Phi_{J/\{l\}} E_{j-1} \\
& \quad - \frac{1}{N} \sum_{i \neq l \in J} [F]_J^{\otimes \{l\}} C_{i,j+1} [F]_{(J/\{l\}) \cup \{j+1\}}^{\otimes \{i\}} \Phi_{(J/\{i,l\}) \cup \{j+1\}} E_{j-1} \\
D_j^{-2} & : \mathbb{L}^{\otimes(j-2)} \rightarrow \mathbb{L}^{\otimes j}, \quad j = 3, \dots, N, \\
E_{j-2} & \mapsto \frac{1}{2N} \sum_{i,s \in J} T_{i,s} [F]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{s\}} \Phi_{J/\{i,s\}} E_{j-2} \\
(138) \quad & \quad - \frac{1}{N} \sum_{i \neq l \in J} [Q(F, F)]_J^{\otimes \{i\}} [F]_{J/\{i\}}^{\otimes \{l\}} \Phi_{J/\{i,l\}} E_{j-2}.
\end{aligned}$$

where, by convention,

$$(139) \quad \begin{cases} D_N^1 := D_1^{-2} := 0 \\ D_1^{-1}(E_0) := -\frac{1}{N} Q(F, F), \\ D_2^{-2}(E_0) := \frac{1}{N} (T_{1,2}(F \otimes F) - Q(F, F) \otimes F - F \otimes Q(F, F)). \end{cases}$$

Note that one has the following estimates:

$$(140) \quad \|D_j\|, \|D_j^1\| \leq j \text{ and } \|D_j^{-1}\|, \|D_j^{-2}\|, \|D_1^{-1}(E_0)\|, \|D_2^{-2}(E_0)\| \leq \frac{j^2}{N}.$$

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