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ON THE COHOMOLOGY OF STOVER SURFACE

AMIR DŽAMBIĆ, XAVIER ROULLEAU

ABSTRACT. We study a surface discovered by Stover which is the surface with minimal Euler number and maximal automorphism group among smooth arithmetic ball quotient surfaces. We study the natural map $\wedge^2 H^1(S, \mathbb{C}) \rightarrow H^2(S, \mathbb{C})$ and we discuss the problem related to the so-called Lagrangian surfaces. We obtain that this surface S has maximal Picard number and has no higher genus fibrations. We compute that its Albanese variety A is isomorphic to $(\mathbb{C}/\mathbb{Z}[\alpha])^7$, for $\alpha = e^{2i\pi/3}$.

1. INTRODUCTION

By the recent work of M. Stover [14], the number of automorphisms of a smooth compact arithmetic ball quotient surface $X = \Gamma \backslash \mathbb{B}_2$ is bounded by $288 \cdot e(X)$, where $e(X)$ denotes the topological Euler number of X .

Furthermore, Stover characterizes the arithmetic ball quotient surfaces X whose automorphism groups attain this bound, which by analogy with Hurwitz curves, he calls *Hurwitz ball quotients*; all such surfaces are finite Galois coverings of the Deligne-Mostow orbifold $\Lambda \backslash \mathbb{B}_2$ corresponding to the quintuple $(2/12, 2/12, 2/12, 7/12, 11/12)$ (see [12, 14]).

Stover constructs also a Hurwitz ball quotient S with Euler number $e(S) = 63$ and automorphism group $\text{Aut}(S)$ isomorphic to $U_3(3) \times \mathbb{Z}/3\mathbb{Z}$, of order $18144 = 2^5 3^{47}$. He shows that S is the unique Hurwitz ball quotient with Euler number $e = 63$, and moreover that $e = 63$ is the minimal possible value for the Euler number of a Hurwitz ball quotient. Having this property the surface S can be seen as the 2-dimensional analog of the Klein's quartic which is the unique curve uniformized by the ball \mathbb{B}_1 with minimal genus and maximal possible automorphism group.

Our aim is to study more closely the cohomology of this particular surface S , which we will call *Stover surface* in the following. This surface S has the following numerical invariants (see [14]):

$e(S)$	$H_1(S, \mathbb{Z})$	q	$p_g = h^{2,0}$	$h^{1,1}$	$b_2(S)$
63	\mathbb{Z}^{14}	7	27	35	89

Let V be a vector space. Let us recall that a 2-vector $w \in \wedge^2 V$ has *rank 1* or is *decomposable* if there are vectors $w_1, w_2 \in V$ with $w = w_1 \wedge w_2$. A vector $w \in \wedge^2 V$ has *rank 2* if there exist linearly independent vectors $w_i \in V$, $i = 1, \dots, 4$ such that $w = w_1 \wedge w_2 + w_3 \wedge w_4$.

Let B be an Abelian fourfold and let $p : S \rightarrow B$ be a map such that $p(S)$ generates B . We say that S is *Lagrangian with respect to p* if there exists a basis w_1, \dots, w_4 of $p^* H^0(B, \Omega_B)$ such that the rank 2 vector $w = w_1 \wedge w_2 + w_3 \wedge w_4$ is in the kernel of the natural map $\phi^{2,0} : \wedge^2 H^0(S, \Omega_S) \rightarrow H^0(S, K_S)$.

Theorem 1. *The surface S has maximal Picard number. The natural map*

$$\phi^{1,1} : H^0(S, \Omega_S) \otimes H^1(S, \mathcal{O}_S) \rightarrow H^1(S, \Omega_S)$$

is surjective with a 14-dimensional kernel. The kernel of the map

$$\phi^{2,0} : \wedge^2 H^0(S, \Omega_S) \rightarrow H^0(S, K_S)$$

is 7-dimensional and contains no decomposable elements. The set of rank 2 vectors in $\text{Ker}(\phi^{2,0})$ is a quadric hypersurface.

There exists an infinite number (up to isogeny) of maps $p : S \rightarrow B$ (where B is an Abelian fourfold) such that S is Lagrangian with respect to p .

The Albanese variety of S is isomorphic to $(\mathbb{C}/\mathbb{Z}[\alpha])^7$, for $\alpha = e^{2i\pi/3}$.

By the Castelnuovo - De Franchis Theorem, the fact that there are no decomposable elements in $\wedge^2 H^0(S, \Omega_S)$ means that S has no fibration $f : S \rightarrow C$ onto a curve of genus $g > 1$. Moreover Theorem 1 implies that S has the remarkable feature that both maps

$$\begin{aligned} \phi^{2,0} &: \wedge^2 H^{1,0}(S) \rightarrow H^{2,0}(S) \\ \phi^{1,1} &: H^{1,0}(S) \otimes H^{0,1}(S) \rightarrow H^{1,1}(S) \end{aligned}$$

have a non-trivial kernel. With Schoen surfaces (see [10, Remark 2.6]), this is the second example of surfaces enjoying such properties. For more on this subject, see e.g. [1, 5, 2, 8, 9].

We obtain these results using Sullivan's theory on the second lower quotient of the fundamental group $\pi_1(S)$ of S (see [4]).

For the motivation and a historic account of surfaces with maximal Picard number we refer to [3].

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2. THE SECOND LOWER CENTRAL QUOTIENT OF THE FUNDAMENTAL GROUP OF S

Let $\Pi := \pi_1(X)$ be the fundamental group of a manifold X . The group $H_1(X, \mathbb{Z})$ is the abelianization of Π : $H_1(X, \mathbb{Z}) = \Pi/\Delta$ where $\Delta := [\Pi, \Pi]$ is the derived subgroup of Π , that is, the subgroup generated by all elements $[h, g] = g^{-1}h^{-1}gh$, $h, g \in \Pi$.

The second group in the lower central series $[\Delta, \Pi]$ is the group generated by commutators $[h, g]$, with $h \in \Delta$, $g \in \Pi$. It is a normal subgroup of the commutator group Δ . According to [4], we have the following results:

Proposition 2. (Sullivan) *Let X be a compact connected Kähler manifold. There exists an exact sequence*

$$0 \rightarrow \text{Hom}(\Delta/[\Delta, \Pi], \mathbb{R}) \rightarrow \wedge^2 H^1(X, \mathbb{R}) \rightarrow H^2(X, \mathbb{R}).$$

(Beauville) *Suppose $H_1(X, \mathbb{Z})$ is torsion free. Then the group $\Delta/[\Delta, \Pi]$ is canonically isomorphic to the cokernel of the map*

$$\mu : H_2(X, \mathbb{Z}) \rightarrow \text{Alt}^2(H^1(X, \mathbb{Z})) \text{ given by } \mu(\sigma)(a, b) = \sigma \cap (a \wedge b),$$

where $\text{Alt}^2(H^1(X, \mathbb{Z}))$ is the group of skew-symmetric integral bilinear forms on $H^1(X, \mathbb{Z})$.

In the case of the Stover surface, computer calculations give us the following result:

Theorem 3. *Let $\Pi = \pi_1(S)$ be the fundamental group of the Stover surface and $\Delta = [\Pi, \Pi]$. The group $\Delta/[\Delta, \Pi]$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}^{28}$.*

Proof. By the construction of S [14], the fundamental group Π is isomorphic to the kernel $\ker(\varphi)$ of the unique epimorphism $\varphi : \Lambda \rightarrow G$ from the Deligne-Mostow lattice Λ corresponding to the quintuple $(2/12, 2/12, 2/12, 7/12, 11/12)$ onto the finite group $G = U_3(3) \times \mathbb{Z}/3\mathbb{Z}$. The lattice Λ is described by Mostow in [12] as a complex reflection group, and by generators and relation by Cartwright and Steger in [7]. This lattice has presentation

$$\Lambda = \langle j, u, v, b|u^4, v^8, [u, j], [v, j], j^{-3}v^2, uvv^{-1}uv^{-1}, (bj)^2(vu^2)^{-1}, [b, vu^2], b^3, (bv u^3)^3 \rangle.$$

MAGMA command `LowIndexSubgroups` is used to identify the unique subgroup $\Gamma \triangleleft \Lambda$ of index 3, which is $\Gamma = \langle u, jb, bj \rangle$. Using the primitive permutation representation of $U_3(3)$ of degree 28, MAGMA is able to identify an homomorphism φ from Γ onto $U_3(3)$ induced from the assignment

$$\begin{aligned}
u &\mapsto (3, 8, 23, 20)(4, 24, 6, 12)(7, 9, 14, 22)(10, 19, 11, 13)(15, 16, 21, 18)(17, 26, 27, 25) \\
jb &\mapsto (1, 9, 20, 12, 19, 23, 6, 16)(2, 27, 14, 17, 13, 26, 15, 25)(3, 24)(4, 5, 10, 21, 7, 11, 28, 8) \\
bj &\mapsto (1, 13, 20, 15, 19, 2, 6, 14)(4, 9, 10, 12, 7, 23, 28, 16)(5, 27, 21, 17, 11, 26, 8, 25)(22, 24).
\end{aligned}$$

This homomorphism extends to an homomorphism φ from Λ onto G such that $\Pi = \ker(\varphi)$ is a torsion-free normal subgroup in Λ , it is the fundamental group of S (see [14]). Let be $\Delta = [\Pi, \Pi]$ and $\Delta_2 = [\Delta, \Pi]$. It is easy to check that that Δ_2 is distinguished into Π . The image of Δ under the quotient map $\Pi \rightarrow \Pi/\Delta_2$ is Δ/Δ_2 , but we observe that it is also equal the commutator subgroup $[\Pi/\Delta_2, \Pi/\Delta_2]$, and therefore, the computation of Δ/Δ_2 is reduced to the one of the derived group $[\Pi/\Delta_2, \Pi/\Delta_2]$.

The MAGMA command `g:=Rewrite(G,g)` is used to have generators and relations of both subgroups $\Gamma < \Lambda$ and $\Pi < \Gamma$. The command `NilpotentQuotient(.,2)` applied to Π describes Π/Δ_2 in terms of a polycyclic presentation. The derived subgroup $[\Pi/\Delta_2, \Pi/\Delta_2]$ is obtained with `DerivedGroup(.)` applied to Π/Δ_2 . Finally, applying the MAGMA function `AQInvariants` to $[\Pi/\Delta_2, \Pi/\Delta_2]$, MAGMA computes that the structure of Δ/Δ_2 is $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}^{28}$. \square

Corollary 4. *The dimension of the kernel of $\wedge^2 H^1(S, \mathbb{R}) \rightarrow H^2(S, \mathbb{R})$ is 28.*

3. COMPUTATION OF THE MAP $\wedge^2 H^1(S, \mathbb{C}) \rightarrow H^2(S, \mathbb{C})$

Let A be the Albanese variety of the Stover surface S . The invariants are:

$$\begin{aligned}
H_1(A, \mathbb{Z}) &= H_1(S, \mathbb{Z}) = \mathbb{Z}^{14}, \quad H_2(A, \mathbb{Z}) = \wedge^2 H_1(A, \mathbb{Z}), \quad H^{2,0}(A) = \wedge^2 H^{1,0}(S) \\
H^{1,1}(A) &= H^{1,0}(S) \otimes H^{0,1}(S), \quad H^{0,2}(A) = \wedge^2 H^{0,1}(S),
\end{aligned}$$

and

$H_1(A, \mathbb{Z})$	q	$h^{2,0}(A)$	$h^{1,1}(A)$	$b_2(A)$
\mathbb{Z}^{14}	7	21	49	91

We have a map respecting Hodge decomposition

$$\begin{array}{c}
H^{2,0}(A) \oplus H^{1,1}(A) \oplus H^{0,2}(A) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S)
\end{array},$$

which is an equivariant map of $\text{Aut}(S)$ -modules. By Corollary 4, the kernel of that map is 28 dimensional ; it is moreover a $\text{Aut}(S)$ -module.

According to the Atlas tables [11], the group $U_3(3)$ has 14 irreducible representations χ_i , $1 \leq i \leq 14$ of respective dimension 1, 6, 7, 7, 7, 14, 21, 21, 21, 27, 28, 28, 32, 32.

The irreducible representations of $\text{Aut}(S) = U_3(3) \times \mathbb{Z}/3\mathbb{Z}$ are the χ_i^t , $i = 1, \dots, 14$, $t = 0, 1, 2$ where $(g, s) \in U_3(3) \times \mathbb{Z}/3\mathbb{Z}$ acts on the same space as χ_i with action $(g, s) \cdot v = \alpha^s g(v)$ with $\alpha = e^{2i\pi/3}$ a primitive third root of unity.

Theorem 5. *The image of S by the Albanese map $\vartheta : S \rightarrow A$ is 2-dimensional.*

The map $H^{1,1}(A) \rightarrow H^{1,1}(S)$ is surjective, with a 14 dimensional kernel isomorphic to χ_6^0 as an $\text{Aut}(S)$ -module. We have $H^1(S, \mathbb{Z}) = \chi_3^1 \oplus \chi_3^2$ and $H^{1,1}(S) = \chi_1^0 \oplus \chi_3^0 \oplus \chi_{10}^0$, as $\text{Aut}(S)$ -modules.

The kernel of the natural map $\wedge^2 H^0(S, \Omega_S) \rightarrow H^0(S, K_S)$ is 7-dimensional, isomorphic to χ_3^0 as a $\text{Aut}(S)$ -module.

The surface S has maximal Picard number.

The Albanese variety A of S is isomorphic to $(\mathbb{C}/\mathbb{Z}[\alpha])^7$, for $\alpha = e^{2i\pi/3}$.

Since A is CM, it follows that S is Albanese standard, that is, the class of its image inside its Albanese variety A sits in the subring of $H^*(A, \mathbb{Q})$ generated by the divisor classes. That contrasts with the above mentioned Schoen surfaces, see [10].

Proof. Suppose that the image of S in A is 1-dimensional. Then there exists a smooth curve C of genus 7 with a fibration $f : S \rightarrow C$ and the map $\wedge^2 H^0(S, \Omega_S) \rightarrow H^0(S, K_S)$ is the 0 map and the kernel of $\wedge^2 H^1(S, \mathbb{C}) \rightarrow H^2(S, \mathbb{C})$ is at least 42 dimensional, which is impossible. Thus the image of S by the Albanese map $\vartheta : S \rightarrow A$ is 2-dimensional.

According to the Atlas character table [11], the possibilities for the $U_3(3)$ -module $H_1(S, \mathbb{Z}) = H_1(A, \mathbb{Z}) = \mathbb{Z}^{14}$ are:

$$\chi_3^{\oplus 2}, \mathcal{R}_{\mathbb{Z}}(\chi_4) = \mathcal{R}_{\mathbb{Z}}(\chi_5) = \chi_4 \oplus \chi_5, \chi_4^{\oplus 2}, \chi_5^{\oplus 2} \text{ or } \chi_6$$

where $\mathcal{R}_{\mathbb{Z}}(\chi_j)$ is the restriction to \mathbb{Z} of the 7-dimensional complex representation χ_j defined over $\mathbb{Z}[i]$. It cannot be $\chi_4^{\oplus 2}$ nor $\chi_5^{\oplus 2}$ because these are not defined over \mathbb{Z} (some traces of elements are in $\mathbb{Z}[i] \setminus \mathbb{Z}$). We cannot have $H^1(S, \mathbb{Z}) = \chi_6$ since χ_6 remains irreducible, but $H^1(S, \mathbb{Z}) \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$ is a Hodge decomposition on which the representation of $U_3(3)$ splits.

By duality, the kernel of $H^{2,0}(A) \rightarrow H^{2,0}(S)$ has same dimension d as the kernel of $H^{0,2}(A) \rightarrow H^{0,2}(S)$. Let k be the dimension of the kernel of the $U_3(3)$ -equivariant map $H^{1,1}(A) \rightarrow H^{1,1}(S)$. We have $28 = k + 2d$, moreover since $h^{1,1}(S) = 35$ and $h^{1,1}(A) = 49$, we get $28 \geq k \geq 14$.

Let us suppose that $H^1(S, \mathbb{Z}) = \chi_4 \oplus \chi_5$. Then the representation $H^{1,1}(A)$ equals to $\chi_4 \otimes \chi_5 = \chi_1 + \chi_7 + \chi_{10}$ (of dimension $1 + 21 + 27$). An Abelian variety on which a finite group G acts possesses a G -invariant polarization (for example $\sum_{g \in G} g^* L$, where L is some polarization). Therefore the one dimensional $\text{Aut}(S)$ -invariant space of $H^{1,1}(A)$ is generated by the class of an ample divisor and the natural map $\vartheta^* : H^{1,1}(A) \rightarrow H^{1,1}(S)$ is injective on that subspace. Therefore the map ϑ^* has a kernel of dimension $k = 21, 27$ or 48 . This is impossible because $k + 2d$ equals 28.

Hence, we have $H^1(S, \mathbb{Z}) = \chi_3^{\oplus 2}$ and moreover

$$H^{2,0}(A) = \wedge^2 \chi_3 = \chi_3 \oplus \chi_6$$

(the dimensions are $21 = 7 + 14$) and

$$H^{1,1}(A) = \chi_3^{\otimes 2} = \chi_1 \oplus \chi_3 \oplus \chi_6 \oplus \chi_{10}$$

($49 = 1 + 7 + 14 + 27$). By checking the possibilities, we obtain $k = 14$, $H^{1,1}(S) = \chi_1 \oplus \chi_3 \oplus \chi_{10}$, and the map $H^{1,1}(A) \rightarrow H^{1,1}(S)$ is surjective. The kernel of the map $H^{2,0}(A) \rightarrow H^{2,0}(S)$ is isomorphic to χ_3 , of dimension 7, the action of $U_3(3)$ on $H^{2,0}(S)$ is then $H^{2,0}(S) = \chi_6 \oplus \chi$, where χ is a 13 dimensional representation.

Let $\sigma \in \text{Aut}(S) = U_3(3) \times \mathbb{Z}/3\mathbb{Z}$ be the order 3 automorphism commuting with every other element. It corresponds to an element $\sigma' \in \Lambda$ normalizing Π in Λ and such that the group Π' generated by Π and σ' contains Π with index 3. Using MAGMA, one find that we can choose $\sigma' = j^4$, where j is the order 12 element described in the proof of Theorem 3.

The quotient surface S/σ of S by σ is equal to \mathbb{B}_2/Π' . The fundamental group of S' is Π'/Π'_{tors} where Π'_{tors} is the subgroup of Π' generated by torsion elements. Using MAGMA, one find that Π' has a set of 8 generators with 7 of them which are torsion elements. Using these elements, we readily compute that Π'/Π'_{tors} is trivial. Therefore the space of one-forms on S that are invariant by σ is 0. Using the symmetries of $U_3(3)$, one see that σ acts on the tangent space $H^0(S, \Omega_S)^*$ as the multiplication by α or α^2 . After possible permutation of σ and σ^2 , we can suppose it is α .

We see that the representation of $\text{Aut}(S)$ on $H_1(S, \mathbb{Z})$ is $\chi_3^1 \oplus \chi_3^2$. The lattice $H_1(S, \mathbb{Z}) \subset H^0(S, \Omega_S)^*$ is moreover a $\mathbb{Z}[\alpha]$ -module. The ring $\mathbb{Z}[\alpha]$ is a principal ideal domain, therefore $H_1(S, \mathbb{Z}) = \mathbb{Z}[\alpha]^7$ (for the choice of a certain basis) and A is isomorphic to $(\mathbb{C}/\mathbb{Z}[\alpha])^7$.

Therefore A has maximal Picard number and all the classes of $H^{1,1}(A)$ are algebraic. These classes remain of course algebraic under the map $H^{1,1}(A) \rightarrow H^{1,1}(S)$, which is surjective. Thus S is a surface with maximal Picard number. \square

4. LAGRANGIAN SURFACES AND STOVER SURFACE

Let B be an Abelian fourfold and let $p : S \rightarrow B$ be a map such that $p(S)$ generates B . Let us recall that S is Lagrangian with respect to p if there exists a basis w_1, \dots, w_4 of $p^*H^0(B, \Omega_B)$ such that the rank 2 vector $w = w_1 \wedge w_2 + w_3 \wedge w_4$ is in the kernel of the natural map $\phi^{2,0} : \wedge^2 H^0(S, \Omega_S) \rightarrow H^0(S, K_S)$. Let us now prove

Theorem 6. *The 7 dimensional space $\text{Ker}(\phi^{2,0})$ contains no decomposable elements. The algebraic set of rank 2 vectors in $\text{Ker}(\phi^{2,0})$ is a quadric $\tilde{Q} \subset \text{Ker}(\phi^{2,0})$.*

There exists an infinite number (up to isogeny) of maps $p : S \rightarrow B$ where B is an Abelian fourfold such that S is Lagrangian with respect to p .

There exists an infinite number (up to isogeny) of maps $p : S \rightarrow B$ where B is an Abelian fourfold such that

$$\tilde{Q} \cap p^*H^0(B, \wedge^2 \Omega_B) = \{0\},$$

*and for some of them we even have $\text{Ker}(\phi^{2,0}) \cap p^*H^0(B, \wedge^2 \Omega_B) = \{0\}$.*

The generic rank 2 element w in $\tilde{Q} \subset \text{Ker}(\phi^{2,0})$ does not correspond to any morphism to an Abelian fourfold.

Proof. We proved in Theorem 5 that

$$H^{2,0}(A) = \wedge^2 \chi_3 = \chi_3 \oplus \chi_6$$

and the kernel of $\phi^{2,0} : H^{2,0}(A) \rightarrow H^{2,0}(S)$ is the 7-dimensional subspace with representation χ_3 . In a basis $\gamma = (e_1, \dots, e_7)$ of $\chi_3 = H^0(S, \Omega_S) = H^{1,0}(S)$, the two following matrices A, B are generators of the group $U_3(3)$:

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Using the basis $\beta = (e_{ij})_{1 \leq i < j \leq 7}$ of $\wedge^2 \chi_3$ ($e_{ij} = e_i \wedge e_j$) with order $e_{ij} \leq e_{st}$ if $i < s$ or $i = s$ and $j \leq t$, one computes that the subspace $\text{Ker}(\phi^{2,0}) = \chi_3 \subset \wedge^2 \chi_3$ is generated by the columns of the matrix $M \in M_{21,7}$, where ${}^t M = (N, 2I_7)$, for

$$N = \begin{pmatrix} 0 & 0 & 2 & -2 & -2 & -2 & 0 & 2 & 2 & -2 & 2 & 2 & 2 & 2 \\ -1 & 0 & 0 & 2 & 4 & 0 & 1 & -3 & -3 & 1 & -3 & -4 & -2 & -4 \\ 0 & -2 & 0 & -2 & -2 & 0 & -2 & 2 & 2 & 0 & 0 & 2 & 2 & 2 \\ -1 & -2 & 2 & 0 & -2 & 0 & -1 & 1 & 3 & 1 & 1 & 0 & 0 & 2 \\ -1 & 1 & -1 & 3 & 1 & 3 & 0 & -4 & -2 & 2 & 0 & -4 & -2 & -2 \\ 0 & 3 & -3 & 1 & -1 & 1 & 1 & -3 & -3 & -1 & 1 & 0 & -2 & -2 \\ 1 & 1 & 1 & 3 & 3 & 1 & 2 & -2 & 0 & 0 & 0 & -2 & 0 & -2 \end{pmatrix} \in M_{7,14}$$

and I_7 the 7×7 identity matrix. Knowing that, we obtain the ideal I_V of the algebraic set V of couples $(w_1, w_2) \in \chi_3 \oplus \chi_3$ such that $w_1 \wedge w_2 \in \text{Ker}(\phi^{2,0}) \subset \wedge^2 \chi_3$. That ideal is generated by 14 homogeneous quadratic polynomials in the variables x_1, \dots, x_{14} . Let W be

the algebraic set of couples $(w_1, w_2) \in \chi_3 \oplus \chi_3$ such that $w_1 \wedge w_2 = 0 \in \wedge^2 \chi_3$. The ideal I_W of W is generated by the 2 by 2 minors of the matrix

$$L = \begin{pmatrix} x_1 & \dots & x_7 \\ x_8 & \dots & x_{14} \end{pmatrix}.$$

Since $W \subset V$, we have $\text{Rad}(I_V) \subset \text{Rad}(I_W)$ where $\text{Rad}(I)$ is the radical of an ideal I . On the other hand, using Maple, one can check that the 21 minors of L are in $\text{Rad}(I_V)$, hence $\text{Rad}(I_W) \subset \text{Rad}(I_V)$, thus $V = W$.

We therefore conclude that the kernel of $\phi^{2,0}$ contains no decomposable elements.

A 2-vector w over a characteristic 0 field can be expressed uniquely as $w = \sum_{i,j} a_{ij} e_i \wedge e_j$ where $a_{ij} = -a_{ji}$. The rank of the vector w is half the rank of the (skew-symmetric) coefficient matrix $A_w := (a_{ij})_{1 \leq i, j \leq 7}$ of w [6, Thm 1.7 & Remark p. 13]. Thus the 2-vector $w = a_1 v_1 + \dots + a_7 v_7$ in $\text{Ker}(\phi^{2,0})$ (where the v_i , $i = 1..7$ are the vectors corresponding to the columns of the matrix M) is a rank 2 vector if and only if the 49 6×6 minors of the matrix A_w are 0. The radical of the ideal generated by these minors is principal, generated by a homogeneous quadric in a_1, \dots, a_7 whose associated symmetric matrix is

$$Q = \begin{pmatrix} 7 & 3 & 3 & 1 & -3 & -3 & -5 \\ 3 & 7 & 3 & 3 & 1 & -3 & -3 \\ 3 & 3 & 7 & 3 & 3 & 1 & -3 \\ 1 & 3 & 3 & 7 & 3 & 3 & 1 \\ -3 & 1 & 3 & 3 & 7 & 3 & 3 \\ -3 & -3 & 1 & 3 & 3 & 7 & 3 \\ -5 & -3 & -3 & 1 & 3 & 3 & 7 \end{pmatrix}.$$

Therefore $w \in \text{Ker}(\phi^{2,0})$ has rank 2 if and only if $(a_1, \dots, a_7) Q^t (a_1, \dots, a_7) = 0$.

The point $(10 + 8\alpha, -7, 0, 0, 7, 0, 0)$ lies on the associated smooth quadric \tilde{Q} , therefore $\tilde{Q}(\mathbb{Q}[\alpha])$ is infinite. Let be w be a 2-vector in $\tilde{Q}(\mathbb{Q}[\alpha])$. The decomposable vector $\wedge^2 w \neq 0$ has coordinates in $\mathbb{Q}[\alpha]$ in the basis $(e_{i1} \wedge \dots \wedge e_{i4})$ of $\wedge^4 H^0(S, \Omega_S)$. The corresponding 4-dimensional vector space W is therefore generated by 4 vectors w_1, \dots, w_4 with coordinates over $\mathbb{Q}[\alpha]$ in the basis $\gamma = (e_1, \dots, e_7)$ of $H^0(S, \Omega_S)$.

One computes that the image of $\mathbb{Q}[\alpha][U_3(3) \times \mathbb{Z}/3\mathbb{Z}]$ in $M_7(\mathbb{Q}[\alpha])$ is 49 dimensional over $\mathbb{Q}[\alpha]$, thus

$$\mathbb{Q}[U_3(3) \times \mathbb{Z}/3\mathbb{Z}] = M_7(\mathbb{Q}(\alpha)) (= \text{End}(A) \otimes \mathbb{Q})$$

in the basis γ , ($H_1(S, \mathbb{Q}[\alpha])$ is the $\mathbb{Q}[\alpha]$ -vector space generated by e_1, \dots, e_k) and therefore there exists a morphism $p : S \rightarrow E^4 = B$ (where $E = \mathbb{C}/\mathbb{Z}[\alpha]$) such that $W = p^* H^0(B, \Omega_B)$. By hypothesis the image $p(S)$ generates B . By construction

$$\wedge^2 p^* H^0(B, \Omega_B) \cap \text{Ker}(\phi^{2,0})$$

is at least one dimensional since it contains w , and therefore S is Lagrangian for p .

A contrario, the trace of an order 2 automorphism $\sigma \in \text{Aut}(S) \subset \text{Aut}(A)$ acting on the tangent space of A at 0 equals to -1 , therefore the image B' of the endomorphism $p : \sigma - 1_A$, where 1_A is the identity of A is an Abelian fourfold. Using Maple, one computes that

$$\wedge^2 p^* H^0(B, \Omega_B) \cap \text{Ker}(f) = \{0\}.$$

Let $\vartheta : S \rightarrow A$ be the Albanese map of S , and let $q : A \rightarrow A$ be an endomorphism with a 4 dimensional image and a representation in $M_7(\mathbb{Q}) \subset M_7(\mathbb{Q}(\alpha))$ in the basis γ . Since the matrix Q is positive definite, we have

$$\wedge^2 p^* H^0(B, \Omega_B) \cap \tilde{Q} = \{0\},$$

where p is the map $p = q \circ \vartheta : S \rightarrow B$. Therefore S is not Lagrangian with respect to p . \square

Remark 7. Let X be a surface and let $\phi^{2,0} : \wedge^2 H^0(X, \Omega_X) \rightarrow H^0(X, K_X)$ be the natural map. Let be $d = \dim \text{Ker}(\phi^{2,0})$ and $q = \dim H^0(X, \Omega_X)$. In the proof of Theorem 6, we saw that the set of rank k vectors in $\text{Ker}(\phi^{2,0})$ is a determinantal variety: the intersection of minors of size $\geq 2k + 1$ of some anti-symmetric matrix of size $q \times q$ with linear entries in d variables. It seems to the authors quite remarkable that for Stover's surface the set of rank 2 vectors (obtained as the zero set of 49 6×6 minors of a size $q = 7$ matrix) is an hypersurface in $\text{Ker}(\phi^{2,0})$. That hypersurface is the only $U_3(3)$ -invariant quadric of $U_3(3)$ acting on $\text{Ker}(\phi^{2,0})$.

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