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# ON THE COHOMOLOGY OF CALOGERO-MOSER SPACES

by

CÉDRIC BONNAFÉ & PENG SHAN

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*Abstract.* — We compute the equivariant cohomology of smooth Calogero-Moser spaces and some associated symplectic resolutions of symplectic quotient singularities.

## 1. Notation and main results

All along this note, we will abbreviate  $\otimes_{\mathbb{C}}$  as  $\otimes$ . By an algebraic variety, we mean a reduced scheme of finite type over  $\mathbb{C}$ .

**1.A. Reflection group.** — Let  $V$  be a  $\mathbb{C}$ -vector space of finite dimension  $n$  and let  $W$  be a finite subgroup of  $\mathbf{GL}_{\mathbb{C}}(V)$ . We set

$$\mathrm{Ref}(W) = \{s \in W \mid \dim_{\mathbb{C}} V^s = n - 1\}$$

and we assume that

$$W = \langle \mathrm{Ref}(W) \rangle.$$

We set  $\varepsilon : W \rightarrow \mathbb{C}^{\times}$ ,  $w \mapsto \det(w)$ .

If  $s \in \mathrm{Ref}(W)$ , we denote by  $\alpha_s^{\vee}$  and  $\alpha_s$  two elements of  $V$  and  $V^*$  respectively such that  $V^s = \mathrm{Ker}(\alpha_s)$  and  $V^{*s} = \mathrm{Ker}(\alpha_s^{\vee})$ , where  $\alpha_s^{\vee}$  is viewed as a linear form on  $V^*$ .

If  $w \in W$ , we set

$$\mathrm{cod}(w) = \mathrm{codim}_{\mathbb{C}}(V^w)$$

and we define a filtration  $\mathcal{F}_{\bullet}(\mathbb{C}W)$  of the group algebra of  $W$  as follows: let

$$\mathcal{F}_i(\mathbb{C}W) = \bigoplus_{\mathrm{cod}(w) \leq i} \mathbb{C}w.$$

Then

$$\mathbb{C}\mathrm{Id}_V = \mathcal{F}_0(\mathbb{C}W) \subset \mathcal{F}_1(\mathbb{C}W) \subset \cdots \subset \mathcal{F}_n(\mathbb{C}W) = \mathbb{C}W = \mathcal{F}_{n+1}(\mathbb{C}W) = \cdots$$

is a filtration of  $\mathbb{C}W$ . For any subalgebra  $A$  of  $\mathbb{C}W$ , we set  $\mathcal{F}_i(A) = A \cap \mathcal{F}_i(\mathbb{C}W)$ , so that

$$\mathbb{C}\mathrm{Id}_V = \mathcal{C}\mathcal{F}_0(A) \subset \mathcal{F}_1(A) \subset \cdots \subset \mathcal{F}_n(A) = A = \mathcal{F}_{n+1}(A) = \cdots$$

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is also a filtration of  $A$ . Let  $\hbar$  be a formal variable, and write

$$\begin{aligned} \text{Rees}_{\mathcal{F}}^{\bullet}(A) &= \bigoplus_{i \geq 0} \hbar^i \mathcal{F}_i(A) \subset \mathbb{C}[\hbar] \otimes A \quad (\text{the Rees algebra}), \\ \text{gr}_{\mathcal{F}}^{\bullet}(A) &= \bigoplus_{i \geq 0} \mathcal{F}_i(A) / \mathcal{F}_{i-1}(A). \end{aligned}$$

Recall that  $\text{gr}_{\mathcal{F}}^{\bullet}(A) \simeq \text{Rees}_{\mathcal{F}}^{\bullet}(A) / \hbar \text{Rees}_{\mathcal{F}}^{\bullet}(A)$ .

**1.B. Rational Cherednik algebra at  $t = 0$ .** — All along this note, we fix a function  $c : \text{Ref}(W) \rightarrow \mathbb{C}$  which is invariant under conjugacy. We define the rational Cherednik algebra  $\mathbf{H}_c$  to be the quotient of the algebra  $\text{T}(V \oplus V^*) \rtimes W$  (the semi-direct product of the tensor algebra  $\text{T}(V \oplus V^*)$  with the group  $W$ ) by the relations

$$(\mathcal{H}_c) \quad \begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \sum_{s \in \text{Ref}(W)} (\varepsilon(s) - 1) c_s \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s, \end{cases}$$

for all  $x, x' \in V^*$ ,  $y, y' \in V$ . The first commutation relations imply that we have morphisms of algebras  $\mathbb{C}[V] \rightarrow \mathbf{H}_c$  and  $\mathbb{C}[V^*] \rightarrow \mathbf{H}_c$ . Recall [4, Theorem 1.3] that we have an isomorphism of  $\mathbb{C}$ -vector spaces

$$(1.1) \quad \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_c$$

induced by multiplication (this is the so-called *PBW-decomposition*).

We denote by  $\mathbf{Z}_c$  the center of  $\mathbf{H}_c$ : it is well-known [4] that  $\mathbf{Z}_c$  is an integral domain, which is integrally closed and contains  $\mathbb{C}[V]^W$  and  $\mathbb{C}[V^*]^W$  as subalgebras (so it contains  $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ ), and that it is a free  $\mathbf{P}$ -module of rank  $|W|$ . We denote by  $\mathcal{Z}_c$  the algebraic variety whose ring of regular functions  $\mathbb{C}[\mathcal{Z}_c]$  is  $\mathbf{Z}_c$ : this is the *Calogero-Moser space* associated with the datum  $(V, W, c)$ .

Using the PBW-decomposition, we define a  $\mathbb{C}$ -linear map  $\Omega_{\mathbf{H}}^c : \mathbf{H}_c \rightarrow \mathbb{C}W$  by

$$\Omega_{\mathbf{H}}^c(fwg) = f(0)g(0)w$$

for all  $f \in \mathbb{C}[V]$ ,  $g \in \mathbb{C}[V^*]$  and  $w \in \mathbb{C}W$ . This map is  $W$ -equivariant for the action on both sides by conjugacy, so it induces a well-defined  $\mathbb{C}$ -linear map

$$\Omega^c : \mathbf{Z}_c \rightarrow Z(\mathbb{C}W).$$

Recall from [3, Corollary 4.2.11] that  $\Omega^c$  is a morphism of algebras, and that

$$(1.2) \quad \mathcal{Z}_c \text{ is smooth if and only if } \Omega^c \text{ is surjective.}$$

The “only if” part is essentially due to Gordon [6, Corollary 5.8] (see also [3, Proposition 9.6.6 and (16.1.2)]) while the “if” part follows from the work of Bellamy, Schedler and Thiel [13, Corollary 1.4].

**1.C. Grading.** — The algebra  $\text{T}(V \oplus V^*) \rtimes W$  can be  $\mathbb{Z}$ -graded in such a way that the generators have the following degrees

$$\begin{cases} \deg(y) = -1 & \text{if } y \in V, \\ \deg(x) = 1 & \text{if } x \in V^*, \\ \deg(w) = 0 & \text{if } w \in W. \end{cases}$$

This descends to a  $\mathbb{Z}$ -grading on  $\mathbf{H}_c$  since the defining relations  $(\mathcal{H}_c)$  are homogeneous. Since the center of a graded algebra is always graded, it follows that  $\mathbf{Z}_c$  also inherits a

$\mathbb{Z}$ -grading. By definition  $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$  is clearly a graded subalgebra of  $\mathbf{Z}_c$ . So the Calogero-Moser space  $\mathcal{Z}_c$  inherits a regular  $\mathbb{C}^\times$ -action.

**1.D. Main results.** — For a complex algebraic variety  $\mathcal{X}$  (equipped with its classical topology), we denote by  $H^i(\mathcal{X})$  its  $i$ -th singular cohomology group with coefficients in  $\mathbb{C}$ . If  $\mathcal{X}$  carries a regular action of a torus  $\mathbf{T}$ , we denote by  $H_{\mathbf{T}}^i(\mathcal{X})$  its  $i$ -th  $\mathbf{T}$ -equivariant cohomology group (still with coefficients in  $\mathbb{C}$ ). Note that  $H^{2\bullet}(\mathcal{X}) = \bigoplus_{i \geq 0} H^{2i}(\mathcal{X})$  is a graded  $\mathbb{C}$ -algebra and  $H_{\mathbf{T}}^{2\bullet}(\mathcal{X}) = \bigoplus_{i \geq 0} H_{\mathbf{T}}^{2i}(\mathcal{X})$  is a graded  $H_{\mathbf{T}}^{2\bullet}(\mathbf{pt})$ -algebra. The following result [4, Theorem 1.8] describes the algebra structure on the cohomology of  $\mathcal{Z}_c$  (with coefficients in  $\mathbb{C}$ ):

**Theorem 1.3 (Etingof-Ginzburg).** — *Assume that  $\mathcal{Z}_c$  is smooth. Then:*

- (a)  $H^{2i+1}(\mathcal{Z}_c) = 0$  for all  $i$ .
- (b) *There is an isomorphism of graded  $\mathbb{C}$ -algebras  $H^{2\bullet}(\mathcal{Z}_c) \simeq \mathfrak{gr}_{\mathcal{F}}^{\bullet}(Z(\mathbb{C}W))$ .*

In this note, we prove an equivariant version of this statement (we identify  $R = H_{\mathbb{C}^\times}^*(\mathbf{pt})$  with  $\mathbb{C}[\hbar]$  in the usual way):

**Theorem A.** — *Assume that  $\mathcal{Z}_c$  is smooth. Then:*

- (a)  $H_{\mathbb{C}^\times}^{2i+1}(\mathcal{Z}_c) = 0$  for all  $i$ .
- (b) *There is an isomorphism of graded  $R$ -algebras  $H_{\mathbb{C}^\times}^{2\bullet}(\mathcal{Z}_c) \simeq \text{Rees}_{\mathcal{F}}^{\bullet}(Z(\mathbb{C}W))$ .*

Note that Theorem A(a) just follows from statement (a) of Etingof-Ginzburg Theorem by Proposition 2.4(a) below. As a partial consequence of Theorem A, we also obtain the following application to the equivariant cohomology of some symplectic resolutions.

**Theorem B.** — *Assume that the symplectic quotient singularity  $(V \times V^*)/W$  admits a symplectic resolution  $\pi : \mathcal{X} \rightarrow (V \times V^*)/W$ . Recall that the  $\mathbb{C}^\times$ -action on  $(V \times V^*)/W$  lifts (uniquely) to  $\mathcal{X}$  (see [11, Theorem 1.3(ii)]). Then:*

- (a)  $H_{\mathbb{C}^\times}^{2i+1}(\mathcal{X}) = 0$  for all  $i$ .
- (b) *There is an isomorphism of graded  $R$ -algebras  $H_{\mathbb{C}^\times}^{2\bullet}(\mathcal{X}) \simeq \text{Rees}_{\mathcal{F}}^{\bullet}(Z(\mathbb{C}W))$ .*

Note that for  $W = \mathfrak{S}_n$  acting on  $\mathbb{C}^n$ , Theorem B describes the equivariant cohomology of the Hilbert scheme of  $n$  points in  $\mathbb{C}^2$ : this was already proved (by different methods) by Vasserot [14]. In [5, Conjecture 1.3], Ginzburg-Kaledin proposed a conjecture for the equivariant cohomology of a symplectic resolution of a symplectic quotient singularity  $E/G$ , where  $E$  is a finite dimensional symplectic vector space and  $G$  is a finite subgroup of  $\mathbf{Sp}(E)$ . However, their conjecture cannot hold as stated, because they considered the  $\mathbb{C}^\times$ -action by dilatation, which is contractible. Theorem B shows that the correct equivariant cohomological realization of the Rees algebra is provided by the symplectic  $\mathbb{C}^\times$ -action, which exists only when  $G$  stabilizes a Lagrangian subspace of  $E$ .

**Remark 1.4.** — Recall from the works of Etingof-Ginzburg [4], Ginzburg-Kaledin [5], Gordon [6] and Bellamy [1] that the existence of a symplectic resolution of  $(V \times V^*)/W$  is equivalent to the existence of a parameter  $c$  such that  $\mathcal{Z}_c$  is smooth, and that it can only occur if all the irreducible components of  $W$  are of type  $G(d, 1, n)$  (for some  $d, n \geq 1$ ) or  $G_4$  in Shephard-Todd classification. ■

**1.E. Conjectures.** — In [3, Chapter 16, Conjectures COH and ECOH], Rouquier and the first author proposed the following conjecture which aims to generalize the above Etingof-Ginzburg Theorem 1.3 into two directions: it includes singular Calogero-Moser spaces and it extends to equivariant cohomology.

**Conjecture 1.5.** — *With the above notation, we have:*

- (1)  $H^{2i+1}(\mathcal{X}_c) = 0$  for all  $i$ .
- (2) We have an isomorphism of graded  $\mathbb{C}$ -algebras  $H^{2\bullet}(\mathcal{X}_c) \simeq \text{gr}_{\mathcal{F}}(\text{Im}(\Omega^c))$ .
- (3) We have an isomorphism of graded  $R$ -algebras  $H_{\mathbb{C}^\times}^{2\bullet}(\mathcal{X}_c) \simeq \text{Rees}_{\mathcal{F}}(\text{Im}(\Omega^c))$ .

By (1.2), when  $\mathcal{X}_c$  is smooth, the image of  $\Omega^c$  coincide with the center of  $\mathbb{C}W$ . So Theorem A proves this conjecture for smooth  $\mathcal{X}_c$ .

We will also prove in Example 3.6 the following result:

**Proposition 1.6.** — *If  $\dim_{\mathbb{C}}(V) = 1$ , then Conjecture 1.5 holds.*

**1.F. Structure of the paper.** — The paper is organized as follows. The proof of Theorem A relies on classical theorems about restriction to fixed points. In the next Section 2, we first recall basic properties on equivariant cohomology and equivariant K-theory and restriction to fixed points. Section 3 explains how these general principles can be applied to Calogero-Moser spaces. Theorem A will be proved in Section 4. The cyclic group case (Proposition 1.6) will be handled in Example 3.6. The proof of Theorem B will be given in Section 5.

## 2. Equivariant cohomology, K-theory and fixed points

**2.A. Equivariant cohomology.** — Let  $\mathcal{X}$  be a complex algebraic variety equipped with a regular action of a torus  $\mathbf{T}$ . Recall that the *equivariant cohomology* of  $\mathcal{X}$  is defined by

$$H_{\mathbf{T}}^{\bullet}(\mathcal{X}) = H^{\bullet}(E_{\mathbf{T}} \times_{B_{\mathbf{T}}} X),$$

where  $E_{\mathbf{T}} \rightarrow B_{\mathbf{T}}$  is a universal  $\mathbf{T}$ -bundle. The pullback of the structural morphism  $x : \mathcal{X} \rightarrow \mathbf{pt}$  yields a ring homomorphism  $H_{\mathbf{T}}^{\bullet}(\mathbf{pt}) \rightarrow H_{\mathbf{T}}^{\bullet}(\mathcal{X})$ , which makes  $H_{\mathbf{T}}^{\bullet}(\mathcal{X})$  a graded  $H_{\mathbf{T}}^{\bullet}(\mathbf{pt})$ -algebra.

Denote by  $X(\mathbf{T})$  the character lattice of  $\mathbf{T}$ . For each  $\chi \in X(\mathbf{T})$ , denote by  $\mathbb{C}_{\chi}$  the one dimensional  $\mathbf{T}$ -module of character  $\chi$ , the first Chern class  $c_{\chi}$  of the line bundle  $E_{\mathbf{T}} \times_{\mathbf{T}} \mathbb{C}_{\chi}$  on  $B_{\mathbf{T}}$  is an element in  $H^2(B_{\mathbf{T}})$ . Identify the vector space  $\mathbb{C} \otimes_{\mathbb{Z}} X(\mathbf{T})$  with the dual  $\mathfrak{t}^*$  of the Lie algebra  $\mathfrak{t}$  of  $\mathbf{T}$  via  $\chi \mapsto d\chi$ . Then the assignment  $\chi \mapsto c_{\chi}$  yields an isomorphism of graded  $\mathbb{C}$ -algebras  $S(\mathfrak{t}^*) = H_{\mathbf{T}}^{2\bullet}(\mathbf{pt})$ .

**2.B. Equivariant K-theory.** — We denote by  $K_{\mathbf{T}}(\mathcal{X})$  the Grothendieck ring of the category of  $\mathbf{T}$ -equivariant vector bundles on  $\mathcal{X}$ . Note that when  $\mathcal{X}$  is a point, a  $\mathbf{T}$ -equivariant vector bundles is the same as a finite dimensional  $\mathbf{T}$ -module. We have a canonical isomorphism  $K_{\mathbf{T}}(\mathbf{pt}) = \mathbb{Z}[X(\mathbf{T})]$  which sends the class of a  $\mathbf{T}$ -module  $M$  to

$$(2.1) \quad \dim^{\mathbf{T}}(M) = \sum_{\chi \in X(\mathbf{T})} \dim_{\mathbb{C}}(M_{\chi})\chi,$$

where  $M_{\chi}$  is the  $\chi$ -weight space in  $M$ .

Let  $\hat{H}_{\mathbf{T}}^{2\bullet}(\mathcal{X})$  be the completion of  $H_{\mathbf{T}}^{2\bullet}(\mathcal{X})$  with respect to the ideal  $\bigoplus_{i>0} H_{\mathbf{T}}^{2i}(\mathcal{X})$ . The equivariant Chern character provides a ring homomorphism

$$\mathrm{ch}_{\mathcal{X}} : K_{\mathbf{T}}(\mathcal{X}) \longrightarrow \hat{H}_{\mathbf{T}}^{2\bullet}(\mathcal{X})$$

with the following properties. First, when  $\mathcal{X}$  is a point  $\mathbf{pt}$ , we have

$$(2.2) \quad \begin{array}{ccc} \mathrm{ch}_{\mathbf{pt}} : K_{\mathbf{T}}(\mathbf{pt}) = \mathbb{Z}[X(\mathbf{T})] & \longrightarrow & \hat{H}_{\mathbf{T}}^{2\bullet}(\mathbf{pt}) = \hat{S}(\mathfrak{t}^*) \\ \chi & \longmapsto & \exp(d\chi). \end{array}$$

Next the Chern character commutes with pullback. More precisely, if  $\mathcal{Y}$  is another variety with a regular action of the same torus  $\mathbf{T}$  and if  $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$  is a  $\mathbf{T}$ -equivariant morphism, then the following diagram commutes

$$(2.3) \quad \begin{array}{ccc} K_{\mathbf{T}}(\mathcal{Y}) & \xrightarrow{\varphi^*} & K_{\mathbf{T}}(\mathcal{X}) \\ \mathrm{ch}_{\mathcal{Y}} \downarrow & & \downarrow \mathrm{ch}_{\mathcal{X}} \\ \hat{H}_{\mathbf{T}}^{2\bullet}(\mathcal{Y}) & \xrightarrow{\hat{\varphi}^*} & \hat{H}_{\mathbf{T}}^{2\bullet}(\mathcal{X}). \end{array}$$

Here  $\varphi^*$  denotes both the pullback map in K-theory or in equivariant cohomology, and  $\hat{\varphi}^*$  is the map induced after completion by the pullback map in equivariant cohomology. In particular, by applying the above diagram to  $\mathcal{Y} = \mathbf{pt}$  and the structural morphism  $\mathcal{X} \rightarrow \mathbf{pt}$ , we may view  $\mathrm{ch}_{\mathcal{X}}$  as a morphism of algebras over  $K_{\mathbf{T}}(\mathbf{pt})$ , with the  $K_{\mathbf{T}}(\mathbf{pt})$ -algebra structure on  $\hat{H}_{\mathbf{T}}^{2\bullet}(\mathcal{X})$  provided by the embedding (2.2).

**2.C. Fixed points.** — We denote by  $\mathcal{X}^{\mathbf{T}}$  the (reduced) variety consisting of fixed points of  $\mathbf{T}$  in  $\mathcal{X}$ . Let  $i_{\mathcal{X}} : \mathcal{X}^{\mathbf{T}} \hookrightarrow \mathcal{X}$  be the natural closed immersion. Since  $\mathbf{T}$  acts trivially on  $\mathcal{X}^{\mathbf{T}}$ , we have  $H_{\mathbf{T}}^{\bullet}(\mathcal{X}^{\mathbf{T}}) = H_{\mathbf{T}}^{\bullet}(\mathbf{pt}) \otimes H^{\bullet}(\mathcal{X}^{\mathbf{T}})$  as  $H_{\mathbf{T}}^{\bullet}(\mathbf{pt})$ -algebras. Recall that the  $\mathbf{T}$ -action on  $\mathcal{X}$  is called *equivariantly formal* if the Leray-Serre spectral sequence

$$E_2^{p,q} = H^p(B_{\mathbf{T}}; H^q(X)) \implies H_{\mathbf{T}}^{p+q}(X)$$

for the fibration  $E_{\mathbf{T}} \times_{\mathbf{T}} X \rightarrow B_{\mathbf{T}}$  degenerates at  $E_2$ . We have the following standard result on equivariant cohomology (see for instance [9, Proposition 2.1]):

**Proposition 2.4.** — Assume that  $H^{2i+1}(\mathcal{X}) = 0$  for all  $i$ . Then  $\mathcal{X}$  is equivariantly formal, and:

- (a) There is an isomorphism of graded  $H_{\mathbf{T}}^{\bullet}(\mathbf{pt})$ -modules  $H_{\mathbf{T}}^{\bullet}(\mathcal{X}) \simeq H_{\mathbf{T}}^{\bullet}(\mathbf{pt}) \otimes H^{\bullet}(\mathcal{X})$ . In particular  $H_{\mathbf{T}}^{2i+1}(\mathcal{X}) = 0$  for all  $i$ .
- (b) The pullback map  $i_{\mathcal{X}}^* : H_{\mathbf{T}}^{\bullet}(\mathcal{X}) \rightarrow H_{\mathbf{T}}^{\bullet}(\mathcal{X}^{\mathbf{T}})$  is injective.
- (c) Let  $\mathfrak{m}$  be the unique graded maximal ideal of  $H_{\mathbf{T}}^{\bullet}(\mathbf{pt})$ . Then we have an isomorphism of algebras  $H^{\bullet}(\mathcal{X}) \simeq H_{\mathbf{T}}^{\bullet}(\mathcal{X}) / \mathfrak{m}H_{\mathbf{T}}^{\bullet}(\mathcal{X})$ .

In particular, this shows that Conjectures 1.5(1) and (3) imply Conjecture 1.5(2).

**Example 2.5 (Blowing-up).** — Let  $\mathcal{Y}$  be an affine variety with a  $\mathbf{T}$ -action and let  $\mathcal{C}$  be a  $\mathbf{T}$ -stable closed subvariety (not necessarily reduced) of  $\mathcal{Y}$ . Let  $\mathcal{X}$  be the blowing-up of  $\mathcal{Y}$  along  $\mathcal{C}$ . Write  $\pi : \mathcal{X} \rightarrow \mathcal{Y}$  for the natural morphism, and we equip  $\mathcal{X}$  with the unique  $\mathbf{T}$ -action such that  $\pi$  is equivariant. *We assume that  $\mathcal{X}^{\mathbf{T}}$  is finite.* We write

$$H_{\mathbf{T}}^{2\bullet}(\mathcal{X}^{\mathbf{T}}) = \bigoplus_{x \in \mathcal{X}^{\mathbf{T}}} S(t^*)\mathbf{e}_x,$$

where  $\mathbf{e}_x \in H_{\mathbf{T}}^0(\mathcal{X}^{\mathbf{T}})$  is the primitive idempotent associated with  $x$  (i.e., the fundamental class of  $x$ ).

Then  $\mathcal{D} = \pi^*(\mathcal{C})$  is a  $\mathbf{T}$ -stable effective Cartier divisor, and we denote by  $[\mathcal{D}]$  the class in  $K_{\mathbf{T}}(\mathcal{X})$  of its associated line bundle (which is  $\mathbf{T}$ -equivariant). We denote by  $\text{ch}_{\mathcal{X}}^1([\mathcal{D}]) \in H_{\mathbf{T}}^2(\mathcal{X})$  its first  $\mathbf{T}$ -equivariant Chern class. We want to compute  $i_{\mathcal{X}}^*(\text{ch}_{\mathbf{T}}^1(\mathcal{D}))$ .

First, let  $I$  be the ideal of  $\mathbb{C}[\mathcal{Y}]$  associated with  $\mathcal{C}$ . As it is  $\mathbf{T}$ -stable, we can find a family of  $\mathbf{T}$ -homogeneous generators  $(a_i)_{1 \leq i \leq k}$  of  $I$ . We denote by  $\lambda_i \in X(\mathbf{T})$  the  $\mathbf{T}$ -weight of  $a_i$ . The choice of this family of generators induces a  $\mathbf{T}$ -equivariant closed immersion  $\mathcal{X} \hookrightarrow \mathcal{Y} \times \mathbb{P}^{k-1}(\mathbb{C})$ . We denote by  $\mathcal{X}_i$  the affine chart corresponding to “ $a_i \neq 0$ ”. If  $x \in \mathcal{X}^{\mathbf{T}}$ , we denote by  $\mathbf{i}(x) \in \{1, 2, \dots, k\}$  an element such that  $x \in \mathcal{X}_{\mathbf{i}(x)}$ . Then

$$(2.6) \quad i_{\mathcal{X}}^*(\text{ch}_{\mathbf{T}}^1(\mathcal{D})) = -\hbar \sum_{x \in \mathcal{D}^{\mathbf{T}}} (d\lambda_{\mathbf{i}(x)})\mathbf{e}_x.$$

Indeed, we just need to compute the local equation of  $\mathcal{D}$  around  $x \in \mathcal{X}^{\mathbf{T}}$ , and this can be done in  $\mathcal{X}_{\mathbf{i}(x)}$ . But  $\mathcal{D} \cap \mathcal{X}_i$  is principal for all  $i$ , defined by

$$\mathcal{D} \cap \mathcal{X}_i = \{(y, \xi) \in \mathcal{Y} \times \mathbb{P}^{k-1}(\mathbb{C}) \mid (y, \xi) \in \mathcal{X}_i \text{ and } a_i(y) = 0\}.$$

So  $\mathcal{D} \cap \mathcal{X}_i$  is defined by a  $\mathbf{T}$ -homogeneous equation of degree  $\lambda_i \in X(\mathbf{T})$ , and so (2.6) follows. ■

### 3. Localization and Calogero-Moser spaces

In this section, we apply the previous discussions to  $\mathcal{X} = \mathcal{X}_c$  and  $\mathbf{T} = \mathbb{C}^\times$ . Denote by  $q : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$  the identity map. Then  $X(\mathbb{C}^\times) = q^{\mathbb{Z}}$ , and we have

$$K_{\mathbb{C}^\times}(\mathbf{pt}) = \mathbb{Z}[q, q^{-1}], \quad H_{\mathbb{C}^\times}^{2\bullet}(\mathbf{pt}) = \mathbb{C}[\hbar],$$

with  $\hbar = c_q$ , following the notation of Section 2.A. So  $\hat{H}_{\mathbb{C}^\times}^{2\bullet}(\mathbf{pt}) = \mathbb{C}[[\hbar]]$  and the Chern map in this case is given by

$$\text{ch}_{\mathbf{pt}} : \mathbb{Z}[q, q^{-1}] \hookrightarrow \mathbb{C}[[\hbar]], \quad q \mapsto \exp(\hbar).$$

Note also that a finite  $\mathbb{C}^\times$ -module is nothing but a finite dimensional  $\mathbb{Z}$ -graded vector space  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  such that  $\mathbb{C}^\times$  acts on  $M_i$  by the character  $q^i$ . The identification  $K_{\mathbb{C}^\times}(\mathbf{pt}) = \mathbb{Z}[q, q^{-1}]$  sends the class of  $M$  to its *graded dimension* (or *Hilbert series*):

$$\dim^{\text{gr}}(M) = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{C}}(M_i)q^i \in \mathbb{N}[q, q^{-1}].$$

**3.A. Fixed points.** — For  $\chi \in \text{Irr}(W)$ , we denote by  $\omega_\chi : Z(CW) \rightarrow \mathbb{C}$  the associated morphism of algebras, i.e.  $\omega_\chi(z) = \chi(z)/\chi(1)$  is the scalar through which  $z$  acts on the irreducible  $CW$ -module with the character  $\chi$ . We denote by  $e_\chi^W$  the unique primitive idempotent of  $Z(CW)$  such that  $\omega_\chi(e_\chi^W) = 1$ . If  $\mathcal{E}$  is a subset of  $\text{Irr}(W)$ , then we set  $e_\mathcal{E}^W = \sum_{\chi \in \mathcal{E}} e_\chi^W$ .

Now, consider the algebra homomorphism

$$\Omega_\chi^c = \omega_\chi \circ \Omega^c : \mathbf{Z}_c \longrightarrow \mathbb{C}.$$

Its kernel is a maximal ideal of  $\mathbf{Z}_c$ , denote by  $z_\chi$  the corresponding closed point in  $\mathcal{Z}_c$ . It follows from [3, Lemma 10.2.3 and (14.2.2)] that  $z_\chi \in \mathcal{Z}_c^{\text{C}^\times}$  and that the map

$$(3.1) \quad \begin{array}{ccc} z : \text{Irr}(W) & \longrightarrow & \mathcal{Z}_c^{\text{C}^\times} \\ \chi & \longmapsto & z_\chi \end{array}$$

is surjective. The fibers of this map are called the *Calogero-Moser  $c$ -families*. They were first considered by Gordon [6] and Gordon-Martino [8]: see also for instance [3, §9.2]. Let  $\text{CM}_c(W)$  be the set of Calogero-Moser  $c$ -families. For  $\mathcal{E} \in \text{CM}_c(W)$ , we denote by  $z_\mathcal{E} \in \mathcal{Z}_c^{\text{C}^\times}$  its image under the map  $z$ . On the other hand, by [3, (16.1.2)] we have

$$(3.2) \quad \text{Im}(\Omega^c) = \bigoplus_{\mathcal{E} \in \text{CM}_c(W)} \mathbb{C} e_\mathcal{E}^W.$$

Hence we get an isomorphism of  $\mathbb{C}$ -algebras

$$H^{2\bullet}(\mathcal{Z}_c^{\text{C}^\times}) \simeq \text{Im}(\Omega^c), \quad e_{z_\mathcal{E}} \mapsto e_\mathcal{E}^W,$$

which extends to an isomorphism of  $\mathbb{C}[\hbar]$ -algebras  $H^{2\bullet}(\mathcal{Z}_c^{\text{C}^\times}) \simeq \mathbb{C}[\hbar] \otimes \text{Im}(\Omega^c)$ . For simplification, we set  $i_c = i_{\mathcal{Z}_c} : \mathcal{Z}_c^{\text{C}^\times} \hookrightarrow \mathcal{Z}$  and, under the above identification, we view the pullback map  $i_c^*$  as a morphism of algebras

$$i_c^* : H_{\mathbb{T}}^{2\bullet}(\mathcal{Z}_c) \longrightarrow \mathbb{C}[\hbar] \otimes \text{Im}(\Omega^c).$$

So, by Proposition 2.4, Conjecture 1.5 is implied by the following one:

**Conjecture 3.3.** — *With the above notation, we have:*

- (1)  $H^{2i+1}(\mathcal{Z}_c) = 0$  for all  $i$ .
- (2)  $\text{Im}(i_c^*) = \text{Rees}_{\mathcal{F}}(\text{Im}(\Omega^c))$ .

**Remark 3.4.** — Set

$$\mathcal{F}_i^{\text{H}}(\text{Im}(\Omega^c)) = \{z \in \text{Im}(\Omega^c) \mid \hbar^i z \in \text{Im}(i_c^*)\}.$$

Then, by construction and Proposition 2.4,  $\mathcal{F}_\bullet^{\text{H}}(\text{Im}(\Omega^c))$  is the filtration of  $\text{Im}(\Omega^c)$  satisfying

$$H_{\mathbb{C}^\times}^{2\bullet}(\mathcal{Z}_c) \simeq \text{Im}(i_c^*) = \text{Rees}_{\mathcal{F}^{\text{H}}}(\text{Im}(\Omega^c))$$

and

$$H^{2\bullet}(\mathcal{Z}_c) \simeq \text{gr}_{\mathcal{F}^{\text{H}}}(\text{Im}(\Omega^c)).$$

So showing Conjecture 3.3 is then equivalent to showing that the filtrations  $\mathcal{F}_\bullet(\text{Im}(\Omega^c))$  and  $\mathcal{F}_\bullet^{\text{H}}(\text{Im}(\Omega^c))$  coincide.

Note also that, since  $\mathcal{Z}_c$  is an affine variety of dimension  $2n$ , we have  $H^i(\mathcal{Z}_c) = 0$  for  $i > 2n$ , so this shows that

$$(b) \quad \mathcal{F}_n(\text{Im}(\Omega^c)) = \text{Im}(\Omega^c) = \mathcal{F}_n^{\text{H}}(\text{Im}(\Omega^c)).$$



On the other hand, since  $\mathcal{X}_c$  is connected, we have

$$(\sharp) \quad \mathcal{F}_0(\mathrm{Im}(\Omega^c)) = \mathcal{F}_0^{\mathrm{H}}(\mathrm{Im}(\Omega^c)) = \mathbb{C}.$$

These two particular cases will be used below. ■

The following result, based on the previous remark, can be viewed as a reduction step for the proof of Conjecture 3.3:

**Proposition 3.5.** — Assume that  $H^{2i+1}(\mathcal{X}_c) = 0$  and

$$\dim_{\mathbb{C}}(H^{2i}(\mathcal{X}_c)) = \dim_{\mathbb{C}}(\mathcal{F}_i(\mathrm{Im}(\Omega^c))/\mathcal{F}_{i-1}(\mathrm{Im}(\Omega^c)))$$

for all  $i$ . Then Conjecture 3.3 holds if and only if  $\mathrm{Rees}_{\mathcal{F}}(\mathrm{Im}(\Omega^c)) \subset \mathrm{Im}(i_c^*)$ .

*Proof.* — Assume that  $H^{2i+1}(\mathcal{X}_c) = 0$  and

$$\dim_{\mathbb{C}}(H^{2i}(\mathcal{X}_c)) = \dim_{\mathbb{C}}(\mathcal{F}_i(\mathrm{Im}(\Omega^c))/\mathcal{F}_{i-1}(\mathrm{Im}(\Omega^c)))$$

for all  $i$ . We keep the notation of Remark 3.4. It then follows from this remark and the hypothesis that

$$\dim_{\mathbb{C}}(\mathcal{F}_i(\mathrm{Im}(\Omega^c))/\mathcal{F}_{i-1}(\mathrm{Im}(\Omega^c))) = \dim_{\mathbb{C}}(H^{2i}(\mathcal{X}_c)) = \dim(\mathcal{F}_i^{\mathrm{H}}(\mathrm{Im}(\Omega^c))/\mathcal{F}_{i-1}^{\mathrm{H}}(\mathrm{Im}(\Omega^c)))$$

for all  $i$ . So, by induction, we get that  $\dim_{\mathbb{C}}(\mathcal{F}_i(\mathrm{Im}(\Omega^c))) = \dim(\mathcal{F}_i^{\mathrm{H}}(\mathrm{Im}(\Omega^c)))$  for all  $i$  (by the equality  $(\sharp)$  of Remark 3.4). This shows that  $\mathrm{Rees}_{\mathcal{F}}(\mathrm{Im}(\Omega^c)) = \mathrm{Im}(i_c^*)$  if and only if  $\mathrm{Rees}_{\mathcal{F}}(\mathrm{Im}(\Omega^c)) \subset \mathrm{Im}(i_c^*)$ , as desired. □

**Example 3.6.** — Assume in this example, and only in this example, that  $\dim_{\mathbb{C}}(V) = 1$ . It is proved in [3, Theorem 18.5.8] that in this case  $H^{2i+1}(\mathcal{X}_c) = 0$  and

$$\dim_{\mathbb{C}}(H^{2i}(\mathcal{X}_c)) = \dim_{\mathbb{C}}(\mathcal{F}_i(\mathrm{Im}(\Omega^c))/\mathcal{F}_{i-1}(\mathrm{Im}(\Omega^c)))$$

for all  $i$ . Since  $\mathcal{X}_c$  is affine of dimension 2, we have  $H^i(\mathcal{X}_c) = 0$  if  $i \notin \{0, 2\}$ . So it follows from the equalities (b) and  $(\sharp)$  that Conjecture 3.3 holds in this case (this proves Proposition 1.6). ■

**3.B. Chern map.** — If  $\mathcal{E}$  is a Calogero-Moser family, we denote by  $\mathfrak{m}_{\mathcal{E}} \subset \mathbf{Z}_c$  the ideal of functions vanishing at  $z_{\mathcal{E}} \in \mathcal{X}_c^{\mathrm{C}^\times}$ . We also set

$$\mathrm{Im}(\Omega^c)_{\mathbb{Z}} = \bigoplus_{\mathcal{E} \in \mathrm{CM}_c(W)} \mathbb{Z}e_{\mathcal{E}}^W.$$

We make the natural identification  $\mathrm{K}_{\mathbb{C}^\times}(\mathcal{X}_c^{\mathrm{C}^\times}) = \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \mathrm{Im}(\Omega^c)_{\mathbb{Z}}$ . Through these identifications, the Chern map  $\mathrm{ch}_{\mathcal{X}_c^{\mathrm{C}^\times}}$  just becomes the natural inclusion  $\mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \mathrm{Im}(\Omega^c)_{\mathbb{Z}} \hookrightarrow \mathbb{C}[[\hbar]] \otimes \mathrm{Im}(\Omega^c)$ . Moreover, if  $P$  is a  $\mathbb{Z}$ -graded finitely generated projective  $\mathbf{Z}_c$ -module, then the commutativity of the diagram (2.3) just says that

$$(3.7) \quad i_c^*(\mathrm{ch}_{\mathcal{X}_c}([P])) = \sum_{\mathcal{E} \in \mathrm{CM}_c(W)} \dim^{\mathrm{gr}}(P/\mathfrak{m}_{\mathcal{E}}P) e_{\mathcal{E}}^W \subset \mathbb{C}[[\hbar]] \otimes \mathrm{Im}(\Omega^c).$$

#### 4. Proof of Theorem A

**Hypothesis and notation.** *We assume in this section, and only in this section, that  $\mathcal{X}_c$  is smooth.*

If  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  and  $N = \bigoplus_{i \in \mathbb{Z}} N_i$  are two finite-dimensional graded  $\mathbb{C}W$ -modules, we set

$$\langle M, N \rangle_W^{\text{gr}} = \sum_{i, j \in \mathbb{Z}} \langle M_i, N_j \rangle_W q^{i+j},$$

where  $\langle E, F \rangle_W = \dim \text{Hom}_{\mathbb{C}W}(E, F)$  for any finite dimensional  $\mathbb{C}W$ -modules  $E$  and  $F$ . We extend this notation to the case where  $M$  or  $N$  is a graded virtual character (i.e. an element of  $\mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z} \text{Irr}(W)$ ). Finally, we denote by  $\mathbb{C}[V^*]^{\text{co}(W)}$  the *coinvariant algebra*, i.e. the quotient of the algebra  $\mathbb{C}[V^*]$  by the ideal generated by the elements  $f \in \mathbb{C}[V^*]^W$  such that  $f(0) = 0$ . Then  $\mathbb{C}[V^*]^{\text{co}(W)}$  is a graded  $\mathbb{C}W$ -module, which is isomorphic to the regular representation  $\mathbb{C}W$  when one forgets the grading (we define similarly  $\mathbb{C}[V]^{\text{co}(W)}$ ).

**4.A. Localization in K-theory.** — Recall from (1.2) that the smoothness of  $\mathcal{X}_c$  implies that  $\text{Im}(\Omega^c) = Z(\mathbb{C}W)$ , so that  $z : \text{Irr}(W) \rightarrow \mathcal{X}_c^{\text{C}^\times}$  is bijective. Let  $e = e_1^W = (1/|W|) \sum_{w \in W} w$ . The smoothness of  $\mathcal{X}_c$  also implies that the functor

$$\begin{aligned} \mathbf{H}_c\text{-mod} &\longrightarrow \mathbf{Z}_c\text{-mod} \\ M &\longmapsto eM = e\mathbf{H}_c \otimes_{\mathbf{H}_c} M \end{aligned}$$

is an equivalence of categories [4, Theorem 1.7]. If  $E$  is a finite dimensional  $\mathbb{Z}$ -graded  $\mathbb{C}W$ -module, the  $\mathbf{H}_c$ -module  $\mathbf{H}_c \otimes_{\mathbb{C}W} E$  is finitely generated,  $\mathbb{Z}$ -graded and projective. Therefore,  $e\mathbf{H}_c \otimes_{\mathbb{C}W} E$  is a finitely generated graded projective  $\mathbf{Z}_c$ -module, which can be viewed as a  $\mathbb{C}^\times$ -equivariant vector bundle on  $\mathcal{X}_c$ . For simplification, we set

$$\text{ch}_c(E) = i_c^*(\text{ch}_{\mathcal{X}_c}(e\mathbf{H}_c \otimes_{\mathbb{C}W} E)) \in \mathbb{C}[[\hbar]] \otimes \text{Im}(\Omega^c).$$

**Proposition 4.1.** — *Assume that  $\mathcal{X}_c$  is smooth. Let  $E$  be a finite dimensional graded  $\mathbb{C}W$ -module. Then*

$$\text{ch}_c(E) = \sum_{\chi \in \text{Irr}(W)} q^{b_\chi} \frac{\langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \otimes E \rangle_W^{\text{gr}}}{\langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \rangle_W^{\text{gr}}} e_\chi^W,$$

where  $b_\chi$  is the valuation of  $\langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \rangle_G^{\text{gr}}$ .

*Proof.* — As the formula is additive, we may, and we will, assume that  $E$  is an irreducible  $\mathbb{C}W$ -module, concentrated in degree 0, and we denote by  $\psi$  its character.

Now, let  $\chi \in \text{Irr}(W)$ : we denote by  $\mathfrak{m}_\chi$  the maximal ideal of  $\mathbf{Z}_c$  corresponding to the fixed point  $z_\chi$ . We set  $\mathfrak{p} = \mathfrak{m}_\chi \cap \mathbf{P}$ : it does not depend on  $\chi$  (it is the maximal ideal of  $\mathbf{P} = \mathbb{C}[V/W \times V^*/W]$  of functions which vanishes at  $(0, 0)$ ; see for instance [3, (14.2.2)]). By (3.7),

$$\text{ch}_c(E) = \sum_{\chi \in \text{Irr}(W)} \dim^{\text{gr}}((e\mathbf{H}_c \otimes_{\mathbb{C}W} E)/\mathfrak{m}_\chi(e\mathbf{H}_c \otimes_{\mathbb{C}W} E)) e_\chi^W.$$

Now, let  $\bar{\mathbf{Z}}_c = \mathbf{Z}_c/\mathfrak{p}\mathbf{Z}_c$  and  $\bar{\mathbf{H}}_c = \mathbf{H}_c/\mathfrak{p}\mathbf{H}_c$ . We set  $\bar{\mathfrak{m}}_\chi = \mathfrak{m}_\chi/\mathfrak{p}\mathbf{Z}_c$ . Then  $\bar{\mathbf{H}}_c$  is a finite dimensional  $\mathbb{C}$ -algebra (called the *restricted rational Cherednik algebra*) and again the bimodule

$e\bar{\mathbf{H}}_c$  induces a Morita equivalence between  $\bar{\mathbf{H}}_c$  and  $\bar{\mathbf{Z}}_c$ . This implies that  $e\bar{\mathbf{H}}_c/\bar{\mathfrak{m}}_\chi\bar{\mathbf{H}}_c = (\bar{\mathbf{Z}}_c/\bar{\mathfrak{m}}_\chi) \otimes_{\bar{\mathbf{Z}}_c} e\bar{\mathbf{H}}_c$  is a simple right  $\bar{\mathbf{H}}_c$ -module (that will be denoted by  $\mathcal{L}_c(\chi)$ ).

Since  $\mathbb{C}W$  is semisimple, the  $\mathbb{C}W$ -module  $E$  is flat, and so

$$\begin{aligned} (e\mathbf{H}_c \otimes_{\mathbb{C}W} E)/\mathfrak{m}_\chi(e\mathbf{H}_c \otimes_{\mathbb{C}W} E) &\simeq (e\mathbf{H}_c/\mathfrak{m}_\chi(e\mathbf{H}_c)) \otimes_{\mathbb{C}W} E \\ &\simeq (e\bar{\mathbf{H}}_c/\bar{\mathfrak{m}}_\chi(e\bar{\mathbf{H}}_c)) \otimes_{\mathbb{C}W} E. \\ &\simeq \mathcal{L}_c(\chi) \otimes_{\mathbb{C}W} E. \end{aligned}$$

But the graded dimension of  $\mathcal{L}_c(\chi) \otimes_{\mathbb{C}W} E$  is known whenever  $\mathcal{Z}_c$  is smooth and is given by the expected formula (see [1, Lemma 3.3 and its proof]).  $\square$

Let  $\mathbb{K}_{\mathbb{C}^\times}(\mathbb{C}W)$  denote the Grothendieck group of the category of finite dimensional graded  $\mathbb{C}W$ -modules. If  $E$  is a finite dimensional graded  $\mathbb{C}W$ -module, we denote by  $[E]$  its class in  $\mathbb{K}_{\mathbb{C}^\times}(\mathbb{C}W)$ . We still denote by  $\text{ch}_c : \mathbb{K}_{\mathbb{C}^\times}(\mathbb{C}W) \rightarrow \mathbb{C}[[\hbar]] \otimes \mathbb{Z}(\mathbb{C}W)$  the map defined by

$$\text{ch}_c([E]) = \text{ch}_c(E).$$

Now, let  $W'$  be a parabolic subgroup of  $W$  and set  $V' = V^{W'}$  and  $r = \text{codim}_{\mathbb{C}}(V')$ . We identify the dual  $V'^*$  of  $V'$  with  $V^{*W'}$  and note that

$$(4.2) \quad V = V' \oplus (V'^*)^\perp.$$

We denote by  $\wedge(V'^*)^\perp$  the element of  $\mathbb{K}_{\mathbb{C}^\times}(\mathbb{C}W')$  defined by

$$\wedge(V'^*)^\perp = \sum_{i \geq 0} (-1)^i [\wedge^i (V'^*)^\perp].$$

Recall also that there exists  $n$  algebraically independent homogeneous polynomials  $f_1, \dots, f_n$  in  $\mathbb{C}[V]^W$  such that  $\mathbb{C}[V]^W = \mathbb{C}[f_1, \dots, f_n]$ , and we denote by  $d_i$  the degree of  $f_i$ .

**Corollary 4.3.** — *Assume that  $\mathcal{Z}_c$  is smooth. Let  $E'$  be a finite dimensional graded  $\mathbb{C}W'$ -module. Then*

$$\text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes [E'])) = \frac{(1-q^{d_1}) \cdots (1-q^{d_n})}{(1-q)^{n-r}} \sum_{\chi \in \text{Irr}(W)} q^{b_\chi} \frac{\langle \chi, \text{Ind}_{W'}^W(E') \rangle_W^{\text{gr}}}{\langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \rangle_W^{\text{gr}}} e_\chi^W.$$

*Proof.* — The group  $W'$  acts trivially on  $V'$  so it acts trivially on  $\wedge^i V'$  for all  $i$ . Therefore,

$$(1-q)^{n-r} \text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes [E'])) = \text{ch}_c(\text{ind}_{W'}^W(\wedge V' \otimes \wedge(V'^*)^\perp \otimes [E'])).$$

But  $\wedge V' \otimes \wedge(V'^*)^\perp = \text{Res}_{W'}^W(\wedge V)$  by (4.2), so, by Frobenius formula,

$$(1-q)^{n-r} \text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes [E'])) = \text{ch}_c(\wedge V \otimes \text{Ind}_{W'}^W(E')).$$

So it follows from Proposition 4.1 that

$$\begin{aligned} (1-q)^{n-r} \text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes [E'])) \\ = \sum_{\chi \in \text{Irr}(W)} \frac{q^{b_\chi}}{\langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \rangle_W^{\text{gr}}} \langle \chi, \mathbb{C}[V^*]^{\text{co}(W)} \otimes (\wedge V) \otimes \text{Ind}_{W'}^W(E') \rangle_W^{\text{gr}} e_\chi^W. \end{aligned}$$

But, if  $w \in W$ , the Molien's formula implies that the graded trace of  $w$  on  $\mathbb{C}[V^*]^{\text{co}(W)}$  is equal to

$$\frac{(1-q^{d_1}) \cdots (1-q^{d_n})}{\det(1-wq)},$$

while its graded trace on  $\wedge V$  is equal to  $\det(1-wq)$ . So the class of  $\mathbb{C}[V^*]^{\text{co}(W)} \otimes \wedge V$  is equal to  $(1-q^{d_1}) \cdots (1-q^{d_n})$  times the class of the trivial module, and the corollary follows.  $\square$

*Proof of Theorem A.* — Assume that  $\mathcal{Z}_c$  is smooth. This implies in particular that Conjecture 1.5(1) and (2) hold (Etingof-Ginzburg Theorem 1.3), and so the hypotheses of Proposition 3.5 are satisfied. Note also that  $\text{Im}(\Omega^c) = Z(\mathbb{C}W)$  by (1.2). It is then sufficient to prove that  $\hbar^i \mathcal{F}_i(Z(\mathbb{C}W)) \subset \text{Im}(i_c^*)$  for all  $i$ .

Let us introduce some notation. If  $G$  is a finite group and  $H$  is a subgroup, we define a  $\mathbb{C}$ -linear map  $\text{Tr}_H^G : Z(\mathbb{C}H) \rightarrow Z(\mathbb{C}G)$  by

$$\text{Tr}_H^G(z) = \sum_{g \in [G/H]} \mathfrak{s}_z = \frac{1}{|H|} \sum_{g \in G} \mathfrak{s}_z$$

(here,  $[G/H]$  denotes a set of representatives of elements of  $G/H$  and  $\mathfrak{s}_z = gzg^{-1}$ ). It is easily checked that

$$(4.4) \quad \text{Tr}_H^G(e_\eta^H) = \frac{\eta(1)}{|H|} \sum_{\gamma \in \text{Irr}(G)} \frac{|G|}{\gamma(1)} \langle \gamma, \text{Ind}_H^G(\eta) \rangle_G e_\gamma^G$$

for all  $\eta \in \text{Irr}(H)$ . Also, if  $h \in H$  and  $\Sigma_H(h) \in Z(\mathbb{C}H)$  denotes the sum of the conjugates of  $h$  in  $H$ , then

$$(4.5) \quad \text{Tr}_H^G(\Sigma_H(h)) = \frac{|C_G(h)|}{|C_H(h)|} \Sigma_G(h).$$

Let  $\mathcal{P}_r(W)$  denote the set of parabolic subgroups  $W'$  of  $W$  such that  $\text{codim}_{\mathbb{C}}(V^{W'}) = r$ . It follows from (4.5) that

$$(4.6) \quad \mathcal{F}_r(Z(\mathbb{C}W)) = \sum_{W' \in \mathcal{P}_r(W)} \text{Tr}_{W'}^W(Z(\mathbb{C}W')).$$

Therefore, by Proposition 3.5, it is sufficient to prove that

$$(\star) \quad \hbar^r \text{Tr}_{W'}^W(e_{\chi'}^{W'}) \in \text{Im}(i_c^*) \text{ for all } W' \in \mathcal{P}_r(W) \text{ and all } \chi' \in \text{Irr}(W').$$

But it turns out that the coefficient of  $\hbar^r$  in

$$\frac{\chi'(1)}{|W'|} \text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes \chi'))$$

is equal, according to Corollary 4.3, to

$$\frac{\chi'(1)}{|W'|} \sum_{\chi \in \text{Irr}(W)} \frac{|W| \langle \chi, \text{Ind}_{W'}^W \chi' \rangle_W}{\chi(1)} e_\chi^W$$

(since  $d_1 \cdots d_n = |W|$ ). Note that by definition  $\frac{\chi'(1)}{|W'|} \text{ch}_c(\text{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes \chi'))$  belongs to  $\mathbb{C}[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} \text{Im}(i_c^*)$ , so each of its homogeneous components belong to  $\text{Im}(i_c^*)$ . By (4.4), this proves that  $(\star)$  holds, and the proof of Theorem A is complete.  $\square$

## 5. Proof of Theorem B

**Hypothesis and notation.** *We assume in this section, and only in this section, that the symplectic quotient singularity  $\mathcal{X}_0 = (V \times V^*)/W$  admits a symplectic resolution  $\mathcal{X} \rightarrow \mathcal{X}_0$ .*

Recall [11, Theorem 1.3(ii)] that the  $\mathbb{C}^\times$ -action on  $\mathcal{X}_0$  lifts uniquely to  $\mathcal{X}$ . As it is explained in Remark 1.4, the existence of a symplectic resolution of  $\mathcal{X}_0$  implies that all the irreducible components of  $W$  are of type  $G(d, 1, n)$  or  $G_4$ . Since the proof of Theorem B can be easily reduced to the irreducible case, we will separate the proof in two cases.

*Proof of Theorem B for  $W = G(d, 1, n)$ .* — Assume here that  $W = G(d, 1, n)$ . Let  $\mathbf{S}^1$  be the group of complex numbers of modulus 1. In this case, it follows from [7] that  $\mathcal{X}$  is diffeomorphic to some smooth  $\mathcal{X}_c$ , and that the diffeomorphism might be chosen to be  $\mathbf{S}^1$ -equivariant. As the  $\mathbf{S}^1$ -equivariant cohomology is canonically isomorphic to the  $\mathbb{C}^\times$ -equivariant cohomology, this proves Theorem B in this case.  $\square$

*Proof of Theorem B for  $W = G_4$ .* — Assume here that  $W = G_4$ . It is possible (probable?) that again  $\mathcal{X}$  is  $\mathbf{S}^1$ -diffeomorphic to some smooth  $\mathcal{X}_c$ , but we are unable to prove it (it is only known that they are diffeomorphic). So we will prove Theorem B in this case by brute force computations.

We fix a primitive third root of unity  $\zeta$  and we assume that  $V = \mathbb{C}^2$  and that  $W = \langle s, t \rangle$ , where

$$s = \begin{pmatrix} \zeta & 0 \\ \zeta^2 & 1 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & -\zeta^2 \\ 0 & \zeta \end{pmatrix}.$$

By the work of Bellamy [2], there are only two symplectic resolutions of  $\mathcal{X}_0 = (V \times V^*)/W$ . They have both been constructed by Lehn and Sorger [12]: one can be obtained from the other by exchanging the role of  $V$  and  $V^*$ , so we will only prove Theorem B for one of them. Let us describe it.

Let  $H = V^s$  and let  $\mathcal{H}$  denote the image of  $H \times V^*$  in  $\mathcal{X}_0$ , with its reduced structure of closed subvariety. We denote by  $\beta : \mathcal{Y} \rightarrow \mathcal{X}_0$  the blowing-up of  $\mathcal{X}_0$  along  $\mathcal{H}$  and we denote by  $\alpha : \mathcal{X} \rightarrow \mathcal{Y}$  the blowing-up of  $\mathcal{Y}$  along its reduced singular locus  $\mathcal{S}$ . Then [12]

$$\pi = \beta \circ \alpha : \mathcal{X} \longrightarrow \mathcal{X}_0$$

is a symplectic resolution.

We now give more details, which all can be found in [12, §1]. First,  $\mathbb{C}[\mathcal{X}_0]$  is generated by 8 homogeneous elements  $(z_i)_{1 \leq i \leq 8}$  whose degrees are given by the following table:

$z$	$z_1$	$z_2$	$z_3$	$z_4$	$z_5$	$z_6$	$z_7$	$z_8$
$\deg(z)$	0	4	-4	2	-2	-6	6	0

The defining ideal of  $\mathcal{H}$  in  $\mathbb{C}[\mathcal{X}_0]$  is generated by 6 homogeneous elements  $(b_j)_{1 \leq j \leq 6}$  whose degrees are given by the following table:

$b$	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$
$\deg(b)$	2	6	0	12	8	4

This defines a  $\mathbb{C}^\times$ -equivariant closed immersion  $\mathcal{Y} \hookrightarrow \mathcal{X}_0 \times \mathbf{P}^5(\mathbb{C})$ . We denote by  $\mathcal{Y}_i$  the affine chart defined by “ $b_i \neq 0$ ”. The equations of the zero fiber  $\beta^{-1}(0)$  given in [12, §1] show that  $\mathcal{Y}^{\mathbb{C}^\times} = \{p_2, p_3, p_4, p_6\}$ , where  $p_i$  is the unique element of  $\mathcal{Y}_i^{\mathbb{C}^\times}$ . We use the notation of Example 2.5. By (2.6) and (5.2), we have

$$(\clubsuit) \quad i_{\mathcal{Y}}^*(\mathrm{ch}_{\mathcal{Y}}^1([\beta^* \mathcal{H}])) = -\hbar(6\mathbf{e}_{p_2} + 12\mathbf{e}_{p_4} + 4\mathbf{e}_{p_6}).$$

Now,  $\mathcal{S}$  is contained in  $\mathcal{Y}_2 \cup \mathcal{Y}_3$ , so  $\alpha$  is an isomorphism in a neighborhood of  $p_4$  and  $p_6$ . So let  $q_4 = \alpha^{-1}(p_4)$  and  $q_6 = \alpha^{-1}(p_6)$ . These are elements of  $\mathcal{X}^{\mathbb{C}^\times}$ .

On the other hand,  $\mathcal{Y}_2$  is a transversal  $A_1$ -singularity, so the defining ideal of  $\mathcal{S} \cap \mathcal{Y}_2$  in  $\mathbb{C}[\mathcal{Y}_2]$  is generated by three homogeneous elements  $a_+$ ,  $a_0$  and  $a_-$  (of degree 6, 0 and  $-6$ , by [12, §1]), and it is easily checked that  $\alpha^{-1}(p_2)^{\mathbb{C}^\times} = \{q_2^+, q_2^-\}$  where  $q_2^\pm$  is the unique  $\mathbb{C}^\times$ -fixed element in the affine chart defined by “ $a_\pm \neq 0$ ”.

Also,  $\mathcal{Y}_3$  is isomorphic to  $(\mathfrak{h} \times \mathfrak{h}^*)/\mathfrak{S}_3$ , where  $\mathfrak{h}$  is the diagonal Cartan subalgebra of  $\mathfrak{sl}_3(\mathbb{C})$  and  $\mathfrak{S}_3$  is the symmetric group on 3 letters, viewed as the Weyl group of  $\mathfrak{sl}_3(\mathbb{C})$ . Let  $\mathbf{Hilb}_3(\mathbb{C}^2)$  denote the Hilbert scheme of 3 points in  $\mathbb{C}^2$ , and let  $\mathbf{Hilb}_3^0(\mathbb{C}^2)$  denote the (reduced) closed subscheme defined as the Hilbert scheme of three points in  $\mathbb{C}^2$  whose sum is equal to  $(0, 0)$ . By [10, Proposition 2.6],

$$(5.3) \quad \mathcal{X}_3 \simeq \mathbf{Hilb}_3^0(\mathbb{C}^2).$$

It just might be noticed that the isomorphism  $\mathcal{Y}_3 \simeq (\mathfrak{h} \times \mathfrak{h}^*)/\mathfrak{S}_3$  becomes  $\mathbb{C}^\times$ -equivariant if one “doubles the degrees” in  $(\mathfrak{h} \times \mathfrak{h}^*)/\mathfrak{S}_3$ , that is, if  $\mathbb{C}^\times$  acts on  $\mathfrak{h}$  (respectively  $\mathfrak{h}^*$ ) with weight 2 (respectively  $-2$ ). Recall that  $\mathbb{C}^\times$ -fixed points in  $\mathbf{Hilb}_3^0(\mathbb{C}^2)$  are parametrized by partitions of 3: we denote by  $q_3^+$ ,  $q_3^0$  and  $q_3^-$  the fixed points in  $\mathcal{X}_3$  corresponding respectively to the partitions  $(3)$ ,  $(2, 1)$  and  $(1, 1, 1)$  of 3, so that

$$(5.4) \quad \mathcal{X}_3^{\mathbb{C}^\times} = \{q_3^+, q_3^0, q_3^-\}.$$

Finally, we have

$$(5.5) \quad \mathcal{X}^{\mathbb{C}^\times} = \{q_2^+, q_2^-, q_3^+, q_3^0, q_3^-, q_4, q_6\}.$$

Also,  $\pi^*(\mathcal{H}) = \alpha^*(\beta^*(\mathcal{H}))$  is an effective Cartier divisor of  $\mathcal{X}$ , and it follows from  $(\clubsuit)$  and the commutativity of the diagram 2.3 that

$$(\diamond) \quad i_{\mathcal{X}}^*(\mathrm{ch}_{\mathcal{X}}^1([\pi^* \mathcal{H}])) = -\hbar(6\mathbf{e}_{q_2^+} + 6\mathbf{e}_{q_2^-} + 12\mathbf{e}_{q_4} + 4\mathbf{e}_{q_6}).$$

We now wish to compute  $i_{\mathcal{X}}^*(\mathrm{ch}_{\mathcal{X}}^1([\alpha^* \mathcal{S}]))$ . As the singular locus  $\mathcal{S}$  is contained in  $\mathcal{Y}_2 \cup \mathcal{Y}_3$ , and contains  $p_2$  and  $p_3$ , there exists  $n_2^+$ ,  $n_2^-$ ,  $n_3^+$ ,  $n_3^0$  and  $n_3^-$  in  $\mathbb{Z}$  such that

$$i_{\mathcal{X}}^*(\mathrm{ch}_{\mathcal{X}}^1([\alpha^* \mathcal{S}])) = -\hbar(n_2^+ \mathbf{e}_{q_2^+} + n_2^- \mathbf{e}_{q_2^-} + n_3^+ \mathbf{e}_{q_3^+} + n_3^0 \mathbf{e}_{q_3^0} + n_3^- \mathbf{e}_{q_3^-}).$$

Since  $a_+$  and  $a_-$  have degree 6 and  $-6$ , it follows from (2.6) that  $n_2^+ = 6$  and  $n_2^- = -6$ .

As we can exchange the roles of  $\mathfrak{h}$  and  $\mathfrak{h}^*$  in the description of  $\mathcal{Y}_3$  and its singular locus (because  $\mathfrak{h} \simeq \mathfrak{h}^*$  as an  $\mathfrak{S}_3$ -module), this shows that  $n_3^- = -n_3^+$  and  $n_3^0 = -n_3^0$ . So  $n_3^0 = 0$  and it remains to compute  $n_3^+$ . So let  $U_+$  be the open subset of  $\mathbf{Hilb}_3^0(\mathbb{C}^2)$  consisting of ideals  $J$  of codimension 3 of  $\mathbb{C}[x, y]$  such that the classes of 1,  $x$  and  $x^2$  form a basis of  $\mathbb{C}[x, y]/J$ . Then we have an isomorphism  $J_+ : \mathbb{C}^4 \xrightarrow{\sim} U_+$  given by

$$(a, b, c, d) \longmapsto J_+(a, b, c, d) = \langle x^3 + ax + b, y - cx^2 - dx - \frac{2}{3}ac \rangle.$$

The form of the generators of the ideal  $J_+(a, b, c, d)$  is here to ensure that  $J_+(a, b, c, d) \in \mathbf{Hilb}_3^0(\mathbb{C}^2)$ . The fixed point  $q_3^+$  is the unique one in  $U_+$ . Through this identification, the action of  $\mathbb{C}^\times$  on  $\mathbb{C}^4$  is given by

$$\xi \cdot (a, b, c, d) = (\xi^4 a, \xi^6 b, \xi^{-6} c, \xi^{-4} d)$$

(remember that we must “double the degrees” of the usual action). The equation of  $\alpha^*(\mathcal{S})$  on this affine chart  $\simeq \mathbb{C}^4$  can then be computed explicitly and is given by  $4a^3 + 27b^2 = 0$ , so is of degree 12. This shows that  $n_3^+ = 12$ . Finally,

$$(\heartsuit) \quad i_{\mathcal{X}}^*(\mathrm{ch}_{\mathcal{X}}^1([\alpha^* \mathcal{S}])) = -\hbar(6\mathbf{e}_{q_2^+} - 6\mathbf{e}_{q_2^-} + 12\mathbf{e}_{q_3^+} - 12\mathbf{e}_{q_3^-}).$$

Let us now conclude. First, recall from [5, Theorem 1.2] that

$$(5.6) \quad \begin{cases} \mathrm{H}^{2i+1}(\mathcal{X}) = 0 & \text{for all } i, \\ \mathrm{H}^{2\bullet}(\mathcal{X}) \simeq \mathrm{gr}_{\mathcal{F}}(Z(\mathbb{C}W)). \end{cases}$$

By Proposition 2.4, we get

$$(\spadesuit) \quad \dim_{\mathbb{C}} \mathrm{H}_{\mathbb{C}^\times}^{2i}(\mathcal{X}) = \begin{cases} 1 & \text{if } i = 0, \\ 3 & \text{if } i = 1, \\ 7 & \text{if } i \geq 2. \end{cases}$$

Now,  $W$  has seven irreducible characters  $1, \varepsilon, \varepsilon^2, \chi, \chi\varepsilon, \chi\varepsilon^2$  and  $\theta$ , where  $\chi$  is the unique irreducible character of degree 2 with rational values and  $\theta$  is the unique one of degree 3. We denote by

$$\Psi : \mathrm{H}_{\mathbb{C}^\times}^{2\bullet}(\mathcal{X}^{\mathbb{C}^\times}) \xrightarrow{\sim} \mathbb{C}[[\hbar]] \otimes Z(\mathbb{C}W)$$

the isomorphism of  $\mathbb{C}[[\hbar]]$ -algebras such that

$$\Psi(\mathbf{e}_{q_4}) = e_1^W, \quad \Psi(\mathbf{e}_{q_6}) = e_\theta^W, \quad \Psi(\mathbf{e}_{q_2^+}) = e_{\chi\varepsilon}^W, \quad \Psi(\mathbf{e}_{q_2^-}) = e_{\chi\varepsilon^2}^W,$$

$$\Psi(\mathbf{e}_{q_3^+}) = e_{\varepsilon^2}^W, \quad \Psi(\mathbf{e}_{q_3^-}) = e_\chi^W \quad \text{and} \quad \Psi(\mathbf{e}_{q_3^-}) = e_\varepsilon^W.$$

By Proposition 2.4,  $\mathrm{H}_{\mathbb{C}^\times}^{2\bullet}(\mathcal{X})$  is isomorphic to its image by  $\Psi \circ i_{\mathcal{X}}^*$  in  $\mathbb{C}[[\hbar]] \otimes Z(\mathbb{C}W)$ . But, by  $(\diamond)$  and  $(\heartsuit)$ , and after investigation of the character table of  $W$ , this image contains

$$\hbar(6\mathbf{e}_{\chi\varepsilon}^W - 6\mathbf{e}_{\chi\varepsilon^2}^W + 12\mathbf{e}_1^W + 4\mathbf{e}_\theta^W) = \hbar(4 + \Sigma_W(s) + \Sigma_W(s^2))$$

$$\text{and} \quad \hbar(12\mathbf{e}_{\varepsilon^2}^W - 12\mathbf{e}_\varepsilon^W + 6\mathbf{e}_{\chi\varepsilon} - 6\mathbf{e}_{\chi\varepsilon^2}) = \hbar((1 + 2\zeta)\Sigma_W(s) + (1 + 2\zeta^2)\Sigma_W(s^2)).$$

So it contains  $\hbar\Sigma_W(s)$  and  $\hbar\Sigma_W(s^2)$ . Also, by  $(\spadesuit)$  and Proposition 2.4, it also contains  $\hbar^2 Z(\mathbb{C}W)$ , so

$$\mathrm{Rees}_{\mathcal{F}}^\bullet(Z(\mathbb{C}W)) \subset \mathrm{Im}(\Psi \circ i_{\mathcal{X}}^*).$$

Using again  $(\spadesuit)$  and Proposition 2.4, a comparison of dimensions yields that

$$\mathrm{Rees}_{\mathcal{F}}^\bullet(Z(\mathbb{C}W)) = \mathrm{Im}(\Psi \circ i_{\mathcal{X}}^*),$$

and the proof is complete.  $\square$

## References

- [1] G. BELLAMY, *On singular Calogero-Moser spaces*, Bull. LMS **41**, 315-326.
- [2] G. BELLAMY, *Counting resolutions of symplectic quotient singularities*, Compos. Math. **152** (2016), 99-114.
- [3] C. BONNAFÉ & R. ROUQUIER, *Cherednik algebras and Calogero-Moser cells*, preprint (2017), arXiv:1708.09764.
- [4] P. ETINGOF & V. GINZBURG. *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Inventiones Mathematicae **147**, no. 2 (2002), 243-348.
- [5] V. GINZBURG & D. KALEDIN, *Poisson deformations of symplectic quotient singularities*, Adv. in Math. **186** (2004), 1-57.
- [6] I. GORDON, *Baby Verma modules for rational Cherednik algebras*, Bull. London Math. Soc. **35** (2003), 321-336.
- [7] I. GORDON, *Quiver varieties, category  $\mathcal{O}$  for rational Cherednik algebras, and Hecke algebras*, Int. Math. Res. Papers **2008** (2008), Art. ID rpn006, 69 pages.
- [8] I. GORDON & M. MARTINO, *Calogero-Moser space, restricted rational Cherednik algebras, and two-sided cells*, Math. Res. Lett. **16** (2009), 255-262.
- [9] M. GORESKY & R. MACPHERSON, *On the spectrum of the equivariant cohomology ring*, Canadian Journal of Mathematics **62**, no. 2 (2010), 262-283.
- [10] M. HAIMAN,  *$t, q$ -Catalan numbers and the Hilbert scheme*, Discrete Math. **193** (1998), 201-224. Selected papers in honor of Adriano Garsia (Taormina, 1994).
- [11] D. KALEDIN, *On crepant resolutions of symplectic quotient singularities*, Selecta Math. **9** (2003), 529-555.
- [12] M. LEHN & C. SORGER, *A symplectic resolution for the binary tetrahedral group*, Geometric methods in representation theory II, 429-435, Sémin. Congr., 24-II, Soc. Math. France, Paris, 2012.
- [13] G. BELLAMY, T. SCHEDLER & U. THIEL, *Hyperplane arrangements associated to symplectic quotient singularities*, preprint (2017), arXiv:1702.04881.
- [14] É. VASSEROT, *Sur l'anneau de cohomologie du schéma de Hilbert de  $\mathbb{C}^2$* , C. R. Acad. Sci. **332** (2001), 7-12.

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